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CONFOCAL CONICS IN SPACE-TIME

GARRETT BIRKHOFF AND ROBERT MORRIS, Harvard University

The nature of confocal conics in the Cartesian plane is very familiar; not so the analogous concepts in relativistic space-time. The relativistic analogs of confocal conics will be derived below.

In one-dimensional special relativity, the *time* interval T between two events E: (t, x) and E': (t', x') is

$$T(E, E') = \sqrt{\{(t - t')^2 - (x - x')^2\}}. \tag{1}$$

This differs from the Cartesian distance formula only in the second minus sign. The *distance* interval or *space* D between E and E' is, dually,

$$D(E, E') = \sqrt{\{(x - x')^2 - (t - t')^2\}}. \tag{1'}$$

If both D and T are real, then $D = T = 0$.

From the preceding formulas it follows immediately that for any fixed *center* F: (t_0, x_0) and *radius* a, the relativistic *time*-like circle $T(F, E) = a$ is the *geometrical hyperbola* $(t-t_0)^2-(x-x_0)^2=a^2>0$ with time-like major axis, the same center F, and *null lines* (paths of light rays) $t-t_0=\pm(x-x_0)$ for asymptotes. Each such pair of null lines is a degenerate "circle" of radius zero. Dually, the relativistic *space*-like circle $D(F, E)=a$ is a geometrical hyperbola with space-like major axis.

Now consider two fixed foci F: $(c, 0)$ and F': $(-c, 0)$ in (t, x)-space; if the segment F, F' is time-like, so that $T(F, F')=2c$ is real, one can always choose axes so as to give F and F' such coordinates. For fixed positive a, consider the relativistic *time*-like ellipse, as the locus of all points E: (t, x) which satisfy

$$T(F, E) + T(F', E) = 2a, \tag{2a}$$

and the relativistic *time*-like hyperbola, whose branches are defined by

$$T(F, E) - T(F', E) = 2a, \tag{2b}$$

$$T(F', E) - T(F, E) = 2a. \tag{2c}$$

The analogous equations in Cartesian space define ellipses and the two branches of a hyperbola, respectively.

To treat the present case, we introduce the definitions

$$u = T(F, E)^2 = (t - c)^2 - x^2, \qquad v = T(F', E)^2 = (t + c)^2 - x^2. \tag{3}$$

Define the algebraic plane curve C as the locus where the (polynomial) product

$$(2a + \sqrt{u} + \sqrt{v})(2a + \sqrt{u} - \sqrt{v})(2a - \sqrt{u} + \sqrt{v})(2a - \sqrt{u} - \sqrt{v}) = 0 \tag{4}$$

of the three conjugate equations (2) with their fourth algebraic conjugate vanishes. Since we have assumed $a>0$ and since $T(E, E')$ refers to the positive square root in (1), the locus C defined by (4) is the *set-union* of the three loci of (2).

Multiplying out the expression (4), we get

$$(4a^2 + 4a\sqrt{u} + (u - v))(4a^2 - 4a\sqrt{u} + (u - v)) = 0$$

which, since $u-v=-4ct$, reduces to $16\{(a^2-ct)^2-a^2u\}=0$. Now substituting for u from (3) and simplifying, we get

$$a^2(a^2 - c^2) = (a^2 - c^2)t^2 - a^2x^2. \tag{5}$$

That is, the equation for C is

$$\frac{t^2}{a^2} - \frac{x^2}{a^2 - c^2} = 1, \qquad a > 0, c > 0, a \neq c. \tag{5*}$$

This is identical with the formula for confocal conics in Cartesian space, except for the reversal of one sign. This proves

THEOREM 1. *Each geometrical conic* (5*) *is the union of pieces consisting of confocal* ***relativistic*** *conics* (2a), (2b), (2c).

If $0<a<c$, then the *relativistic* conic (5*) is a *geometrical ellipse*

$$\frac{t^2}{a^2} + \frac{x^2}{c^2 - a^2} = 1, \qquad 0 < a < c. \tag{6a}$$

However, the geometrical foci of E differ from the relativistic foci F, F': $(\pm c, 0)$. If $c^2<2a^2$, then the ellipse is prolate; the geometrical foci are at $(\pm\sqrt{(2a^2-c^2)}, 0)$. In the oblate case $c^2>2a^2$, the geometrical foci are not even on the time axis, but are at the points $(0, \pm\sqrt{(c^2-2a^2)})$. If $c^2=2a^2$, the relativistic conic (5*) is a geometric circle.

If $0<c<a$, then the space-time conic (5*) is a *geometrical hyperbola*

$$\frac{t^2}{a^2} - \frac{x^2}{a^2 - c^2} = 1, \qquad 0 < c < a. \tag{6b}$$

Since $a^2>a^2-c^2$, this geometrical hyperbola is always acute: not every geometrical hyperbola is a space-time conic. The relativistic foci $(\pm c, 0)$ again fail to coincide with the geometric foci, which are* at $(\pm\sqrt{(2a^2-c^2)}, 0)$.

It is well known† that the usual triangle inequality is replaced in space-time by its opposite:

$$T(A, B) + T(B, C) \leqq T(A, C) \qquad \text{(Minkowski inequality)} \tag{7}$$

if $T(A, B)<T(A, C)$ and $T(B, C)<T(A, C)$. As a result, various inequalities on a and c are reversed, in passing from geometrical to relativistic conics. Thus,

* W. F. Osgood and W. C. Graustein, Plane and Solid Analytic Geometry, New York, 1935, Ch. VIII.

† Alfred Schild, The clock paradox in relativity theory, this MONTHLY, vol. 66, 1959, pp. 1–18. The inequality (7) asserts that the longest side of a triangle is greater than the sum of the other two sides.

the geometric ellipse (6a) lies *between* its relativistic foci, while the geometrical hyperbola (6b) lies outside its relativistic foci.

THEOREM 2. *The relativistic conics* (5*) *are geometrically tangent to the null lines through the foci F and F'.*

Proof. First consider the geometric ellipses (6a), with slope satisfying

$$\lambda = \frac{dx}{dt}\,\frac{(a^2 - c^2)t}{a^2 x}. \tag{8}$$

The points $(t, c-t)$ on the null line $x+t=c$ through F satisfy (6a) with $a^2=ct$; hence the slope $\lambda = \{(ct-c^2)t\}/\{ct(c-t)\} = -1$ of the ellipse (6a) equals that of the null line at their point of contact. Tangency with the other null lines through the foci follows by symmetry in the axes. The case of the geometric hyperbolas (6b) follows similarly.

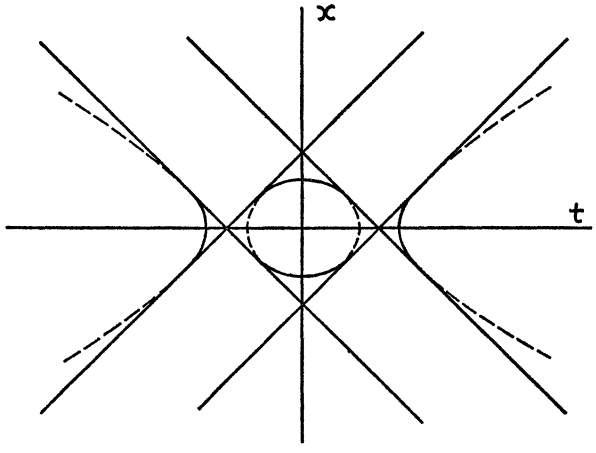


FIG. 1

COROLLARY 1. *The relativistic foci of any conic are located at the intersections of the tangent null lines.*

COROLLARY 2. *Two time-like conics are* **confocal** *if and only if their tangent null lines coincide.*

Surprisingly, one can pass from a (relativistic) time-like ellipse (2a) to a time-like hyperbola by analytic continuation. More precisely, the following is true.

THEOREM 3. *On the geometrical ellipses* (6a), *the segments where the slope is less than* 45° *are relativistic time-like ellipses; those having a slope of more than* 45° *are time-like hyperbolas. On the geometrical hyperbolas* (6b), *the reverse is true.*

The preceding facts are depicted graphically in Figure 1, in which the solid lines represent time-like ellipses and the dotted lines represent time-like hyperbolas.

The proof is by continuity. From Theorem 2, it is apparent that the loci $d(\sqrt{u}+\sqrt{v})=0$ and $d(\sqrt{u}-\sqrt{v})=0$ have different slopes except where $du/\sqrt{u}=0$ or $dv/\sqrt{v}=0$—*i.e.*, on the null lines through F and F'. This is where the slope changes from less than 45° to more than 45° on any conic.

On the other hand, the segment $\overline{FF'}$ is a time-like ellipse, and its continuations beyond F and F' are hyperbolas, as in the geometric case. Whereas the x-axis is a time-like hyperbola. Interchanging x and t, we have the

COROLLARY. *On the geometrical ellipses* (6a), *the segments corresponding to time-like ellipses are relativistic space-like hyperbolas, with* ***dual foci*** $(0, \pm c)$. *Likewise, the segments corresponding to time-like hyperbolas are relativistic space-like ellipses, also with foci at* $(0, \pm c)$.

On the other hand, the geometrical hyperbolas (6b) do not constitute relativistic space-like conics, because their asymptotes have slopes less than one—and (rotating through 90°) the geometrical asymptotes of relativistic space-like conics have slopes greater than one.

From the facts proved above (especially Theorem 2), and known properties* of solutions of Euler's quadratic differential equation

$$(1 - x^2)y'^2 = (1 - y^2), \tag{9}$$

we obtain the following result.

COROLLARY 3. *Rotation through* 45° *carries the solutions of Euler's quadratic DE* (9) *into confocal relativistic conics, whose foci are at the intersections of the singular solutions. The singular solutions are null lines* (*paths of light rays*).

Which shows what good taste Euler had in choosing significant problems!

* G. Birkhoff and G. C. Rota, Ordinary Differential Equations, Ch. I, Ex. D6.

PREFERENTIAL ARRANGEMENTS

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This paper is concerned with the total number of distinct rational preferential arrangements available to a person faced with n distinguishable decisions or objects, allowing indifference; *i.e.*, each object is to be placed in exactly one indifference set with the sets themselves totally ranked in order of strict transitive preference. We shall denote this total number by f_n.

In [1] (Proposition XXII, p. 87), Whitworth derives the following formula:

The number of ways in which n different things can be distributed into r different parcels (without blank lots) is n! times the coefficient of x^n in the expansion of $(e^x-1)^r$. Denoting this number by $A(n, r)$ and d^n/dx^n by D^n, we have

$$A(n, r) = D^n(e^x - 1)^r\big|_{x=0}. \tag{1}$$

One then obtains the total number of preferential arrangements by summing the above expression on r. Thus,

$$f_n = \sum_{r=1}^{n} A(n, r) = \sum_{r=1}^{n} D^n(e^x - 1)^r\Big|_{x=0}.$$

Note that $A(n, r)$ vanishes for $r=0$ and for $r>n$, and hence that we may equally validly sum from 0 to ∞; *i.e.*,

$$f_n = \sum_{r=0}^{\infty} D^n(e^x - 1)^r\Big|_{x=0} = D^n \sum_{r=0}^{\infty} (e^x - 1)^r\Big|_{x=0} \quad \text{(permuting operators)}, \tag{2}$$

$$f_n = D^n(2 - e^x)^{-1}\Big|_{x=0} \quad \text{(summing a geometric series)}$$

$$= \tfrac{1}{2}D^n(1 - \tfrac{1}{2}e^x)^{-1}\Big|_{x=0}$$

$$= \tfrac{1}{2}D^n \sum_{v=0}^{\infty} (\tfrac{1}{2}e^x)^v\Big|_{x=0} = \tfrac{1}{2} \sum_{v=0}^{\infty} D^n(\tfrac{1}{2}e^x)^v\Big|_{x=0}; \tag{3}$$

$$f_n = \tfrac{1}{2} \sum_{v=0}^{\infty} v^n 2^{-v}. \tag{4}$$

Note that this last formula can be used to define f_n for nonintegral n. (All of the preceding steps can easily be made rigorous.)

Equation (4) suggests an asymptotic formula for f_n; for, if we replace summation by integration, we obtain

$$g_n = \frac{1}{2}\int_0^{\infty} x^n 2^{-x} dx = \frac{\Gamma(n+1)}{2\ln^{n+1} 2} = \frac{n!}{2\ln^{n+1} 2}. \tag{5}$$

The writer has found, on computing the first nine values of g_n, that f_n is the nearest integer to g_n for $n=1, \cdots, 9$. One might be tempted to conjecture that this is true for every positive integer n. That such a conjecture is false can be shown by elementary analysis. For suppose the conjecture were true. This would require that $|f_n - g_n| < \frac{1}{2}$ for all positive integral n. We shall show, however, that $\{|f_n - g_n|\}$ is actually unbounded.

We readily obtain from (5):

$$g_n = \tfrac{1}{2}D^n(\ln 2 - z)^{-1}\big|_{z=0}. \tag{6}$$

From (2) and (6) we obtain

$$(7) \qquad f_n - g_n = \phi^{(n)}(0), \quad \text{where} \quad \phi(z) = \frac{1}{2 - e^z} - \frac{1}{2(\ln 2 - z)}.$$

We remark that ϕ has a simple pole at $z = \ln 2 + 2\pi i$. Now suppose $|f_n - g_n| \leqq B$ for all n. Since ϕ is regular in a neighborhood of the origin, it has a representative Taylor series given by

$$\phi(z) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} \frac{f_k - g_k}{k!} z^k.$$

Whence, by virtue of our supposed bound,

$$|\phi(z)| \leqq \sum_{k=0}^{\infty} \frac{|f_k - g_k|}{k!} |z|^k \leqq Be^{|z|},$$

thus contradicting the existence of the singularity, and the conjecture is disproved.

Nevertheless, (5) is a rather good asymptotic expression for f_n. To see this, we need merely note that the residues of the two terms in ϕ at $z = \ln 2$ cancel. In fact, $\lim_{z \to \ln 2} \phi(z) = \frac{1}{4}$, as an easy calculation verifies. The singularities nearest the origin are poles at $\ln 2 \pm 2\pi i$. Thus ϕ is regular within any circle about the origin of radius less than $\sqrt{(\ln^2 2 + 4\pi^2)}$. Choose R in the open interval $(\ln 2, \sqrt{(\ln^2 2 + 4\pi^2)})$. Then ϕ has, by continuity, a maximum modulus M, say, on the circle $|z| = R$. Applying Cauchy's inequality, we have $|\phi^{(n)}(0)| \leqq (Mn!)/R^n$, *i.e.*, by (7), $|f_n - g_n| \leqq (Mn!)/R^n$. Dividing by $|g_n|$ and noting (5), we obtain $|(f_n/g_n) - 1| \leqq 2(\ln 2)M\{(\ln 2)/R\}^n$. But, since $\ln 2 < R$, this last inequality asserts that the ratio of f_n to g_n converges geometrically to unity as $n \to \infty$, and, in particular, that $g_n \sim f_n$.

There is another interesting formula that is useful in proving certain number theoretic properties of f_n. By the binomial theorem,

$$\begin{aligned} D^n(e^x - 1)^r\Big|_{x=0} &= D^n \sum_{j=0}^{r} \binom{r}{j} (-1)^j e^{(r-j)x}\Big|_{x=0} \\ &= \sum_{j=0}^{r} \binom{r}{j} (-1)^j D^n e^{(r-j)x}\Big|_{x=0} \\ &= \sum_{j=0}^{r} \binom{r}{j} (-1)^j (r - j)^n. \end{aligned}$$

This last expression is recognizable as the rth retarded difference with respect to r (at the unit intervals) of the function r^n (evaluated at r), and, in view of (3), we write

$$(8) \qquad f_n = \sum_{r=0}^{\infty} \nabla^r r^n.$$

A recursion formula useful for computing f_n will now be derived. From (4):

$$\binom{n}{j} f_{n-j} = \frac{1}{2} \sum_{v=0}^{\infty} \binom{n}{j} v^{n-j} 2^{-v}.$$

Summing on j and making use of the binomial theorem yields

$$\sum_{j=0}^{n-1} \binom{n}{j} f_{n-j} = \frac{1}{2} \sum_{v=0}^{\infty} (v+1)^n 2^{-v} - \frac{1}{2} \sum_{v=0}^{\infty} 2^{-v},$$

where the second term on the right is due to the fact that we have summed only to $n-1$ instead of to n. Now, by (4)

$$\frac{1}{2} \sum_{v=0}^{\infty} (v+1)^n 2^{-v} = \sum_{v=0}^{\infty} (v+1)^n 2^{-v-1} = \sum_{v=0}^{\infty} v^n 2^{-v} = 2f_n$$

and $\frac{1}{2} \sum_{v=0}^{\infty} 2^{-v} = 1$. Hence

$$\sum_{j=0}^{n-1} \binom{n}{j} f_{n-j} = 2f_n - 1,$$

(9)

$$f_n = 1 + \sum_{j=1}^{n-1} \binom{n}{j} f_{n-j}.$$

Equation (9) may be written symbolically in the form $(1+f)^n - 2f^n = 0$, where we have replaced subscripts by exponents. The numbers f_n are thus seen to bear a resemblance to the Bernoulli numbers.

Using (9), the writer has computed the first 14 values of f_n. They are:

n	f_n
1	1
2	3
3	13
4	75
5	541
6	4,683
7	47,293
8	545,835
9	7,087,261
10	102,247,563
11	1,622,632,573
12	28,091,567,595
13	526,858,348,381
14	10,641,342,970,443

Observe that the final digits of f_n appear to follow a cyclic sequence of period 4, the cycle consisting of the digits 1, 3, 3, and 5. One wonders if this sequence continues indefinitely in this manner. That this is, in fact, the case, is very easy to prove. It would suffice to show that

$$f_{n+4} \equiv f_n \pmod{10}. \tag{10}$$

We proceed to do this. Writing (8) in the form of a congruence yields $f_n \equiv \sum_{r=0}^{\infty} \nabla^r r^n \pmod{10}$. Similarly, $f_{n+4} \equiv \sum_{r=0}^{\infty} \nabla^r r^{n+4} \pmod{10}$, and by subtraction,

$$f_{n+4} - f_n \equiv \sum_{r=0}^{\infty} \nabla^r \{r^n(r^4 - 1)\}. \tag{11}$$

Consider the expression

$$u = r^n(r^4 - 1) = r^{n-1}(r^5 - r),$$

with n a positive integer. By Fermat's theorem, $r^5 - r$ is divisible by 5, and hence, so is u. Moreover, r^n and $r^4 - 1$ are of different parity, and hence u is divisible by 2. Consequently, u is divisible by $2 \cdot 5 = 10$; *i.e.*, $r^n(r^4 - 1) \equiv 0 \pmod{10}$. Substituting this last into (11) yields the desired result (10).

The above result is, of course, a special case of a more general theorem; in particular, if p is an odd prime,

$$f_{n+p-1} \equiv f_n \pmod{2p}.$$

Several other interesting results concerning properties of associated polynomials, recurrence formulae deducible from combinatorial considerations, and the like, have been obtained.

Of perhaps greater interest, however, is a geometric interpretation of f_n. To obtain this interpretation we define a certain $n-1$ dimensional compact convex set in Euclidean n-space as follows:

Let P_n denote the set of $n!$ points in E_n with the property that each point $x \in P_n$ has as its coordinates the integers $1, \cdots, n$ arranged in some order. Let C_n denote the convex hull of P_n. It is easy to show that C_n is $(n-1)$-dimensional. f_n is related to the boundary of C_n and, in fact, is precisely the total number of bounding elements of C_n; *i.e.*, f_n = the number of extreme points + the number of edges on the boundary + the number of 2-dimensional bounding faces, $+ \cdots +$ finally, a count of 1 for the $n-1$ dimensional set itself.

Remarks. The author is indebted to the referee for observing that the infinite series for f_n enables one to interpret it in still another context, namely, as the nth moment of the geometric distribution on the nonnegative integers with $p_j = 2^{-j-1}$, $j = 0, 1, \cdots$. The referee noted also that the methods used in this paper can be exploited to find the number of points in the metric space proposed by Kemeny [2] for studying preferences.

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1. W. A. Whitworth, Choice and Chance, with 1000 Exercises, New York, 1934.
2. John G. Kemeny, Mathematics without numbers, Daedalus, vol. 88, 1959, pp. 577–591.

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept. Accordingly, in order for a college to receive q acceptances, it will generally have to offer to admit more than q applicants. The problem of determining how many and which ones to admit requires some rather involved guesswork. It may not be known (a) whether a given applicant has also applied elsewhere; if this is known it may not be known (b) how he ranks the colleges to which he has applied; even if this is known it will not be known (c) which of the other colleges will offer to admit him. A result of all this uncertainty is that colleges can expect only that the entering class will come reasonably close in numbers to the desired quota, and be reasonably close to the attainable optimum in quality.

The usual admissions procedure presents problems for the applicants as well as the colleges. An applicant who is asked to list in his application all other colleges applied for in order of preference may feel, perhaps not without reason, that by telling a college it is, say, his third choice he will be hurting his chances of being admitted.

One elaboration is the introduction of the "waiting list," whereby an applicant can be informed that he is not admitted but may be admitted later if a vacancy occurs. This introduces new problems. Suppose an applicant is accepted by one college and placed on the waiting list of another that he prefers. Should he play safe by accepting the first or take a chance that the second will admit him later? Is it ethical to accept the first without informing the second and then withdraw his acceptance if the second later admits him?

We contend that the difficulties here described can be avoided. We shall describe a procedure for assigning applicants to colleges which should be satisfactory to both groups, which removes all uncertainties and which, assuming there are enough applicants, assigns to each college precisely its quota.

2. The assignment criteria. A set of n applicants is to be assigned among m colleges, where q_i is the quota of the ith college. Each applicant ranks the colleges in the order of his preference, omitting only those colleges which he would never accept under any circumstances. For convenience we assume there are no ties; thus, if an applicant is indifferent between two or more colleges he is nevertheless required to list them in some order. Each college similarly ranks the students who have applied to it in order of preference, having first eliminated those appli-

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cants whom it would not admit under any circumstances even if it meant not filling its quota. From these data, consisting of the quotas of the colleges and the two sets of orderings, we wish to determine an assignment of applicants to colleges in accordance with some agreed-upon criterion of fairness.

Stated in this way and looked at superficially, the solution may at first appear obvious. One merely makes the assignments "in accordance with" the given preferences. A little reflection shows that complications may arise. An example is the simple case of two colleges, A and B, and two applicants, α and β, in which α prefers A and β prefers B, but A prefers β and B prefers α. Here, no assignment can satisfy all preferences. One must decide what to do about this sort of situation. On the philosophy that the colleges exist for the students rather than the other way around, it would be fitting to assign α to A and β to B. This suggests the following admittedly vague principle: other things being equal, students should receive consideration over colleges. This remark is of little help in itself, but we will return to it later after taking up another more explicit matter.

The key idea in what follows is the assertion that—whatever assignment is finally decided on—it is clearly desirable that the situation described in the following definition should *not* occur:

DEFINITION. *An assignment of applicants to colleges will be called* ***unstable*** *if there are two applicants α and β who are assigned to colleges A and B, respectively, although β prefers A to B and A prefers β to α.*

Suppose the situation described above did occur. Applicant β could indicate to college A that he would like to transfer to it, and A could respond by admitting β, letting α go to remain within its quota. Both A and β would consider the change an improvement. The original assignment is therefore "unstable" in the sense that it can be upset by a college and applicant acting together in a manner which benefits both.

Our first requirement on an assignment is that it not exhibit instability. This immediately raises the mathematical question: will it always be possible to find such an assignment? An affirmative answer to this question will be given in the next section, and while the proof is not difficult, the result seems not entirely obvious, as some examples will indicate.

Assuming for the moment that stable assignments do exist, we must still decide which among possibly many stable solutions is to be preferred. We now return to the philosophical principle mentioned earlier and give it a precise formulation.

DEFINITION. *A stable assignment is called* ***optimal*** *if every applicant is at least as well off under it as under any other stable assignment.*

Even granting the existence of stable assignments it is far from clear that there are optimal assignments. However, one thing that is clear is that the

optimal assignment, if it exists, is unique. Indeed, if there were two such assignments, then, at least one applicant (by our "no tie" rule) would be better off under one than under the other; hence one of the assignments would not be optimal after all. Thus the principles of stability and optimality will, when the existence questions are settled, lead us to a unique "best" method of assignment.

3. Stable assignments and a marriage problem. In trying to settle the question of the existence of stable assignments we were led to look first at a special case, in which there are the same number of applicants as colleges and all quotas are unity. This situation is, of course, highly unnatural in the context of college admissions, but there is another "story" into which it fits quite readily.

A certain community consists of n men and n women. Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner. We seek a satisfactory way of marrying off all members of the community. Imitating our earlier definition, we call a set of marriages *unstable* (and here the suitability of the term is quite clear) if under it there are a man and a woman who are not married to each other but prefer each other to their actual mates.

QUESTION: *For any pattern of preferences is it possible to find a stable set of marriages?*

Before giving the answer let us look at some examples.

Example 1. The following is the "ranking matrix" of three men, α, β, and γ, and three women, A, B, and C.

	A	B	C
α	1, 3	2, 2	3, 1
β	3, 1	1, 3	2, 2
γ	2, 2	3, 1	1, 3

The first number of each pair in the matrix gives the ranking of women by the men, the second number is the ranking of the men by the women. Thus, α ranks A first, B second, C third, while A ranks β first, γ second, and α third, etc.

There are six possible sets of marriages; of these, three are stable. One of these is realized by giving each man his first choice, thus α marries A, β marries B, and γ marries C. Note that although each woman gets her last choice, the arrangement is nevertheless stable. Alternatively one may let the women have their first choices and marry α to C, β to A, and γ to B. The third stable arrangement is to give everyone his or her second choice and have α marry B, β marry C, and γ marry A. The reader will easily verify that all other arrangements are unstable.

Example 2. The ranking matrix is the following.

	A	*B*	*C*	*D*
α	1, 3	2, 3	(3, 2)	4, 3
β	1, 4	4, 1	3, 3	(2, 2)
γ	(2, 2)	1, 4	3, 4	4, 1
δ	4, 1	(2, 2)	3, 1	1, 4

There is only the one stable set of marriages indicated by the circled entries in the matrix. Note that in this situation no one can get his or her first choice if stability is to be achieved.

Example 3. A problem similar to the marriage problem is the "problem of the roommates." An even number of boys wish to divide up into pairs of roommates. A set of pairings is called *stable* if under it there are no two boys who are not roommates and who prefer each other to their actual roommates. An easy example shows that there can be situations in which there exists no stable pairing. Namely, consider boys α, β, γ and δ, where α ranks β first, β ranks γ first, γ ranks α first, and α, β and γ all rank δ last. Then regardless of δ's preferences there can be no stable pairing, for whoever has to room with δ will want to move out, and one of the other two will be willing to take him in.

The above examples would indicate that the solution to the stability problem is not immediately evident. Nevertheless,

THEOREM 1. *There always exists a stable set of marriages.*

Proof. We shall prove existence by giving an iterative procedure for actually finding a stable set of marriages.

To start, let each boy propose to his favorite girl. Each girl who receives more than one proposal rejects all but her favorite from among those who have proposed to her. However, she does not accept him yet, but keeps him on a string to allow for the possibility that someone better may come along later.

We are now ready for the second stage. Those boys who were rejected now propose to their second choices. Each girl receiving proposals chooses her favorite from the group consisting of the new proposers and the boy on her string, if any. She rejects all the rest and again keeps the favorite in suspense.

We proceed in the same manner. Those who are rejected at the second stage propose to their next choices, and the girls again reject all but the best proposal they have had so far.

Eventually (in fact, in at most n^2-2n+2 stages) every girl will have received a proposal, for as long as any girl has not been proposed to there will be rejections and new proposals, but since no boy can propose to the same girl more than once, every girl is sure to get a proposal in due time. As soon as the last girl

gets her proposal the "courtship" is declared over, and each girl is now required to accept the boy on her string.

We assert that this set of marriages is stable. Namely, suppose John and Mary are not married to each other but John prefers Mary to his own wife. Then John must have proposed to Mary at some stage and subsequently been rejected in favor of someone that Mary liked better. It is now clear that Mary must prefer her husband to John and there is no instability.

The reader may amuse himself by applying the procedure of the proof to solve the problems of Examples 1 and 2, or the following example which requires ten iterations:

	A	*B*	*C*	*D*
α	1, 3	2, 2	3, 1	4, 3
β	1, 4	2, 3	3, 2	4, 4
γ	3, 1	1, 4	2, 3	4, 2
δ	2, 2	3, 1	1, 4	4, 1

The condition that there be the same number of boys and girls is not essential. If there are b boys and g girls with $b<g$, then the procedure terminates as soon as b girls have been proposed to. If $b>g$ the procedure ends when every boy is either on some girl's string or has been rejected by all of the girls. In either case the set of marriages that results is stable.

It is clear that there is an entirely symmetrical procedure, with girls proposing to boys, which must also lead to a stable set of marriages. The two solutions are not generally the same as shown by Example 1; indeed, we shall see in a moment that when the boys propose, the result is optimal for the boys, and when the girls propose it is optimal for the girls. The solutions by the two procedures will be the same only when there is a unique stable set of marriages.

4. Stable assignments and the admissions problem. The extension of our "deferred-acceptance" procedure to the problem of college admissions is straightforward. For convenience we will assume that if a college is not willing to accept a student under any circumstances, as described in Section 2, then that student will not even be permitted to apply to the college. With this understanding the procedure follows: First, all students apply to the college of their first choice. A college with a quota of q then places on its waiting list the q applicants who rank highest, or all applicants if there are fewer than q, and rejects the rest. Rejected applicants then apply to their second choice and again each college selects the top q from among the new applicants and those on its waiting list, puts these on its new waiting list, and rejects the rest. The procedure terminates when every applicant is either on a waiting list or has been rejected by every college to which he is willing and permitted to apply. At this point each college

admits everyone on its waiting list and the stable assignment has been achieved. The proof that the assignment is stable is entirely analogous to the proof given for the marriage problem and is left to the reader.

5. Optimality. We now show that the "deferred acceptance" procedure just described yields not only a stable but an optimal assignment of applicants. That is,

THEOREM 2. *Every applicant is at least as well off under the assignment given by the deferred acceptance procedure as he would be under any other stable assignment.*

Proof. Let us call a college "possible" for a particular applicant if there is a stable assignment that sends him there. The proof is by induction. Assume that up to a given point in the procedure no applicant has yet been turned away from a college that is possible for him. At this point suppose that college A, having received applications from a full quota of better-qualified applicants $\beta_1, \cdots, \beta_q$, rejects applicant α. We must show that A is impossible for α. We know that each β_i prefers college A to all the others, except for those that have previously rejected him, and hence (by assumption) are impossible for him. Consider a hypothetical assignment that sends α to A and everyone else to colleges that are possible for them. At least one of the β_i will have to go to a less desirable place than A. But this arrangement is unstable, since β_i and A could upset it to the benefit of both. Hence the hypothetical assignment is unstable and A is impossible for α. The conclusion is that our procedure only rejects applicants from colleges which they could not possibly be admitted to in any stable assignment. The resulting assignment is therefore optimal.

Parenthetically we may remark that even though we no longer have the symmetry of the marriage problem, we can still invert our admissions procedure to obtain the unique "college optimal" assignment. The inverted method bears some resemblance to a fraternity "rush week"; it starts with each college making bids to those applicants it considers most desirable, up to its quota limit, and then the bid-for students reject all but the most attractive offer, and so on.

6. Concluding remarks. The reader who has followed us this far has doubtless noticed a certain trend in our discussion. In making the special assumptions needed in order to analyze our problem mathematically, we necessarily moved further away from the original college admission question, and eventually in discussing the marriage problem, we abandoned reality altogether and entered the world of mathematical make-believe. The practical-minded reader may rightfully ask whether any contribution has been made toward an actual solution of the original problem. Even a rough answer to this question would require going into matters which are nonmathematical, and such discussion would be out of place in a journal of mathematics. It is our opinion, however, that some of the ideas introduced here might usefully be applied to certain phases of the admissions problem.

Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which non-mathematicians believe mathematics to be concerned with.

Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with "a head for figures," or that they "know a lot of formulas." At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter.

What, then, to raise the old question once more, is mathematics? The answer, it appears, is that any argument which is carried out with sufficient precision is mathematical, and the reason that your friends and ours cannot understand mathematics is not because they have no head for figures, but because they are unable to achieve the degree of concentration required to follow a moderately involved sequence of inferences. This observation will hardly be news to those engaged in the teaching of mathematics, but it may not be so readily accepted by people outside of the profession. For them the foregoing may serve as a useful illustration.

GRADUATED INTEREST RATES IN SMALL LOANS

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Many small loan companies charge a graduated interest rate in accordance with various state laws. For example, 3% per month is charged on the first \$150 of a loan, and 2% on the portion of the loan in excess of \$150. Rates may be graduated in two, three or more brackets. A three-bracket loan might be at the rate of $2\frac{1}{2}\%$ on that part of the loan or loan balance which is \$100 or less, at the rate of 2% on that part of a loan which is in excess of \$100 but less than \$200, and at the rate of 1% on that part of a loan which is in excess of \$200. Such a graduated rate is written: $2\frac{1}{2}\%/2\%/1\%/\$100/\$200$.

The main problem considered in this paper is that of finding the level monthly rent payment which will amortize a loan in a given time at a graduated rate.

1. Two-bracket case. We consider first the graduated rate for the two bracket case. Given the loan of B_0 for n months, what equal monthly rent R will amortize the loan at a graduated rate of $r\%/s\%/L$? ($B_0 > L$). The interest charge on the first L of the loan each month for the first m months (m the breakpoint) is Lr. This leaves a payment of $R-Lr$ to be applied to the loan balance above L which is at $s\%$. Now B_0-L, the amount above the breakpoint, is increased at the interest rate s for m periods. This is then reduced by payments of $R-Lr$, accumulated as an annuity at $s\%$. By an examination of the amortization schedule, the balance after one month is

$$B_1 = L(1 + r) + (B_0 - L)(1 + s) - R$$

or

$$B_1 - L = (B_0 - L)(1 + s) - (R - Lr)s_{\overline{1}|(s)}.$$

After the second payment, the balance B_2 is given by

$$B_2 - L = (B_0 - L)(1 + s)^2 - (R - Lr)s_{\overline{2}|(s)},$$

and, after the mth payment, the balance B_m is given by

$$B_m - L = (B_0 - L)(1 + s)^m - (R - Lr)s_{\overline{m}|(s)}, \tag{1}$$

where $B_{m-1} \geqq L \geqq B_m$ (m an integer). Formula (1) gives the number of months m required to reduce the loan so that it is below the break point between the two brackets.

Now after B_m is reached, remembering that $B_{m-1} \geqq L \geqq B_m$, the rate s is no longer operative. In fact,

$$B_m = Ra_{\overline{n-m}|(r)}. \tag{2}$$

Equating (1) and (2) and solving for R gives

$$R = \frac{L + (B_0 - L)(1 + s)^m + Lrs_{\overline{m}|(s)}}{a_{\overline{n-m}|(r)} + s_{\overline{m}|(s)}}, \tag{3}$$

or

$$R = \frac{L(r - s) + B_0/a_{\overline{m}|(s)}}{1 + a_{\overline{n-m}|(r)}/s_{\overline{m}|(s)}}. \tag{4}$$

Thus we are able to find R by (3) or (4) provided m can first be obtained. Now,

$$B_0 - L = (R - Lr)a_{\overline{\alpha}|(s)}. \tag{5}$$

Again, since the balance L is to be amortized at rate r, we have

$$L = Ra_{\overline{n-\alpha}|(r)}. \tag{6}$$

Eliminating R between (5) and (6), we obtain what might be called an auxiliary equation

$$(7) \qquad (B_0 - L)s_{\overline{n-\alpha}|(r)} - La_{\overline{\alpha}|(s)} = 0.$$

Now, α as given by (7) is not usually integral while m in the schedule is an integer. Moreover, it is not necessary to determine α except to the nearest integer, for $m = [\alpha]+1$, where $[\alpha]$ denotes the largest integer contained in α. We represent the left member of (7) as $f(\alpha)$ and proceed to use tables to find $[\alpha]+1$. A first trial may be made by taking $[\alpha] \doteq n\{1-(L/B_0)\}$, which assumes the breakpoint is the ratio $(B_0-L)/B_0$ or without interest. If, for a given value of $[\alpha]$, $f([\alpha])>0$, but $f([\alpha]+1)<0$, then $m=[\alpha]+1$.

The value of R obtained by (3) or (4) can be checked by (1) or (2) or by an amortization schedule.

Example 1. Find the level payment required to repay a loan of \$250 in 8 months at a rate of 3%/2%/\$100.

Here, $f(5) = 150s_{\overline{3}|(3\%)} - 100a_{\overline{5}|(2\%)} = -$, $f(4) = 150s_{\overline{4}|(3\%)} - 100a_{\overline{4}|(2\%)} = +$. Note that it is not necessary to compute values of $f([\alpha])$. All that is needed is the sign of the result. Using $m=5$,

$$R = \frac{1 + 250/a_{\overline{5}|(2\%)}}{1 + a_{\overline{3}|(3\%)}/s_{\overline{5}|(2\%)}} = 35.01.$$

The equivalent single rate i is given by the solution of $250 = 35.01a_{\overline{8}|(i)}$. By interpolation, $i = 2.60\%$.

TABLE OF TWO-BRACKET EXAMPLES

B_0	$r\%$	$s\%$	L	n	m	R	$i\%$
250	3%	2%	100	1	1	256.00	2.4%
				8	5	35.01	2.60%
				15	10	20.33	2.595%
				20	14	16.17	2.59%
				30	21	12.06	2.57%
				100	81	6.77	2.53%
400	3%	2%	100	1	1	409.00	2.25%
				8	6	55.55	2.40%
				100	89	10.26	2.32%

The value of R can be determined only by using the correct value of m. However, it is helpful to consider a variation of m as it affects R while holding

the remaining arguments fixed. Thus if we are given 3%/2%/\$100, with $B_0=250$, $n=15$, then

$$R(m) = \frac{1 + 250/a_{\overline{m}|\,(2\%)}}{1 + a_{\overline{15-m}|\,(3\%)}/s_{\overline{m}|\,(2\%)}}.$$

The following values are computed

m	8	9	10	11	12
$R(m)$	20.3518	20.3354	20.3289	20.3332	20.3484

Thus corresponding to values of m, $R(m)$ is a flat curve with relatively small variations.

This fact presents an alternative method for finding m. That is, instead of using (7) to find m, we could estimate m, use (4) to compute R, then use (1) to compute the true m and use (4) to compute the true R. This procedure works well in the three- and four-bracket cases.

An examination of the above example shows that $R(m)$ has a minimum. This value is obtained by finding dR/dm from (3) and placing the result equal to 0. This gives

$$(8)\qquad g(m) = B^m \ln B\left(Ca_{\overline{n-m}|\,(r)} - \frac{B_0}{s}\right) + A^{m-n} \ln A\left(\frac{Cs}{r}s_{\overline{m}|\,(s)} + \frac{B_0}{r}\right) = 0,$$

where $A=1+r$, $B=1+s$, $C=B_0-L+(Lr/s)$.

Example 1 above applied to (8) gives

$$g(9) = -2.78, \qquad g(10) = -0.26, \qquad g(11) = +2.45.$$

This indicates a root between 10 and 11. Generally $g(m)=0$ may be used to find the integer m which may be used to find R.

THEOREM. *The true value of m (the breakpoint) which must be used to compute R is the integral value nearest m which makes R(m) a minimum.*

2. Three-bracket case. Given the loan B_0 for n months, what equal monthly rent R will amortize the loan at a graduated rate $r\%/s\%/t\%/L/M$, where $B_0>M$?

Similar to the two-bracket case, the balance after m payments have been made and the first breakpoint has been passed is given by

$$(9)\qquad B_m = M + (B_0 - M)(1 + t)^m - \{R - Lr - (M - L)s\}s_{\overline{m}|\,(t)},$$

where $B_{m-1} \geqq M \geqq B_m$ (m an integer). Again, passing the second breakpoint,

$$(10)\qquad B_{p+m} = L + (B_m - L)(1 + s)^p - (R - Lr)s_{\overline{p}|\,(s)} = Ra_{\overline{n-m-p}|\,(r)},$$

where $B_{p+m-1} \geqq L \geqq B_{p+m}$ ($p+m$ an integer).

Eliminating B_m from (9) and (10) and solving for R gives

$$(11) \qquad R = \frac{L(r-s) + [B_0(1+t)^m + (M(s-t) + L(r-s))s_{\overline{m}|(t)}]/a_{\overline{p}|(s)}}{1 + a_{\overline{n-m-p}|(r)}/s_{\overline{p}|(s)} + s_{\overline{m}|(t)}/a_{\overline{p}|(s)}},$$

where n, m, p are integers.

Corresponding to (5) and (6), we have the auxiliary equations for determining m and p

$$(12) \qquad \begin{aligned} B_0 - M &= \{R - Lr - (M-L)s\}a_{\overline{\alpha}|(t)}, \\ M - L &= (R - Lr)a_{\overline{\beta}|(s)}, \qquad L = Ra_{\overline{n-\alpha-\beta}|(r)}. \end{aligned}$$

Eliminating R from (12) gives

$$(13) \qquad f(\alpha, \beta) = (M-L)s_{\overline{n-\alpha-\beta}|(r)} - La_{\overline{\beta}|(s)} = 0,$$

$$(14) \qquad g(\alpha, \beta) = (B_0 - M)s_{\overline{\beta}|(s)} - (M-L)a_{\overline{\alpha}|(t)} = 0.$$

In general the solution of (13) and (14) for α and β will result in nonintegral values. Since m and p are integers, it is necessary only to find the nearest integral solutions of (13) and (14). These equations will always give the correct value of $m = [\alpha]+1$, but actually with m substituted for α, B_m should replace L. Since $L \geqq B_m$, this means that β might be one too large. Hence $p = [\beta]+1$ or $p = [\beta]$. The situation where $p = [\beta]$ is more apt to occur for small values of n.

A crude estimate may be made by using $\alpha \doteq n(B_0 - M)/B_0$ and $\beta \doteq n(M-L)/B_0$. We then find the smallest values $[\alpha]$, $[\beta]$ for which $f(\alpha, \beta) < 0$, $g(\alpha, \beta) < 0$. Then $m = [\alpha]+1$, and $p = [\beta]+1$ or $p = [\beta]$.

It may be noted empirically that a change of one in p will give a very small change in R. Thus for values of m and p determined by (13) and (14), compute R by (11), then use this value of R to compute the true values of m and p in (9) and (10). Finally use the values of m and p to compute R by (11). All of these steps may not be necessary. This procedure checks the value of R as to its correctness.

Example 1. Given $2\frac{1}{2}\%/2\%/1\%/\$100/\$200$, $n = 18$, $B_0 = 270$. Then $\alpha \doteq 18[1 - (200/270)] = 4\frac{2}{3}$, $\beta \doteq 18(100/270) = 6\frac{2}{3}$,

$$\begin{aligned} f(5, 7) &= 100s_{\overline{6}|(2\frac{1}{2}\%)} - 100a_{\overline{7}|(2\%)} = 639 - 647 = -8, \\ g(5, 7) &= 70s_{\overline{7}|(2\%)} - 100a_{\overline{5}|(1\%)} = 520 - 485 = +35. \end{aligned}$$

Try $m = 6$, $p = 7$; then

$$R = \frac{270(1.02)^7(1.01)^6 + (1.02)^7 s_{\overline{6}|(1\%)}(2.5) - s_{\overline{7}|(2\%)}(0.5)}{a_{\overline{5}|(2\frac{1}{2}\%)} + s_{\overline{6}|(1\%)}(1.02)^7 + s_{\overline{7}|(2\%)}} = 18.32.$$

Check: $B_5 = 200 + 70(1.01)^6 - (18.32 - 2.5 - 2)s_{\overline{6}|(1\%)} = 189.28$, $B_{13} = 100 + 89.28(1.02)^7 - 15.82s_{\overline{7}|(2\%)} = 84.94$ or $B_{13} = 18.32a_{\overline{5}|(2\frac{1}{2}\%)} = 85.11$. Although there is a difference of 17¢ in the two values for B_{13}, the value of R cannot be changed by as much as 1¢.

TABLE OF THREE-BRACKET EXAMPLES

$r\%/s\%/t\%/$	L/M	B_0	n	m	p	R	$i\%$
$2\frac{1}{2}\%/2\%/1\%/$	100/200	270	18	6	7	18.32	2.19%
		280		7	6	18.95	2.16%
		210		2	9	14.43	2.34%

TABLE OF FOUR-BRACKET EXAMPLES

$r\%/s\%/t\%/u\%$	$L/M/N$			B_0	n	m	p	q	R
4%/3%/2%/1%	100	200	300	400	12	4	3	3	40.22
					40	14	11	8	17.09
					100	46	27	17	11.75

3. Generalization of the formulas for R. Formulas (3), (11), (17) may be written in a general form as follows: Let

$$r = r_1,\ s = r_2,\ t = r_3,\ u = r_4,\ \cdots; \qquad L = L_1,\ M = L_2,\ N = L_3,\ \cdots;$$

$$X_p = L_p(r_p - r_{p+1}), \qquad p = 1, \cdots, k-1;$$

n_1 = number of periods where balance is below L_1;

n_p = number of periods where balance is above L_{p-1} but below L_p,

$$p = 2, \cdots, k-1;$$

n_k = number of periods where balance is above L_{k-1} but below B_0.

Then, for the case of k brackets,

$$R = \frac{B_0 \prod_{p=1}^{k} (1 + r_p)^{n_p} + \sum_{p=2}^{k} s_{\overline{n_p}|(r_p)} \left(\prod_{q=1}^{p-1} (1 + r_q)^{n_q} \right) \left(\sum_{r=1}^{p-1} X_r \right)}{\sum_{p=1}^{k} s_{\overline{n_p}|(r_p)} \left(\prod_{q=1}^{p-1} (1 + r_q)^{n_q} \right)}.$$

4. Length of time to repay a loan. If α is eliminated between (5) and (6), and if we solve for n, we have

$$(19) \qquad n = \frac{\log R - \log (R - Lr)}{\log (1 + r)} + \frac{\log (R - Lr) - \log (R - Lr + Ls - B_0 s)}{\log (1 + s)}.$$

In the general case the formula for n may be written as

$$n = \sum_{p=1}^{k} \left\{ \frac{\log\left(R - L_{p-1}r_{p-1} - \sum_{q=1}^{p-2} X_q\right) - \log\left(R - L_p r_p - \sum_{q=1}^{p-1} X_q\right)}{\log\,(1 + r_p)} \right\},$$

where $L_0 = 0$, $L_k = B_0$.

5. Equivalent rate of interest. If we consider the balance B_1 in the two-bracket case, the interest for the first period on B_0 is $B_0 i$, while for the graduated rate it is $(B_0 - L)s + Lr$. Equating these expressions for interest gives $i = s + (L/B_0)(r - s)$. Again, if

$$R = \frac{B_0}{a_{\overline{n}|(i)}} = \frac{L(r - s) + B_0/a_{\overline{m}|(s)}}{1 + a_{\overline{n-m}|(r)}/s_{\overline{m}|(s)}},$$

let $n \to \infty$. This requires that $m \to \infty$ and taking the limit gives $B_0 i = L(r - s) + B_0 s$, the same result as for a single payment. The examples for the two-bracket case show that for intermediate values i is larger and increases up to a certain value of n and then decreases. The equivalent single rate thus satisfies $s < i < r$ and depends largely on the size of the second bracket $B_0 - L$ as compared to L. The formula [2]

$$i = \frac{2I}{B(n + 1)}\left(\frac{3B_0(n + 1) + (n + 2)I}{3B(n + 1) + (2n + 1)I}\right),$$

where $I = Rn - B_0$, can be used for an accurate determination of i.

For the three-bracket case, if $n = 1$ or $n = \infty$, the equivalent rate i is given by

$$i = t + (M/B_0)(s - t) + (L/B_0)(r - s) \qquad (r > i > t).$$

Again, for intermediate values of n, i is larger, increasing with n up to a maximum and then decreasing as n increases.

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A THREE-POINT PROPERTY

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In a complete metric space in which each two points may be joined by a unique metric segment a set is said to be convex if each pair of its points may be joined by a metric segment lying in the set. This is equivalent to the demand that all three of the segments determined by three pairwise distinct points of the set be in the set. This definition of convexity in terms of triples of points suggests a weakening of the requirement that all three segments of each triple of points be in the set. A weaker requirement would be that only two or one of the segments determined must be in the set. If a set containing at least three points is closed and if the set contains two of the three segments determined by each triple of its points, then the set is convex. Further weakening of the hypothesis to the new demand that only one of the segments need lie in the set leads not to convexity, but to a property which Valentine [2] has chosen to call P_3. He has shown that a closed, connected subset of the plane possessing the property P_3 is representable as the union of three or fewer convex sets.

In this note we show that if a closed subset of the Euclidean n-space has the property that it contains two of the segments determined by each "isosceles triple" of its points, then the set is convex. We also show that for certain subsets of the Euclidean n-space the hypothesis that one of the segments from each isosceles triple lies in the set implies the set has Valentine's property P_3. In this connection, we need the following definitions.

DEFINITION 1. *If x, y, and z are distinct points of a metric space S such that at least two of the distances xy, yz, or xz are equal, then the points x, y, and z are said to form an isosceles triple in S.*

DEFINITION 2. *Let S be a metric space in which every pair of points is joined by a unique metric segment. A subset M of S is said to have the single-isosceles three-point property if each of its isosceles triples contains a pair of points whose connecting segment is contained in M. The subset M has the double-isosceles three-point property if two connecting segments of each isosceles triple belong to M.*

DEFINITION 3. *A subset M of a metric space S which has unique segments is said to possess the three-point convexity property P_3 if for each triple of points x, y, z in M at least one of the closed segments $S(x, y)$, $S(x, z)$, $S(y, z)$ is in M.*

These definitions will now be applied to obtain a theorem concerning convexity of certain sets.

THEOREM 1. *Let M be a closed, connected subset of a Euclidean or hyperbolic space S. If M has the double-isosceles property, then M is convex.*

Proof. Let x and y be an arbitrary pair of distinct points of M. Let L denote the hyperplane perpendicular to the closed segment $S(x,y)$ which passes through the midpoint m of $S(x, y)$. Since M is closed there exists a point z of

$L \cap M$ nearest the segment $S(x, y)$. Since M has the double-isosceles property, at least two of the segments $S(x, y)$, $S(x, z)$, and $S(y, z)$ lie in M. If the segment $S(x, y)$ lies in M, the proof is complete. Therefore, assume $S(x, y)$ does not lie in M. Then both segments $S(x, z)$ and $S(y, z)$ must lie in M. Let d denote the length of the segment $S(x, z)$, and let w denote the point of $S(x, z)$ which is $\frac{1}{4}d$ units from z. In the plane of x, y, and z let the reflection of w with respect to $S(z, m)$ be called v. The point v lies on $S(z, y)$ and $zv = \frac{1}{4}d$. Since $wv < \frac{1}{2}d$, there exists a point u of $S(x, w)$ such that $uw = wv$. At least one of the segments $S(w, v)$ or $S(u, v)$ determined by the isosceles triple u, v, w must lie in M. Since each of these segments intersects $S(z, m)$ and since such an intersection is closer to $S(x, y)$ than z is, a contradiction has been reached.

The remaining results establish a relationship between the single-isosceles property and the property P_3 for some subsets of the n-dimensional Euclidean space E^n.

THEOREM 2. *Let M be a closed subset of E^n such that M has the single-isosceles property and such that every pair of points of M can be connected by a rectifiable arc in M. Then every pair of points of M can be connected by a polygonal arc which is contained in M and which consists of at most two line segments* [3].

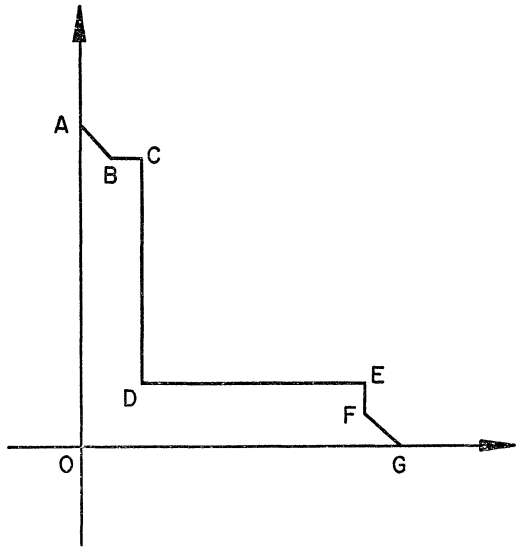
Proof. Let x and y be arbitrary distinct points of M, and let $A(x, y)$ be a shortest arc in M connecting x and y (cf. [1], p. 70). If $A(x, y)$ is a line segment or a polygonal arc consisting of two segments, the proof is complete. Therefore, assume $A(x, y)$ does not consist of two or fewer line segments. Consider the locus $B(x, y)$ of points equidistant from x and y. Since $A(x, y)$ is connected, there exists a point z_1 of $A(x, y) \cap B(x, y)$. Since M has the single-isosceles property, either the segment $S(x, z_1)$ or the segment $S(z_1, y)$ lies in M. Suppose $S(x, z_1)$ lies in M. Now $S(x, z_1) \subset A(x, y)$, for otherwise $A(x, y)$ would not be the shortest arc from x to y. Let $S(x, w_1)$ be the longest segment in $A(x, y)$ which contains $S(x, z_1)$. Now consider $B(w_1, y)$. As before there exists a point $z_2 \in A(x, y) \cap B(w_1, y)$. Also as before, one of the segments $S(w_1, z_2)$ or $S(z_2, y)$ lies in M and is a subarc of $A(x, y)$. If both of these line segments lie in $M \cap A(x, y)$, then $A(x, y)$ contains a polygonal arc consisting of three noncollinear line segments. If only one of the segments lies in $M \cap A(x, y)$, consider the longest segment containing it and lying in $A(x, y)$. Repeating this procedure at most three more times we find that $A(x, y)$ contains a polygonal arc consisting of three noncollinear line segments.

Let the vertices of such a polygonal arc be u_1, u_2, u_3, and u_4. Now there exist points $v_1 \in S(u_1, u_2)$ and $v_2 \in S(u_3, u_4)$, where $u_1 \neq v_1 \neq u_2$ and $u_3 \neq v_2 \neq u_4$, such that $B(v_1, v_2)$ intersects $S(u_2, u_3)$ at r with $u_2 \neq r \neq u_3$. Since M has the single-isosceles property one of the segments $S(v_1, r)$, $S(v_2, r)$, or $S(v_1, v_2)$ lies in M. Such a segment is a subarc of $A(x, y)$. But this contradicts the assumption that $A(x, y)$ was the shortest arc lying in M from a to b. Thus the polygonal arc $A(x, y)$ cannot contain a subarc consisting of three noncollinear segments.

COROLLARY. *If M satisfies the hypotheses of Theorem 2 and if the shortest arc connecting x and y in M consists of two noncollinear line segments, then their common end point z is a point of local nonconvexity of M.*

Proof. Suppose the contrary; *i.e.*, M is locally convex at z. Then in any neighborhood $U(z)$, there exist points $u \in S(x, z)$ and $v \in S(z, y)$ such that $u \neq z \neq v$ and $S(u, v)$ lies in $U \cap M$. But this contradicts the assumption that the shortest arc from x to y in M was $S(x, z) \cup S(z, y)$.

THEOREM 3. *Suppose M is a closed subset of E^n and that every pair of points of M can be connected by a rectifiable arc in M. If M has the single-isosceles property and exactly one isolated point of local nonconvexity, then M has the property P_3.*



$A(0, 10)$, $B(1, 9)$, $C(2, 9)$, $D(2, 2)$, $E(9, 2)$, $F(9, 1)$, $G(10, 0)$

FIG. 1

Proof. Assume M does not have the property P_3. Then there exist points x, y, and w of M such that none of the segments $S(x, y)$, $S(x, w)$, or $S(y, w)$ lie in M. Denote the point of local nonconvexity of M by the letter z. By Theorem 2 and its corollary the shortest arcs from x and y to w are of the form $S(x, z) \cup S(z, w)$ and $S(y, z) \cup S(z, w)$. In addition the shortest arc from x to y is $S(x, z) \cup S(z, y)$. If z coincides with any of the points x, y, or w, we have an immediate contradiction. Otherwise there exist points x_1 interior to $S(x, z)$, y_1 interior to $S(y, z)$, and w_1 interior to $S(z, w)$ which form an isosceles triple. Since M has the single-isosceles property, one of the segments $S(x_1, y_1)$, $S(x_1, w_1)$, or $S(y_1, w_1)$ lies in M. If $S(x_1, y_1)$ lies in M, then $S(x, z) \cup S(z, y)$ is not the shortest arc from x to y. In a similar manner, if either $S(x_1, w_1)$ or $S(y_1, w_1)$ lie in M, a con-

tradiction is reached. Hence the assumption is false, and the theorem has been proved.

The hypotheses of Theorem 1 may not be weakened to read "double-equilateral" instead of "double-isosceles." The set M of points on two unit line segments forming a V with interior angle of 60° has the property that each equilateral triple of its points contains two pairs of points whose connecting segments lie in M. But M is not convex.

Obviously if a set has the property P_3, then it has the single-isosceles property. On the other hand the single-isosceles property alone does not imply P_3 even for closed, connected subsets of the plane. The example illustrated verifies this (Fig. 1). The set M consists of all of the interior and boundary points of the polygon $OABCDEFG$. This set M has the single-isosceles property, but the non-isosceles triple A, C, E has none of its segments in M.

The set of points on three unit segments in the plane having a common end point (forming a Y) shows that possession of an isolated point of local non-convexity is not sufficient to assure that the set has property P_3.

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A CHARACTERIZATION OF CONVEX BODIES

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1. Introduction. We denote by R^n the n-dimensional euclidean space; $\|x\|$ stands for the euclidean norm of $x \in R^n$. We introduce the following usual notations

$$E^{l+1} = \{x: x \in R^{l+1},\ \|x\| \leqq 1\} \qquad (l \geqq 0); \tag{1}$$

$$S^l = \{x: x \in R^{l+1},\ \|x\| = 1\} \qquad (l \geqq 0). \tag{2}$$

The first is, of course, the $(l+1)$-ball, or $(l+1)$-element, the second is its boundary sphere.

A space X is called k-connected, $k \geqq 0$, if every continuous map $f: S^l \to X$ can be extended into a continuous map $F: E^{l+1} \to X$, for $l = 0, 1, \cdots, k$. Clearly, if X is k-connected, then it is k'-connected, for every $k' \leqq k$. In particular, a k-

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connected space is always connected by arcs, because any pair of points in the space can be taken as the image of S^0, and it is then connected by the arc $F: E^1 \to X$. Thus on the line the only k-connected sets are the intervals, the empty set, the one point set, closed, open, and half-open intervals included. For $k \geqq 1$, a k-connected space is simply connected, that is, roughly stated, every closed curve in the space can be contracted into a point. The sphere S^k, $k \geqq 2$. is trivially $(k-1)$-connected; it is a rather deep result, that S^k is *not* k-connected. In fact, the identity map $S^k \to S^k$ cannot be extended to a *continuous* map $F: E^{k+1} \to S^k$ (see, for example, [3], p. 40).

A coset of a k-dimensional sub-vector-space of R^n will be termed k-flat. A subset, A, of R^n is convex, if $a_0, a_1 \in A$ implies that $(1-\alpha)a_0+\alpha a_1$ belongs to A for $0 \leqq \alpha \leqq 1$. In other words, convexity of A means that every 1-flat intersects A in a 0-connected set. This is the property we will generalize below, replacing 1-flat by k-flat and 0-connectedness by $(k-1)$-connectedness; the title of the paper refers to this characterization of convex bodies.

THEOREM 1. *Let A be a compact subset of R^n having the following property. There exists an integer k, $1 \leqq k \leqq n-1$, such that every k-flat L of R^n intersects A in a $(k-1)$-connected set $L \cap A$. Then A is a convex body. Conversely, every convex body (i.e., compact convex set in R^n) has this property for every $k \geqq 1$.*

The statement of the theorem is trivial, if the said property holds true with $k=1$. There are n-connected, compact sets in R^n which are not convex; thus the restriction $k \leqq n-1$ is necessary.

Theorem 1 follows easily from a special case, which we state explicitly below.

THEOREM 2. *Let A be a compact subset of R^n such that every $(n-1)$-flat L intersects A in an $(n-2)$-connected set $L \cap A$. Then A is a convex body.*

The main results of this note are new elementary proofs of the theorems above, which are due to G. Aumann [4]. Our proofs are based on properties of convex hulls in R^n. As these properties may be useful in other connections, we formulate them in five lemmas below. Lemma 6 is, of course, just an appropriate form of well known facts from algebraic topology; all topological tools needed in this note are explicitly stated in connection with that lemma. As a point of fact, we need only one deep result, ([3], p. 40), of which a simple direct proof is given in [3].*

2. On the convex hull of a subset of a euclidean space. The sets

$$L = \left\{x: x \in R^n, \sum_{i=1}^{n} a_i x_i = p\right\} \qquad ((a_i) \neq (0)), \tag{3}$$

* Antoni A. Kosinski found a new proof of Theorem 1 based on his theorem concerning fixed points of multivalued transformations and hence using deeper results of algebraic topology. (To appear in Pacific J. Math.) We remark that the conditions of Theorem 1 may be stated in terms of homology or cohomology groups of $L \cap A$, but we do not dwell on this to avoid technicalities of algebraic topology.

$$\left\{x: x \in R^n, \sum_{i=1}^{n} a_i x_i \leqq p\right\}, \tag{4}$$

$$\left\{x: x \in R^n, \sum_{i=1}^{n} a_i x_i < p\right\}, \tag{5}$$

are respectively an $(n-1)$-flat, sometimes called hyperplane L of R^n, the negative closed half-space, and the negative open half-space determined by L. An $(n-r)$-flat is defined by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= p_1, \\ \cdot \qquad\qquad\quad \cdot \quad &\cdot \\ \cdot \qquad\qquad\quad \cdot \quad &\cdot \\ a_{r1}x_1 + \cdots + a_{rn}x_n &= p_r, \end{aligned} \tag{6}$$

where the r vectors $(a_{i1}, \cdots, a_{in})$, $i=1, \cdots, r$, are linearly independent.

If A is a closed set in R^n, we say that (3) is a supporting hyperplane of A, if A is contained in the closed half-space (4), and $L \cap A \neq \varnothing$. In particular, we speak about supporting hyperplanes of a closed convex set A. If A is *convex and compact*, we call it a *convex body*; in this case, A is the intersection of all such half-spaces (4). If A is again an arbitrary compact set, the intersection of all closed half-spaces containing A is a convex body C, called the *convex hull* of A. This is the smallest convex body containing A. Every supporting plane of A is a supporting plane of its convex hull C, and conversely.

If C is the convex hull of the compact set A in R^n, then, for every choice of the $n+1$ points $a_0, \cdots, a_n$ in A, the point

$$x = \sum_{i=0}^{n} \alpha_i a_i \qquad \left(\alpha_i \geqq 0, \sum_{i=0}^{n} \alpha_i = 1\right) \tag{7}$$

belongs to C. This statement is trivial, but its converse, stated in Lemma 1 below, is not entirely evident.

LEMMA 1. *Let A be a compact subset of R^n, and C its convex hull. Every point x of C can be written in the form* (7) *with appropriately chosen points $a_0, \cdots, a_n$ belonging to A.*

Proof. Let us suppose first that A is a finite set, $A = \{b_1, \cdots, b_\nu\}$. By induction on the dimension n, it is easy to see then that the statement of the lemma is true in this case.

In the general case, we choose a countable set $\{b_1, b_2, \cdots\}$ of points of A, which is everywhere dense in A. We set $A_\nu = \{b_1, \cdots, b_\nu\}$; we denote by C the convex hull of A and by C_ν the convex hull of A_ν. Then $C_1 \subset C_2 \subset \cdots \subset C$, hence the limit $C' = \lim_{\nu \to \infty} C_\nu$ exists. Furthermore, C' is a convex body containing A and contained in C, thus $C' = C$. Every point of C_ν can be represented in the form (7), by the first part of this proof. As every point of C is a limit of such points, (7) holds true for every $x \in C$, by continuity.

LEMMA 2. *Let A be a compact subset of R^n, C the convex hull of A, and L a supporting $(n-1)$-flat of C. We set $A'=L\cap A$, and $C'=L\cap C$. Then C' is the convex hull of A'.*

Proof. Let us denote by C'' the convex hull of A'; clearly $C''\subset L$. As C' is a convex body containing all points of A', we have $C'\supset C''$. We have to show that C' is, in fact, not larger than C''.

Let x be a point of C'. Then x is of the form (7), where $a_0, \cdots, a_n$ are on the negative side of L, and $x\in L$. Hence all points a_i, for which $\alpha_i\neq 0$, belong to L, thus to A'. This proves $x\in C''$; hence $C''=C'$, as stated in the lemma.

Definitions and notations. Every convex body C is contained in a smallest r-flat M; $r=n$, and $M=R^n$, if and only if C is proper, *i.e.*, contains inner points in R^n. This $M\supset C$ is unique and C is proper in M. Thus C contains inner points *in* M, though not necessarily in R^n, and its boundary *in* M is different from C. The boundary of C in M will be denoted by ∂C. Then C is homeomorphic to E^r, $r=\dim M$, and ∂C is homeomorphic to S^{r-1}, ([2], p. 3).

We will introduce now the concept of supporting k-flag of a convex body C. By definition, this is a sequence of subspaces

$$L = M_1 \supset \cdots \supset M_k \qquad (\dim M_i = n - i,\ i = 1, \cdots, k), \tag{8}$$

such that $L=M_1$ is a supporting $(n-1)$-flat of C, and, if we set

$$C_i = M_i \cap C \qquad (i = 1, \cdots, k-1), \tag{9}$$

then M_{i+1} is a supporting $(n-i-1)$-flat of C_i in M_i, $i=1, \cdots, k-1$. The flats of the sequence (8) will be represented as follows. We take a set of k $(n-1)$-flats $L_1, \cdots, L_k$, $L_1=L$, given by (6), say, and such that M_i in (8) are $L_1\cap \cdots \cap L_i$, $i=1, \cdots, k$.

LEMMA 3. *Let A be a compact subset of R^n, and C its convex hull. If A does not contain ∂C, then there is a supporting k-flag* (8) *such that $\partial(M_k\cap C)$ is contained in A.*

Proof. The statement is trivial for $n=1$. If A is a compact subset of the plane, and ∂C is not contained in A, there will be a segment on the boundary of C whose endpoints belong to A; this proves the statement in this case. Let us suppose now that the lemma is true for compact sets in spaces of dimensions $\leqq n-1$; let us use the notations of the lemma. We take a point x_0 on the boundary of C, which does not belong to A, and we call L a supporting plane of C containing x_0. If we apply now Lemma 2 to these data, we see that either $\partial(L\cap C)\subset A$, or the conditions of Lemma 3 are satisfied for the compact set $A'=L\cap A$ contained in a space of dimension $n-1$. Thus the induction hypothesis applies, and this completes the proof of the lemma.

Lemmas 4 and 5 below will show that such an $M_k\cap C$, found by Lemma 3, is the limit of sections of C by $(n-1)$-flats. These last lemmas on the convex

hull of a compact set occupy a key position in the proof of Theorem 2. They are difficult to word, however, and are not interesting in themselves, thus the reader may prefer to leave them out for the present and come back when reading the proof of Theorem 2.

Lemma 4 is a special case of Lemma 5, but we state it separately, because it is used in the proof of Lemma 5.

LEMMA 4. *Let C be a convex body, L a supporting $(n-1)$-flat of C, $M \subset L$ a supporting $(n-2)$-flat of $L \cap C$ in L. Hence $L \supset M$ is a supporting 2-flag of C. Then there is a sequence L_ν of $(n-1)$-flats going through M, and such that*

$$\lim_{\nu \to \infty} (L_\nu \cap C) = M \cap C. \tag{10}$$

Proof. By our conventions concerning supporting hyperplanes, C is on the negative side of L, and, similarly, $L \cap C$ is on the negative side L^- of M in L. Let x_ν, $\nu = 1, 2, \cdots$, be points on the positive side of L, at a distance $\geqq 1$ to M, and such that $y = \lim_{\nu \to \infty} x_\nu$ exists, and be in L on the negative side of M in L. Let us denote by L the $(n-1)$-flat going through M and containing x_ν. M divides L_ν in two open halves; we denote by L_ν^- the one containing x_ν. Clearly, $L_\nu^- \cap C = \varnothing$ and $\lim L_\nu^- = L^-$, *i.e.*, the half-plane whose closure contains $L \cap C$. Thus $\limsup (L_\nu \cap C) \subset M \cap C$. But $L_\nu \cap C \supset M \cap C$ for all ν, thus the limit exists and (10) holds true.

LEMMA 5. *Let C be a given convex body, L a supporting $(n-1)$-flat of C, and (8) a given supporting k-flag of C. Then there exists a sequence L_ν of $(n-1)$-flats containing M_k and such that*

$$\lim_{\nu \to \infty} (L_\nu \cap C) = M_k \cap C. \tag{11}$$

Proof. We will use again the convention concerning the positive half of M_i determined by M_{i+1}, $i = 0, \cdots, k-1$ (we set here $M_0 = R^n$ and $M_1 = L$). Let us choose $x_{i\nu}$, $i = 1, \cdots, k-1$, $\nu = 1, 2, \cdots$, in M_{i-1}, on the positive side of M_i, at a distance $\geqq 1$ to M_k, and such that, for a fixed i,

$$\lim_{\nu \to \infty} x_{i\nu} = y_i \qquad (1 \leqq i \leqq k-1)$$

exists, belongs to M_i, and is on the negative side of M_{i+1} in M_i; we suppose furthermore that $y_1, \cdots, y_{k-1}$ are linearly independent. Let us denote by L_ν the $(n-1)$-flat containing M_k, and the points $x_{1\nu}, \cdots, x_{k-1,\nu}$; $L_{i\nu}^-$ stands for the open half of $L_\nu \cap M_i$ which is on the positive side of M_{i+1}. Clearly, $L_{i\nu} \cap C = \varnothing$.

We use now induction in k, that is, in the length of the supporting flag. For $k = 2$, we have the statement of Lemma 4. Hence, we may suppose that the statement of Lemma 5 is true, if we replace, in the construction above, k by $k-1$. Let us denote by L_ν' the $(n-1)$-flat containing M_{k-1} and the points $x_{1\nu}, \cdots, x_{k-2,\nu}$. By the inductive hypothesis:

$$\lim_{\nu\to\infty}(L_\nu' \cap C) = M_{k-1} \cap C.$$

Hence, for a given μ and a sufficiently large ν, the point $x_{k-1,\mu}$ is separated from $M_k \cap C$ by L_ν'. This shows that no point of lim sup$(L_\nu \cap C)$ is in the positive half of M_{k-1} determined by M_k. In other words, we have lim sup$(L_\nu \cap C) \subset M_k \cap C$. Also, $L_\nu \cap C \supset M_k \cap C$; thus the limit in (11) exists and is equal to the right-hand side.

3. Topological preliminaries. For reference purposes, we formulate below well known facts from algebraic topology. Let us recall that a continuous map $f: S^l \to X$ is said to be homotopic to zero, if it has an extension $F: E^{l+1} \to X$.

LEMMA 6. *Let M be an l-flat in R^n, B a convex body in M, containing inner points in M, $D = \partial B$ the boundary of B in M; we also consider a specific homeomorphism $h: S^{l-1} \to D$. Let B_ν be a sequence of compact sub-sets of R^n, such that*

$$B_\nu \supset D \ (\nu = 1, 2, \cdots), \qquad \lim_{\nu\to\infty} B_\nu \subset B.$$

If the map $S^{l-1} \to D$ is homotopic to zero in B_ν for every ν, then $\lim_{\nu\to\infty} B_\nu = B$.

Proof. Let B_ν' be the image of B_ν under the orthogonal projection into M. We claim that $B_\nu' \supset B$ for every ν. If this would be false, we could find an $x_0 \in B - D$ and B_μ' such that $x_0 \notin B_\mu'$. Let us project B_μ' from x_0 into D. Now $S^{l-1} \to D$ is homotopic to zero in B_ν'; thus, if this projection exists, $S^{l-1} \to D$ would be homotopic to zero in D, which contradicts the fact that the identity map of S^{l-1} is not homotopic to zero (see, for example, [3], p. 40).

4. Proof of the main results. *Proof of Theorem* 2. We use the notations of Theorem 2, and we denote by C the convex hull of A. If the smallest flat containing C is of dimension $m \leqq n-1$, then we have the statement of the theorem in R^m. Hence, in what follows we will suppose that C is proper.

Let us suppose now that $\partial C \subset A$, and a point x_0 of C does not belong to A. Let L be an $(n-1)$-flat through x_0. The $(n-1)$-sphere $L \cap (\partial C)$ is then not homotopic to zero in $L \cap A$; as a point of fact, such a homotopy could be projected from x_0 into $L \cap (\partial C)$, which is a contradiction to the well known fact that the identity map $S^{n-2} \to S^{n-2}$ is not homotopic to zero. This contradiction shows that, if $\partial C \subset A$, then $A = C$.

Let us suppose now that ∂C is not contained in A. By Lemma 3, there is a supporting k-flat (8) such that $\partial(M_k \cap C) \subset A$, but some point of $M_k \cap C$ does not belong to A. By Lemma 5, there is a sequence L_ν of $(n-1)$-flats, such that (11) holds true. By hypothesis, the set $B_\nu = L_\nu \cap A$ are $(n-1)$-connected. Thus Lemma 6 applies, and shows that lim $B_\nu = M_k \cap C$; hence $M_k \cap C \subset A$, contrary to the hypothesis.

This contradiction shows that every point of C belongs to A; in other words, A is a convex body. Thus the proof of Theorem 2 is complete.

Proof of Theorem 1. If $k=n-1$ in Theorem 1, then we have the statement of Theorem 2, which has already been proved. Hence, we may suppose that $k\leqq n-2$. We consider now a $(k+1)$-flat U, and the compact set $A'=U\cap A$. Taking only k-flats contained in U, we have for A' and $n=k+1$ the data of Theorem 2. Hence A' is a convex body. This implies that A is convex, thus the proof of the theorem is complete.

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MATHEMATICAL NOTES

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A LEMMA ON MONOTONIC FUNCTIONS

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Let $f(x)$, for $x\geqq x_0$, be a positive and twice-differentiable function which tends monotonically to zero as x becomes infinite and such that its ordinary derivative, $f'(x)$, and its "bilogarithmic derivative"

$$\frac{d\log f(x)}{d\log x}=\frac{xf'(x)}{f(x)} \tag{1}$$

are monotonic for all $x\geqq x_1\geqq x_0$.

By definition (1), the bilogarithmic derivative of the positive and monotonic function $y=f(x)$ is the ordinary derivative of the same function represented in bilogarithmic coordinates: $\xi=\log x$, $\eta=\log y$; that is: $e^{\eta}=f(e^{\xi})$ or $\eta=\log f(e^{\xi})$.

There are three typical classes of functions whose bilogarithmic derivatives may be mentioned for better illustration:

Example 1. The bilogarithmic derivatives of functions of reciprocal monomial growth, $f(x)=a/x^p$, (a and p are arbitrary positive numbers), are given by $d\log f/d\log x=-p$.

Example 2. The bilogarithmic derivatives of functions of reciprocal exponential growth, $f(x)=e^{-ax^p}$, (a and p arbitrary positive numbers), are given by

$$\frac{d\log f}{d\log x}=-apx^p.$$

Example 3. The bilogarithmic derivatives of functions of reciprocal logarithmic growth, $f(x)=1/(\log ax)^p$, (a and p arbitrary positive numbers), are given by

$$\frac{d\log f}{d\log x} = -\frac{p}{\log ax}.$$

It may be noted that the ordinary and bilogarithmic derivatives of the positive and twice-differentiable function $f(x)$ are continuous for $x \geqq x_0$. Since $f(x)$ is monotonically decreasing, both derivatives of $f(x)$ are negative.

From the assumption $\lim_{x\to\infty} f(x)=0$ follows the existence of the infinite integral

$$f(x) = \int_{\infty}^{x} f'(x)dx. \tag{2}$$

Because of the monotonic integrand $f'(x)$ we see that*

$$\lim_{x\to\infty} f'(x) = 0. \tag{3}$$

It also follows, for the same reason, that the "logarithmic derivative of the first kind" of $f(x)$, that is,

$$\frac{df(x)}{d\log x} = xf'(x),$$

increases to zero, *i.e.*, that

$$\lim_{x\to\infty} \frac{df(x)}{d\log x} = \lim_{x\to\infty} xf'(x) = 0.$$

A counterpart of these limits does not exist for the bilogarithmic derivative of $f(x)$ and for the "logarithmic derivative of the second kind" of $f(x)$, that is,

$$\frac{d\log f(x)}{dx} = \frac{f'(x)}{f(x)},$$

which is commonly referred to as simply the "logarithmic derivative" of $f(x)$. Since the bilogarithmic derivative of $f(x)$ is monotonic, only the existence of its limit

$$\lim_{x\to\infty} \frac{d\log f}{d\log x} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = L$$

can be concluded (where $-\infty \leqq L \leqq 0$).

Although the bilogarithmic derivative of $f(x)$ can really approach any non-

* See, *e.g.*, G. H. Hardy, A Course of Pure Mathematics (10th ed.), Cambridge, 1952, p. 361, Problem 10.

positive value L for various functions $f(x)$, (as can be seen from Examples 1, 2, and 3), the following lemma holds:

LEMMA. *If the limit L is:*
(a) $L<0$, *then*

$$\lim_{x\to\infty} \frac{d}{d\log x}\left(\frac{d\log x}{d\log f}\right) = \lim_{x\to\infty}\left(1 - \frac{f}{xf'} - \frac{ff''}{f'^2}\right) = 0;$$

(b) $L>-\infty$, *then*

$$\lim_{x\to\infty} \frac{d}{d\log f}\left(\frac{d\log f}{d\log x}\right) = \lim_{x\to\infty}\left(1 - \frac{xf'}{f} + \frac{xf''}{f'}\right) = 0,$$

provided the second-order derivatives concerned are monotonic for sufficiently large values of x.

In order to prove the lemma, note that both derivatives in question do exist, because $f(x)$ is assumed twice differentiable, and because $f'(x)$ is negative as mentioned before. Now since $1/L$ is finite in case (a) and L finite in case (b), the existence of the infinite integrals

$$\frac{d\log x}{d\log f} = \frac{1}{L} - \int_{\log x}^{\infty} \frac{d}{d\log x}\left(\frac{d\log x}{d\log f}\right) d\log x$$

in case (a), and

$$\frac{d\log f}{d\log x} = L - \int_{\log f}^{\infty} \frac{d}{d\log f}\left(\frac{d\log f}{d\log x}\right) d\log f$$

in case (b) is guaranteed. Thus the stated lemma is proved in the same manner as equation (3) was derived from the infinite integral (2).

As was observed by M. P. Jarnagin, Jr., the statement (b) of the lemma can also be obtained by directly applying L'Hospital's* rule to the bilogarithmic derivative of $f(x)$. Indeed it is

$$L = \lim_{x\to\infty} \frac{xf'}{f} = \lim_{x\to\infty} \frac{f' + xf''}{f'},$$

because $f(x)$ is monotonic and the limit on the right side exists. Naturally also the statement (a) can be obtained by applying L'Hospital's rule to the reciprocal bilogarithmic derivative of $f(x)$. It is

$$\frac{1}{L} = \lim_{x\to\infty} \frac{f}{xf'} = \lim_{x\to\infty} \frac{f/f'}{x} = \lim_{x\to\infty} \frac{1 - (ff'')/f'^2}{1},$$

where x is monotonic and the limit on the right side exists.

* See, *e.g.*, T. J. I'a. Bromwich, An Introduction to the Theory of Infinite Series (2nd ed.) London, 1926, p. 415.

A CLASS OF ADDITIVE ARITHMETICAL FUNCTIONS

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We shall consider a certain class of additive arithmetical functions, *i.e.*, functions $f(n)$ for which $f(mn)=f(m)+f(n)$ whenever $(m, n)=1$. Let $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ be the representation of n as a product of powers of distinct primes and define $\Omega_k(n)=\alpha_1^k+\cdots+\alpha_r^k$, where k is an arbitrary nonnegative integer. It is obvious that $\Omega_k(n)$ is additive. Also, the properties of the functions $\Omega_0(n)=\omega(n)$ and $\Omega_1(n)=\Omega(n)$ are well known ([1], Sec. 22.10).

Our object is to derive an asymptotic formula for $\sum_{n\leq x}\Omega_k(n)$. An immediate consequence of this result is that the average order of $\Omega_k(n)$ is $\log\log n$ for every k. It also follows from this and other known results that $\Omega_k(n)$ has normal order $\log\log n$ ([1], Sec. 22.11).

Define $G_k(x)=\theta^k\{x/(1-x)\}$, where θ is the differential operator $x(d/dx)$. Thus, $G_k(x)=\sum_{n=1}^{\infty}n^kx^n$. It is necessary to interpret θ^0 as the identity operator in order that the following theorem be true for $k=0$. Also, the convention $0^0=0$ is followed in the proof.

THEOREM. $\sum_{n\leq x}\Omega_k(n)=x\log\log x+B_kx+O(x/\log x)$, *where* $B_k=\gamma+\sum_p\{(1-p^{-1})G_k(p^{-1})+\log(1-p^{-1})\}$ *and* γ *is Euler's constant.*

Proof.

$$\sum_{n\leq x}\Omega_k(n)=\sum_{n\leq x}\sum_{p^m\mid n}\{m^k-(m-1)^k\}=\sum_{p^m\leq x}\{m^k-(m-1)^k\}[x/p^m]$$

$$=\sum_{p\leq x}[x/p]+\sum_{p^m\leq x,\,m\geq 2}\{m^k-(m-1)^k\}[x/p^m]$$

$$=\sum_{p\leq x}[x/p]+x\sum_{p^m\leq x,\,m\geq 2}\{m^k-(m-1)^k\}p^{-m}+O\Big[\sum_{p^m\leq x}\{m^k-(m-1)^k\}\Big].$$

Now

$$\sum_{p^m>x,\,m\geq 2}\{m^k-(m-1)^k\}p^{-m}\leqq\sum_{p^m>x,\,m\geq 2}m^kp^{-m}$$

$$\leqq\sum_{p>\log x}\sum_{m=2}^{\infty}m^kp^{-m}+\sum_{p\leq\log x}\sum_{m>\log x/\log p}m^kp^{-m}.$$

But

$$\sum_{m=2}^{\infty}\sum_{p>\log x}m^kp^{-m}\leqq\sum_{m=2}^{\infty}\sum_{n>\log x}m^kn^{-m}$$

$$=O\Big[\sum_{m=2}^{\infty}m^k\Big(\frac{1}{\log x}\Big)^{m-1}\Big]=O\Big[\frac{1}{\log x}\sum_{m=0}^{\infty}(m+2)^k\Big(\frac{1}{\log x}\Big)^m\Big]=O\Big(\frac{1}{\log x}\Big).$$

If we let $\mu=\log x/\log\log x$, then

$$\sum_{p\leqq \log x}\sum_{m>\log x/\log p} m^k p^{-m} \leqq \sum_{p\leqq \log x}\sum_{m>\mu} m^k 2^{-m}$$

$$= O\left(\mu \sum_{m>\mu} m^k 2^{-m}\right) = O\left(\mu \sum_{m>\mu} e^{-\alpha m}\right) = O(\mu e^{-\alpha\mu}),$$

where $\alpha>0$. But $\mu e^{-\alpha\mu}=O(1/\mu^2)=O(1/\log x)$. Also,

$$\sum_{p^m\leqq x}\{m^k-(m-1)^k\} = \sum_{p\leqq x}\left[\frac{\log x}{\log p}\right]^k \leqq \log^k x \sum_{p\leqq x} 1/\log^k p$$

$$= \log^k x \sum_{n\leqq x}\{\pi(n)-\pi(n-1)\}/\log^k n.$$

By partial summation and Tschebycheff's inequality we have

$$\sum_{n\leqq x}\frac{\pi(n)-\pi(n-1)}{\log^k n} = \frac{\pi(x)}{\log^k x} + k\int_2^x \frac{\pi(t)dt}{t\log^{k+1} t}$$

$$= O\left(\frac{x}{\log^{k+1} x}\right) + O\left(\int_2^x \frac{dt}{\log^{k+2} t}\right) = O\left(\frac{x}{\log^{k+1} x}\right).$$

Hence,

$$\sum_{p^m\leqq x}\{m^k-(m-1)^k\} = O(x/\log x).$$

It is also known [1, 22.10] that

$$\sum_{p\leqq x}[x/p] = \sum_{n\leqq x}\Omega_0(n) = x\log\log x + B_0 x + O(x/\log x),$$

where $B_0=\gamma+\sum_p\{\log(1-p^{-1})+p^{-1}\}$. Hence,

$$\sum_{n\leqq x}\Omega_k(n) = x\log\log x + \left\{B_0 + \sum_p\sum_{m=2}^{\infty}\{m^k-(m-1)^k\}p^{-m}\right\}x + O(x/\log x).$$

But

$$\sum_{m=2}^{\infty}\{m^k-(m-1)^k\}p^{-m} = \sum_{m=1}^{\infty} m^k(p^{-m}-p^{-m-1}) - p^{-1}$$

$$= \left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty} m^k p^{-m} - p^{-1} = (1-p^{-1})G_k(p^{-1}) - p^{-1}$$

and the desired result follows.

Finally, we consider briefly the average

$$\alpha_k(n) = \frac{1}{\tau(n)}\sum_{d|n}\Omega_k(d).$$

Since,

$$\sum_{d|n} \Omega_k(d) = \sum_{0\leqq\beta_\nu\leqq\alpha_\nu} \Omega_k(p_1^{\beta_1}\cdots p_r^{\beta_r})$$

$$= \sum_{0\leqq\beta_\nu\leqq\alpha_\nu} (\beta_1^k + \cdots + \beta_r^k) = \tau(n) \sum_{\nu=1}^{r} (1^k + \cdots + \alpha_\nu^k)/(\alpha_\nu + 1)$$

we have $\alpha_k(n) = \sum_{\nu=1}^{r}(1^k + \cdots + \alpha_\nu^k)/(\alpha_\nu+1)$. Thus it is obvious that $\alpha_k(n) \leqq \frac{1}{2}\Omega_{k+1}(n)$ for $k\geqq 0$; $\alpha_k(n)\geqq\frac{1}{2}\Omega_{k-1}(n)$ for $k\geqq 1$ and $\alpha_0(n)\geqq\frac{1}{2}\Omega_0(n)$. It follows from this and the theorem that $\sum_{n\leqq x}\alpha_k(n)\sim\frac{1}{2}x\log\log x$, *i.e.*, the average order of $\alpha_k(n)$ is $\frac{1}{2}\log\log n$ for all k. It can also be shown that $\sum_{n\leqq x}\alpha_k(n) = \frac{1}{2}x\log\log x + c_k x + O(x/\log x)$. We omit the proof since it is similar to that for the case $k=0$ [2] and the above theorem.

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ON THE RATIO OF N TO π(N)

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THEOREM. *Let $\pi(n)$ be the number of primes $\leqq n$. Then for $n>1$, the ratio $n/\pi(n)$ takes on every integer value $m>1$.*

Proof. From the most elementary results of Euler on the distribution of the prime numbers, the ratio $\pi(x)/x$ ultimately becomes and remains less than any assigned $\epsilon>0$, as $x\to\infty$. It starts at $\pi(2)/2=\frac{1}{2}$.

For any $m\geqq 2$, there is thus a *unique largest prime* $p_k = p_{k(m)}$ for which $\pi(p_k) = k \geqq p_k/m$. Thus, $m\pi(p_k) = mk \geqq p_k$. Either $mk < p_{k+1}$ or $mk \geqq p_{k+1}$.

If $mk < p_{k+1}$, and since $p_k \leqq mk$, $\pi(p_k) \leqq \pi(mk) < \pi(p_{k+1})$, from which $\pi(mk) = k$, and $mk/\pi(mk) = m$, so that $n = mk$ is an integer for which $n/\pi(n) = m$.

If $mk \geqq p_{k+1}$, then $\pi(p_{k+1}) = k+1 > k = mk/m \geqq p_{k+1}/m$ which contradicts the choice of p_k as the *largest* prime for which $\pi(p)\geqq p/m$.

Using exactly the same proof, the following more general result is obtained.

COROLLARY. *Let $\{b_n\}_{n=1}^{\infty}$ be an infinite subsequence of the integers, let $\beta(x)$ be the number of terms of this sequence not exceeding x, and suppose that $\beta(x) = o(x)$ as $x\to\infty$. Then the integer values taken on by the ratio $n/\beta(n)$ are precisely the set of all integers $\geqq g_0$, for some integer g_0, $1\leqq g_0\leqq b_1$ with $g_0 = 1$ if and only if $b_1 = 1$.*

The result $1\leqq g_0\leqq b_1$ comes from $\beta(n)\leqq n$ for all n, and when $n = b_1$, $n/\beta(n) = b_1/1 = b_1$. If $b_1\neq 1$, then $\beta(n) < n$ for all n, and the ratio $n/\beta(n) = 1$ cannot occur.

Table I contains a list of the first 22 integers n for which $\pi(n)$ divides n.

TABLE I

The first 22 integers n for which $\pi(n)$ divides n.

n	$\pi(n)$	Ratio
2	1	2
4	2	2
6	3	2
8	4	2
27	9	3
30	10	3
33	11	3
96	24	4
100	25	4
120	30	4
330	66	5
335	67	5
340	68	5
350	70	5
355	71	5
360	72	5
1080	180	6
1092	182	6
1116	186	6
1122	187	6
1128	188	6
1134	189	6

EIGENVALUES OF JORDAN PRODUCTS*

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1. The following Research Problem has been proposed by Professor Taussky-Todd: *Given Hermitian matrices A and B which have their eigenvalues in the intervals*

$$m \leqq \lambda(A) \leqq M, \qquad n \leqq \lambda(B) \leqq N, \tag{1}$$

what can be said about the eigenvalues of the Jordan product $AB+BA$?

A look at the case $m=0=n$, $M=1=N$, will bring out the point of the question. The obvious estimate $|\lambda(AB+BA)| \leqq \|AB+BA\| \leqq 2\|A\|\,\|B\| \leqq 2$ will then ensure that the eigenvalues lie in the interval $[-2, 2]$. The upper bound cannot be improved (take $A=B=I$) but -2 seems extravagantly low, since the

* This research was supported by the United States Air Force under contract No. AF-49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

assumption that A and B are nonnegative definite lies unused. In fact, it would be easy to show that AB and BA each have nonnegative eigenvalues. Unfortunately, it doesn't follow that 0 is an acceptable lower bound, as the following example shows:

Let $y=(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $z=(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$ and let A and B be projections: $Av=(v, y)y$ and $Bv=(v, z)z$. Then if $x=(1, 0)$, we have $Ax=\frac{1}{2}y$, $Bx=\frac{1}{2}z$, $ABx=-\frac{1}{4}y$, $BAx=-\frac{1}{4}z$, so $(AB+BA)x=-\frac{1}{4}(y+z)=-\frac{1}{4}x$. Actually, this is an extreme case, as Theorem I will establish.

We compute below, for the general case, the best possible bounds on the eigenvalues of $AB+BA$, and simultaneously solve the same problem for the commutator $AB-BA$. Then for the choice $m=0=n$, $M=1=N$, we describe the set of all possible (complex) values of the inner product (Ax, Bx).

2. We assume now that A and B (of any order exceeding one) satisfy the inequalities (1), and admit throughout the paper only vectors of unit length. An elementary lemma is the key to our method.

LEMMA. *To any admissible A and x there correspond a vector y and a real constant $0\leqq\rho\leqq 1$ such that $Ax=mx+\rho(M-m)(x, y)y$.*

Proof. If $Ax=mx$, put $\rho=0$ and then any y will do. Otherwise, choose y in the direction of $Ax-mx$, so that $Ax-mx=\alpha y$ with $\alpha\neq 0$. Notice that $D=(A-mI)/(M-m)$ has eigenvalues in the interval $[0, 1]$, so that $D-D^2$ is nonnegative definite; this implies

$$0\leqq (x, (D-D^2)x)=(x, Dx)-(Dx, Dx)=\frac{\bar{\alpha}(x, y)}{M-m}-\frac{|\alpha|^2}{(M-m)^2}.$$

It follows that $(x, y)\neq 0$, and if we set $\alpha=\rho(M-m)(x, y)$, that $0<\rho\leqq 1$. The extreme value $\rho=1$ is assumed when D is the projection $Dx=(x, y)y$.

3. Before applying the lemma to both A and B, we point out that for any triad x, y, z, we can choose an orthogonal triad e_1, e_2, e_3 (omit e_3 if the space has dimension two) so that

$$x=e_1, \qquad y=e_1\cos\theta e^{i\alpha}+e_2\sin\theta,$$
$$z=e_1\cos\phi e^{i\beta}+e_2\sin\phi\cos\psi e^{i\gamma}+e_3\sin\phi\sin\psi.$$

Our object is to bound the eigenvalues of $AB+BA$, or, in other words, to find the *extreme values* of

$$\begin{aligned}((AB+BA)x, x) &= 2\,\mathrm{Re}(Ax, Bx)\\ &= 2\,\mathrm{Re}(mx+\rho(M-m)(x, y)y, nx+\rho'(N-n)(x, z)z)\\ &= 2[mn+\rho(M-m)n\cos^2\theta+\rho'(N-n)m\cos^2\phi\\ &\quad +\rho\rho'(M-m)(N-n)\cos^2\theta\cos^2\phi]\\ &\quad +2\,\mathrm{Re}[\rho\rho'(M-m)(N-n)\cos\theta\cos\phi\sin\theta\sin\phi\cos\psi e^{i(\beta-\alpha-\gamma)}].\end{aligned} \tag{2}$$

The extrema clearly occur when all of ψ, α, β, γ, vanish, and the problem actually lies in real 2-space. Furthermore, because the expression is linear in ρ and ρ', we need consider only their extreme values 0 and 1. In case $\rho=1=\rho'$, we define

$$a = (M+m)(N+n), \qquad b = (M-m)(N+n),$$
$$c = (M+m)(N-n), \qquad d = (M-m)(N-n),$$

and the expression (2) whose extreme values we want simplifies to

$$\tfrac{1}{2}(a + b\cos 2\theta + c\cos 2\phi + d\cos 2(\theta-\phi)).$$

The partial derivatives with respect to θ and ϕ vanish first of all if θ and ϕ both are multiples of $\frac{1}{2}\pi$; this gives rise to the four possible extrema

$$E_1 = \tfrac{1}{2}(a+b+c+d) = 2MN, \qquad E_2 = \tfrac{1}{2}(a+b-c-d) = 2Mn,$$
$$E_3 = \tfrac{1}{2}(a-b+c-d) = 2mN, \qquad E_4 = \tfrac{1}{2}(a-b-c+d) = 2mn.$$

The interest in the problem derives from the other point where the partial derivatives vanish, namely

$$\cos 2\theta = \frac{1}{2}\left(\frac{cd}{b^2} - \frac{c}{d} - \frac{d}{c}\right), \qquad \cos 2\phi = \frac{1}{2}\left(\frac{bd}{c^2} - \frac{b}{d} - \frac{d}{b}\right).$$

If both of these unwieldy expressions have magnitude less than one, there is a fifth candidate

$$E_5 = \frac{1}{4}\left(\frac{bc}{d} - \frac{db}{c} - \frac{cd}{b}\right) = \frac{16MmNn - (M-m)^2(N-n)^2}{4(M+m)(N+n)}.$$

A glance at (2) shows that the possible extrema contributed by the three cases $\rho=0=\rho'$, $\rho=0$, $\rho'=1$, and $\rho=1$, $\rho'=0$, come from among E_2, E_3, and E_4. The answer to Professor Taussky-Todd's question then is

THEOREM I. *If A and B satisfy* (1), *the best possible bounds for the eigenvalues of $AB+BA$ are the least and greatest of E_1, E_2, E_3, E_4, and in certain cases, E_5.*

4. The corresponding problem for the commutator $AB-BA$ is now easy. Since $i(AB-BA)$ is Hermitian, we want the extreme values of

$$(i(AB-BA)x, x) = 2\,\mathrm{Im}(Ax, Bx)$$
$$= 2\,\mathrm{Im}[\rho\rho'(M-m)(N-n)\cos\theta\cos\phi\sin\theta\sin\phi\cos\psi e^{i(\beta-\alpha-\gamma)}].$$

THEOREM II. *The best possible bounds for the eigenvalues of $i(AB-BA)$ are* $\pm\frac{1}{2}(M-m)(N-n)$.

5. Finally, for the special choice $m=0=n$, $M=1=N$, one can solve the more inclusive problem of finding all possible values of $(Ax, Bx) = \rho\rho'(x, y)(y, z)(z, x)$. The extreme cases lie, as before, in 2-dimensional space

(now necessarily complex), but the calculations are too tedious to record here, and we simply state their result.

THEOREM III. *The values $re^{i\theta}$ which can be assumed by the inner product (Ax, Bx) are those such that*

$$0 \leqq r \leqq \left(\frac{\cos \frac{1}{3}\pi}{\cos \frac{1}{3}(\pi - |\theta|)}\right)^3, \quad -\pi \leqq \theta \leqq \pi.$$

Each of these theorems is valid in Hilbert space, since a counterexample there would yield a counterexample in the space spanned by x, Ax, and Bx.

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MATRICES WITH TRACE ZERO

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1. Introduction. One of the results of this note is an alternative proof of the following theorem which can be proved by quite elementary means. It is reproved here in Section 2 via unitary matrices.

THEOREM. *The equations*

$$\sum_1^n \lambda_i = 0, \qquad |\lambda_i| = 1 \tag{1}$$

imply for $n=2$ that the λ_i are the end points of a diameter; for $n=3$ that the λ_i are the vertices of an equilateral triangle; for $n=4$ that the λ_i are the endpoints of two diameters.

Also the remainder of the note deals mainly with unitary matrices and with alternative proof for facts discovered previously.

In Section 3, it is shown how a unitary matrix with trace zero can be transformed into a unitary matrix with only zeros in the diagonal, via a unitary similarity.*

It was pointed out by H. Zassenhaus that a unitary matrix with a zero element in the diagonal has its eigenvalues on an arc at least a semicircle [2, 3]. An alternative proof for this is given here in Section 4, using the concept of the field of values of a matrix.

Finally, an alternative proof is given in Section 5 for the theorem mentioned in [4] that an arbitrary (not necessarily unitary) matrix with complex numbers as elements, of trace zero has zero as an inner point of the field of values, when "inner" is meant in the sense of convex bodies.

* It was shown by K. Shoda [1] that every matrix with trace zero is similar to a matrix with a zero diagonal. It is further known that a unitary similarity can always be used to achieve this (oral communication from T. Kato, H. Wielandt).

2. Let $\sum_1^n \lambda_i = 0$, $|\lambda_i| = 1$. Consider a unitary matrix U with λ_i as eigenvalues. Such a matrix certainly exists since the diagonal matrix formed by the λ_i is unitary. The characteristic polynomial of U has zero as coefficient of x^{n-1}. Use next the fact that in a unitary matrix every element is equal to the product of the complex conjugate of its cofactor and the determinant of the whole matrix. It follows that also the coefficient of x in the characteristic polynomial of U vanishes. This implies the above Theorem for $n = 2, 3, 4$.

3. Let Λ be the diagonal matrix diag $(\lambda_1, \cdots, \lambda_n)$ and $\sum_1^n \lambda_i = 0$. Interpret this equation as the following set of n equations:

$$\sum_{i=1}^{n} \lambda_i x_{ik} \bar{x}_{ik} = 0, \qquad (k = 1, \cdots, n) \tag{2}$$

where $x_{ik}\bar{x}_{ik} = 1/n$ for all i, $k = 1, \cdots, n$. Choose the complex numbers x_{ik} further so that the matrix $X = (x_{ik})$ is unitary. Such a matrix can be given explicitly, independently of the λ_i, *e.g.*,

$$X = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & \xi & \xi^2 & \cdots \\ 1 & \xi^2 & \xi^4 & \cdots \\ 1 & \xi^3 & \xi^6 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

where ξ is a primitive nth root of unity.

The equations (2) imply that $(\Lambda x, x) = 0$ for every column vector $x = (x_{1k}, \cdots, x_{nk})$, $k = 1, \cdots, n$. Consider then the matrix $X^{-1}\Lambda X$. The equations (2) and the fact that X is unitary imply that $(X^{-1}\Lambda X \cdot X^{-1}x, X^{-1}x) = (X^{-1}\Lambda x, X^{-1}x) = 0$. Since $X^{-1}x$ is a coordinate unit vector for $k = 1, \cdots, n$, it follows that the matrix $X^{-1}\Lambda X$ has its diagonal elements zero. Hence a *unitary matrix with trace zero is unitarily similar to a unitary matrix with only zeros in the diagonal.*

4. Let U be a unitary matrix. Its field of values is, by definition, the set of complex numbers (Ux, x) for $(x, x) = 1$. If U has a zero element in the diagonal then (Ux, x) includes the zero point. On the other hand, it is known that for a unitary (even more generally, normal) matrix the field of values coincides with the convex hull of the eigenvalues. Hence 0 lies in the convex hull of a set of points on the unit circle. This implies *that the eigenvalues must include an arc at least as large as a semicircle.**

5. It will now be shown that the field of values of a matrix $A \neq 0$ of trace zero contains zero as an inner point, *i.e.* that zero is a positive combination of at least two other points in the field of values. Since the trace is zero the arithmetic

* The possibility of a connection with convex sets was also noticed by R. D. Ryan.

mean of the eigenvalues is zero. Since the eigenvalues are in the field of values it follows that zero is included in it too. If at least one of the eigenvalues is different from 0, this implies that at least two are different from 0 and hence zero is an inner point because trace $A=0$. Assume then that all eigenvalues are 0 and transform A unitarily to (upper) triangular form. The field of values remains invariant. If then the matrix contains an element $\alpha_{ik}=0$ in the (i, k) place, we take the vector x with $x_i=x_k=1/\sqrt{2}$ and zeros elsewhere. This gives the value $(Ax, x)=\frac{1}{2}\alpha_{ik}$. Then take the vector x with $x_i=1/\sqrt{2}$, $x_k=-1/\sqrt{2}$, and zeros elsewhere. This gives the value $(Ax, x)=-\frac{1}{2}\alpha_{ik}$. Hence 0 is an inner point.*

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ELEMENTARY OPERATIONS AND SYSTEMS OF LINEAR EQUATIONS

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By an *elementary operation* (eo) on a system of linear equations in $x_1, \cdots, x_n$ with coefficients in a field F, is meant one of the following:

I. The interchange of two equations of the system.

II. The multiplication of an equation of the system by a nonzero element of F.

III. The addition to any equation of the system of a constant multiple of another equation of the system.

IV. The adjunction to or deletion from the system of the equation $0x_1+\cdots+0x_n=0$.

It is well known that the application of a finite sequence of eo to a system of linear equations yields an equivalent system, that is, one with the identical solution set. This note is concerned with a direct proof of a converse of this result.

THEOREM. *If two consistent systems of linear equations are equivalent, then one is obtainable from the other by a finite sequence of eo.*

Proof. (i) The reduction to the case of homogeneous systems. Let $AX=B$ and $A'X=B'$ be two consistent and equivalent systems. Then the respective associated homogeneous systems $AX=0$ and $A'X=0$ are equivalent. If $A'X=0$ is obtainable from $AX=0$ by a sequence of eo then $A'X=B'$ is obtainable from $AX=B$ by the same sequence. Indeed, if τ denotes the resultant of a sequence

* As M. P. Drazin pointed out to the author, the argument could also be carried out via the diagonal elements.

$i=1, \cdots, r$, so that (H_3) and (H_3') are identical. Thus the equivalence of (H) and (H′) implies that both are reducible by eo to a common system. Since eo are reversible, it follows that each of the given systems can be transformed into the other by eo, thereby completing the proof.

In conclusion it is of interest to note that the assumption of consistency is necessary. For example, the following pair of inconsistent (thus *a priori* equivalent) systems cannot be obtained from each other by eo:

$$\text{(A)}\ x = 1,\quad x = 2; \qquad \text{(B)}\ 0x = 3.$$

COMMENTS ON SUCCESSIVE APPROXIMATION APPLIED TO QUADRATURE FORMULAS

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In a recent note, *Successive approximation applied to quadrature formulas*,* Struble and Miller established a *formal* series solution for the differential equation $dy/dx=f(x)$ by using Simpson's rule in an iterative fashion. For $f(x)=x^n$, it was verified that this formal solution is a valid solution for $n<10.96$, approximately. For larger values of n, the formal series diverges, and it was concluded that formulas such as Simpson's rule do not yield arbitrarily good results when the increment of the independent variable is chosen sufficiently small, rather than drawing the conclusion that the formal procedure yields no explicit information in certain cases.

For $f(x)$ continuous and x finite, Simpson's rule gives, for a subdivision into $2j$ equal intervals, $S_{2j}=\frac{1}{3}\Delta[f(0)+4f(\Delta)+2f(2\Delta)+\cdots+f(2j\Delta)]$, where $\Delta=x/(2j)$, for $\int_0^x f(t)dt$. Assuming a value of A for the definite integral, we have

$$2\cdot\Delta[f(\Delta)+f(2\Delta)+\cdots+f(2j\Delta)]\to 2A,\ (2\Delta)[f(\Delta)+f(3\Delta)+\cdots+f((2j-1)\Delta)]\to A,$$

and hence $S_{2j}\to\frac{1}{3}(0+2A+A)=A$.

Their result does bring out the interesting point that the absolute value of the relative error obtained by using $\frac{2}{3}x[f(0)+4f(x/2)+f(x)]$ as an approximation to $\int_0^x f(t)dt$, for $f(x)=x^n$, is independent of x and tends to infinity with n.

However, it is somewhat misleading to associate the procedure followed with that ordinarily used in applying Simpson's rule. Here the procedure is confined solely to the use of a 3-point rule. Thus, beginning with $dy/dx=Dy=f(x)$, y is approximated by $y_1=Sf(x)\equiv\frac{1}{6}x[f(0)+4f(\frac{1}{2}x)+f(x)]$. Letting $y_2=y-y_1$, it is found that $dy_2/dx=(1-DS)f(x)$, and in turn y_2 is approximated by the same procedure as y_1 with $(1-DS)f(x)$ replacing $f(x)$, etc. Thus the procedure gives rise to a completely different method of approximation. Further, if the operator $1-DS$ is applied to a power series, the coefficient of x^n in the resulting series becomes numerically larger than the corresponding coefficient in the original series for $n>10$, and repeated application leads to the conclusion that such coefficients are unbounded. Thus it seems unlikely that the procedure will converge for a very large class of functions.

* This MONTHLY, vol. 67, 1960, pp. 661–664.

CLASSROOM NOTES

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A METHOD FOR EVALUATING $\int x^n e^{-kx}dx$

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We will show how the exponential shift [1] yields a method for evaluating $\int x^n e^{-kx}dx$, n a positive integer, $k \neq 0$, avoiding successive integrations by parts. The operational methods used can be verified directly and therefore can be given in an elementary calculus course.

Let

$$\int x^n e^{-kx}dx = u + C, \qquad y = e^{kx}u,$$

where u is the particular continuous integral of $Du = x^n e^{-kx}$ that vanishes for $kx = +\infty$. Then by the exponential shift we have

$$x^n = e^{kx}Du = (D - k)(e^{kx}u) = (D - k)y. \tag{1}$$

A particular solution y of (1) is

$$y = \frac{1}{D - k}(x^n) = -\frac{1}{k}\frac{1}{1 - D/k}(x^n) = -\frac{1}{k}\sum_{r=0}^{\infty}\left(\frac{D}{k}\right)^r(x^n) \tag{2}$$

provided the latter series converges. But since $D^r x^n = 0$ for $r > n$, the series in (2) has only $n+1$ nonvanishing terms; we have

$$\int x^n e^{-kx}dx = e^{-kx}y + C = -e^{-kx}\sum_{r=0}^{n}\frac{D^r(x^n)}{k^{r+1}} + C.$$

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A NATURAL AUXILIARY FUNCTION FOR THE MEAN VALUE THEOREM

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The usual classroom proof of the mean value theorem is based on the auxiliary function g: $g(x) = f(x) - f(a) - m(x - a)$, where $m = \{f(b) - f(a)\}/(b - a)$, to which Rolle's theorem is applied. Consideration of g is motivated by the observation that x_0 is likely to occur as a value of x at which the vertical directed distance over x from the chord containing $(a, f(a))$ and $(b, f(b))$ to the graph of f is an extreme (Fig. 1).

There are a number of pedagogical disadvantages to this approach. In the first place, it seems more natural to rotate the xy-coordinate system than to look at the space between the graph and the chord. Consider the result of rotating the xy-system so that the x'-axis (the new x-axis) is parallel to the chord. Then Rolle's theorem applies to the function h: $h(x)$ = the directed distance from the x'-axis to the point $(x, f(x))$ (Fig. 2).

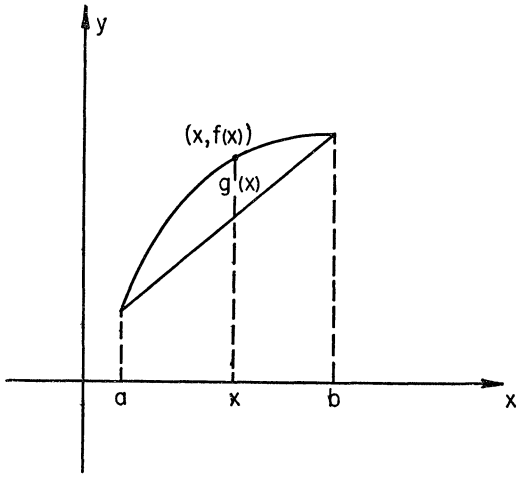


FIG. 1

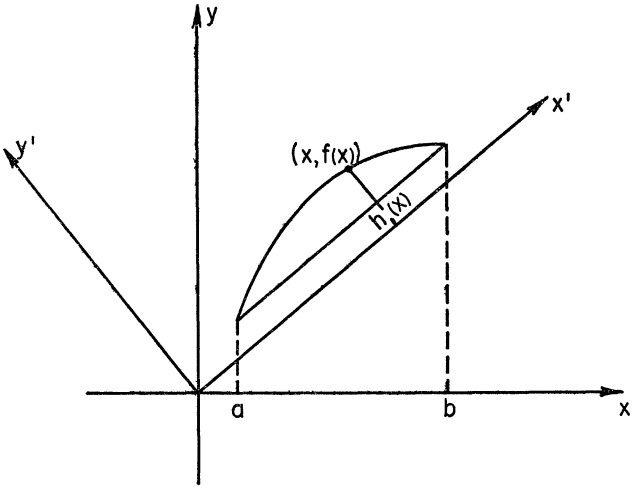


FIG. 2

It is easy to express h in terms of f: the directed distance from a line $y=mx+b$ to a point (x_1, y_1) is $(y_1-mx_1-b)/\sqrt{(m^2+1)}$, and an equation of the x'-axis is $y=mx$, so that $h(x)=f(x)-mx/\sqrt{(m^2+1)}$.

The student will find it obvious geometrically (and easy enough algebraically) that $h(a)=h(b)$. Note also that he will encounter no difficulty in differentiating h, since the denominator is a constant (and he will not have to contend with $f(a)$, which occurs in g and gives him pause). The main pedagogical advantage in h, I think, is that it gives a very easy rigorous proof along lines that parallel perfectly the natural intuitive approach.

Although $\sqrt{(m^2+1)}$ is not really needed, retention of this denominator presents no complication whatever, and gives the natural geometric interpretation to h. Some students will realize this, and might be encouraged to investigate alternative auxiliary functions; others are apt to be confused if the dispensability of $\sqrt{(m^2+1)}$ is discussed before the proof is completed.

A suitable modification H of h can be used to similar advantage as an auxiliary function for Cauchy's theorem: *If f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(t)\neq 0$ on (a, b) then there is a number t_0 in (a, b) such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(t_0)}{g'(t_0)}.$$

Here $H(t)=\{f(t)-mg(t)\}/\sqrt{(m^2+1)}$ and, with the parametric equations $y=f(t)$, $x=g(t)$, the geometric interpretation is the same.

BOUNDARY CONDITIONS FOR THE LINEAR DIFFERENTIAL EQUATION

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1. Introduction. We consider the differential equation

$$y^{(n)}=\sum_{h=0}^{n-1}P_h(x)y^{(h)}+Q(x), \qquad y^{(0)}=y, \tag{1}$$

where, throughout this paper, $P_h(x)$ and $Q(x)$ represent continuous functions over an interval $[a, b]$. Boundary conditions of the type $y^{(h)}(\alpha_h)=\beta_h$ are desired where the α_h are restricted only to the requirement that they belong to $[a, b]$ and where the β_h are arbitrarily assigned real numbers.

It is easily seen that the conditions $y(\alpha_0)=\beta_0$, $y'(\alpha_1)=\beta_1$ cannot be satisfied uniquely by a solution of $y''=-k^2y$, $k>0$, unless $|\alpha_1-\alpha_2|<\frac{1}{2}\pi/k$. Hence, if the α_h are to remain unrestricted, conditions on the $P_h(x)$ must be imposed. In the present paper such conditions are established which assure a unique solution of (1) satisfying two-point and three-point boundary conditions.

Nicolletti [1] obtained results similar to the results of this note, but he placed strong limitations on the α_i. Also, similar results have been obtained for the nonlinear differential system (see [2]) and for the linear differential system (see [3]). However, the results here cannot be obtained from [3] by replacing (1) by the equivalent differential system.

2. A two-point boundary condition. For points α_1, $\alpha_2(\alpha_1<\alpha_2)$ of $[a, b]$ we require the following over $[\alpha_1, \alpha_2)$:

A. $P_k(x)$, $k=0, \cdots, n-2$, do not all have a common zero.

B. $P_{n-j}(x) \geqq 0$, j even; $P_{n-j}(x) \leqq 0$, j odd, $j=2, \cdots, n$.

THEOREM 1.* *Let the $P_k(x)$ satisfy* (A), (B) *for some α_1, α_2 of $[a, b]$ and, for a given m, $0 \leqq m \leqq n-1$, let $P_m(\alpha_2) \neq 0$ if $m \neq n-1$. Then there exists a unique solution of* (1) *satisfying*

$$y^{(m)}(\alpha_1) = \beta_m, \qquad y^{(h)}(\alpha_2) = \beta_h, \qquad h = 0, 1, \cdots, n-1; h \neq m. \tag{2}$$

Proof. Let $y_i(x)$, $i=1, \cdots, n$, be the solutions of

$$y^{(n)} = \sum_{h=0}^{n-1} P_h(x) y^{(h)}$$

satisfying

$$y_i^{(h)}(\alpha_2) = \begin{cases} 1 & h = i-1, \\ 0 & h \neq i-1, \end{cases} \tag{3}$$

and let $y_p(x)$ be a particular solution of (1). Then the general solution $y(x)$ of (1) is

$$y(x) = \sum_{i=1}^{n} c_i y_i(x) + y_p(x). \tag{4}$$

Imposing the conditions (2) and (3) on (4) and the equations obtained by differentiating (4) $n-1$ times, we obtain

$$c_{h+1} = \beta_h - y_p^{(h)}(\alpha_2), \qquad h = 0, \cdots, n-1; h \neq m;$$
$$c_1 y_1^{(m)}(\alpha_1) + \cdots + c_{m+1} y_{m+1}^{(m)}(\alpha_1) + \cdots + c_n y_n^{(m)}(\alpha_1) = \beta_m - y_p^{(m)}(\alpha_1).$$

This system has a unique solution for the c_i, hence the theorem follows, if $y_{m+1}^{(m)}(\alpha_1) \neq 0$.

Assume that at least one of $y_{m+1}^{(h)}(x)$, $h=0, 1, \cdots, n-1$, has a zero on $[\alpha_1, \alpha_2)$. By using the hypotheses and (3) we find that $y_{m+1}^{(n)}(\alpha_2) \neq 0$; hence there exists a largest zero of all the zeros of these functions on this interval. We denote it by c and consider

Case 1. $m=n-j_1$, j_1 even. We find $y_{m+1}^{(n)}(\alpha_2) > 0$ which, with (3), implies

$$\begin{cases} y_{m+1}^{(n-j)}(x) \begin{cases} > 0, & j \text{ even} \\ < 0, & j \text{ odd} \end{cases} & j = 2, \cdots, n, \; x \in [c, \alpha_2) \\ y_{m+1}^{(n-1)}(x) < 0, & x \in (c, \alpha_2). \end{cases} \tag{5}$$

* A dual theorem is obtained for $\alpha_2<\alpha_1$ by imposing (A) over $(\alpha_2, \alpha_1]$ and requiring $P_k(x) \geqq 0$ over $(\alpha_2, \alpha_1]$ for $k=0, 1 \cdots, n-2$.

Hence $y_{m+1}^{(n-1)}(c)=0$ and $y_{m+1}^{(n)}(c)\leqq 0$. But, using (A), (B) and (5), we find

$$y_{m+1}^{(n)}(c)=\sum_{h=0}^{n-2}P_h(c)y_{m+1}^{(h)}(c)>0$$

and, thus, a contradiction to the choice of c.

Case 2. $m=n-j_2$, j_2 odd. A contradiction to the choice of c is obtained for this case in a similar manner as that for Case 1. Hence $y_{m+1}^{(h)}(x)\neq 0$ on $[\alpha_1, \alpha_2)$ for $h=0, 1, \cdots, n-1$, and the theorem follows.

COROLLARY 1. *Let the* $P_k(x)$ *satisfy* (A), (B) *over* $[a, b]$ *and let* $P_m(x)\neq 0$ *on* $[a, b]$. *Then Theorem* 1 *is valid without restricting the boundary points* α_1, α_2 *further than requiring them to belong to* $[a, b]$.

3. A three-point boundary condition. We now consider (1) with the boundary conditions

$$\begin{cases} y^{(e)}(\alpha_1)=\beta_e, \\ y^{(h)}(\alpha_2)=\beta_h, \qquad h=0, 1, \cdots, n-1, h\neq e, f, \\ y^{(f)}(\alpha_3)=\beta_f, \end{cases} \tag{6}$$

where $f(1\leqq f\leqq n-1)$ and $e(e<f)$ are such that $f=n-j_1$, j_1 odd and $e=n-j_2$, j_2 even, and where $\alpha_1, \alpha_2, \alpha_3(\alpha_1\leqq\alpha_2\leqq\alpha_3)$ are points of $[a, b]$. We impose the following over $[\alpha_1, \alpha_3]$:

C. $P_h(x)$, $h=0, 1, \cdots, f$, do not all have a common zero.

D. $P_{n-j}(x)\geqq 0$, j even; $P_e(\alpha_2)>0$,

$$P_{n-j}(x)\begin{cases}\leqq 0, & x\in[\alpha_1, \alpha_2), \\ \geqq 0, & x\in(\alpha_2, \alpha_3], \end{cases} \quad j \text{ odd},$$

$$P_h(x)\equiv 0,\ h=f+1, \cdots, n-1.$$

THEOREM 2. *Let the* $P_h(x)$ *of* (1) *satisfy* (C), (D) *for some* e *and* f *and for some* $\alpha_1, \alpha_2, \alpha_3$ *of* $[a, b]$. *Then there exists a unique solution of* (1), (6).

Proof. Let the $y_i(x)$ and $y(x)$ be defined as in the proof of Theorem 1. Imposing (2) and (3) as in the preceding proof, we obtain

$$c_{h+1}=\beta_h-y_p^{(h)}(\alpha_2),\ h=0, 1, \cdots, n-1, h\neq e, f,$$

$$\sum_{i=1}^{n}c_iy_i^{(e)}(\alpha_1)=\beta_e-y_p^{(e)}(\alpha_1), \qquad \sum_{i=1}^{n}c_iy_i^{(f)}(\alpha_3)=\beta_f-y_p^{(f)}(\alpha_3).$$

There is a unique solution for the c_i, hence (6) can be satisfied uniquely, if

$$y_{e+1}^{(e)}(\alpha_1)y_{f+1}^{(f)}(\alpha_3)-y_{f+1}^{(e)}(\alpha_1)y_{e+1}^{(f)}(\alpha_3)\neq 0.$$

We now establish this fact.

a) $y_{e+1}^{(e)}(x)>0$ over $[\alpha_1, \alpha_2)$. Since $e=n-j_2$, j_2 even, this inequality follows in a manner similar to the proof of Theorem 1, Case 1.

b) $y_{f+1}^{(e)}(x)<0$ over $[\alpha_1, \alpha_2)$. Since $y_{f+1}^{(f)}(\alpha_2)=1$, the inequalities

$$y_{f+1}^{(f-j)}(x) < 0, j \text{ odd}; \quad y_{f+1}^{(f-j)}(x) > 0, j \text{ even};$$

hold to the immediate left of $x=\alpha_2$. This, with (C), (D) and the condition on f, gives $y_{f+1}^{(n)}(x)<0$ to the left of α_2. This inequality and (3) imply that $y_{f+1}^{(f+1)}(x)$, $\cdots$, $y_{f+1}^{(n-1)}(x)$ are also nonzero to the immediate left of α_2. Assume that at least one of $y_{f+1}^{(h)}(x)$, $h=0, 1, \cdots, n-1$, has a zero on $[\alpha_1, \alpha_2)$ and let $x=c$ denote the largest. Then $y_{f+1}^{(k)}(c)\neq 0$, $k=0, 1, \cdots, n-2$, and $y_{f+1}^{(n-1)}(c)=0$. Now (3) and the sign of $y_{f+1}^{(n)}(x)$ imply that $y_{f+1}^{(n-1)}(x)>0$ on (c, α_2); hence $y_{f+1}^{(n)}(c)\geqq 0$. But

$$y_{f+1}^{(n)}(c) = \sum_{h=0}^{f} a_h(c) y^{(h)}(c) < 0.$$

Hence we have a contradiction to the choice of c and the above inequalities, in particular $y_{f+1}^{(e)}(x)<0$, hold over $[\alpha_1, \alpha_2)$.

Proved in a similar manner to b) are the inequalities

$$y_{f+1}^{(f)}(x) > 0, \qquad y_{e+1}^{(f)}(x) > 0, \qquad x \in (\alpha_2, \alpha_3].$$

Uniqueness of the solution for the c_i now follows for $\alpha_1<\alpha_2<\alpha_3$, $\alpha_1=\alpha_2<\alpha_3$, or $\alpha_1<\alpha_2=\alpha_3$. This gives a unique solution of (1), (6).

COROLLARY 2. *Let the $P_h(x)$ satisfy* (C), (D) *over* $[a, b]$ *and let* $P_e(x)>0$ *on* $[a, b]$. *Then Theorem 2 is valid without restricting the points* $\alpha_1, \alpha_2, \alpha_3$ *further than requiring them to belong to* $[a, b]$.

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METHODS OF PROVING MEAN VALUE THEOREMS

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The primary purpose of this note is to emphasize the equivalence of various proofs of the extended law of the mean, and to indicate yet more general results.

Let a curve C be defined parametrically by $x(t)$, $y(t)$, where these functions are continuous on $a\leqq t\leqq b$ and differentiable on $a<t<b$. Designate the constant vector from A to B by $\overrightarrow{K}=[x(b)-x(a), y(b)-y(a)]$ and the variable vector from A to P by $\overrightarrow{V}(t)=[x(t)-x(a), y(t)-y(a)]$ (Fig. 1). The magnitude of $\overrightarrow{K}\times\overrightarrow{V}(t)$ is equal to the area of parallelogram $APBR$ and may be expressed in terms of a

determinant [1] which is proportional to the length of PQ [2]. Thus, it is geometrically evident that Rolle's theorem may, in general, be applied to

$$\phi(t) = k_1 \left| \vec{K} \times \vec{V}(t) \right| + k_2, \qquad k_1, k_2 \text{ constants}, \qquad k_1 \neq 0. \tag{1}$$

As an immediate consequence, we obtain the extended law of the mean which asserts that there exists at least one point T such that $a < T < b$ and

$$x'(T)[y(b) - y(a)] = y'(T)[x(b) - x(a)]. \tag{2}$$

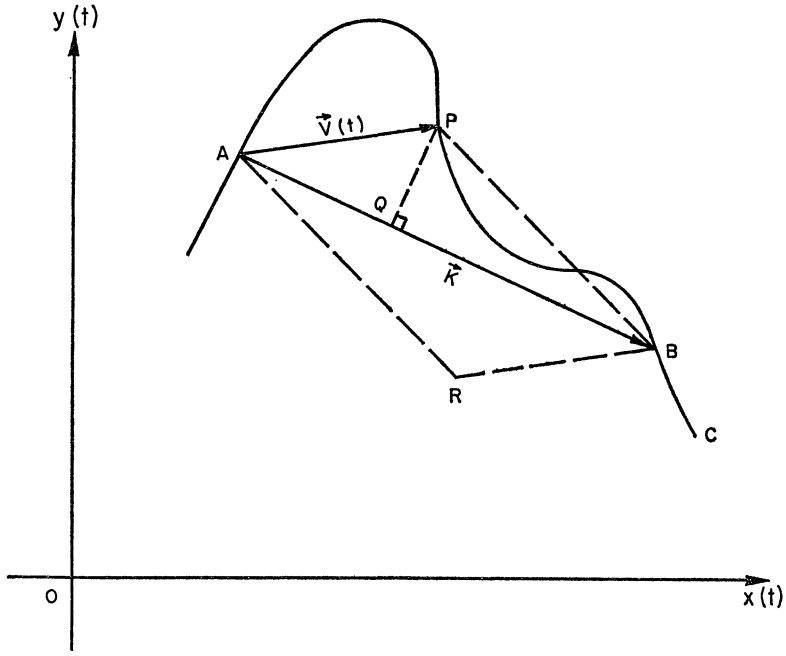


FIG. 1

An equivalence of the usual analytic, geometric, vector, and determinant methods of deriving (2) is now apparent, for the success of each approach ultimately depends upon an application of Rolle's theorem to a function of type (1). But the use of (1) to derive (2) is just a special instance of the following result.

Let $F(t) = f[x(t), y(t)]$ be any t-differentiable function such that $F(a) = F(b)$. Then Rolle's theorem may be applied to $F(t)$ and we find that there exists at least one value T such that $a < T < b$ and

$$x'(T)[f_x]_{t=T} = -y'(T)[f_y]_{t=T}. \tag{3}$$

If $F(t) = x(t) - ky(t)$ is a linear function of x and y (as is ϕ), then (3) particularizes to (2) at once.

Clearly, many other generalized laws of the mean may be generated by extending Rolle's theorem to suitably modified differentiable functions of three or more variables.

References

1. L. C. Barrett and R. A. Jacobson, Extended laws of the mean, this MONTHLY, vol. 67, 1960, pp. 1005–1007.
2. R. C. Yates, The law of the mean, this MONTHLY, vol. 66, 1959, pp. 579–580.

CAUCHY'S CRITERION FOR CONVERGENCE OF SEQUENCES

W. J. FIREY AND M. S. KNEBELMAN, Washington State University

1. Cauchy's criterion is usually omitted or stated without proof in introductory calculus courses. The proofs given in books on advanced calculus are usually based on notions that are rather too sophisticated for beginners. The purpose of this note is to furnish a reasonably simple proof that is based on the completeness of the real number system and its immediate consequences. Since the proof of the necessity is straightforward we shall prove only its sufficiency.

2. We state the following lemmas without proof as the proofs are direct consequences of definitions.

A bounded nondecreasing sequence converges to its l.u.b.

A bounded nonincreasing sequence converges to its g.l.b.

(1) *If* $\{x_n\}$ *is a bounded sequence so is any subsequence* $\{x_{n_i}\}$ *of it and* l.u.b.$\{x_{n_i}\} \leqq$ l.u.b.$\{x_n\}$, g.l.b.$\{x_{n_i}\} \geqq$ g.l.b.$\{x_n\}$.

(2) *If* $\{x_n\}$ *is nondecreasing,* $\{y_n\}$ *is nonincreasing and* $y_n \geqq x_n$ *while* $\lim_{n\to\infty}(y_n - x_n) = 0$, *then both sequences converge to the same limit.*

(3) *If* $m_n \leqq x_n \leqq M_n$ *and* $\lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n = L$, *then* $\lim_{n\to\infty} x_n = L$.

3. We state Cauchy's criterion as follows:

If $\{x_n\}$ *is a sequence for which* $|x_{n+p} - x_n| < \epsilon/3$ *for all indices* $p > 0$ *and* $n > N(\epsilon)$, *the sequence converges.*

Let n_0 be a fixed index; then $x_{n_0} - \frac{1}{3}\epsilon < x_{n_0+p} < x_{n_0} + \frac{1}{3}\epsilon$ so the subsequence $\{x_{n_0+p}\}$ is bounded. Let M_{n_0} and m_{n_0} be its l.u.b. and g.l.b.; then $M_{n_0} \leqq x_{n_0} + \frac{1}{3}\epsilon$, $m_{n_0} \geqq x_{n_0} - \frac{1}{3}\epsilon$ so that $0 < M_{n_0} - m_{n_0} \leqq \frac{2}{3}\epsilon < \epsilon$. If we choose any subsequence $\{x_{n_0+q+p}\}$ then by (1) $M_{n_0+q} \leqq M_{n_0}$ and $m_{n_0+q} \geqq m_{n_0}$. Hence for all $n > n_0$, $|M_n - m_n| < \epsilon$. Since $\{M_n\}$ is nonincreasing and $\{m_n\}$ is nondecreasing, the two sequences satisfy all conditions of (2) and therefore converge to a common limit. Then by (3) the sequence $\{x_n\}$ converges.

NOTE

It has come to my attention that the example in *Simple constructions of nondifferentiable functions and space-filling curves* by William C. Swift, this MONTHLY, vol. 68, 1961, pp. 653–655, is a special case of the example in *Continuous functions without derivatives*, K. A. Bush, this MONTHLY, vol. 59, 1952, pp. 222–225.

C. O. OAKLEY

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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FACULTY EMPLOYMENT AND GRADUATE ENROLLMENTS

DONALD J. DESSART, State University College, Oneonta, New York

Two recent statistical reports* discussed faculty employment and graduate enrollments in colleges and universities of the United States. The first report, the *Scientific Manpower Bulletin* of the National Science Foundation, concerned the employment of scientists and engineers at 1,916 institutions of higher education during 1957–1958. The second report, *Resources Analysis*, dealt with trends in graduate enrollments and Ph.D. output in selected science fields at 80 schools during 1959–1960 and 1960–1961.

Professional personnel. About 311,000 professional personnel were employed in 1957–1958 including about 157,000 scientists and engineers and 154,000 persons in other fields. "Scientists and engineers" were defined to include personnel employed in the physical, life, and social sciences in addition to engineering. Nearly 26% of this group were employed in the areas of physical sciences, which presumably included pure and applied mathematics. The remaining 74% were categorized as engineers (16%), life scientists (39%), and social scientists (19%). These groups were further subdivided into faculty and non-faculty, in which the engineers, physical, and life scientists had a distribution of approximately 60% faculty and 40% non-faculty, and the social scientists had an 80% faculty-20% non-faculty ratio.

Research and development. The *Scientific Manpower Bulletin* further considered the numbers of personnel engaged in research and development. About 45% of the combined group of faculty and non-faculty physical scientists were reported to be engaged in this activity. This figure compared with 49% of the engineers, 53% of the life scientists, and 25% of the social scientists. In the physical science research group, nearly 63% were non-faculty, in engineering about 61%, and approximately 40% of the life and social science researchers were of a non-faculty status.

As is probably expected, about 85% of the faculty research and development personnel were employed at the university level. The breakdown for various

* *Scientific Manpower Bulletin, No. 13* and *Resources Analysis, Memo No. 3*, Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C.

areas showed that 76% of the faculty engineers, 80% of the faculty physical scientists, 91% of the faculty life scientists, and 85% of the faculty social scientists engaged in research were located at universities. Considering more closely the remaining 20% of the faculty physical scientists engaged in research, it is found that the technological schools employed 11% and the liberal arts colleges about 9%. Outside of the universities the greatest number of faculty science and engineering researchers, about 12%, were employed by liberal arts colleges or technological schools.

Faculty personnel. In examining the types of institutions that employed all faculty personnel, whether or not they are engaged in research, it is found that 49% of the faculty physical scientists were employed by universities, 26% by liberal arts colleges, 10% by junior colleges, 6% by teachers colleges, and the remaining 9% by various other schools. It is interesting to note that about 75% of the faculty physical scientists were employed by universities and liberal arts colleges; whereas 71% of the engineers, 87% of the life scientists, and 79% of the social scientists were employed by the same institutions. All together, the universities and liberal arts colleges employed nearly 80% of the faculty scientists and engineers.

Graduate enrollments. Turning to graduate enrollments in the 80 selected schools, which enrolled 70% of all science students and granted 91% of all doctorates in 1959–60, there was an overall increase of 11% in science enrollments from 1959–60 to 1960–61. During this period, mathematics and statistics enjoyed the greatest increase, 20%; and social work had the smallest increase, 6%. A total of 7,358 students out of a graduate population of more than 180,000 students, studied mathematics and statistics in 1960–61.

Examining more closely the 1960–61 intermediate graduate level enrollments (students with at least 1 year of graduate training but not expected to attain the Ph.D. in the current academic year), it is found that 2,647 students were enrolled in mathematics and statistics. In this category, about 51% were doctoral candidates and 49% were master's degree students. Almost half of these students were reported to be studying on a part-time basis; so it is questionable what portion of these students will attain the master's or doctor's degree.

The actual number of doctorates awarded in mathematics and statistics has remained relatively constant in the 80 schools. In 1957–58, 216 doctorates were granted followed by 268 and 248 in the next two academic years. These schools estimated that 358 doctorates would be awarded in 1959–60, and the actual figure was 248. The estimate for 1960–1961 was 408.

These reports revealed that both the employment of scientists and engineers and the size of the graduate student population had increased over the past several years. It seems that such kinds of increases, however, will not be adequate to meet our future scientific manpower needs.

INTEGRATED SCIENCE AND MATHEMATICS IN JUNIOR HIGH SCHOOL*

ALEXANDER CALANDRA, Washington University

For the past four years a number of schools in the St. Louis area have been experimenting with integrated courses in science and mathematics in grades 7 through 9 [with accelerated classes, this approach has been used at grade levels as low as 5]. In 1961 there were 42 classes with a total of 1,400 students in the program.

The purpose of this program is:

[I] To take maximum advantage of the opportunities of using mathematics in science instruction in a systematic fashion.

[II] To provide additional motivation for the study of mathematics.

[III] To introduce science instruction in a manner that will be useful for later work in this area.

In most of the classes, the integrated program in science and mathematics covers, at this time, only about half of the class time ordinarily allotted to these fields, and it includes only those areas of science and mathematics that can be advantageously treated in an integrated fashion. The program starts with a series of units relating some simple geometry and ratio and proportion to some very elementary geometric optics. The second series of units covers some basic ideas on motion, force, work, and energy, in connection with an introduction to tables, graphs and equations. Most of the experimental materials in these two units have been worked out in considerable detail. Related units in chemistry and biology are in the process of development.

The effectiveness of this program depends even more than usual on the availability of printed material and the preparation of the teachers. The former problem is on its way to a satisfactory solution. Significant progress has been made on the second problem in different ways by using one or more of the following approaches in various schools:

[1] An in-service training program at Washington University for the Junior High School teachers involved in the work.

[2] Use of highly select university undergraduate science and mathematics students as ancillary instructors in the Junior High School classroom.

[3] Use of some of the regular teachers as science specialists.

[4] Collaboration of the mathematics and science teachers within a given school.

During the first year of the program in the St. Louis Public Schools, the same introductory materials were given to all the accelerated students in grades 5 through 8 [the St. Louis Public Schools put their accelerated students in grades 5 through 8 in separate sections]. The following year, these first materials

* A brief review of the work during 1959 and 1960 in the development of an integrated program in science and mathematics for the junior high school.

were left at the fifth grade level, and the second stage of the program was introduced simultaneously to students in grades 6 through 8. Next year the second year materials will be left at the sixth grade level, etc.

As a result of this economical and efficient pattern, all teachers learn all of the materials that have been taught to their students at earlier stages in the program. Volume I, dealing with geometric optics, ratio and proportion, and related topics was available for general distribution on January 1st. These materials have been found to require about 50 hours of class time in most class situations, but there is a considerable variability in the amount of time, depending upon the preparation of the teacher, the amount of individual laboratory work as compared to demonstrations, etc. Volume II, dealing with motion, force and energy requires about 75 hours of class time, and this volume will be available for general distribution on September 1st, 1962.

COMPUTER AGE MATHEMATICS

A freshman-level course entitled "Computer Age Mathematics" was presented to crews of the Navy Polaris submarine fleet while on patrol duty. Offered by the Commission on Extension Courses of Harvard University, the course involved:

1. Thirty half-hour kinescopes.
2. Associated readings and problems in the text.
3. An electrical device called a problem solver which in multiple-choice fashion leads the student through more than 1200 questions and answers designed to assist him in solving all assigned problems.
4. Shore school and examinations at the end of the cruise.

The course was first run in the spring term of 1961 on Boston's educational channel WGBH-TV (which prepared videotapes and kinescopes) and then as an early morning summer course by WNBC-TV New York. The lecturer was Professor Francis Scheid, mathematics chairman at Boston University. The text ($1+1=?$) and the problem-solver were also by Professor Scheid.

The course was divided into two parts. Part I dealt with Boolean arithmetic and number fields with related topics such as machines and programming. Part II involved elementary probability, including games of chance and strategy.

Both parts were aimed at the student whose main interest was probably not mathematics, so the content was chosen in an effort to minimize dependence upon earlier experiences in mathematics. The course was experimental. Revisions of materials are anticipated when first results are in.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1496. *Proposed by T. J. Kearns, University of Santa Clara*

Let a and b be relatively prime integers. Show that there exist integers m and n such that $a^m+b^n\equiv 1 \bmod ab$.

E 1497. *Proposed by A. W. Knapp and Albert Whitcomb, Dartmouth College*

Suppose x, y, z, n are integers such that $x^n+y^n=z^n$. (1) Show that if $2n+1$ is prime and n is sufficiently large, then $2n+1$ divides at least one of x, y, or z. (2) Show that if $4n+1$ is prime and n is sufficiently large, then $4n+1$ divides at least one of x, y, or z. (3) Can (1) and (2) be generalized to all primes of the form 2^qn+1?

E 1498. *Proposed by Joseph Hammer, University of Sydney, Australia*

The six rectangular coordinates of two points P and Q in space are the integers 1 to 6 taken in some order. If O is the origin, which ordering will maximize (1) the area, (2) the perimeter, of triangle OPQ?

E 1499. *Proposed by José Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela*

Denote the vertices of a triangle by A_1, B_1, C_1, and the midpoints of the respective opposite sides by A_2, B_2, C_2. Continue the process and let A_n, B_n, C_n be the vertices of the nth triangle. Let I_n be the incenter of triangle $A_nB_nC_n$. Prove that the sequence $\{I_n\}$ has a limit point and find its position.

E 1500. *Proposed by N. A. Court, University of Oklahoma*

Show that a given point M and its three harmonic conjugates with respect to the three pairs of opposite edges of a given tetrahedron (T) are the vertices of a tetrahedron harmonic with respect to the cevian tetrahedron of M for (T).

SOLUTIONS

Property of a Cyclic Polygon

E 1466 [1961, 506]. *Proposed by H. Y. Shee, Huwei Middle School, Huwei, Taiwan, China*

Let $A_1 \cdots A_n$ be a cyclic polygon. Let B_i be the foot of the perpendicular from A_i on $A_{i-1}A_{i+1}$, C_i the point of intersection of A_iA_{i+2} and $A_{i-1}A_{i+1}$, and D_i the foot of the perpendicular from C_i on A_iA_{i+1}, where it is understood that all

subscripts are to be reduced to the least positive residue modulo n. Show that

$$\prod_{i=1}^{n} A_{i-1}B_i/B_iA_{i+1} = \prod_{i=1}^{n} A_iD_i/D_iA_{i+1} = 1.$$

Solution by D. C. B. Marsh, Colorado School of Mines. (1) We have

$$\begin{aligned}\prod_{i=1}^{n} A_{i-1}B_i/B_iA_{i+1} &= \prod_{i=1}^{n} (A_{i-1}B_i/A_iB_i)/(B_iA_{i+1}/A_iB_i)\\ &= \prod_{i=1}^{n} \cot(A_{i+1}A_{i-1}A_i)/\cot(A_{i-1}A_{i+1}A_i)\\ &= \prod_{i=1}^{n} \cot(A_{i+1}A_{i-1}A_i)/\cot(A_iA_{i+2}A_{i+1}),\end{aligned}$$

by a subscript shift in the denominator. But the inscribed angles $A_{i+1}A_{i-1}A_i$ and $A_iA_{i+2}A_{i+1}$ are equal since they both subtend arc $A_{i+1}A_i$. Hence the above product of ratios is 1.

(2) By the same form of argument as in (1),

$$\begin{aligned}\prod_{i=1}^{n} A_iD_i/D_iA_{i+1} &= \prod_{i=1}^{n} (A_iD_i/D_iC_i)/(D_iA_{i+1}/D_iC_i)\\ &= \prod_{i=1}^{n} \cot(A_{i+2}A_iA_{i+1})/\cot(A_iA_{i+1}A_{i-1})\\ &= \prod_{i=1}^{n} \cot(A_{i+2}A_iA_{i+1})/\cot(A_{i+2}A_{i+3}A_{i+1}) = 1.\end{aligned}$$

Also solved by Michael Goldberg, Robert Goldberg, L. D. Goldstone, Oliver Gross, Harry Langman, R. C. Lyness, David Sachs, Leon Steinberg, Guy Torchinelli, and the proposer.

Writing Down a Remainder

E 1467 [1961, 506]. *Proposed by D. G. de Figueiredo, New York University*

Write down the remainder of the division of $(x+1)^n$ by $(x-1)^3$.

I. *Solution by C. F. Pinzka, University of Cincinnati.* (1) Let $(x+1)^n = P(x)(x-1)^3+Ax^2+Bx+C$. Two differentiations and the substitution $x=1$ yields $A=2^{n-3}n(n-1)$, $B=2^{n-2}n(3-n)$, $C=2^{n-3}(n^2-5n+8)$.

(2) The Taylor expansion of $(x+1)^n$ about $x=1$ and the binomial expansion of $[2+(x-1)]^n$ each yield $2^n+2^{n-1}n(x-1)+2^{n-3}n(n-1)(x-1)^2+$terms divisible by $(x-1)^3$, giving the same result as (1).

II. *Solution by W. C. Waterhouse, Harvard University.* To find the remainder of $(x+a)^n/(x+b)^m$, put $y=x+b$; then it is clear we want simply the first m terms of $(y+a-b)^n$, or $\sum_{i=0}^{m-1} \binom{n}{i}(a-b)^{n-i}(x+b)^i$.

Also solved by A. N. Aheart, Winifred Asprey, J. J. Bailey, Jr. and J. B. Bohac (jointly), C. B. Barfoot, Merrill Barnebey, Robert Bart, Robert Bowen, Brother Joseph Heisler, J. A. Brown, R. J. Cormier, G. C. Dodds, Underwood Dudley, Ragnar Dybvik, A. D. Egendorf, S. J. Einhorn, William Emerson, H. M. Feldman, John Freund, David Friedman, N. V. Glick and Richard Katz (jointly), Michael Goldberg, Robert Goldberg, L. D. Goldstone, S. W. Golomb, S. H. Greene, Cornelius Groenewoud, Oliver Gross, Corinne Hattan, G. A. Heuer, J. C. Hickman, J. T. Humphrey, A. R. Hyde, Erwin Just and Norman Schaumberger (jointly), M. S. Klamkin, J. D. E. Konhauser, Harry Langman, T. J. Lee, Viktors Linis, Jiang Luh, R. C. Lyness, C. R. MacCluer and Gary Wilson (jointly), B. B. Mapenen, D. C. B. Marsh, Otto Mond, D. A. Moran, J. B. Muskat, Hugh Noland, R. J. Oberg, M. J. Pascual, Walter Penney, K. R. Rajagopalan, David Rothman, H. D. Ruderman, David Sachs, J. K. Siberz, S. J. Sidney, Sister Mary Denis Baldwin, E. L. Spitznagel, Jr., Leon Steinberg, R. W. Sturgeon, Wu Ta-Sun, R. P. Tapscott, Guy Torchinelli, Kermit Uggen, R. J. Wagner, E. W. Wallace, Harry Weingarten, J. E. Wilkins, Jr., Dale Woods, David Zeitlin, and the proposer.

Zeitlin considered the more general problem of finding the remainder after division of $P(x)=\sum_{k=0}^{n} b_k x^k$, $b_n \neq 0$, by $(x-a)^m$.

A Theorem on Rearrangements

E 1468 [1961, 506]. *Proposed by B. H. Bissinger, Lebanon Valley College*

Let $b_1, \cdots, b_n$ be any rearrangement of the positive numbers $a_1, \cdots, a_n$. Then $a_1/b_1 + \cdots + a_n/b_n \geqq n$.

I. *Solution by Julius Vogel, Prudential Insurance Company.* Since the arithmetic mean of n positive numbers is at least as large as their geometric mean, we have

$$(1/n) \sum (a_i/b_i) \geqq [\prod (a_i/b_i)]^{1/n}.$$

But, by the conditions of the problem, $\prod a_i = \prod b_i$. So $\sum (a_i/b_i) \geqq n$.

II. *Solution by Oliver Gross, The RAND Corporation.* More generally, if $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are similarly monotone sequences of real numbers and π is a permutation on the integers $1, \cdots, n$, then

$$\sum_{i=1}^{n} a_i b_i \geqq \sum_{i=1}^{n} a_i b_{\pi(i)} \geqq \sum_{i=1}^{n} a_i b_{n-i+1},$$

as a pairwise interchange argument shows. However, perhaps the easiest direct proof of the above special case uses the concavity property of the logarithm. Thus

$$\log\left[(1/n) \sum_{i=1}^{n} (a_i/b_i)\right] \geqq (1/n) \sum_{i=1}^{n} \log(a_i/b_i)$$

$$= (1/n)\left(\sum_{i=1}^{n} \log a_i - \sum_{i=1}^{n} \log b_i\right) = 0,$$

and the desired conclusion follows by exponentiation.

III. *Solution by David Zeitlin, Remington Rand Univac.* The result is trivial

for $n=1$ and true for $n=2$ since $r+r^{-1}\geqq 2$ for $r>0$. Suppose now the inequality is true for any set of n positive integers. Consider any set of $n+1$ positive integers, $a_1, \cdots, a_{n+1}$, and let $b_1, \cdots, b_{n+1}$ be any rearrangement of the a_i. Then, for some k, $a_k/b_k \geqq 1$; for if all $a_i/b_i<1$, this would contradict the inequality for any set of n positive integers. Thus

$$\sum_{i=1}^{n+1}(a_i/b_i) = \left\{\sum_{i=1}^{n+1}(a_i/b_i) - a_k/b_k\right\} + a_k/b_k \geqq n+1,$$

and the conclusion follows by mathematical induction.

Also solved by D. W. Bailey, Robert Bart, P. T. Bateman, William Becker, E. E. Bosman, D. A. Breault, J. L. Brown, Jr., Leonard Carlitz, Charles Carniglia, E. W. Cheney and E. M. Scheuer (jointly), J. L. Cline, D. T. Cottingham, Dennis Couzin, C. P. Donahoe, Jr., Herb Doughty, Underwood Dudley, William Emerson, H. M. Feldman, David Forslund, David Friedman, Fred Galvin and Mahmoud Sayrafiezadeh (jointly), N. V. Glick, Michael Goldberg, L. J. Goldstein, L. D. Goldstone, S. W. Golomb, S. H. Greene, A. S. Gregory, M. Handel, A. R. Hyde, R. D. James, D. M. Johnson, Erwin Just and Norman Schaumberger (jointly), P. G. Kirmser, M. S. Klamkin, L. C. Kurtz, Harry Langman, Joseph Lehner, Viktors Linis, D. M. Long, Adolph Lu, Jiang Luh, R. C. Lyness, W. M. McKeeman, B. B. Mapenen, D. C. B. Marsh, H. F. Mattson and R. J. Turyn (jointly), D. S. Mitrinović, J, B. Muskat, P. M. Pepper, C. F. Pinzka, Harsh Pittie, Thomas Porsching, David Rothman, Allen Rubenstein, H. D. Ruderman, Charles Ryavec, David Sachs, S. J. Sidney, E. L. Spitznagel, Jr., Leon Steinberg, E. A. Strawbridge, R. W. Sturgeon, Paul Stygar, R. P. Tapscott, Wu Ta-Sun, Guy Torchinelli, R. J. Wagner, R. M. Warten, W. C. Waterhouse, Albert Wilansky, Clem Winston, Arthur Wouk, and the proposer. Late solution by H. S. Cash.

Editorial Note. The more general inequalities given in Solution II may be found in Hardy, Littlewood, and Pólya, *Inequalities*, 2nd ed., pp. 261–2.

A Functional Equation

E 1469 [1961, 506]. *Proposed by P. G. Kirmser, Kansas State University*

Find a function $f(x)$ such that $f(2x)=e^{x^2}f(x)\cos x$ with $f(0)=1$.

I. *Solution by Leon Steinberg, Remington Rand Univac.* Set

$$f(x) = x^{-1}e^{x^2/3}\sin x\, h(x).$$

Then $h(0)=f(0)=1$. Moreover

$$f(2x) = (2x)^{-1}e^{4x^2/3}\sin 2x\, h(2x) = x^{-1}e^{4x^2/3}\sin x\cos x\, h(2x).$$

But also

$$f(2x) = e^{x^2}f(x)\cos x = x^{-1}e^{4x^2/3}\sin x\cos x\, h(x).$$

Hence the original functional equation will be satisfied for any $h(x)$ such that $h(2x)=h(x)$ for $x\neq n\pi/2$ (n a nonzero integer) and $h(0)=1$. There are infinitely many such solutions, four of which are:

(1) $h(x)=1$, yielding the only continuous solution of the functional equation.

(2) $h(0)=1$, $h(x)=0$ if $x\neq 0$, yielding the trivial solution $f(0)=1$, $f(x)=0$ if $x\neq 0$.

(3) $h(0)=1$, $h(x)=c$ if $x\neq 0$.
(4) $h(0)=1$, $h(x)=a$ for nonzero rational x, $f(x)=b$ for irrational x.

II. *Solution by Kermit Uggen, St. Olaf College.* From $f(2x)=e^{x^2}f(x)\cos x$ one obtains $f(x/2^k)=e^{x^2/4^{k+1}}f(x/2^{k+1})\cos(x/2^{k+1})$. Thus, if one assumes $f(x)$ to be analytic, one has $f(x)=\prod_{k=0}^{\infty}f(x/2^k)/f(x/2^{k+1})=e^{x^2/3}g(x)$, where $g(x)$ has the property that $g(2x)=g(x)\cos x$, and is analytic. Upon expressing each side of this last equation in its Maclaurin series expansion and equating coefficients one finds that $g(x)$ has the Maclaurin series expansion of $(\sin x)/x$. Thus $f(x)=e^{x^2/3}(\sin x)/x$, $x\neq 0$, $f(0)=1$, satisfies the conditions of the problem and is the only analytic function to do so. Clearly there are infinitely many nonanalytic functions which satisfy the conditions of the problem.

Also solved by D. W. Bailey, Merrill Barnebey, Robert Bart, William Becker, H. G. Bray and E. A. Rutter, Jr. and A. W. Zechmann (jointly), J. A. Brown, J. L. Brown, Jr., Kenneth Brownstein, Leonard Carlitz, D. I. A. Cohen, Dennis Couzin, G. C. Dodds and E. L. Magnuson (jointly), Underwood Dudley, E. S. Eby, William Emerson, R. L. Farrell, J. A. Faucher, H. M. Feldman, Daniel Giesy and C. R. MacCluer (jointly), N. V. Glick and Richard Katz (jointly), Michael Goldberg, S. W. Golomb, Cornelius Groenewoud, S. H. Greene, Oliver Gross, M. Handel, G. A. Heuer, J. C. Hickman, J. B. Johnston, Erwin Just and Norman Schaumberger (jointly), M. S. Klamkin, J. D. E. Konhauser, T. J. Lee, Viktors Linis, D. B. Loftus, R. C. Lyness, D. C. B. Marsh, B. B. Mapenen, D. A. Moran, R. W. Neufeld, Hugh Noland, R. J. Oberg, Walter Penney, D. J. Peterson, C. F. Pinzka, K. R. Rajagopalan, David Rothman, Allen Rubenstein, H. D. Ruderman, David Sachs, Marlow Sholander, S. J. Sidney, Barry Simon, C. S. Smith, E. L. Spitznagel, Jr., Paul Stygar, Guy Torchinelli, K. R. Unni, Julius Vogel, W. C. Waterhouse, R. J. Wagner, W. J. Wayne, J. E. Wilkins, Jr., David Zeitlin, the proposer, and an anonymous solver.

Covering a 2×n Rectangle with Dominoes

E 1470 [1961, 506]. *Proposed by W. E. Patten, South Boston, Virginia*

It is desired to form a $2\times n$ rectangle from 1×2 rectangles (dominoes), or we may say, to cover the rectangle with dominoes. In how many distinct ways can this be done, where two solutions are distinct when they cannot be brought into coincidence by rotations and reflections?

Solution by S. W. Golomb, California Institute of Technology. We first enumerate all $2\times n$ rectangles of dominoes, without regard to possible symmetries among them. For $n=1$, the number of possibilities is $f_1=1$, and for $n=2$, the two dominoes are either horizontal or vertical, so that $f_2=2$. For $n>2$, starting at the left, the $2\times n$ rectangle begins with either a vertical domino, which can be "extended" to fill out the $2\times n$ rectangle in f_{n-1} ways, or with a pair of horizontal dominoes, which can be "extended" to fill out the rectangle in f_{n-2} ways. Hence $f_n=f_{n-1}+f_{n-2}$, and $\{f_n\}$ is the Fibonacci sequence.

Let c_m refer to the number of solutions to the $2\times m$ rectangle in which left-to-right mirror images are not regarded as distinct. Then $c_m=\frac{1}{2}(f_m+s_m)$, where s_m is the number of solutions which are left-to-right symmetric.

Consider even and odd m separately. For $m=2n+1$, a symmetric solution must have a vertical domino in the center, leaving a $2\times n$ rectangle on each

side. One side can be completed in f_n ways, and then the other side is completely specified as the mirror image. Thus $s_{2n+1}=f_n$.

For $m=2n$, a vertical line down the middle either cuts a horizontal pair of dominoes, or cuts no dominoes. In the former case, there are f_{n-1} ways to specify the $2\times(n-1)$ rectangle to the left of the horizontal pair; in the latter case, f_n ways to specify the $2\times n$ rectangle to the left of the midline; and having specified the left half, the right half is determined by symmetry. Hence $s_{2n}=f_{n-1}+f_n=f_{n+1}$.

Thus $c_{2n+1}=\frac{1}{2}(f_{2n+1}+f_n)$ and $c_{2n}=\frac{1}{2}(f_{2n}+f_{n+1})$, which is the answer to the given problem except for the case $m=2$, since the 2×2 rectangle is a square and admits of a 90° rotational symmetry, which reduces $c_2=2$ to $c_2=1$, without affecting any of the other cases.

Also solved by Robert Bart, Dennis Couzin, N. V. Glick, Michael Goldberg, S. H. Greene, Oliver Gross, J. B. Herreshoff, A. R. Hyde, Harry Langman, R. C. Lyness, D. C. B. Marsh, D. A. Moran, Walter Penney, David Rothman, David Sachs, Leon Steinberg, Paul Stygar, Guy Torchinelli, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS AND SOLUTIONS

5001. *Proposed by Melvin Hausner, New York University*

A well known result of Erdös is that a sequence of n^2+1 terms contains a monotonic sequence of $n+1$ terms. Prove that if $pq<r$, then a sequence of r terms contains an increasing subsequence of $p+1$ terms or a decreasing subsequence of $q+1$ terms.

5002. *Proposed by I. I. Kolodner, University of New Mexico*

For every $a>0$, let the function $f: R^+\to R$ be square integrable on $[0, a]$, and let the composition $h\circ f$ be defined by $h\circ f(t)=\int_0^t f^2-(\int_0^t f)^2/t$. Show that

1. $h\circ f(t)\leqq A$ if and only if there is a number α such that $(f-\alpha)\in L_2(R^+)$;
2. if $f\in L_2(R^+)$ then as $t\to\infty$, $(\int_0^t f)^2/t\to 0$.

5003. *Proposed by D. R. Morrison, Sandia Corporation, Albuquerque, N. M.*

A crew of m men is to be partitioned into teams, each team to include at least one man, and the teams to be arranged in cyclic order. Show that for $m>1$,

the number of ways in which this can be done using an even number of teams is equal to the number of ways it can be done using an odd number of teams. (Example: 3 men, say A, B, C. Even number of teams: (A, BC), (B, AC), (C, AB); odd number: (ABC), (A, B, C), (A, C, B).)

5004. *Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan*

If $0 \leqq \beta \leqq \alpha \leqq \pi$, prove that

$$\frac{1 - \cos\beta + 2\cos\alpha}{1 + \cos\beta} \leqq \cosh\sqrt{(\alpha^2 - \beta^2)} \leqq \frac{1 - \cos\alpha + 2\cos\beta}{1 + \cos\alpha}.$$

5005. *Proposed by Reuben Hersh, New York University*

Suppose A is any non-singular n by n matrix with elements a_{ij} and its inverse is B with elements b_{kl}. Then form a new matrix C in the following way: Take the ordered pairs composed of the natural numbers up to n, and in any arbitrary manner number them from 1 to n^2. Let the symbol $[i, j]$ stand for the number of the ordered pair i, j. Then the elements of C can be indexed by the bracket symbol. Now define C by $c_{[i,j],[k,l]} = a_{il}b_{jk}$. Prove that the matrix $(C - I)$ has rank $n(n-1)/2$.

5006. *Proposed by E. Ehrhart, Lycée Kléber, Strasbourg, France*

Is the following proposition valid: Every plane section of a tetrahedron is smaller (in area) than the largest face?

SOLUTIONS

A Non-vanishing Determinant

4935 [1960, 1033]. *Proposed by H. Schwerdtfeger, McGill University*

Prove that if c is a rational number such that the equation $x^m = c$ is irreducible over the rational field, then the determinant

$$\begin{vmatrix} a_0 & ca_{m-1} & \cdots & ca_2 & ca_1 \\ a_1 & a_0 & \cdots & ca_3 & ca_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m-2} & a_{m-3} & \cdots & a_0 & ca_{m-1} \\ a_{m-1} & a_{m-2} & \cdots & a_1 & a_0 \end{vmatrix}$$

is different from zero for all sets of m rational numbers $a_0, a_1, \cdots, a_{m-1}$ which are not all equal to zero.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Consider the matrix

$$M = \begin{pmatrix} I & C \\ 0 & A \end{pmatrix},$$

where I is the $m-1$ by $m-1$ identity, 0 is the m by $m-1$ zero matrix, C is the

$m-1$ by m matrix with $-c$ in the $(i, i+1)$ positions and zeros elsewhere, and A is the given m by m array. At once, $\det(M) = \det(A)$.

For all integers r, s satisfying $1 \leqq s \leqq r \leqq m-1$, multiply the rth row of M by a_{m-s} and add to the $(r+m-s)$th row. The determinant of the result, while unchanged in value, is recognized as the eliminant of $x^m - c$ and $a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0$. Since $x^m - c$ is irreducible over the rationals, it will have no factor in common with a rational polynomial of lesser degree, and thus the eliminant will not vanish (unless, trivially, $a_{m-1} = \cdots = a_0 = 0$).

II. *Solution by Harley Flanders, Purdue University.* Let Q denote the field of rationals and K the extension $K = Q(\theta)$ where $\theta^m = c$, so that $[K:Q] = m$. If $\alpha = a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1} \neq 0$, then the given determinant is, up to sign, nothing but the norm $N_{K/Q}(\alpha)$, which is non-zero. The simplest way to see this: the mapping $\beta \to \alpha\beta$ is a non-singular linear transformation over Q on K onto K. The norm is merely its determinant, here computed with respect to the linear basis $1, \theta, \theta^2, \cdots, \theta^{m-1}$.

Also solved by J. L. Brenner, L. Carlitz, Virginia Hanly, James P. Jans, R. H. C. Newton, F. W. Ponting, R. C. Thompson, and the proposer.

A Finite Weierstrass Theorem

4936 [1960, 1033]. *Proposed by R. C. Buck, University of Wisconsin*

Let A be an algebra of functions defined on a space X and taking values in an arbitrary (not necessarily commutative) field k. Suppose that A separates X: given x and y in X, $x \neq y$, there is $f \in A$ with $f(x) \neq f(y)$. How close does A come to being dense in the algebra of all functions on X to k? Given a finite set $S \subset X$, and a function g on X, is there $f \in A$ so that $f(x) = g(x)$ for all $x \in S$?

Solution by the proposer. One says that a function algebra A is n-transitive on X if, given n distinct points $x_1, x_2, \cdots, x_n$ in X and n values $c_i \in k$, there is a function $f \in A$ such that $f(x_i) = c_i$ for $i = 1, 2, \cdots, n$. A is fully transitive on X if it is n-transitive for each n. Under the hypotheses on A in the problem, there are only two possibilities: either A is fully transitive on X, or it becomes fully transitive when one special point is removed from X. To prove this, first examine the special algebra $K = k \times k$ consisting of pairs (a, b) with a and b in k. It is easily seen that there are exactly three proper subalgebras of K, namely those (a, b) with $a = 0$, those with $b = 0$, and those with $a = b$; for, if B is any other subalgebra of K, we can assume that B contains elements of the form (a, b) with $ab \neq 0$ and $a \neq b$, from which we see that B contains $(a^2 - a, b^2 - b)$ and must therefore be K itself.

For a fixed pair x, y in X, $x \neq y$, let B be the set of all pairs $(f(x), f(y))$ for $f \in A$. This is then a subalgebra of K. Since A separates X, B is either K, or for one of the two points, say x, we have $f(x) = 0$ for all $f \in A$. Since A separates X, there cannot be two such points in X. Deleting this point, if it is present, we find that A is 2-transitive on the remaining set, which we may call X^*. We complete the proof by showing that 2-transitivity implies full transitivity. Given

$S=\{x_1, x_2, \cdots, x_n\}$ we select functions $f_2, f_3, \cdots, f_n$ in A so that $f_j(x_1)=1$ and $f_j(x_j)=0$. Set $F_1=f_2f_3f_4\cdots f_n$, obtaining a function in A such that $F_1(x_1)=1$ and $F_1(x_i)=0$ for $i\neq 1$. In the same way, construct functions $F_2, \cdots, F_n$ so that $F_j(x_j)=1$ but $F_j(x_i)=0$ for $i\neq j$. Finally, given values $c_1, c_2, \cdots, c_n$, put $F=\sum c_jF_j$; this is in A, and $F(x_i)=c_i$, $i=1, 2, \cdots, n$.

In general, it is not possible to show that A has a stronger approximation property on X; in particular, it might be noted that when k is the complex field, one needs to assume that A is closed under the conjugate operation—an assumption not needed here.

Also solved by Harley Flanders, and Albert Wilansky.

Product of Magic Matrices

4937 [1960, 1033]. *Proposed by G. P. Sturm, Jr., Oklahoma State University*

Prove that the matrix product of any two third order magic squares is a doubly symmetric matrix. A magic square is defined as a square array of n^2 elements with the property that the sum of the elements of any line (both diagonals included) is a constant; double symmetry is defined as symmetry in both diagonals.

Solution by D. C. B. Marsh, Colorado School of Mines. Any third order magic square with line-sum $3S$ may be written in matrix form as

$$M=\begin{bmatrix} S+a+b & S-2a & S+a-b \\ S-2b & S & S+2b \\ S-a+b & S+2a & S-a-b \end{bmatrix}$$

with parameters a, b. By setting $(1, 1, 1)=x$, $(1, 0, -1)=y$ and $(1, -2, 1)=z$, we have $M=Sx'x+ay'z+bz'y$, where $'$ denotes transpose. For a second such square, $M_1=S_1x'x+a_1y'z+b_1z'y$, it is easy to compute $MM_1=3SS_1x'x+6ab_1y'y+2a_1bz'z$. Since each term is doubly symmetric, MM_1 is doubly symmetric.

Also solved by Bernard Anderson, F. P. Callahan, Jr., Donald Knuth, J. B. Muskat, F. D. Parker, F. W. Ponting, Brother Louis Zirkel, and the proposer.

Representations with Non-integral Radix

4938 [1960, 1933]. *Proposed by R. W. McChesney, Albion College*

It is known that every real number $x\in[0, 1)$ has at most two representations of the form $x=\sum_{m=1}^{\infty} a_mN^{-m}$, where N is any integer greater than 1 and each a_i is one of the set $0, 1, \cdots, N-1$. Now consider a representation of the form $x=\sum a_mb^{-m}$, where $x\in[0, 1)$, and a_i is one of $1, 2, \cdots, n-1$; n an integer greater than 1 and $n-1\leqq b<n$.

(a) Show that an x can exist having more than two such representations.

(b) Prove that at most two representations of x exist if

$$\tfrac{1}{2}[n-1+\sqrt{(n^2+2n-3)}]\leqq b<n.$$

Solution by J. H. van Lint, Technical University, Eindhoven, Netherlands.

(a) *Example*: Instead of $\sum a_m b^{-m}$ we write $.a_1a_2a_3\cdots$. Take $b=\frac{1}{2}(1+\sqrt{5})$. Then $.0110000\cdots=.01001111\cdots=.00111111\cdots$ because the sums of these series are $1/b^2+1/b^3$, $1/b^2+1/b^4(b-1)$, and $1/b^2(b-1)$, all three expressions equalling $b-1$.

(b) Suppose $b>3/2$, and consider the three different expressions:

$$x=\sum_{m=1}^{\infty} a_m b^{-m}=x_1+\sum_{m=k+1}^{\infty} a_m b^{-m},$$

$$x=\sum_{m=1}^{\infty} a'_m b^{-m}=x_2+\sum_{m=k+1}^{\infty} a'_m b^{-m},$$

$$x=\sum_{m=1}^{\infty} a''_m b^{-m}=x_3+\sum_{m=k+1}^{\infty} a''_m b^{-m}.$$

If $a_i\neq a'_i$ then these numbers differ by 1 because $(n-1)\sum b^{-m}<2$. Hence there are at least two places where the above expressions differ. Let the second of these be the kth place and suppose $x_1<x_2<x_3$. Then $\sum_{m=k+1}^{\infty} a_m b^{-m}$ must be at least $1/b^{k-1}-(n-2)/b^k$. Hence $\sum_{m=k+1}^{\infty}(n-1)/b^k\geqq 1/b^{k-1}-(n-2)b^k$. Therefore there are at most two different expressions for a number x if $n>b>\frac{1}{2}[(n-1)+\sqrt{(n^2+2n-3)}]$. As the example shows, the equals sign in the proposal cannot hold.

Also solved by the proposer.

Open Subsets

4939 [1960, 1034]. *Proposed by C. N. Campopiano, Radio Corp., of America, Camden, N. J.*

Let E be a topological space. Let p be the collection of all those (and only those) subsets A of E such that $A\cap\overline{B}\subset\overline{A\cap B}$ for every $B\subset E$. ($\overline{X}$ is the closure of X for $X\subset E$.) The following facts are known: if A is an open subset of E, then $A\in p$; p is non-empty; there exist topological spaces having non-open subsets which are not in p. (See Bourbaki, *General Topology*, Book III, Chap. I, p. 16, Proposition 3 and p. 18, Exercise 6.)

Prove that, if $A\in p$, then A is open; thus, $A\in p$ if and only if A is open.

Solution by Dennison R. Brown, Louisiana State University, New Orleans. For B take $E-A$, the complement of A. Then $A\cap\overline{E-A}\subset\overline{A\cap E-A}=\varnothing$. Hence $x\in\overline{E-A}$ implies $x\in E-A$, and therefore $E-A$ is closed. A is thus open.

Also solved by G. D. Adams, Don Anderson, Anders Bager, Manuel Berri, E. W. Cheney, P. R. Chernoff, H. D. Coomes, Helen F. Cullen, T. N. Delmer, R. M. Fischler, Harley Flanders, Joseph Geiser, E. R. Gentile, Melvin Henriksen, D. W. Hullinghorst, G. D. Johnson, M. J. Z. Kascic, Jr., Seymour Kase, G. Laman, Annie L. Laurer, Gerald Leibowitz, L. E. Minning, B. Riečan, Benjamin Sapolsky, B. T. Sims, Fred Suvarov, S. Swaminathan, D. R. Taylor, K. R. Unni, E. Vogel, W. C. Waterhouse, Joseph A. Wolf, and the proposer. Later solution by F. J. Wagner.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Analytic Geometry and Calculus. By Herbert Federer and Bjarni Jónsson. Ronald Press, New York, 1961. vi+671 pp. $8.75.

One of the outstanding features of this textbook is the adherence to the framework of set theory throughout the book. For students with a background of high school mathematics similar to that developed by the School Mathematics Study Group, this text would be a natural at the college and university level. Although the authors claim that the reader needs only "the usual background of high school algebra and geometry, not necessarily including trigonometry," the generally sophisticated treatment of the classical topics in analytic geometry, logarithms, and trigonometry would not be wasting a student's time who happened to have, say, four years of mathematics in high school.

For the benefit of the students who may not have been exposed to the modern approach in the high school, the first chapter does take up the topics in real numbers, sets, relations and functions. Ample diagrams supplement the detailed discussions throughout the book. A particularly fine chapter on limits and continuity accounts for one-eighth of the book and differentiation is not introduced until the fourth of the twelve chapters. Integration is introduced in the next to the last chapter followed by a final chapter on "Geometric Aspects of Integration." Mention should be made of the unique presentation of functions of functions through the use of arrow diagrams in the eighth chapter entitled, "Superpositions and Inverses."

In summary, this text presents a modern and mature approach to the elementary concepts in analytic geometry and calculus which is classical in content, thorough in treatment, and not likely to be out-of-date soon.

LAWRENCE WAHLSTROM
Wisconsin State College at Eau Claire

Theory of Formal Systems. Raymond M. Smullyan. Annals of Mathematics Studies, Number 47. Princeton University Press, Princeton, 1961. 142 pp. $3.00.

This is a rather technical study purporting to be an introduction to recursive function theory founded on an analysis of the nature of an "elementary formal system." Godel theorems and the work of Church and Rosser on undecidability are integral parts of the development. Recursive function theory is treated in Chapter IV using the theory of elementary formal systems, and Quine's theory of concatenation. Some new results of Smullyan's own investigations are presented in the concluding chapter.

It should be remarked that the study of formal systems is quite analogous

to the study of language-structures, *i.e.*, linguistics. Indeed, it could be called mathematical linguistics.

The reviewer suggests that although Smullyan states he has kept in view the needs of the "mature mathematician" with no knowledge of mathematical logic, the study would be more easily read and its importance recognized if the "mature mathematician" were acquainted with (*e.g.*) Kleene's *Introduction to Metamathematics* and/or Church's *Introduction to Mathematical Logic.*

L. O. KATTSOFF
Boston College

Principles of Regression Analysis. By R. L. Plackett. Clarendon Press, Oxford 1960. 173 pp. $5.25.

Chapter headings are: Linear equations, Quadratic forms in normal variables, Least squares, Linear hypotheses, Departures from standard test conditions, Polynomial regression, Stationary error processes, Symmetrical factorial experiments, and Randomization. As the author forewarns, the level of the book is such that the reader needs considerable background in "statistics, matrices, complex variables, and groups." The treatment is concise, painstaking, and heavily documented. The source material is largely articles and books published since 1950. A person sincerely interested in pursuing the development of any of the topics treated would surely profit by working his way through the text material and problems, and by taking advantage of the pinpoint references.

V. V. LATSHAW
Lehigh University

Quantum Mechanics. By Eugen Merzbacher. Wiley, New York, 1961. x+544 pp. College Ed. $9.95; Professional Ed. $12.00.

This book presents, in ample detail, the principal nonrelativistic aspects of quantum theory. Like most elementary books on the subject, it develops the Schrödinger method before proceeding to the abstract formulation. It is intended for graduate students, and while it assumes that the reader has a reasonable mathematical proficiency in analysis, it gives careful guidance through the technical details. The abstract and group-theoretic portions are self-contained and easily intelligible to students lacking preparation in combinatorial mathematics. The exercises are many and excellent, and should prove a boon both to student and instructor. The author has a knack of anticipating students' difficulties and removing them deftly by brief, candid remarks. Occasional ineptitudes are all the more surprising in view of the excellence of the book as a whole. The author's deep knowledge of the subject, and his gift for exposition, make his book particularly welcome as an introductory text on quantum mechanics.

BANESH HOFFMANN
Queens College

Calculus: An Introductory Approach. By Ivan Niven. Van Nostrand, Princeton, N. J., 1961. viii+172 pp. $4.75.

This book is intended as a text for a brief course in calculus for students who understand algebra and trigonometry. The first two of the eight chapters are devoted to preparatory topics, including slope, sigma notation, inequalities, absolute value, radian measure, functions, and limits. These chapters are somewhat inappropriately headed, "What is Calculus?" and "Limits." Chapter 3 is on integration and Chapter 4 on differentiation; examples and exercises are limited to integral powers and simple trigonometric functions. Chapters 5, 6, 7 are devoted strictly to theory; namely, the fundamental theorem, trigonometric functions, and logarithmic and exponential functions. The latter two chapters consist chiefly of series expansions. The last chapter describes these "Further Applications": series for arctan, computation of pi, escape velocity of a missile, law of growth, reflective property of parabola, Newton's method.

By the publisher's count, there are only 310 problems posed for students. These problems are generally of two types; either they require routine manipulation or they involve extension of theory. The text would be improved by enlarging the lists of Problems with appropriate exercises.

Emphasis throughout is on theory, rather than on formal processes or applications of calculus. In the reviewer's opinion, the sub-title should be "An Experimental Approach" rather than "An Introductory Approach." Actually, the text could be studied with profit by a mathematics major who has completed a conventional course in calculus.

H. S. KALTENBORN
Memphis State University

Concepts from Tensor Analysis and Differential Geometry. By T. Y. Thomas. Academic Press, New York, 1961. 115 pp. $5.00.

This is Volume I of a new series of monographs and textbooks on Mathematics in Science and Engineering. It is composed of brief sketches of twenty-five topics in the theory of tensors and their use in metric differential geometry. As would be expected from this author, one finds a masterfully written work. But it is brief to a fault. For instance, each of the three topics: scalars, vector product and curl, and Gaussian and mean curvature of a surface, is disposed of in one page. Less than three pages are devoted to the concept of absolute differentiation, and developable and minimal surfaces are treated descriptively in little more than one page by employment of the usual phase "it can be shown that." There is a mine of information in this little volume but the prospect of its use as a textbook is disappointing because of its brevity and the total lack of exercises.

C. E. SPRINGER
The University of Oklahoma

Introductory Topology. By Stewart Scott Cairns. Ronald Press, New York, 1961. ix+244 pp. $8.75.

This book is the result of repeated revisions of class notes used by the author in teaching introductory topology. It can be used as a text for such a course at the advanced undergraduate and graduate levels.

The first three chapters present illustrative geometric examples, the topological classification of surfaces, and introductory set-theoretic topology. Chapter 4 is devoted to the study of complexes, including the development of machinery prerequisite to the exposition of homology and cohomology theory of complexes in Chapter 5. In Chapter 6 a strengthened version of Sperner's lemma is used as the principal tool in proving the Brouwer fixed-point theorem, the invariance of regionality, and the invariance of homology properties. In Chapter 7 homology properties of manifolds are presented. Included are the duality theorems of Poincare, Lefschetz, and Alexander. The subject matter of Chapter 8 is the theory of the fundamental group of an arcwise connected topological space.

The book is carefully written and attractively presented. It is, especially in view of the difficult typography, remarkably free of errors.

JOHN KEESEE
The University of Arkansas

Modern Plane Trigonometry. By William L. Hart. D. C. Heath, Boston, 1961. viii+239 pp.+124 pp. of tables. $5.25.

This textbook is intended for students who may desire to study calculus later. While it is basically a traditional trigonometry, much effort has been made to use the modern approaches which are so helpful in giving greater meaning to the mathematics. In particular, the function concept is clearly set forth and the frequent reference to domain and range of the functions throughout the book helps keep this concept in mind. The inverse functions take on greater meaning because of this approach.

The trigonometric functions are defined for general angles before the special case of acute angles is treated. Much emphasis is put on the distinction between functions of angles and functions of numbers, the latter of which are approached through angles measured in radians. Unfortunately, the winding function approach is found only in the appendix. Analytic geometry proofs of the addition formulas and the law of cosines are used to advantage.

There is occasional lack of precision in the use of terminology, but more serious is the whole style of presentation, which is mostly just giving facts and formulas, proving the formulas, and telling the student how to proceed. Rarely is the student taken into confidence and told why we are doing what we are doing, what it is leading to, what principles underlie it.

Considerable attention is given to approximate numbers and how to use

them, but some of the explanation is vague. For a while it is assumed that all numbers in problems are exact—which is unrealistic and leads to absurd answers—and later, when approximate numbers are recognized, the answers repeatedly violate the agreements which have been made about approximate numbers.

There may be more computational trigonometry in the book than some teachers want, but it is arranged so that parts may be omitted. There are 3-place, 4-place, and 5-place tables. A section on the algebra of sets in the appendix seems rather irrelevant.

HENRY SWAIN
New Trier Township High School, Winnetka, Illinois

Partial Differential Equations and Continuum Mechanics. xv+397 pp. R. E. Langer, Ed. University of Wisconsin Press, Wisconsin, 1961. $5.00.

The nineteen lectures published in this volume were given by invitation at an International Conference conducted by the Mathematics Research Center at the University of Wisconsin. It is a remarkable collection in that the invited speakers are eminent researchers in their field and expert speaks to brother expert without fear or favour.

The general reader should not conclude that the book is not for him. It is useful to lift eyes beyond our immediate research and to survey, even briefly, the frontier problems in the present volume. For example, Hörmander reviews the existence and nonexistence theorems which have been obtained for large classes of differential equations in the past ten years. Kampé de Fériet develops a statistical fluid mechanics for two-dimensional gravity waves and Lewy discusses atypical differential equations (*i.e.* equations which possess solutions contrasting markedly in properties with the behaviour of solutions of elliptic, hyperbolic, or partial differential equations).

Nine lectures relate to the theory of partial differential equations and nine are concerned with problems of continuum mechanics. In addition to the lectures, abstracts of forty-five papers presented at the conference are given; some of the abstracted papers may be published in the periodical literature and it seems excessive to devote fifty-eight pages of the volume to abstracts.

The nineteen lectures appear to have been printed as delivered and the wisdom of this policy may be questioned since the spoken and the written word are quite different media of exposition. Worse, the reader is often tantalized by the overworked phrase "a more detailed version of this paper will be published elsewhere." It has for some time been accepted practice to regard publication in conference proceedings as a preliminary canter to journal publication; editorial review of this practice is long overdue.

JOHN MCNAMEE
University of Alberta

Applied Statistical Decision Theory. By Howard Raiffa and Robert Schlaifer. Graduate School of Business Administration, Harvard University, Boston, 1961. xxviii+356 pp. $9.50.

This book is the first of a series, *Studies in Managerial Economics*, of the Division of Research of the Harvard Business School and was published by that school.

In the words of the authors, "This book is an introduction to the mathematical analysis of decision-making when the state of the world is uncertain but further information about it can be obtained by experimentation. The authors take as given that the objective of such analysis is to identify a course of action (which may or may not include experimentation) that is logically consistent with the decision-maker's own preferences for consequences, as expressed by numerical utilities, and with the weights he attaches to the possible states of the world, as expressed by numerical probabilities. The logical and philosophical justification for this statement of the problem has been fully developed by Savage in his *Foundations of Statistics* (New York, 1954); the purpose of the present book is not to discuss these basic principles but to contribute to the body of analytical techniques and numerical results that are needed if practical decision problems are to be solved in accordance with them." And again, "This book . . . is addressed to persons who are interested in using statistics as a tool in practical problems . . . and who also have the necessary training in mathematics and statistics to employ these analytical techniques. It is not written for the general businessman"

This is a fair statement of the objective of the book and the objective is well realized as an admirable point of departure for those interested in this field. The authors are to be applauded for making the effort.

The approach is, as indicated above, Bayesian in nature. There are three sections: Experimentation and Decision—General Theory, Extensive-Form Analysis When Sampling and Terminal Utilities Are Additive, and Distribution Theory.

In the section on general theory, there is a chapter on two modes of analysis, normal and extensive form, a chapter on sufficient statistics and noninformative stopping and a chapter on conjugate prior distributions.

In the section on extensive-form analysis, there is a chapter on additive utility, opportunity loss and the value of information, on linear terminal analysis, on selection of the best of several processes and on problems in which the act and state spaces coincide.

The third section, on distribution theory, covers normal, binomial, Poisson and other functions, putting them in the framework and terminology of prior and posterior analysis. Some multivariate and regression processes are also covered.

It is good to see the beginnings of serious attempts to grapple with the problems of applying decision theory: devising a workable terminology subsuming

all facets of whatever approach one prefers, putting known theory into this terminology, filling in gaps in the theory, etc. This is a good beginning, especially in view of the accelerating interest in such work on the part of econometricians, business statisticians, operations analysts and others.

PAUL D. MINTON
Southern Methodist University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor J. P. Russell, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Rev. E. J. Burke as President of St. John's University on November 1, 1961.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Dr. A. O. Davidson as President of Wagner College on November 12, 1961.

Professor E. P. Starke, Rutgers University, represented the Association at the inauguration of Dr. R. F. Oxnam as President of Drew University on October 12, 1961.

Auburn University: Dr. P. D. Hill, Institute for Advanced Study, has been appointed Assistant Professor; Mr. W. T. Ingram, Louisiana State University, Miss Faye E. Jenkins, Hanna Senior High School, Mr. J. F. Stokes, Kansas State University, and Mr. R. O. Fulp, University of North Carolina, have been appointed Instructors; Associate Professor N. C. Perry has been promoted to Professor; Assistant Professor Ben Fitzpatrick, Jr. has been promoted to Associate Professor.

Beloit College: Mr. C. P. Seguin, University of Wisconsin, has been appointed Instructor; Assistant Professor E. F. Wilde has been promoted to Associate Professor; Professor R. C. Huffer retired June 30, 1961 with the title Professor Emeritus.

Boston College: Dr. S. S. Holland, Jr., Harvard University, has been appointed Assistant Professor; Mr. Armand Brumer, Princeton University, has been appointed Lecturer; Assistant Professor S. J. Bezuszka has been promoted to Associate Professor.

Boston University: Dr. Paul Shields, Massachusetts Institute of Technology, has been appointed Assistant Professor; Mr. C. D. Dixon, Brandeis University, has been appointed Instructor; Assistant Professor P. T. Maker has been promoted to Associate Professor; Professor F. J. Scheid is on leave as Fulbright Lecturer at the University of Rangoon, Burma.

Brigham Young University: Messrs. R. A. Hansen, University of Utah, J. C. Higgins, Church College of Hawaii, and L. K. Tolman have been appointed Instructors; Mr.

F. E. Haupt has returned from leave at the Naval Ordnance Training Station, China Lake, California, and has been appointed Instructor; Mr. H. E. Wickes has returned from leave at Harvard University and has been appointed Instructor; Associate Professor D. W. Robinson has been appointed Chairman of the Department of Mathematics.

Carnegie Institute of Technology: Professor A. J. Perlis has been appointed Head of the Department of Mathematics; Professor J. H. Neelley retired June 30, 1961 with the title Professor Emeritus.

College of William and Mary: Messrs. Sidney Lawrence, University of Virginia, and J. E. Shockley, University of North Carolina, have been appointed Assistant Professors; Miss Carol Cobb, University of South Carolina, has been appointed Instructor; Assistant Professor B. R. Cato, Jr. has been promoted to Associate Professor; Associate Professor Emily E. Calkins retired with the title of Professor Emeritus.

Case Institute of Technology: Associate Professor Robert Archer, Massachusetts Institute of Technology, has been appointed Associate Professor; Professor S. W. McCuskey was awarded the third annual Case Achievement Award on September 21, 1961.

Colorado College: Assistant Professor R. B. Paine, Walla Walla College, has been appointed Assistant Professor; Mr. Ronald Bierstedt, University of Notre Dame, has been appointed Instructor; Assistant Professor William Gately is on leave to study at Oklahoma State University under a grant from the National Science Foundation.

Duquesne University: Miss Patricia Gauthier, University of Pittsburgh, has been appointed Assistant Professor; Mrs. Joanne O'Donnell, Versailles Township Public School, has been appointed Instructor; Associate Professor C. F. Sebesta has been promoted to Professor; Mr. E. G. Zdinak has been promoted to Assistant Professor.

East Carolina College: Professor F. W. Saunders, Coker College, has been appointed Professor; Mr. F. D. Johnson, Rhode Island College, has been appointed Associate Professor; Mr. C. A. Webber, College of Holy Names, has been appointed Assistant Professor; Mr. Robert Woodside, State College of Agriculture and Engineering, Raleigh, North Carolina, and Mrs. Stella Daugherty, Chowan College, have been appointed Instructors; Assistant Professors J. B. Davis and F. M. Johnson have been promoted to Associate Professors; Mrs. Mildred Derrick has been promoted to Assistant Professor; Associate Professor A. C. Holland retired August 31, 1961.

East Tennessee State College: Associate Professor Charles Taylor, Maryville College, has been appointed Assistant Professor; Mr. John Kinloch, George Peabody College for Teachers, has been appointed Instructor.

Florence State College: Mr. H. B. Prendergast, Bendix Corporation, has been appointed Assistant Professor; Assistant Professor J. L. Locker has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics; Miss Orpha A. Culmer retired June 2, 1961 with the title Professor Emeritus.

Henry Ford Community College: Mr. Bernard Anderson, Wayne State University, has been appointed to the teaching staff; Mr. P. W. Edmonson has been appointed Chairman of the Department of Mathematics; Mr. Henry Navarro is on leave for the academic year at the National Science Foundation Institute at Washington University.

Fordham University: Dr. M. I. Aissen, Johns Hopkins University, has been appointed Associate Professor; Dr. Nosup Kwak, Duke University, and Messrs. Robert Craig, Ohio State University, and Guillermo Owen, Princeton University, have been appointed Assistant Professors; Mr. R. F. Vivona has been appointed Instructor; Rev. Arthur Clarke is on leave of absence for the academic year as a National Science Foundation Faculty Fellow at Yeshiva University; Assistant Professor Peter Curran is on leave of absence as an National Science Foundation Faculty Fellow at Columbia University.

Fort Hays Kansas State College: Mr. M. E. Rolfs and Miss Kay W. Dundas have been appointed Instructors.

Franklin and Marshall College: Associate Professor V. H. Haag has been promoted to Professor and will be on leave at the University of Wisconsin under a National Science Foundation Science Faculty Fellowship; Assistant Professors P. E. Bedient, Bernard Jacobson, and Clifford Marburger have been promoted to Associate Professors.

Gonzaga University: Messrs. John Firkins, University of Washington, and J. W. Robinson have been appointed Instructors.

Harvard University: Assistant Professor John Thompson, University of Chicago, has been appointed Assistant Professor; Drs. G. L. Stolzenberg, L. M. Young, Newcomb Greenleaf, and H. M. Edwards have been appointed Instructors; Professor Oscar Zariski has been promoted to Dwight Parker Robinson Professor.

Idaho State College: Mrs. Verna L. Engstrom-Heg, Oregon State College, and Mr. T. G. Burgess, Colorado State University, have been appointed Instructors.

Indiana State College: Mr. R. E. Rector, University of Illinois has been appointed Assistant Professor; Mr. R. A. Gardiner has been appointed Instructor; Mr. J. A. Berton has been promoted to Assistant Professor.

Indiana University: Drs. Jack Hachigian, L. J. Ratcliff, Jr., and W. P. Ziemer have been appointed Lecturers; Assistant Professors Louis Auslander and R. E. MacKenzie have been promoted to Associate Professors; Professor H. E. Wolfe retired July 1, 1961 with the title Professor Emeritus; Associate Professor A. H. Wallace was a winner of the Whitaker Memorial Prize given by the Edinburgh Mathematical Society, Scotland, for meritorious work in mathematics.

Iowa State University: Drs. Ward Cheney, Space Technology Laboratories, Los Angeles, California, and R. H. Sprague, University of Kentucky, have been appointed Assistant Professors.

Lake Forest College: Dr. Samuel Pasiencier, Northwestern University, has been appointed Assistant Professor; Messrs. Kevin Keenan, Peter Mundy, A. M. Lemos, and Randall Dunn have been appointed Instructors.

Michigan State University: Dr. W. T. Sledd, University of Kentucky, has been appointed Assistant Professor; Assistant Professors J. E. Adney, P. H. Doyle, and Ti Yen have been promoted to Associate Professors; Professor J. S. Frame is on leave for the academic year at the Conference Board of the Mathematical Sciences, Washington, D. C.; Assistant Professor R. P. Gilbert is on leave for the academic year at the Institute for Applied Mathematics and Fluid Dynamics at the University of Maryland.

Michigan State University—Department of Statistics: Drs. B. R. Bhat, Karnatak University, India, K. M. Daniel and Dorian Feldman, University of California, have been appointed Assistant Professors; Associate Professor K. J. Arnold has been promoted to Professor; Professor W. D. Baten retired June 1961 with the title Professor Emeritus.

Missouri School of Mines and Metallurgy: Messrs. W. J. Kabeiseman, University of South Dakota, P. D. Thomas, Stephan F. Austin State College, D. L. Rogier, G. G. Robinson, and K. C. Woodruff have been appointed Instructors; Assistant Professor C. S. Cave has been promoted to Associate Professor.

New Mexico State University: Drs. E. O. Thorpe and J. D. Thomas, Massachusetts Institute of Technology, have been appointed Assistant Professors.

North Carolina State College: Drs. Morton Lowengrub, Duke University, and L. B. Martin, Jr., Service Bureau Corporation, New York, New York, have been appointed Assistant Professors; Mr. D. L. George, Duke University, has been appointed Instructor; Assistant Professor Jacob Burlak, University of Glasgow, Scotland, has been appointed Visiting Assistant Professor; Mr. J. H. Bishir has been promoted to Assistant Professor.

Northern Illinois University: Associate Professor H. C. McKenzie, South Dakota State College, has been appointed Associate Professor; Messrs. E. D. Bender, University of Chicago, and M. J. Behr have been appointed Instructors.

Ohio State University: Assistant Professor F. W. Carroll, Jr., University of Wisconsin,

Milwaukee, and Dr. D. J. Troy, St. Louis University, have been appointed Assistant Professors; Associate Professor L. H. Miller has been promoted to Professor; Assistant Professors Arno Cronheim and J. P. Tull have been promoted to Associate Professors.

Ohio University: Mr. R. F. Craggs, Johns Hopkins University, and Miss Lois Lackner, American College for Women, Istanbul, Turkey, have been appointed Instructors.

Our Lady of the Lake College: Sister M. Annunciata and Sister M. Matthew have been appointed to the teaching staff.

Pacific University: Mr. Edmund Geyer, University of Washington, has been appointed Instructor; Associate Professor Andrewa R. Noble has been promoted to Professor.

Purdue University: Professors Daniel Pedoe, University of Malaya, Singapore. Malaya, Harry Pollard, Cornell University, Eugene Schenkman, Louisiana State University, and O. F. G. Schilling, University of Chicago, have been appointed Professors; Dr. M. L. Keedy, University of Maryland, has been appointed Associate Professor; Assistant Professor J. S. Maybee, University of Oregon, and Drs. D. R. Lick, Michigan State University, M. S. Patel, Research Triangle Institute, Durham, North Carolina, and F. I. John have been appointed Assistant Professors; Messrs. B. W. Paget, U. Hunter, J. J. Masterson, S. J. Cohen, D. L. Prullage, M. Prullage, L. E. Synder, J. J. Deely, P. S. Conn, and P. E. Dress have been appointed Instructors; Professor Samuel Kaplan, Wayne State University, has been appointed Visiting Professor; Associate Professors Edward Silverman, A. H. Smith, and Henry Teicher have been promoted to Professors; Assistant Professors Helen Bozivich, Donald Greenspan, G. L. Krabbe, J. C. Lille, and Imanuel Marx have been promoted to Associate Professors; Assistant Professor N. L. Ailing is on leave for the academic year as a National Science Foundation Post-doctoral Fellow at Harvard University; Assistant Professor J. A. Norton is on leave and has been appointed Principal Investigator in Biostatistics and Research Consultant in Biostatistical Problems at the Institute of Psychiatric Research at the Indiana University Medical Center; Associate Professor A. H. Copeland is on leave and has been appointed Visiting Associate Professor at Northwestern University; Associate Professor G. L. Krabbe is on leave for the academic year as a NATO Fellow at the University of Rennes, France; Associate Professor Sam Perlis is on leave and has been appointed Acting Associate Program Director in the Mathematics Division of the National Science Foundation; Associate Professor G. J. Rieger is on leave for the academic year at the University of Munich, Germany; Associate Professor M. Skibinsky is on sabbatical leave as a National Science Foundation Science Faculty Fellow at the University of California, Berkeley; Associate Professor R. E. Zink is on leave and has been appointed Visiting Professor at Wabash College.

Reed College: Assistant Professor Malcolm Goldman, University of Michigan, Dr. V. C. Williams, Harvard University, and Mr. R. C. Mjolsness have been appointed Assistant Professors; Assistant Professors H. E. Chrestenson, J. A. Dudman, and J. D. Leadley have been promoted to Associate Professors; Associate Professor H. E. Chrestenson is on leave for the academic year on a National Science Foundation Science Faculty Fellowship at Yale University; Associate Professor Burrowes Hunt is on leave for the academic year at the University College of North Staffordshire, England.

Rice University: Dr. R. C. O'Neil, De Paul University, has been appointed Assistant Professor; Associate Professor Jim Douglas, Jr. has been promoted to Professor; Assistant Professor Guy Johnson, Jr. has been promoted to Associate Professor; Professor Emeritus H. E. Bray has been appointed Distinguished Trustee Professor for the academic year.

Rutgers, The State University: Visiting Associate Professor Rafael Artzy, University of North Carolina, has been appointed Professor; Drs. John Mariani, Fairleigh Dickinson University and Frank Levin, University of Manchester, England, have been appointed Assistant Professors; Messrs. W. E. Gould and W. H. Richter, Princeton University, have been appointed Instructors; Associate Professors J. C. E. Dekker and

Solomon Leader have been promoted to Professors; Assistant Professors Harry Gonshor and R. E. Heaton have been promoted to Associate Professors; Mr. T. S. MacHenry has been promoted to Assistant Professor; Professor E. P. Starke retired June 30, 1961 with the title Professor Emeritus; Professor K. G. Wolfson has been appointed Chairman of the Department of Mathematics.

St. John's University: Messrs. E. J. Miranda, Richard Morgan, Norbert Oldani, Peter Stebe, and J. J. Sullivan have been appointed Instructors.

St. Joseph's College, Philadelphia: Messrs. W. W. Kuhn, Drexel Institute of Technology, and B. E. Gaffney have been appointed Instructors; Mr. J. J. Costello has been promoted to Assistant Professor.

St. Louis University: Drs. E. G. Eigel, Jr., U. S. Army, Arlington, Virginia, and R. W. Freese, University of Missouri, have been appointed Assistant Professors; Mr. G. P. Murphy, McDonnell Aircraft, St. Louis, Missouri, has been appointed Lecturer.

St. Mary's University: Messrs. James Dowdy, University of Notre Dame, and Philip Kielpinski, Purdue University, have been appointed Instructors.

St. Xavier College: Rev. J. D. Nash, Sister Mary Carlita, and Sister Mary Magdala have been appointed Instructors; Sister Mary Justina has been appointed Director of Freshman Mathematics and Director of student teaching at the secondary level; Sister Mary Carlita has been appointed Assistant Director of student teaching at the secondary level.

State College of Iowa: Mr. Carl Wehner has been appointed Instructor; Associate Professors F. W. Lott, Jr. and E. W. Hamilton have been promoted to Professors.

State University of New York—Long Island Center: Professor Robert Schatten, University of Kansas, has been appointed Professor; Assistant Professors W. C. Fox, Tulane University, and W. D. Barcus, Brown University, have been appointed Associate Professors; Mr. C. L. Gape, Syracuse University, has been appointed Instructor.

Tulane University: Mr. A. A. Armendariz, Rice University, has been appointed Assistant Professor and Research Associate; Associate Professor P. F. Conrad has been promoted to Professor; Assistant Professor A. C. Woods has been promoted to Associate Professor.

United States Military Academy: Assistant Professor J. H. Cabaniss, Jr. has been promoted to Associate Professor; Mr. L. W. Caffey has been promoted to Assistant Professor and executive officer of the Department of Mathematics.

University of British Columbia: Assistant Professor Hsin Chu, Northwestern University, and Dr. Takashi Ono, Institute for Advanced Study, have been appointed Assistant Professors; Dr. Mario Borelli, Indiana University, and Messrs. K. J. Travers and J. E. Smith have been appointed Lecturers; Associate Professor T. E. Hull has been promoted to Professor; Assistant Professors Mario Benedicty and Maurice Sion have been promoted to Associate Professors; Drs. J. V. Whittaker and D. W. Bressler have been promoted to Assistant Professors; Assistant Professor May Barclay retired October 1, 1961; Professor D. C. Murdoch has been awarded a Canada Council Senior Research Fellowship and is on leave for the academic year at Harvard University; Professor Douglas Derry is on sabbatical leave in Italy; Assistant Professor Rimhak Ree is on leave for the academic year at Yale University.

University of California, Berkeley: Drs. P. N. Burgoyne, Princeton University, Leonard Evens, Paul Fong, C. C. Moore, University of Chicago, and D. A. Ludwig, New York University, have been appointed Assistant Professors; Drs. Hans Läuchli, Swiss Federal Institute of Technology, J. D. Monk, and Dr. Haim Gaifman have been appointed Instructors and Junior Research Mathematicians; Drs. Gertrude Heller, RIAS, Baltimore, Maryland, Barry Johnson, University of Cambridge, England, and David Shale, University of Toronto, have been appointed Instructors; Professors Marcel Berger, University of Strasbourg, France, and F. E. P. Hirzebruch, University of Bonn,

Germany, have been appointed Visiting Professors; Assistant Professor M. O. Rabin, Hebrew University, Israel, has been appointed Visiting Associate Professor; Drs. Manfred Breuer, University of Bonn, Germany, and Jan Mycielski, Wroclaw University, Poland, and Mr. T. P. Srinivasan, Panjab University, Pakistan, have been appointed Visiting Assistant Professors; Assistant Professors Verena H. Dyson, San Jose State College, and E. M. Zaustinsky, University of California, Santa Barbara, and Drs. Robert Hermann, Massachusetts Institute of Technology, S. T. Kuroda, University of Tokyo, Japan, and G. M. Wing, Sandia Corporation, Albuquerque, New Mexico, have been appointed Lecturers; Associate Professors Henry Helson and Bertram Kostant have been promoted to Professors; Assistant Professor R. S. Lehman has been promoted to Associate Professor; Visiting Assistant Professors Lester Dubins and Antoni Kosinski, and Acting Assistant Professor Marvin Greenberg have been promoted to Assistant Professors; Professor C. B. Morrey, Jr. is on sabbatical leave in Paris, France; Professors A. P. Morse and E. H. Spanier have been awarded Research Professorships under the Miller Foundation; Professor Henry Helson will be on leave for the spring semester as Visiting Professor at the Universities of Uppsala, Stockholm and Lund, Sweden; Associate Professor E. A. Bishop in on sabbatical leave at Princeton University; Assistant Professor Marvin Greenberg has been awarded a National Science Foundation Postdoctoral Fellowship to study at Harvard University and in Paris, France.

University of California, Santa Barbara: Dr. J. M. Sloss, University of California, Berkeley, has been appointed Acting Assistant Professor; Assistant Professor E. H. Ostrow, University of California, Riverside, has been appointed Lecturer; Associate Professor D. H. Potts, Long Beach State College, has been appointed Visiting Associate Professor; Associate Professor P. J. Kelly has been promoted to Professor; Assistant Professor D. M. Merriell has been promoted to Associate Professor.

University of Kansas: Associate Professor P. J. McCarthy, Florida State University, has been appointed Associate Professor; Dr. R. G. Hetherington, University of Wisconsin, has been appointed Assistant Professor and Director of the Computation Center; Professor Emilio Gagliardo, University of Genoa, Italy, has been appointed Research Associate; Dr. Pawel Szeptycki, University of South Africa, Union of South Africa, has been appointed Visiting Assistant Professor.

University of Maine: Mr. R. A. Altenberger, Iowa State University, has been appointed Assitant Professor and Director of the Computation Center; Messrs. Stuart Goff, National Engineering Science Company, Pasadena, California, and R. W. Packard, Lehigh University, have been appointed Instructors.

University of Massachusetts: Assistant Professor Stanley Kertzner, University of Rhode Island, and Mr. Paul Dormont, Brown University, have been appointed Assistant Professors; Messrs. Javier Bunuel, University of Chicago, and L. J. Simonoff, Syracuse University, have been appointed Instructors; Associate Professor David Dickinson is on leave on a Fulbright appointment at Aligarh, India; Assistant Professor Ronald McHaffey is on leave on a Smith-Mundt appointment at Baghdad, Iraq.

University of Minnesota: Drs. C. A. McCarthy, Massachusetts Institute of Technology, P. L. Crawley, California Institute of Technology, and Harry Furstenberg have been appointed Assistant Professors; Dr. C. F. Koch has been appointed Instructor; Drs. C. L. Miracle and J. T. Joichi, and Mr. Frank Knight have been promoted to Assistant Professors; Assistant Professors David Storvick and Adriano Garsia have been promoted to Associate Professors; Professor W. L. Hart retired June 1961.

University of Nevada: Miss Lee Halligan, University of Nebraska, has been appointed Instructor; Mr. D. C. Pfaff, University of California, has been appointed Lecturer.

University of New Hampshire: Dr. A. R. Jacoby, Stanford Research Institute, Menlo Park, California, has been appointed Associate Professor; Mr. H. E. Perkins has been

appointed Assistant Professor; Assistant Professor J. C. Mairhuber has been promoted to Associate Professor.

University of Oklahoma: Dr. B. A. Fusaro, University of Maryland, has been appointed Assistant Professor; Mr. C. M. Maudlin, Rose Polytechnic Institute, has been appointed Institute Lecturer; Associate Professor R. V. Andree has been promoted to Professor and Chairman of the Department of Mathematics; Associate Professor W. N. Huff has been promoted to Professor; Assistant Professor Gene Levy has been promoted to Associate Professor.

University of South Carolina: Assistant Professor C. A. Nicol, University of Oklahoma, has been appointed Associate Professor; Mr. Anwar Nissan, Central State College, Edmond, Oklahoma has been appointed Assistant Professor.

University of Southern California: Dr. Geoffrey Sheppard, University of Birmingham, England, has been appointed Visiting Assistant Professor; Associate Professor Henry Autosiewicz has been promoted to Professor; Assistant Professor W. T. Kyner has been promoted to Associate Professor; Mr. Edgar Kann has been promoted to Assistant Professor.

University of Tennessee: Associate Professor J. H. Barrett, University of Utah, has been appointed Professor; Dr. Lida K. Barrett, University of Utah, has been appointed Associate Professor; Professor O. G. Harrold, Jr. has been appointed Head of the Department of Mathematics; Assistant Professor Eckford Cohen has been promoted to Associate Professor.

University of Toledo: Miss Donna J. Decker has been appointed Instructor; Miss Lucy Fellows has been appointed Lecturer.

University of Utah: Assistant Professor J. H. Case, University of Rochester, has been appointed Associate Professor; Dr. H. G. Ellis, University of Texas, has been appointed Assistant Professor; Dr. S. K. Kaul, University of Rochester, has been appointed Instructor; Professor Alexander Peyerimhoff, University of Marburg, Germany, has been appointed Visiting Professor; Dr. Otto Koerner, University of Marburg, Germany, has been appointed Visiting Lecturer; Associate Professor C. E. Burgess has been promoted to Professor; Associate Professor S. S. Smith has retired with the title Associate Professor Emeritus; Assistant Professor R. C. Bzoch is on leave as Research Assistant Professor at Louisiana State University; Associate Professor E. A. Davis is on leave with the National Science Foundation as Associate Program Director; Assistant Professor M. B. Smith is on leave as Visiting Associate Professor at the Extension Division of the University of Wisconsin.

University of Washington: Professor Roy Dubisch, Fresno State College, has been appointed Professor; Adjunct Assistant Professor N. D. Ylvisaker, New York University, Assistant Professor J. W. Woll, Jr., University of California, Berkeley, Visiting Assistant Professor Gunter Lumer, Stanford University, and Drs. C. R. Hobby, Institute for Defense Analyses, and H. H. Johnson, Princeton University, have been appointed Assistant Professors; Dr. D. S. Newman, Cornell University, has been appointed Instructor; Dr. Heinz Bauer, University of Hamburg, Germany, has been appointed Visiting Associate Professor; Assistant Professor E. M. Alfsen, University of Oslo, Norway, and Dr. Z. W. Semadeni, Michiewicz University, Poland, have been appointed Visiting Assistant Professors; Associate Professors M. G. Arsove, F. H. Brownell, and Albert Nijenhuis have been promoted to Professors; Assistant Professors R. M. Blumenthal, and R. F. Tate have been promoted to Associate Professors; Drs. T. K. Boehme, and Jack Segal have been promoted to Assistant Professors; Professor Albert Nijenhuis has been awarded a Guggenheim Fellowship and is on leave at the Institute for Advanced Study; Professor R. S. Pierce is on leave at the University of California, Berkeley; Professor J. L. Selfridge is on leave at the University of California, Los Angeles.

Utah State University: Mr. E. E. Underwood, University of Illinois, has been ap-

pointed Assistant Professor; Mr. L. O. Cannon, University of Wisconsin, has been appointed Instructor.

Valparaiso University: Mr. M. G. Mundt, Iowa State University, has been appointed Assistant Professor; Mr. John Sorenson, Purdue University, has been appointed Instructor; Mrs. Miriam H. Bretscher, Mrs. Diane H. Krebs, and Mrs. Ruth K. Deters have been promoted to Assistant Professors; Professor A. E. Hallerberg has been appointed Assistant Chairman of the Department of Mathematics.

Wake Forest College: Drs. W. G. May and J. G. May, Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland, have been appointed Assistant Professors; Messrs. J. B. Linder, University of North Carolina, and P. M. Hendricks, North Carolina State College, have been appointed Instructors; Associate Professor J. W. Sawyer has been promoted to Professor; Associate Professor K. T. Raynor retired July 1, 1961.

West Texas State College: Messrs. B. L. Fargason, Boeing Airplane Company, Wichita, Kansas, and R. G. Huffstutler, Colorado State University, have been appointed Instructors.

Wisconsin State College, Eau Claire: Messrs. L. L. Hofeldt, University of South Dakota, and A. C. Mudrich, Pennsylvania State University, have been appointed Instructors.

Wisconsin State College and Institute of Technology: Mr. M. D. Gray, Platteville High School, Platteville, Wisconsin, has been appointed Instructor; Mr. Donald Weber has been appointed to Assistant Professor.

Woman's College of the University of North Carolina: Mr. J. L. Steinmetz, Duke University has been appointed Instructor; Assistant Professor Whitfield Cobb has been promoted to Associate Professor.

Mr. P. J. Arpaia, Woodland Junior High School, East Meadow, New York, has been appointed Instructor at Clarkson College of Technology.

Dr. Hack Arroe, University of Denver, has been appointed Professor at Montana State College.

Mrs. Claire T. Bowen, University of Michigan, has been appointed Instructor at Eastern Michigan University.

Miss Fay B. Burras, Smith College, has been appointed Instructor at Randolph-Macon Woman's College.

Dr. Ronald Butler, Manchester College of Science and Technology, England, has been appointed Visiting Associate Professor at the University of Saskatchewan.

Dr. A. T. Butson, Boeing Airplane Company, Seattle, Washington, has been appointed Professor at the University of Miami.

Assistant Professor G. T. Cargo, Syracuse University, is on leave for the academic year at the National Bureau of Standards, Washington, D. C., as a National Academy of Sciences National Research Council Post-doctoral Resident Research Associate.

Assistant Professor J. B. Chicarelli, Fordham University, has been appointed Assistant Professor at Stonehill College.

Miss Mildred B. Cole, Waldo Junior High School, Aurora, Illinois, has been appointed Visiting Lecturer at the University of Maryland.

Assistant Professor D. E. Dupree, Auburn University, has been appointed Head of the Department of Mathematics at Northeast Louisiana State College.

Mr. T. S. Ellis, Sylvan Hills High School, North Little Rock, Arkansas, has been appointed Instructor at Henderson State Teachers College.

Mr. H. R. Emory, University of Detroit, has been appointed Director of the Computation Laboratory.

Mr. Joaquin Galvan-Pumarejo, St. Mary's University, has been appointed Instructor at Mount Angel College.

Dr. A. B. Gray, Jr., White Sands Missile Range, New Mexico, has been appointed Associate Professor at Arizona State College.

Mr. Roger Grobe, Michigan State University, has been appointed Instructor at General Motors Institute.

Dr. D. J. Henderson, University of Utah, has been appointed Assistant Professor at the University of Idaho.

Mr. M. O. Holoien, North Dakota State University, has been promoted to Assistant Professor.

Mr. C. M. Hughey, Western Electric Company, Winston-Salem, North Carolina, has accepted a position as Engineer with the Lockheed Missile and Space Company, Charleston, South Carolina.

Assistant Professor D. H. Husemuller, Pennsylvania State University, has been appointed Assistant Professor at Haverford College.

Mr. N. W. Johnson, Carleton College, has been appointed Assistant Professor at Geneva College.

Assistant Professor Hewitt Kenyon, University of Rochester, has been appointed Assistant Professor at George Washington University.

Associate Professor Joseph Kerec, Lewis College, has been appointed Assistant Professor at Canisius College.

Mr. Churl Suk Kim, Southern Illinois University, has been appointed Assistant Professor at Lindenwood College.

Mr. W. A. Kirby, University of Texas, has been appointed Instructor at Bowling Green State University.

Assistant Professor A. H. Kruse, University of Kansas, has been appointed Research Professor at the Research Center of New Mexico State University.

Mr. D. A. Kuehner, University of Maine, has been appointed Instructor at Bridgeport College.

Dr. C. L. Lawson, University of California, Los Angeles, has accepted a position as Research Engineer in the Jet Propulsion Laboratory of the California Institute of Technology.

Associate Professor J. R. Lee, College of William and Mary, has been granted a National Science Foundation Post-doctoral Fellowship and is on leave of absence to study at the University of California, Berkeley.

Mr. J. F. Leetch, Ohio State University, has been appointed Assistant Professor at Bowling Green State University.

Professor Solomon Lefschetz, Director of the Center for Differential Equations at RIAS, Baltimore, Maryland, has been named a foreign member of the Royal Society of London.

Mr. J. D. Lifsey, University of Georgia, has accepted a position in the Data Processing Division of the Climatic Center, USAF Air Weather Service, Asheville, North Carolina.

Miss Mary Lou Lyons, University of Kentucky, has accepted a position as Mathematician at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Miss Mary N. Manning, Arlington County School System, Arlington, Virginia, has been appointed Instructor at Tennessee Polytechnic Institute.

Miss Janice Marsh, University of Oklahoma, has been appointed Instructor at the State University of New York, College of Education at Oswego.

Dr. Beckham Martin, Owens-Illinois Glass Company, Toledo, Ohio, has been appointed Associate Professor at the University of Dayton.

Professor K. O. May, Carleton College, has received a four year grant from the Course Content Improvement Section of the National Science Foundation for the devel-

opment of an experimental pre-graduate training program in mathematics involving undergraduate colloquia and a local journal.

Mr. S. S. Mitra, University of Washington, has been appointed Acting Assistant Professor at the University of Idaho.

Mr. M. E. Moore, West Texas State College, has been appointed Instructor at the University of New Mexico.

Associate Professor W. K. Moore, Albion College, has been appointed Head of the Department of Mathematics.

Assistant Professor W. C. Nemitz, University of Kansas, has been appointed Assistant Professor at Southwestern at Memphis.

Mr. R. A. Newcomb, United States Navy, has been appointed Instructor at Whittier College.

Mr. V. D. Norum, Space Technology Laboratories, Inc., Los Angeles, California, has accepted a position as a member of the Technical Staff of Space-General Corporation, Glendale, California.

Mr. K. D. Parakh, University of Cincinnati, has accepted a position as Programmer with the National Cash Register Company, London, England.

Associate Professor W. A. Pierce, Syracuse University, has been appointed Professor at West Virginia University.

Assistant Professor E. C. Posner, Harvey Mudd College, has accepted a position as Research Specialist with the Jet Propulsion Laboratory, Pasadena, California.

Dr. Gordon Raisbeck, Bell Telephone Laboratories, Murray Hill, New Jersey, has accepted a position as mathematician and communications specialist with Arthur D. Little, Inc. Cambridge, Massachusetts.

Assistant Professor P. C. Rogers, St. Anselm's College, has been recalled to active duty with the United States Navy with the rank of Lieutenant Commander and has been appointed Professor at the Naval Postgraduate School.

Assistant Professor R. F. Shortt, San Fernando Valley State College, has been appointed Assistant Professor at Bates College.

Associate Professor O. E. Taulbee, Michigan State University Oakland, has accepted a position in the Avionics and Electronics Research and Development Section of the Goodyear Aircraft Corporation, Akron, Ohio.

Mrs. Birdie D. Van Hooser, George Peabody College for Teachers, has been appointed Instructor at the University of Louisville.

Mr. J. E. Vinson, Oregon State University, has accepted a position with the Aerojet-General Corporation, Sacramento, California.

Assistant Professor R. R. Welland, Ohio State University, has been appointed Research Associate at the University of Chicago.

Miss V. Agnes Wessling, Western Illinois University, has been appointed Instructor at the College of Saint Teresa.

Mrs. Erna B. Yackel, Wisconsin State College, River Falls, has been appointed Instructor at Hamline University.

Assistant Professor B. J. Yozwiak, Youngstown University, has been promoted to Associate Professor and has been awarded the Watson Foundation Distinguished Professor Award.

Assistant Professor R. E. Zink, Purdue University, has been appointed Visiting Professor at Wabash College.

Professor E. A. Hedberg, University of South Carolina, died September 11, 1961. He was a member of the Association for 16 years.

Professor H. D. Larsen, Albion College, died August 9, 1961. He was a member of the Association for 26 years.

Professor E. G. Olds, Carnegie Institute of Technology, died October 9, 1961. He was a member of the Association for 36 years.

Associate Professor Ruth M. Peters, University of New Hampshire, died May 12, 1961. She was a member of the Association for 24 years.

NEW STATISTICS DEPARTMENT AT SMU

Southern Methodist University announces the opening of a new Department of Mathematical and Experimental Statistics and Statistical Laboratory in the Graduate School. Faculty members include Drs. Paul D. Minton and Vanamamalai Seshadri, assisted part-time by Mr. Del West and Mr. Mack Usher.

Coursework leading to the degree of Master of Science will be offered. Departmental assistantships are available. For further information write to Dr. Paul D. Minton, Department of Mathematical and Experimental Statistics, Southern Methodist University, Dallas 22, Texas.

NYU EXPANDS MATHEMATICS PROGRAM

A major expansion of New York University's research and graduate educational program in mathematics, including construction of a new building at the University's Washington Square Center, has been made possible by a matching grant of $2,750,000 from the Alfred P. Sloan Foundation.

Two distinguished scientists who have made major contributions in mathematics are to be honored. The University's Institute of Mathematical Sciences, which was founded by Dr. Richard Courant in 1938 is to be renamed the Courant Institute of Mathematical Sciences, and the new building that will house the Institute is to be named Warren Weaver Hall. Dr. Weaver is vice president of the Alfred P. Sloan Foundation and a leading figure in national science affairs.

Of the total, $2,000,000 will go toward the construction of Warren Weaver Hall. The grant also will provide $750,000 for fellowships and research at the Courant Institute. This sum will finance fellowships for about 20 additional pre-doctoral students and for eight to ten new post-doctoral fellows annually. The funds also will support new research programs in statistics and mathematical physics and will provide for the establishment of a $300,000 special fund for research.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held April 29, 1961 at Western Kentucky State College, Bowling Green, Kentucky. Professor A. G. Anderson, Western Kentucky State College, presided at both the morning and afternoon sessions. There were 54 persons in attendance, including 37 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor A. W. Goodman, University of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor A. G. Anderson, Western Kentucky State College.

By invitation of the Committee, Professor M. L. Curtis of Florida State University delivered an hour address at the afternoon session entitled, "Geometrically-realized group theory."

The following papers were presented at the morning session:

1. *An undergraduate seminar in analytic geometry at the University of Louisville*, by Professor W. L. Moore, University of Louisville.

2. *Pairs of distinct primes belonging to N*, by Professor E. E. Mayo, Morehead State College. Material related to the Goldbach conjecture was discussed.

3. *Some remarks concerning power series*, by Professor V. F. Cowling, University of Kentucky. This was a discussion of the form of entire functions in terms of the numerical properties of the coefficients and the rate of growth of the function.

4. *An infinite product for* π, by Professor A. W. Goodman, University of Kentucky.

5. *Remarks on certain second order linear differential equations*, by Professor T. J. Pignani, University of Kentucky.

6. *On Faber polynomials*, by Professor W. C. Royster, University of Kentucky. This was an expository talk concerning the development of the Faber polynomials together with applications.

V. F. COWLING, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962,

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, May 5, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, U. S. Naval Weapons Laboratory, Dahlgren, Virginia, April 28, 1962.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA, College of St. Teresa, Winona, May 12, 1962.

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, November 24, 1962.

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Northwestern State College, Alva, April 13–14, 1962.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA

ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, May 4–5, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

SOUTHERN CALIFORNIA, Long Beach State College, March 10, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April 1962

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, May 5, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

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SOME EXAMPLES IN WEAK SEQUENTIAL CONVERGENCE

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1. Introduction. Let μ be a nonnegative and countably additive measure in the nonempty point set X. Any μ-measurable real function $f(x)$ on X for which the integral $\int_X |f|\, d\mu$ exists as a finite number will be called a μ-summable function. Instead of μ-measurable, μ-summable or μ-almost everywhere, we shall say, shortly, measurable, summable or almost everywhere, respectively.

The Hahn-Saks theorem (sometimes called the Fubini-Hahn-Saks theorem) states essentially that if $f_n(x)$ is a sequence of summable functions on X such that for every measurable subset E of X the sequence $\int_E f_n d\mu$ converges to a finite real number $\nu(E)$, then there exists a summable function $f(x)$ such that $\nu(E) = \int_E f d\mu$ for all E. (A proof may be found in [2], Section 40, Example 12 or in [3], Section 40, Theorem 4.) It is usual to say in this case that the sequence f_n converges weakly to f. Assuming that we have this situation, a question which immediately arises is whether there exists any pointwise relation between the sequence f_n and its weak limit f. More specifically, we may ask the following questions:

(i) If f_n converges weakly to f, is it true then that $f_n \to f$ pointwise almost everywhere on X, or at least $f_{n_k} \to f$ (as $k \to \infty$) pointwise almost everywhere on X for some subsequence f_{n_k}?

(ii) If the answer to the first question is no in general, but if in some particular case it is given that $f_{n_k} \to g$ pointwise (almost everywhere) for some subsequence f_{n_k}, is it true then that $g = f$ almost everywhere?

(iii) Is it true that, although f_n may not contain any subsequence converging pointwise to f, we have at least that $\liminf f_n \leqq f \leqq \limsup f_n$ holds almost everywhere on X?

Note that if the answer is yes, then the answer to the preceding question is yes.

(iv) If the answer to the preceding question is yes, can it happen then that $\liminf f_n = f < \limsup f_n$ holds almost everywhere?

Some other questions are:

(v) Does weak convergence of $f_n(x)$ to $f(x)$ always imply weak convergence of $|f_n(x)|$ to $|f(x)|$?

(vi) If the answer to the preceding question is no in general, can it happen then that f_n converges to f and $|f_n|$ converges to g, where f and g are arbitrarily prescribed summable functions subject only to the condition that $|f(x)| \leqq g(x)$ almost everywhere?

It is the purpose of this paper to present a few simple theorems and examples which throw some light upon the questions above. Thanks are due to Professor W. A. J. Luxemburg for several stimulating discussions.

* The preparation of this paper was supported by the National Science Foundation under grant NSF-G14002 to the California Institute of Technology.

2. A standard example. A standard example for weak convergence of a sequence $f_n(x)$ is the example in which μ is Lebesgue measure in $X=[0, 2\pi]$, and $f_n(x)=\sin nx$. As is well known, $f_n(x)=\sin nx$ converges weakly to the function which is identically zero. Since the integral of $\sin nx$ over any interval of length $2\pi/n$ is zero, it is immediately evident that $\int_a^b \sin nxdx\to 0$ for any subinterval $[a, b]\subset X$, and it is not astonishing, therefore, that the same holds if $[a, b]$ is replaced by an arbitrary measurable set $E\subset X$. The proof of this fact, however, is not easier than the proof of the well-known Riemann-Lebesgue theorem that $\int_0^{2\pi} f(x) \sin nxdx\to 0$ as $n\to\infty$ for any summable f. This shows then, incidentally, that the sequence $g_n(x)=f(x) \sin nx$ converges weakly to zero for any summable f. There exist several methods to prove the Riemann-Lebesgue theorem, and we shall say a few words about some of these methods. One way to proceed is by means of the substitution $x=y-\pi/n$. Writing $b_n=\int_0^{2\pi} f(x) \sin nxdx$ and assuming, as we may, that f is of period 2π, the substitution yields

$$b_n = -\int_{\pi/n}^{2\pi+\pi/n} f(y - \pi/n) \sin nydy,$$

so $2b_n$ is, apart from two integrals which obviously tend to zero as $n\to\infty$, equal to

$$\int_{\pi/n}^{2\pi} \{f(x) - f(x - \pi/n)\} \sin nxdx,$$

and this also tends to zero since

$$\int_0^{2\pi} |f(x) - f(x - \pi/n)|\, dx \to 0 \quad \text{as} \quad n \to \infty.$$

This particular proof uses essentially the special property of $\sin x$ that $\sin(x+\pi) = -\sin x$ for all x. Another popular proof is obtained by assuming first that f is square summable, i.e., $\int_0^{2\pi} |f|^2 dx < \infty$. Observing then that the system $\phi_n(x) = \pi^{-\frac{1}{2}} \sin nx$, $n=1, 2, \cdots$, is an orthonormal system in $[0, 2\pi]$ and applying Bessel's inequality for orthonormal systems, it follows that $\sum_1^\infty b_n^2 < \infty$, hence $b_n\to 0$. If $f(x)$ is summable, but not necessarily square summable, there exists for any $\epsilon>0$ a square summable $g(x)$ satisfying $\int_0^{2\pi} |f-g|\, dx<\epsilon$, and it follows immediately that

$$\left| \int_0^{2\pi} \{f(x) - g(x)\} \sin nxdx \right| < \epsilon$$

for all n, hence $\int_0^{2\pi} f(x) \sin nxdx\to 0$ as $n\to\infty$. As observed already, this second proof makes essential use of the orthonormality of the system $\phi_n(x)=\pi^{-\frac{1}{2}} \sin nx$. Both proofs are "unfair" in the sense that some special property of the trigonometrical function system is drawn in unnecessarily. The proof of this statement is contained in the following theorem.

THEOREM 1. *Let $\phi(x)$ be real, Lebesgue measurable, bounded and of period* 1 *on the real line, and let $\int_0^1 \phi(x)dx = \alpha$. Then*

$$\int_0^1 f(x)\phi(nx)dx \to \alpha \int_0^1 f(x)dx \qquad (n \to \infty)$$

for any f which is Lebesgue summable over $[0, 1]$.

Proof. Set $\psi(x) = \phi(x) - \alpha$. Then $\psi(x)$ satisfies the same conditions as $\phi(x)$, except for the fact that $\int_0^1 \psi(x)dx = 0$. Furthermore, the conclusion of the theorem is equivalent to the statement that $\int_0^1 f(x)\psi(nx)dx \to 0$. Hence, we may as well assume from the beginning on that $\alpha = 0$. Under this assumption, the integral of $\phi(nx)$ over any interval of length n^{-1} is zero, and it follows easily that for any interval $[a, b]$ contained in $[0, 1]$ we have $\int_a^b \phi(nx)dx \to 0$. More precisely, $\left|\int_a^b \phi(nx)dx\right| \leqq \beta n^{-1}$, where $\beta = \int_0^1 |\phi(x)|\,dx$.

Any finite linear combination (with real coefficients) of characteristic functions of intervals will be called a step function. Given the Lebesgue summable function $f(x)$ on $[0, 1]$ and the number $\epsilon > 0$, there exists a step function $s(x)$ on $[0, 1]$ such that $\int_0^1 |f - s|\,dx < \epsilon/M$, where $M = \sup |\phi(x)|$. This is an essential theorem in the theory of Lebesgue integration, and in some of the modern approaches to integration theory it is even no longer a theorem but serves as the definition of Lebesgue summability of a function f. It follows then immediately that

$$\left|\int_0^1 f(x)\phi(nx)dx - \int_0^1 s(x)\phi(nx)dx\right| \leqq M \int_0^1 |f - s|\,dx < \epsilon$$

for all n. It remains to prove, therefore, that

$$\int_0^1 s(x)\phi(nx)dx \to 0 \quad \text{as} \quad n \to \infty$$

for any step function $s(x)$. This, however, is immediately derived from the fact, proved above, that $\int_a^b \phi(nx)dx \to 0$ as $n \to \infty$ for any interval $[a, b]$.

Some examples are: $\phi(x) = \sin 2\pi x$, $\alpha = 0$; $\phi(x) = \sin^2 2\pi x$, $\alpha = \frac{1}{2}$; $\phi(x) = |\sin 2\pi x|$, $\alpha = 2/\pi$. Hence, in the interval $[0, 2\pi]$, the sequences $\sin nx$, $\sin^2 nx$ and $|\sin nx|$ converge weakly to 0, $\frac{1}{2}$ and $2/\pi$ respectively.

3. Extension to unbounded functions. The result in the preceding section can still be regarded as rather simple. It can be extended to the case that $\phi(x)$ is no longer bounded, and this extension is not completely trivial.

THEOREM 2. *Let $\phi(x)$ be real, Lebesgue measurable, of period* 1, *and Lebesgue summable over the interval* $[0, 1]$ *with $\int_0^1 \phi(x)dx = \alpha$. Then*

$$\int_0^1 f(x)\phi(nx)dx \to \alpha \int_0^1 f(x)dx$$

for any f which is Lebesgue measurable and essentially bounded on $[0, 1]$. In particular, the sequence $f_n(x)=\phi(nx)$ converges weakly to the constant α.

Proof. Exactly as in Theorem 1 we may assume without loss of generality that $\alpha=0$. We set $\int_0^1|\phi(x)|\,dx=\beta$; hence $\beta<\infty$. For $\beta=0$ the theorem is trivial; we shall assume therefore that $\beta>0$. Lebesgue measure will be denoted by μ. Given $\epsilon>0$, there exists a number $\delta>0$ such that if $E\subset[0, 1]$ and $\mu(E)\leqq\delta$, then $\int_E|\phi(x)|\,dx\leqq\epsilon$. More precisely, if we let E run through all subsets of $[0, 1]$ of measure δ, and we denote by $P(\delta)$ the supremum of the so obtained numbers $\int_E|\phi(x)|\,dx$, then $P(\delta)\leqq\epsilon$. It is not difficult to show that the supremum $P(\delta)$ is actually attained. This is done by considering for variable $\lambda\geqq0$ the sets $A_\lambda=\{x: x\in[0, 1],\ |\phi(x)|\geqq\lambda\}$. If λ descends from $+\infty$ to 0, then A_λ ascends from a set of measure zero to the whole interval $[0, 1]$, and so $\mu(A_\lambda)$ increases from 0 to 1. Although $\mu(A_\lambda)$ may have jumps at the points λ where the set $\{x: |\phi(x)|=\lambda\}$ is of positive measure, it is evident that if δ equals $\mu(A_{\lambda_0})$ for a certain value λ_0, then the integral of $|\phi|$ over this A_{λ_0} yields the maximum value $P(\delta)$, while if $\delta<\mu(A_{\lambda_0})$ but $\delta>\mu(A_\lambda)$ for all $\lambda>\lambda_0$ then there is a subset of A_{λ_0} of measure δ such that the integral of $|\phi|$ over this subset yields the maximum value $P(\delta)$. Let us consider now the function $|\phi(nx)|$ on $[0, 1]$ and try to determine, for the same value δ as before, the corresponding maximum of the numbers $\int_E|\phi(nx)|\,dx$ for all $E\subset[0, 1]$ satisfying $\mu(E)=\delta$. It will be evident from the above argument and the periodicity of $\phi(x)$ that this maximum can be obtained by splitting up the available measure δ into n parts δ/n, and allocating an amount δ/n in each of the intervals $[k/n, (k+1)/n]$, $k=0, 1, \cdots, n-1$, with the purpose of making $\int_F|\phi(nx)|\,dx$ maximal in such an interval under the condition that $\mu(F)=\delta/n$. But evidently $F\subset[k/n, (k+1)/n]$ and $\mu(F)=\delta/n$ implies $\int_F|\phi(nx)|\,dx\leqq\epsilon/n$, and so the final result is that if $E\subset[0, 1]$ and $\mu(E)\leqq\delta$, then $\int_E|\phi(nx)|\,dx\leqq\epsilon$. In other words, the integrals $\int_E|\phi(nx)|\,dx$ are uniformly absolutely continuous with respect to n.

Note the analogy with the argument needed to solve the following problem: If one has decided on the "best" way to spend a thousand dollars in a large department store, determine then the best way to spend the same amount in the "union" of ten separate smaller stores under the conditions that each item is priced the same in each store, and the stock of each item in each smaller store is one tenth of what it is in the larger store. In the measure problem above each store may have infinitely small quantities of infinitely expensive items for sale.

Having obtained this preliminary result we shall make no further use of the fact that the nth function is of the special form $\phi(nx)$. Denoting the nth function by $f_n(x)$, we shall use only that the integrals $\int_E|f_n|\,dx$ are uniformly absolutely continuous, the integral of $f_n(x)$ over an interval of length n^{-1} is zero, and $\int_0^1|f_n|\,dx\leqq\beta$ for all n.

We turn our attention now towards the given function f, measurable and essentially bounded on $[0, 1]$. The last condition means that there exists a number $M\geqq0$ such that $|f(x)|\leqq M$ on $[0, 1]$, except perhaps on a set of measure

zero. We recall that β is supposed to be positive, and we select $\epsilon>0$ such that $\epsilon\leqq\beta$, and then $\delta>0$ such that $\mu(E)\leqq\delta$ implies $\int_E|f_n|\,dx\leqq\epsilon$ for all n. It may be assumed that $\delta\leqq\epsilon/\beta$; if not, we replace δ by $\min(\delta, \epsilon/\beta)$. Similarly as in the proof of Theorem 1 there exists a step function $s(x)$ (i.e. a finite linear combination of characteristic functions of intervals) satisfying $\int_0^1|f-s|\,dx\leqq\delta^2$; it may be assumed without loss of generality that $|s(x)|\leqq M$, and so $|f(x)-s(x)|\leqq 2M$ for all $x\in[0, 1]$. Let

$$E_1 = \{x: x\in[0, 1],\ |f(x)-s(x)|>\delta\},$$
$$E_2 = \{x: x\in[0, 1],\ |f(x)-s(x)|\leqq\delta\}.$$

Then $\delta\mu(E_1)\leqq\int_{E_1}|f-s|\,dx\leqq\delta^2$, so $\mu(E_1)\leqq\delta$, and it follows that

$$\left|\int_{E_1}(f-s)f_n dx\right| \leqq 2M\int_{E_1}|f_n|\,dx \leqq 2M\epsilon,$$

$$\left|\int_{E_2}(f-s)f_n dx\right| \leqq \delta\int_0^1|f_n|\,dx \leqq \delta\beta.$$

Hence, by addition,

$$\left|\int_0^1 ff_n dx-\int_0^1 sf_n dx\right| \leqq 2M\epsilon+\delta\beta\leqq 2M\epsilon+\epsilon=(2M+1)\epsilon,$$

and this holds for all n. It remains to prove, therefore, that for any step function $s(x)$ the integral $\int_0^1 sf_n dx$ tends to zero as $n\to\infty$, and it is sufficient for this purpose to show that $\int_a^b f_n dx\to 0$ as $n\to\infty$ for any interval $[a, b]$. This, however, follows immediately from the fact that the integral of f_n over any interval of length n^{-1} is zero and from the uniform absolute continuity of the integrals $\int_E|f_n|\,dx$.

Some examples are:

$$\phi(x) = x^{-\frac{1}{2}}-1 \quad \text{on} \quad [0, 1],\ \alpha=1;$$
$$\phi(x) = \log\sin x \quad \text{on} \quad [0, \tfrac{1}{2}\pi],\ \alpha=-\tfrac{1}{2}\pi\log 2.$$

That the decisive factor is indeed the uniform absolute continuity of the functions $\int_E|f_n|\,dx$ is shown clearly by the following example:

Let $f_n(x)$, for $0\leqq x\leqq 1$ and $n=1, 2, \cdots$, be a nonnegative function whose graph (or more precisely, the part of the graph where $f_n(x)>0$) consists of 2^n (narrow and high) congruent isosceles triangles standing upon the x-axis around the centers of the intervals $[k/2^n, (k+1)/2^n]$, $k=0, 1, \cdots, 2^n-1$. Evidently, these 2^n triangles may be chosen such that $\int_0^1 f_n dx=1$ and the sum of the lengths of their bases equals $1/2^{n+1}$. Since the integral of f_n over any interval of length $1/2^n$ equals now $1/2^n$, it is easily seen that $\int_a^b f_n dx\to b-a=\int_a^b dx$ as $n\to\infty$ for any interval $[a, b]\subset[0, 1]$. Considering only intervals it seems, therefore, as if the sequence f_n converges weakly to the constant 1. This, however, is not true. Indeed, the set $\{x: f_n(x)>0\}$ is of measure $1/2^{n+1}$ and hence the set on which at least one of the f_n is positive is of measure not exceeding $1/2^2+1/2^3+\cdots$

$=\frac{1}{2}$. This shows that if $E=\{x: f_n(x)=0 \text{ for all } n\}$, then $\mu(E)\geqq\frac{1}{2}$. Hence, $\lim \int_E f_n dx=\int_E dx=\mu(E)$ is not satisfied for this particular set E, since $\int_E f_n dx=0$ for all n.

4. Pointwise equalities and inequalities. Now that we have available some examples of weakly converging sequences $f_n(x)$, it is possible to answer the questions raised in Section 1.

THEOREM 3. *Let $\phi(x)$ be a real function on the real line, Lebesgue measurable and of period 1, and let λ be the essential supremum of $\phi(x)$; it may be that $\lambda=-\infty$ or $\lambda=+\infty$. Finally, let $f_n(x)=\phi(nx)$ for $n=1, 2, \cdots$. Then* $\limsup f_n(x)=\lambda$ *at almost every point x. More generally,* $\limsup f_{n_k}(x)=\lambda$ *at almost every point x for any subsequence f_{n_k}. A similar statement holds for* $\liminf f_n(x)$ *and the essential infimum of $\phi(x)$.*

Proof. If $\lambda=-\infty$, then $\phi(x)=-\infty$ almost everywhere, and hence $f_n(x)=-\infty$ almost everywhere, for every n. It follows that $\limsup f_n(x)=-\infty$ almost everywhere in this case. We may assume, therefore, from here on that $\lambda>-\infty$. Lebesgue measure will be denoted by μ again. Let α be a finite number satisfying $\alpha<\lambda$, and let $E_\alpha=\{x: x\in[0, 1], \phi(x)\geqq\alpha\}$. Then $\mu(E_\alpha)=\beta$ for some β such that $0<\beta\leqq 1$. If $E_{n,\alpha}=\{x: x\in[0, 1], \phi(nx)\geqq\alpha\}$ for the same α and $n=1, 2, \cdots$, then $\mu(E_{n,\alpha})=\beta$ where β is the same number as above. Let $S_\alpha=\limsup_n E_{n,\alpha}$, and let $I=[a, b]$ be an arbitrary subinterval of $[0, 1]$. Since it is evident that $\mu(E_{n,\alpha}\cap I)\to\beta(b-a)$ as $n\to\infty$, we have

$$\mu(S_\alpha\cap I)\geqq\limsup\mu(E_{n,\alpha}\cap I)>\tfrac{1}{2}\beta(b-a).$$

Let us assume now that $\limsup f_n(x)<\lambda$ holds on a subset F_1 of $[0, 1]$ of positive measure. Then there exists a finite number $\alpha<\lambda$ such that $\limsup f_n(x)<\alpha$ on a subset $F\subset F_1$ with $\mu(F)>0$. Let β be the number corresponding to this α according to the preceding paragraph, and choose γ such that $0<\gamma<1$ and $\frac{1}{2}\beta+\gamma>1$. Since F is a set of positive Lebesgue measure there exists an interval $I=[a, b]$ in $[0, 1]$ such that $\mu(F\cap I)>\gamma(b-a)$ (cf. [2], Section 16, Theorem A or [3], Section 10, Lemma α). On $F\cap I$ we have $\limsup f_n(x)<\alpha$, and on $S_\alpha\cap I$ we have $\limsup f_n(x)=\limsup\phi(nx)\geqq\alpha$; hence, $F\cap I$ and $S_\alpha\cap I$ are disjoint. On the other hand, their measures exceed $\gamma\mu(I)$ and $(\frac{1}{2}\beta)\mu(I)$ respectively with $\gamma+\frac{1}{2}\beta>1$. This yields a contradiction. Hence, $\limsup f_n(x)=\lambda$ holds almost everywhere.

Finally, it is evident from the proof that the same argument is valid for any subsequence of $f_n(x)$.

Some examples are:

$$\limsup \sin nx=1 \quad\text{and}\quad \liminf \sin nx=-1 \text{ almost everywhere;}$$

$$\limsup|\sin nx|=1 \quad\text{and}\quad \liminf|\sin nx|=0 \text{ almost everywhere;}$$

if $\phi(x)$ is of period 1 and equal to $x^{-\frac{1}{2}}-1$ on $[0, 1]$, then $\limsup\phi(nx)=+\infty$ and $\liminf\phi(nx)=0$ almost everywhere;

if $\phi(x)$ is of period $\frac{1}{2}\pi$ and equal to $\log \sin x$ on $[0, \frac{1}{2}\pi]$, then $\limsup \phi(nx) = 0$ and $\liminf \phi(nx) = -\infty$ almost everywhere. In all these examples the same result holds for any subsequence.

We recall that the first and third questions in Section 1 were as follows:

(i) If f_n converges weakly to f, is it true then that $f_n \to f$ pointwise almost everywhere, or at least $f_{n_k} \to f$ pointwise almost everywhere for some subsequence f_{n_k}?

(iii) Is it true that, although f_n may not contain any subsequence converging pointwise to f, we have at least that $\liminf f_n \leqq f \leqq \limsup f_n$ holds almost everywhere?

The examples above show that the answer to the first question is no in general. In all examples the considered sequence f_n converges weakly (to a finite constant function), but for every subsequence f_{n_k} the functions $\limsup f_{n_k}$ and $\liminf f_{n_k}$ are different almost everywhere.

We shall prove now that the answer to the third question is yes.

THEOREM 4. *Let μ be a measure in X, and let the μ-summable functions $f_n(x)$ converge weakly to the μ-summable function $f(x)$. Then*

$$\liminf f_n(x) \leqq f(x) \leqq \limsup f_n(x)$$

holds μ-almost everywhere on X.

Proof. Since all $f_n(x)$ are simultaneously zero outside some subset of X of σ-finite measure, we may assume without loss of generality that X is of σ-finite measure. Hence, $X = \bigcup_1^\infty X_n$ with $\mu(X_n) < \infty$ for all n, and it will be sufficient to prove that $\liminf f_n \leqq f \leqq \limsup f_n$ holds on each X_n separately. In other words, we may and shall assume that $\mu(X)$ is finite. We set $p(x) = \limsup f_n(x)$ and $g_n(x) = \sup\{f_n(x), f_{n+1}(x), \cdots\}$, and we consider first the subset $P = \{x: p(x) = -\infty\}$. Since $g_n(x) \downarrow p(x)$, we have $P = \bigcup_1^\infty P_n$, where $P_n = \{x: x \in P, g_n(x) < 0\}$. On each P_n the theorem on integration of monotone sequences may be applied to the sequence $g_n, g_{n+1}, \cdots$, hence $\int_F g_k d\mu \to \int_F p d\mu$ as $k \to \infty$ for all measurable sets $F \subset P_n$. Since, therefore,

$$\int_F f d\mu = \lim \int_F f_k d\mu \leqq \lim \int_F g_k d\mu = \int_F p d\mu$$

for all $F \subset P_n$, we obtain the result that $f(x) \leqq p(x)$ holds almost everywhere on P_n, and hence on $P = \bigcup_1^\infty P_n$. It follows incidentally that $\mu(P) = 0$, since $f = p = -\infty$ on P and f is summable. Obviously, $f \leqq p$ on the set where $p(x) = +\infty$, and it remains to consider the set on which $-\infty < p(x) < +\infty$. This set is the union (for $N = 1, 2, \cdots$) of the sets $Q_N = \{x: -N \leqq p(x) \leqq N\}$, and it is sufficient to prove that $f \leqq p$ on each Q_N. Assume, therefore, that we have a fixed Q_N. The set Q_N is the union of the sets $S_n = \{x: x \in Q_N, g_n(x) - p(x) < 1\}$, and again it is sufficient to prove that $f \leqq p$ on each S_n. Any S_n is of finite measure, and the functions $p(x)$, $g_n(x)$, $g_{n+1}(x)$, $\cdots$ are uniformly bounded (and, of

course, μ-measurable) on S_n, so $\int_F g_k d\mu \to \int_F p d\mu$ as $k \to \infty$ for all measurable sets $F \subset S_n$. But then, again,

$$\int_F f d\mu = \lim \int_F f_k d\mu \leqq \lim \int_F g_k d\mu = \int_F p d\mu$$

for all these $F \subset S_n$, and it follows that $f \leqq p$ holds almost everywhere on S_n. This finishes the proof that $f(x) \leqq \limsup f_n(x)$ holds almost everywhere on X. The proof of the inequality $\liminf f_n(x) \leqq f(x)$ is then immediately derived by observing that $\liminf f_n(x) = -\limsup \{-f_n(x)\}$.

In order to give an answer to the fourth question in Section 1, we present now an example of a sequence f_n converging weakly to f such that $\liminf f_n = f < \limsup f_n$ holds almost everywhere. Let μ be Lebesgue measure in the real line, and let $\phi(x)$ be a real function of period 1, Lebesgue summable over $[0, 1]$, such that the essential supremum and infimum of $\phi(x)$ are $+\infty$ and 0 respectively (e.g. $\phi(x) = x^{-\frac{1}{2}} - 1$ on $[0, 1]$). For $k = 1, 2, \cdots$ and $l = 1, \cdots, k$ the function $g_{kl}(x)$ is defined on $[0, 1]$ as follows: $g_{kl}(x) = \phi(kx)$ on $[(l-1)/k, l/k]$ and $g_{kl}(x) = 0$ elsewhere. Hence, the functions $g_{k1}, \cdots, g_{kk}$ are formed by splitting up $\phi(kx)$ into k "disjoint" parts. Arranging all the functions $g_{kl}(x)$ into a single sequence, we obtain a sequence $f_n(x)$, $n = 1, 2, \cdots$. Since

$$\int_0^1 g_{kl}(x) dx = k^{-1} \int_0^1 \phi(kx) dx \leqq k^{-1} \int_0^1 |\phi(x)| dx,$$

we have $\int_0^1 f_n dx \to 0$ as $n \to \infty$, and so the sequence $f_n(x)$ converges weakly to the function f which is identically zero. It is obvious now from the construction of the sequence f_n and from Theorem 3 that we have

$$0 = \liminf f_n(x) = f(x) < \limsup f_n(x) = +\infty$$

at almost every point $x \in [0, 1]$.

5. Weak convergence of the sequence of absolute values. The fifth question in Section 1 is: (v) Does weak convergence of $f_n(x)$ to $f(x)$ imply weak convergence of $|f_n(x)|$ to $|f(x)|$?

The answer is no in general as the example $f_n(x) = \sin nx$ on $[0, 2\pi]$ shows. It is, therefore, something special when a sequence f_n converging weakly to f has the property that $|f_n|$ converges weakly to $|f|$, and one might believe for a moment that this has consequences for pointwise convergence. This belief is false, however. If $f_n(x)$ is defined as $\sin^2 nx$ on $[0, \pi]$ and $-\sin^2 nx$ on $[\pi, 2\pi]$, then f_n converges weakly on $[0, 2\pi]$ to the function which is $\frac{1}{2}$ on $[0, \pi]$ and $-\frac{1}{2}$ on $[\pi, 2\pi]$, and $|f_n|$ converges weakly to $|f|$ on $[0, 2\pi]$, but there is no subsequence of f_n converging pointwise almost everywhere to f.

It will be shown, next, that the answer to the sixth question in Section 1 is yes.

THEOREM 5. *Let μ be Lebesgue measure in the real line, and let $f(x)$ and $g(x)$ be real and Lebesgue summable over $[0, 2\pi]$, and such that $|f(x)| \leqq g(x)$ on $[0, 2\pi]$. Then there exists a sequence of summable functions $f_n(x)$ such that f_n converges weakly to f and $|f_n|$ converges weakly to g on $[0, 2\pi]$.*

Proof. Set $h_1(x) = \frac{1}{4}\pi\{g(x)+f(x)\}$ and $h_2(x) = \frac{1}{4}\pi\{g(x)-f(x)\}$ on $[0, 2\pi]$; note that h_1 and h_2 are nonnegative in view of $|f| \leqq g$. The sequence $f_n^+(x) = h_1(x)\{|\sin nx| + \sin nx\}$ converges weakly to $(2/\pi)h_1(x)$ by Theorem 1, and similarly the sequence $-f_n^-(x) = h_2(x)\{|\sin nx| - \sin nx\}$ converges weakly to $(2/\pi)h_2(x)$. The functions $f_n^+(x)$ and $-f_n^-(x)$ are nonnegative, and at each point $x \in [0, 2\pi]$ one at least of them is zero. Hence, if we define $f_n = f_n^+ + f_n^-$, then $|f_n| = f_n^+ - f_n^-$. It follows that f_n converges weakly to $(2/\pi)(h_1 - h_2) = f$ and $|f_n|$ converges weakly to $(2/\pi)(h_1 + h_2) = g$.

In the last example both sequences f_n and $|f_n|$ are weakly convergent. We may ask whether weak convergence of f_n always implies weak convergence of $|f_n|$. The answer is no as shown by the example (for Lebesgue measure on $[0, 2\pi]$) in which $f_n(x) = \sin nx$ for n odd and $f_n(x) = 2 \sin nx$ for n even. Here, f_n converges weakly to zero, and $|f_n|$ does not converge weakly at all. However, $|f_n|$ has evidently weakly converging subsequences. A more reasonable question to ask is, therefore, whether there can exist a weakly converging sequence f_n such that no subsequence of $|f_n|$ converges weakly. The answer is no; if f_n converges weakly on the set X, then it can be shown first by means of the uniform absolute continuity of the functions $\int_E |f_n|\, d\mu$ (which is part of the general conclusion of the Hahn-Saks theorem) that the sequence of numbers $\int_X |f_n|\, d\mu$ is bounded (even in the case that the measure μ contains so-called atoms), and this fact (together with the stated uniform absolute continuity again) implies the existence of a weakly converging subsequence of $|f_n|$. The proof is somewhat complicated (for further details we refer to [1], Ch. IV, Section 8, Theorem 9).

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A PROPERTY OF THE FOURIER COEFFICIENTS OF AN INTEGRABLE FUNCTION

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1. Introduction. Let $f(x)$ be a real or complex valued function of x which is 2π-periodic and Lebesgue integrable over a period and let its Fourier series be denoted as usual by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

Some time ago S. Hartman [1], announced without proof the following interesting properties of the coefficients a_n and b_n:

$$\text{(i)} \qquad \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{b_{kn}}{n} = \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{b_{k(2n+1)}}{2n+1} = \lim_{k\to\infty}\sum_{n=1}^{\infty}(-1)^n\frac{a_{k(2n+1)}}{2n+1} = 0;$$

(ii) if $|f|\log|f|$ is integrable over a period, then for any positive integer l we have

$$\begin{aligned}\lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{a_{kn+l}}{n} &= \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{b_{kn+l}}{n} = \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{a_{k(2n+1)+l}}{2n+1}\\ &= \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{b_{k(2n+1)+l}}{2n+1} = \lim_{k\to\infty}\sum_{n=1}^{\infty}(-1)^n\frac{a_{k(2n+1)+l}}{2n+1}\\ &= \lim_{k\to\infty}\sum_{n=1}^{\infty}(-1)^n\frac{b_{k(2n+1)+l}}{2n+1} = 0.\end{aligned}$$

The object of this note is to prove some general theorems of this type which imply the above mentioned results of Hartman. For this purpose we shall first prove a theorem which may be considered as a generalization of the well-known Riemann-Lebesgue lemma. For a more thorough discussion of such generalizations we would like to refer the reader to the preceding article of A. C. Zaanen [2].

2. A generalization of the Riemann-Lebesgue lemma. Let I be an arbitrary interval and let f be a real- or complex-valued measurable function defined on I and Lebesgue integrable over I. For such a function f the Riemann-Lebesgue lemma asserts that

$$\lim_{|\lambda|\to\infty}\int_I f(x)\sin\lambda x dx = \lim_{|\lambda|\to\infty}\int_I f(x)\cos\lambda x dx = 0.$$

The following theorem generalizes this result. It is contained in a more general theorem given in [2]. In order to make this note independent of [2] we shall include a proof of this theorem.

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THEOREM 2.1. *Let $\phi(x)$ be a bounded and real- or complex-valued measurable function defined for all real values of x and let ϕ be periodic with period $p>0$. If I is an arbitrary interval and f a real or complex-valued measurable function defined on I and integrable over I, then*

$$\lim_{|\lambda|\to\infty} \int_I f(x)\phi(\lambda x)dx = \left(\frac{1}{p}\int_0^p \phi(x)dx\right)\left(\int_I f(x)dx\right).$$

Proof. Without loss of generality we may assume that $\int_0^p \phi(x)dx = 0$. Indeed, if $\int_0^p \phi(x)dx \neq 0$ we replace ϕ by the function

$$\psi(x) = \phi(x) - \frac{1}{p}\int_0^p \phi(t)dt,$$

for all real values of x, which is easily seen to be periodic with period p, and has the property that $\int_0^p \psi(x)dx = 0$.

Let $\Phi(x) = \int_0^x \phi(t)dt$; then, since ϕ is p-periodic and $\int_0^x \phi(x)dx = 0$, we have $|\Phi(x)| \leqq \int_0^p |\phi(t)|\,dt$ for all x, i.e. $\Phi(x)$ is uniformly bounded in x. Now let $J = J(a, b)(a<b)$ be an arbitrary but finite interval, then

$$\int_J \phi(\lambda x)dx = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} \phi(t)dt = \frac{1}{\lambda}\left(\Phi(\lambda b) - \Phi(\lambda a)\right).$$

Hence, since Φ is bounded,

$$\lim_{|\lambda|\to\infty} \int_J \phi(\lambda x)dx = 0.$$

From this result we conclude immediately

$$\lim_{|\lambda|\to\infty} \int_{-\infty}^{+\infty} s(x)\phi(\lambda x)dx = 0$$

for all functions s which are finite linear combinations of characteristic functions of finite intervals.

To complete the proof let f be an arbitrary real- or complex-valued measurable function defined on an arbitrary interval I and integrable over I. If $\epsilon>0$ is given, then there exists a function s defined on I which is a finite linear combination of characteristic functions of finite intervals such that $\int_I |f-s|\,dx < \epsilon/\|\phi\|_\infty$, where $\|\phi\|_\infty$ denotes the essential supremum of ϕ.

Hence

$$\left|\int_I f(x)\phi(\lambda x)dx\right| \leqq \int_I |f(x) - s(x)|\,|\phi(\lambda x)|\,dx + \left|\int_I s(x)\phi(\lambda x)dx\right|$$

$$\leqq \epsilon + \left|\int_I s(x)\phi(\lambda x)dx\right|.$$

We conclude that

$$\lim_{|\lambda|\to\infty} \int_I f(x)\phi(\lambda x)dx = 0,$$

which finishes the proof.

Remark. If we take for $\phi(x)$ either the function $\sin x$ or the function $\cos x$ we obtain the Riemann-Lebesgue lemma.

3. A property of the Fourier coefficients of an integrable function. In this section f will always denote a 2π-periodic, real- or complex-valued, measurable function which is integrable over a period. The Fourier series of f will always be denoted by (1). On the other hand, ϕ will always denote a 2π-periodic real- or complex-valued function, which is of bounded variation over a period. Its Fourier series will always be denoted by

$$\frac{1}{2}c_0 + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx).$$

THEOREM 3.1. *Using the notation defined above, we have*

$$\lim_{k\to\infty} \sum_{n=1}^{\infty}(a_{nk}c_n + b_{nk}d_n) = 0.$$

Proof. It is well known ([3], p. 57, Theorem 8.1, and p. 90, Theorem 3.7) that the Fourier series of a 2π-periodic function of bounded variation converges boundedly to that function except at an at most countable number of points. Hence, the series

$$\frac{1}{2}c_0 + \sum_{n=1}^{\infty}(c_n \cos nkx + d_n \sin nkx)$$

converges boundedly to $\phi(kx)$ for almost all x. We conclude that if this series is multiplied by an integrable function we may integrate the resulting series term by term (Lebesgue's dominated convergence theorem). Thus

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\phi(kx)dx = \frac{1}{2}c_0a_0 + \sum_{n=1}^{\infty}(a_{nk}c_n + b_{nk}d_n).$$

However, from Theorem 2.1 it follows that

$$\lim_{k\to\infty} \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\phi(kx)dx = \frac{1}{\pi}\cdot\frac{1}{2\pi}\left(\int_{-\pi}^{+\pi}\phi(x)dx\right)\left(\int_{-\pi}^{+\pi} f(x)dx\right) = \frac{1}{2}c_0a_0.$$

Hence,

$$\lim_{k\to\infty} \sum_{n=1}^{\infty}(a_{nk}c_n + b_{nk}d_n) = 0.$$

COROLLARY. *We have*

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{nk}c_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} b_{nk}d_n = 0.$$

Proof. In order to obtain this result observe that the 2π-periodic function of bounded variation $\phi_1(x) = \frac{1}{2}[\phi(x)+\phi(-x)]$ has as its Fourier series the series $\frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nx$ and that the 2π-periodic function of bounded variation $\phi_2(x) = \frac{1}{2}[\phi(x)-\phi(-x)]$ has as its Fourier series the series $\sum_{n=1}^{\infty} b_n \sin nx$. The result follows then immediately from the preceding result.

If the function f is required to satisfy a stronger integrability condition we can prove the following stronger result.

THEOREM 3.2. *If the function* $|f| \log |f|$ *is integrable over a period, then*

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{nk}d_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} b_{nk}c_n = 0.$$

Proof. As is well known the integrability of $|f| \log |f|$ implies the integrability of the conjugate function $\tilde{f}$ ([3], p. 254, Theorem 2.8). Hence the trigonometric series

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \tag{2}$$

is the Fourier series of the integrable function $\tilde{f}$. If we now apply the Corollary of Theorem 3.1 to (2) the required result follows.

COROLLARY. *If the function* $|f| \log |f|$ *is integrable over a period, then for any positive integer* l *we have*

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{nk+l}c_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{nk+l}d_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} b_{nk+l}c_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} b_{nk+l}d_n = 0.$$

Proof. Let $g(x) = f(x) \cos lx$ and let $h(x) = f(x) \sin lx$. We denote by $a_n(g)$ $(n = 0, 1, 2, \cdots)$ and $a_n(h)$ $(n = 0, 1, 2, \cdots)$ the Fourier cosine coefficients of g and h respectively, and by $b_n(g)$ $(n = 1, 2, \cdots)$ and $b_n(h)$ $(n = 1, 2, \cdots)$ the Fourier sine coefficients of g and h respectively. It is easily verified that $a_{nk+l} = a_{nk}(g) - b_{nk}(h)$ and $b_{nk+l} = b_{nk}(g) + a_{nk}(h)$. Since f has the property that $|f| \log |f|$ is integrable over a period, it is easily verified that g and h have this property too. Hence the result follows from Theorem 3.2 and the Corollary of Theorem 3.1.

Remark. From the way we applied Theorem 2.1 in the proofs of the main theorems of Section 3 it follows that these theorems remain true when k is replaced by any positive integer valued function $m(k)$ which has the property that $m(k)$ tends to infinity as k tends to infinity.

4. Examples. The results of S. Hartman, mentioned in the introduction of

this paper, can now be seen to be easy consequences of the results proved in Section 3. For instance, in order to prove (i) we may apply Theorem 3.1 to the functions

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad \sum_{n=1}^{\infty} \frac{\sin(2n+1)x}{2n+1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n+1)x}{2n+1}$$

respectively, since they all are Fourier series of functions of bounded variation. The results mentioned in (ii) follow in the same way from the Corollary of Theorem 3.2.

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FORBIDDEN AREA

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1. Introduction. The following problem was proposed by Norman Anning in [1]: *Consider points on the median of a triangle. Through the centroid no lines can be drawn which will cut off one-third of the area. Through a point, four-fifths of the distance from the vertex to the base, four such lines may be drawn: find points on the median at which the number of possible lines changes.*

The problem remained unsolved until interest in it was stimulated by a talk by Howard Eves on *Unsolved problems in the elementary problem section* [2]. A solution, using ideas of C. S. Ogilvy and Free Jamison, was given by Hoggatt [3].

Free Jamison [3] raised the issue of a *forbidden region* which may be explained as follows: In a circle C every diameter D cuts the circle into two regions of equal area. A chord L of a circle, such that $L<D$, cuts the circle into two regions of unequal areas whose areal ratio is $k<1$. For each k there is a chord L and a well-defined circle C_k such that:

(1) Through every point in the plane outside or on the boundary of C_k, there may be drawn at least one line dividing the area of circle C into two regions with an areal ratio of $k<1$.

(2) No such lines may be drawn through any interior point of circle C_k.

DEFINITION. *The interior of circle C_k is a forbidden region of circle C, relative to the areal ratio k.*

Due to symmetry, the above forbidden region is relatively simple and so is the proof. In a triangle the forbidden region is more complicated and more interesting. Our main problem is:

Given triangle ABC of area E draw a line L through an interior point P forming a triangle (which is part of triangle ABC) with any two sides (not extended) of triangle ABC, having an area kE, or $(1-k)E$, where $0<k\leqq\frac{1}{2}$. For a given k, do points P exist for which this is impossible?

DEFINITION. *The set of such points P in the plane of triangle ABC shall be a forbidden region of triangle ABC relative to areal ratio k. Clearly, for $k=\frac{1}{2}$, such a region contains no points.*

2. **Solution of the problem.** Using oblique cartesian coordinates, let vertex C of triangle ABC be at the origin and the sides a and b fall along the positive x and y axes, respectively. The triangles with vertices $(0, 0)$, $(ra, 0)$ and $(0, kb/r)$ all have area kE, and vertices $(ra, 0)$ and $(0, kb/r)$ lie on the sides a and b (unextended) if $0<k\leqq r\leqq 1$. The sides extended of these triangles are the x and y-axes and the family of lines

$$\text{(AB1)}\qquad \tau_1(x, y, r, k) = kbx + ayr^2 - kabr = 0, \quad \text{where} \quad 0 < k \leqq r \leqq 1.$$

Equating to zero the partial of the right side of (AB1) with respect to the parameter r, and then eliminating r between the resulting equation and (AB1), we obtain an equation of the envelope (the justification is given by Howard Eves in [4])

$$\text{(AB2)}\qquad xy = \tfrac{1}{4}kab.$$

Replacing k by $(1-k)$ in (AB1) yields

$$\text{(AB3)}\qquad \tau_1(x, y, r, 1-k) = (1-k)bx + ayr^2 - (1-k)abr = 0,$$

where $0<1-k\leqq r\leqq 1$, with envelope

$$\text{(AB4)}\qquad xy = \tfrac{1}{4}(1-k)ab.$$

For a given k, the *closed hyperbolic segment*, *CHS*, associated with the first quadrant branch of (AB2) has continuously turning tangents which are generated by the family (AB1) as r varies $0<k\leqq r\leqq 1$. The endpoint tangent lines to the *CHS* of (AB2) are

$$\text{(AB5)}\qquad \tau_1(x, y, 1, k) = kbx + ay - kab = 0$$

and

$$\text{(AB6)}\qquad \tau_1(x, y, k, k) = k(bx + ayk - kab) = 0,$$

where the *CHS* end-point tangencies are in points $(\frac{1}{2}a, \frac{1}{2}kb)$ and $(\frac{1}{2}ka, \frac{1}{2}b)$ on (AB5) and (AB6), respectively. Similarly, the *CHS* of (AB4) gives lines

(AB7) $\tau_1(x, y, 1, 1-k) = (1-k)bx + ay - (1-k)ab = 0$

and

(AB8) $\tau_1(x, y, 1-k, 1-k) = (1-k)(bx + (1-k)ay - (1-k)ab) = 0,$

where the *CHS* end-point tangencies are in points $(\frac{1}{2}a, \frac{1}{2}(1-k)b)$ and $(\frac{1}{2}(1-k)a, \frac{1}{2}b)$ on (AB7) and (AB8), respectively.

Next, consider lines cutting sides a and c proper, again forming triangles of area kE or $(1-k)E$. The corresponding equations are:

(AC1) $\tau_2(x, y, r, k) = kbx - ay(r^2 - k) + kab(r-1) = 0$

(AC2) $xy = ay - ay^2/b - \frac{1}{4}kab$

(AC3) $\tau_2(x, y, r, 1-k) = 0$

(AC4) $xy = ay - ay^2/b - \frac{1}{4}(1-k)ab$

(AC5) $\tau_2(x, y, 1, k) = kbx - ay(1-k) = 0$

(AC6) $\tau_2(x, y, k, k) = k(bx + ay(1-k) - ab(1-k)) = 0$

(AC7) $\tau_2(x, y, 1, 1-k) = (1-k)bx - ayk = 0$

(AC8) $\tau_2(x, y, 1-k, 1-k) = (1-k)[bx + ayk - abk] = 0.$

The *CHS* of (AC2) has tangencies at points $(\frac{1}{2}(1-k)a, \frac{1}{2}kb)$ and $(\frac{1}{2}(1-k)a, \frac{1}{2}b)$ on (AC5) and (AC6), respectively. The *CHS* of (AC4) has tangencies at points $(\frac{1}{2}ka, \frac{1}{2}(1-k)b)$ and $(\frac{1}{2}ka, \frac{1}{2}b)$ on (AC7) and (AC8), respectively.

Finally, consider lines cutting sides b and c. One obtains the equations by interchanging x with y and b with a in (AC1–8) to obtain

(BC1) $\tau_3(x, y, r, k) = kay - bx(r^2 - k) + kab(r-1) = 0$

(BC2) $xy = bx - bx^2/a - \frac{1}{4}kab$

(BC3) $\tau_3(x, y, r, 1-k) = 0$

(BC4) $xy = bx - bx^2/a - \frac{1}{4}(1-k)ab$

(BC5) $\tau_3(x, y, 1, k) = kay - bx(1-k) = 0$

(BC6) $\tau_3(x, y, k, k) = k(ay + bx(1-k) - (1-k)) = 0$

(BC7) $\tau_3(x, y, 1, 1-k) = (1-k)ay - bkx = 0$

(BC8) $\tau_3(x, y, 1-k, 1-k) = (1-k)[ay + bkx - kab] = 0.$

The *CHS* of (BC2) has tangencies at points $(\frac{1}{2}ka, \frac{1}{2}(1-k)b)$ and $(\frac{1}{2}a, \frac{1}{2}(1-k)b)$ on (BC5) and (BC6), respectively. The *CHS* of (BC4) has tangencies at points $(\frac{1}{2}(1-k)a, \frac{1}{2}kb)$ and $(\frac{1}{2}a, \frac{1}{2}kb)$ on (BC7) and (BC8) respectively. The twelve lines (AB5, 6, 7, 8), (AC5, 6, 7, 8) and (BC5, 6, 7, 8) are coincident in pairs: (AC5) = (BC7), (AC6) = (AB8), (AC7) = (BC5), (AC8) = (AB6), (BC6) = (AB7) and (BC8) = (AB5). The twelve points of tangency are coincident in pairs. We can now break the problem into three cases.

Case I, $0<k<\frac{4}{9}$: In this case, closed hyperbolic segments on hyperbolas (AB2), (BC2), and (AC2), bounding the horizontally-shaded area of Figure 1, actually enclose a forbidden region V. Through the interior points of V can pass no tangents from the *CHS* of hyperbolas (AB2), (AB4), (BC2), (BC4), (AC2), or (AC4). In Figure 1, which is drawn for $k=\frac{1}{3}$, the dotted lines are the medians and lines such as (AC5), (AC6), (AC7), (AC8), (BC6), and (BC8).

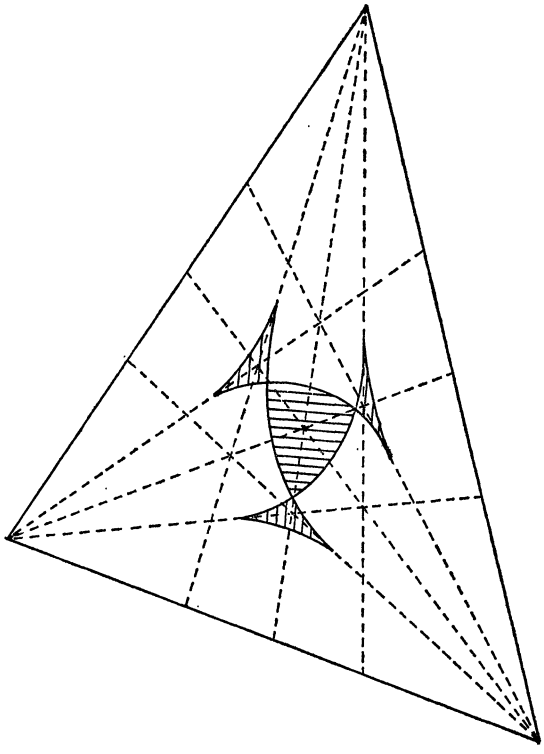


FIG. 1. $k=\frac{1}{3}$

The area A of V is analytically

$$A = \tfrac{1}{2}E\left[3\sqrt{(1-2k)} - 1 - 3k\ln\left|\frac{1-k+\sqrt{(1-2k)}}{2k}\right|\right]$$

for $0<k<\frac{1}{2}$. It will be noted that for $k=\frac{4}{9}$, $A=0$, for then hyperbolas (AB2), (BC2), and (AC2) intersect at the centroid G. Also $A\to E$ as $k\to 0$ as should be expected since $V\to\Delta\, ABC$ as $k\to 0$.

In Figure 1, let V be the horizontally-shaded region and U consist of the three disjoint vertically shaded regions. Then we can make the following comments.

1. Through each interior point of V, there exist no lines which cut off areas kE or $(1-k)E$. (Hereafter referred to as *lines*.)

2. Through each nonvertex boundary point of V, only *one* such *line* exists.

3. Through each interior point of U, exactly *four lines* exist.

4. Through each nonvertex boundary point of U, exactly *three* such *lines* exist.

5. Through each vertex point of U and V and through each point exterior to U and V exactly *two* such *lines* exist.

In Figure 2, let W be the horizontally-shaded region and X consist of the three disjoint vertically shaded regions.

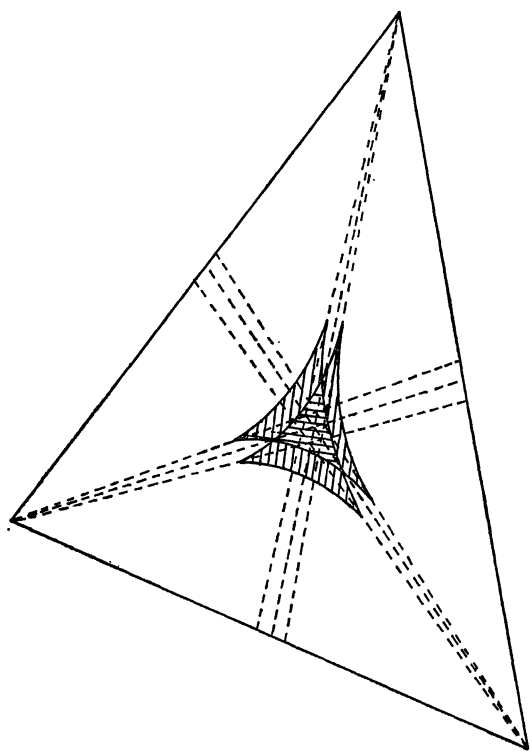


FIG. 2. $k=\frac{17}{36}$

Case II, $\frac{4}{9}<k<\frac{1}{2}$. We can make the following comments:

1. Through each interior point of W, exactly *six lines* exist.

2. Through each nonvertex boundary point of W, exactly *five lines* exist.

3. Through each vertex boundary point of W and each interior point of X, exactly *four lines* exist.

4. Through each nonvertex boundary point of X, not a boundary point of W, exactly *three lines* exist.

5. Through each of six vertex boundary points of X, not a vertex boundary point of W, and through each point exterior to both W and X, exactly *two lines* exist.

Let the vertically-shaded region in Figure 3 be Y.

Case III, $k=\frac{1}{2}$.

1. Through each interior point of Y, exactly *three lines* exist.

2. Through each nonvertex boundary point of Y, exactly *two lines* exist.

3. Through each of the three vertex boundary points of Y and each point exterior to Y, exactly *one line* exists.

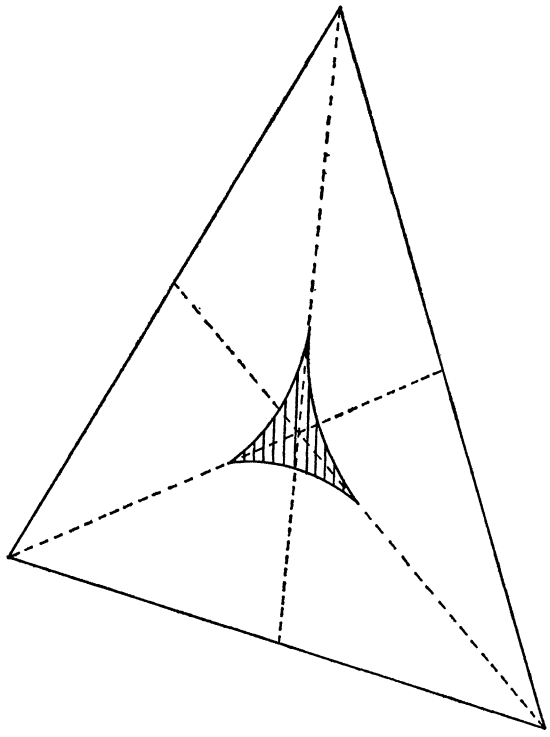


FIG. 3. $k=\frac{1}{2}$

3. Additional Remarks. The equations for the medians of triangle ABC are:

$$\text{(MA)} \qquad 2bx + ay - ab = 0,$$

$$\text{(MB)} \qquad bx + 2bx - ab = 0,$$

$$\text{(MC)} \qquad bx - ay = 0,$$

where MA, MB, MC are the medians from vertices A, B, and C, respectively. If $m=\frac{1}{2}\sqrt{(1-2k)}$, for $0<k\leqq\frac{1}{2}$, then:

MA intersects (AB2) and (AC2) in $(\frac{1}{2}a(1-m), \frac{1}{2}b(1-m))$ and (BC2) in $(\frac{1}{2}a\sqrt{k}, b(1-\sqrt{k}))$,

MB intersects (AB2) and (BC2) in $(am, \frac{1}{2}b(1-m))$ and (AC2) in $(a(1-\sqrt{k}), \frac{1}{2}b\sqrt{k})$.

MC intersects (AC2) and (BC2) in $(\frac{1}{2}a(1-m), bm)$ and (AB2) in $(\frac{1}{2}a\sqrt{k}, \frac{1}{2}b\sqrt{k})$.

Centroid G is $(\frac{1}{3}a, \frac{1}{3}b)$.

1. The six points of tangency lie on the sides of the median pedal triangle of triangle ABC, independently of the value of $0<k\leqq\frac{1}{2}$.

2. The centroid of region V is at G, the centroid of triangle ABC.

3. The three medians cut region V into 6 regions equal in area.

4. The medians of triangle ABC cut the boundary of V in 6 points, the vertices of a convex polygon W. The alternate vertices of W form two triads of points such that each triad form a triangle with centroid at G, and homothetic to triangle ABC with respect to centroid G. One triangle is directly similar and the other inversely. If polygon W is removed from region V, the 6 remaining hyperbolic area segments are equal in area.

5. The six points of tangency lie on an ellipse homothetic, with respect to the centroid of triangle ABC, to the Steiner ellipse [5] of triangle ABC. As $0<k\leqq\frac{1}{2}$, these ellipses cover the area between the inner Steiner ellipse and outer Steiner ellipse of the median pedal triangle.

6. The three vertices of region V, the first triad of (3), also lie on an ellipse homothetic to the Steiner ellipse of triangle ABC, with centroid G as the center of homothety.

7. The second triad of (3) also lies on an ellipse homothetic to the Steiner ellipse of triangle ABC.

Comment. The above remarks follow easily from affine geometry.

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ON A CURVILINEAR COORDINATE SYSTEM

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A certain Perseus [1], as quoted by Proclus in his commentary on Euclid's Elements, discussed the sections which a plane cuts in a torus. He called these spiric curves. Following this ancient lead, I shall use the adjective "spiric" in matters pertaining to that particular geometric form.

I define spiric coordinates as follows: A line of fixed length b rotates in a reference plane, about one of its end points, and makes an angle θ with a reference position. A second line, originating at the moving end of b, lies in the plane through b and perpendicular to the reference plane, has the length r and makes an angle β with the reference plane. The spiric coordinates consist of r, θ, β.

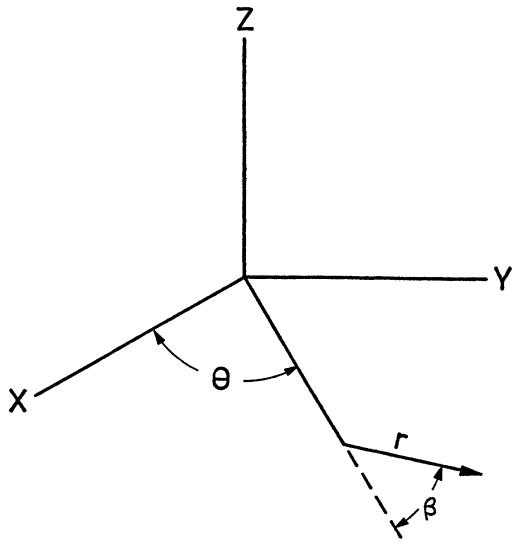


FIG. 1. Spiric coordinate system.

Let $\theta = 0$ coincide with the x-axis. Then Figure 1 at once reveals the formulas linking x, y, z and r, θ, β:

$$x = b\cos\theta + r\cos\theta\cos\beta, \qquad y = b\sin\theta + r\sin\theta\cos\beta, \qquad z = r\sin\beta.$$

Following conventional analysis procedures [2], let $\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. Then

$$\partial\mathbf{R}/\partial r = \mathbf{i}\cos\theta\cos\beta + \mathbf{j}\sin\theta\cos\beta + \mathbf{k}\sin\beta,$$

$$\partial\mathbf{R}/\partial\theta = -\mathbf{i}(b\sin\theta + r\sin\theta\cos\beta) + \mathbf{j}(b\cos\theta + r\cos\theta\cos\beta),$$

$$\partial\mathbf{R}/\partial\beta = -\mathbf{i}r\cos\theta\sin\beta - \mathbf{j}r\sin\theta\sin\beta + \mathbf{k}r\cos\beta.$$

Easy calculations yield the orthogonality conditions, the three dot products equal zero. Equally easy algebra gives the results:

$$|\partial \mathbf{R}/\partial r| = 1, \quad |\partial \mathbf{R}/\partial \theta| = b + r\cos\beta, \quad |\partial \mathbf{R}/\partial \beta| = r.$$

$$J = br + r^2\cos\beta, \quad dV = (br + r^2\cos\beta)drd\theta d\beta,$$

$$ds^2 = dr^2 + r^2d\beta^2 + (b + r\cos\beta)^2d\theta^2.$$

The unit tangent vectors work out to

$$\mathbf{e}_r = \mathbf{i}\cos\theta\cos\beta + \mathbf{j}\sin\theta\cos\beta + \mathbf{k}\sin\beta,$$

$$\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta,$$

$$\mathbf{e}_\beta = -\mathbf{i}\cos\theta\sin\beta - \mathbf{j}\sin\theta\sin\beta + \mathbf{k}\cos\beta.$$

Solving these for $\mathbf{i}, \mathbf{j}, \mathbf{k}$ yields the vector conversion formulas:

$$\mathbf{i} = \mathbf{e}_r\cos\theta\cos\beta - \mathbf{e}_\theta\sin\theta - \mathbf{e}_\beta\cos\theta\sin\beta,$$

$$\mathbf{j} = \mathbf{e}_r\sin\theta\cos\beta + \mathbf{e}_\theta\cos\theta - \mathbf{e}_\beta\sin\theta\sin\beta,$$

$$\mathbf{k} = \mathbf{e}_r\sin\beta + \mathbf{e}_\beta\cos\beta.$$

Area elements have the forms:

$$dA(\beta, \theta) = (br + r^2\cos\beta)d\beta d\theta, \quad dA(r, \beta) = rdrd\beta, \quad dA(r, \theta) = (b + r\cos\beta)drd\theta.$$

The usual vector operators have the forms:

$$\nabla\phi = (\partial\phi/\partial r)\mathbf{e}_r + (b + r\cos\beta)^{-1}(\partial\phi/\partial\theta)\mathbf{e}_\theta + r^{-1}(\partial\phi/\partial\beta)\mathbf{e}_\beta,$$

$$\nabla\cdot\mathbf{A} = (br + r^2\cos\beta)^{-1}\partial(br + r^2\cos\beta)A_r/\partial r + \partial(rA_\theta)/\partial\theta + \partial(b + r\cos\beta)A_\beta/\partial\beta,$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_r & (b + r\cos\beta)\mathbf{e}_\theta & r\mathbf{e}_\beta \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\beta \\ A_r & (b + r\cos\beta)A_\theta & rA_\beta \end{vmatrix},$$

$$\nabla^2\phi = \frac{1}{br + r^2\cos\beta}\left[\frac{\partial}{\partial r}\left((br + r^2\cos\beta)\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r}{b + r\cos\beta}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\beta}\left(\frac{b + r\cos\beta}{r}\frac{\partial\phi}{\partial\beta}\right)\right].$$

A few elementary problems may prove interesting. In spiric coordinates, a torus or ring has the form $r = a$. Thus the ring, in spiric coordinates, has the simplicity of the cylinder in cylindrical coordinates or the sphere in spherical coordinates.

If $r = a$, $\theta =$ constant, then $ds = ad\beta$, defining the arc along the circle $r = a$, $\theta =$ constant. Again, if $r = a$, $\theta =$ constant,

$$\int dA = \int_0^\alpha \int_0^a rdrd\beta = \tfrac{1}{2}a^2\alpha,$$

the area of a sector from that same circle. The limits of integration can vary, of

course, over any patch of that circle. If r is any function of β, limits varying from 0 to α and 0 to $r(\beta)$, the integral obviously yields:

$$\int dA = \int_0^{\alpha} \tfrac{1}{2}r^2(\beta)d\beta,$$

as we expect from cylindrical coordinates.

Again, if $r=a$,

$$\begin{aligned} A &= \int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2}(ba + a^2\cos\beta)d\beta d\theta \\ &= (\alpha_2 - \alpha_1)[ba(\lambda_2 - \lambda_1) + a^2(\sin\lambda_2 - \sin\lambda_1)] \end{aligned}$$

for any part of the ring. Once more $r=a$, and

$$\begin{aligned} V &= \int dV = \int_{\lambda_1}^{\lambda_2}\int_{\alpha_1}^{\alpha_2}\int_0^{a}(br + r^2\cos\beta)drd\theta d\beta \\ &= \tfrac{1}{2}ba^2(\alpha_2 - \alpha_1)(\lambda_2 - \lambda_1) + \tfrac{1}{3}a^3(\alpha_2 - \alpha_1)(\sin\lambda_2 - \sin\lambda_1). \end{aligned}$$

If $\alpha_2=\lambda_2=2\pi$, and $\alpha_1=\lambda_1=0$, then $A=4\pi^2ba$, $V=2\pi^2ba^2$.

If $\beta=$constant, $ds=\sqrt{\{dr^2+(b+r\cos\beta)^2d\theta^2\}}$, the arc of a conical helix. Let $r=a$, $\beta=$constant; then $ds=(b+a\cos\beta)d\theta$, the arc of a parallel circle. Again for $r=a$, let $\beta=n\theta$; then $ds=\sqrt{\{a^2n^2+(b+a\cos n\theta)^2\}}d\theta$, the arc of a toroidal helix.

The spiric system teems with interesting, if commonplace, forms. If thus emerges as a worthy competitor of cylindrical and spherical coordinates for a place in textbooks. My interest in spiric coordinates lies in their potential application to pneumatic tire problems. A discussion of certain of these follows.

Areas. The formulas given above for the areas of toroidial patches have an application, for example, in cost estimation. In certain cases their use will give a more accurate estimate of the weight and cost of special material located at odd places around the tire.

The drum method of tire building. The tire builder lays innerliner, fabric piles, tread and sidewall, and certain other components down on a cylindrical drum, as shown in Figure 2. Now, modern manufacturing processes yield most of these components in flat sheet form. Even the tread-sidewall composite, as it leaves the extruder, has one flat surface. Geometrically, then, the tire building and expansion process consists of two transformations, first from flat sheet to cylindrical forms and then to the molded shape as indicated in Figure 3. Steam pressure, inside a suitable bladder, accomplishes the second step and holds the tire against the mold during cure.

Specification calculations begin with a desired finished section, as based on engineering requirements. A body of rules, partly empirical, transform that section back down to the drum.

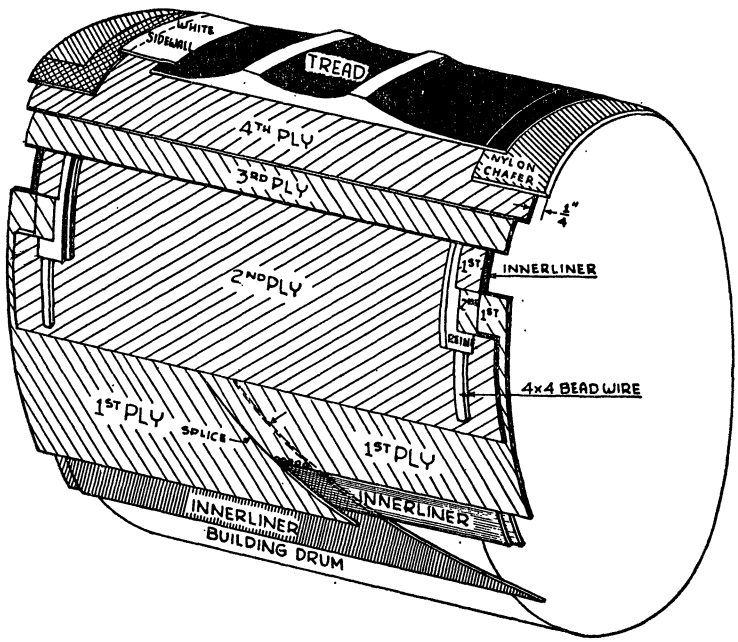


FIG. 2. Drum layout.

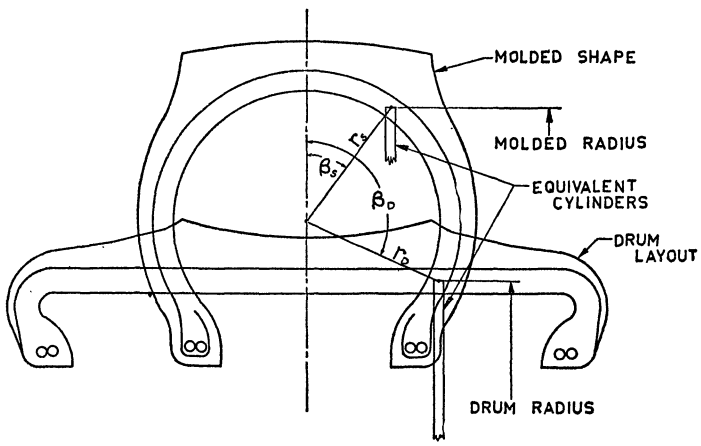


FIG. 3. Drum-to-mold expansion.

The pantograph law. The transformation from flat to cylindrical form presents little of mathematical interest, and follows prosaic paths. Efforts to develop improved building methods may lead to a closer look at this, but currently we take it more or less for granted.

The more interesting expansion stage involves considerable distortion of the tire components. The spec calculation must, in effect, successfully predict those distortions. A mathematical model, called the pantograph law, has proven useful in making such predictions.

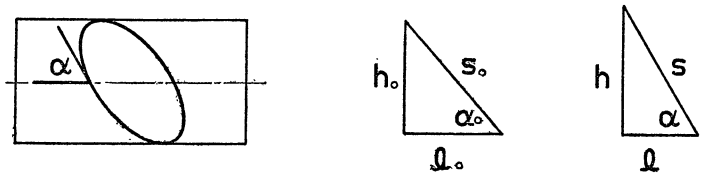


FIG. 4. Cord on a cylinder.

Figure 4 illustrates one method of deriving the pantograph relationship. It shows a cord wrapped around a cylinder, the cord lying in a plane which makes an angle α with the centerline of the tire. The cord forms one, endless loop around the cylinder.

Let the cylinder have an initial radius R_0, and the cord-plane make an initial angle α_0 with the cylinder axis. Now, the cylinder expands from R_0 to R. One or both of two things happen to the cord. The angle α may change, and the cord may stretch. Let the angle change from α_0 to α, and the cord stretch in the ratio $e/1$.

The two triangles of Figure 2 represent front elevations of the cord-cylinder setup. Under the stated conditions,

$$s = es_0, \qquad h_0/h = R_0/R$$

and the following derived relations hold:

$$h_0/s_0 = \sin \alpha_0, \qquad h/s = h/(es_0) = \sin \alpha, \qquad \sin \alpha = (R \sin \alpha_0)/(R_0 e),$$

$$l = \frac{l_0 \tan \alpha_0}{R_0} \sqrt{\left\{\left(\frac{eR_0}{\sin \alpha_0}\right)^2 - R^2\right\}} = l_0 TD,$$

where TD is the pantograph factor.

A more common derivation [3] makes a rhombus of 4 triangles in the subscripted state, and extends this out into a rhombus consisting of 4 of the unsubscripted triangles. The cords of the plies, where they cross, form the corners of the rhombi. The pantograph theory says that the cords do not slip over one another at these points.

The letters h and l identify the diagonals of the crossed-cord rhombi. The diagonal h lies parallel to a circumference of the drum (Fig. 2), while l parallels the drum axis. In any narrow band, two adjacent plies act like a folding gate, or pantograph.

Tire expansion. With the ends of the cylinder of Figure 2 sealed off, let internal pressure act, while the beads move toward one another. Figure 3 symbolizes the geometric events taking place during this expansion. A typical point P_D on the drum, moves to P_S in the section.

Now as P_D rotates around the drum, it sweeps out a certain distance, as does P_S sweep out a certain larger distance when rotating around the same, common axis. The swept-out distances have to each other the ratio of the two associated radii, namely the *drum radius* and *molded radius*. In the terminology of Figures 3 and 4, the drum layout becomes a series of equivalent cylinders, each expanding in the ratio of drum to molded radius.

The fabric body elongates, along parallel circles, to accommodate the new conditions. Most of that adjustment takes the form of a pantographic elongation of the crossed-cord rhombi. The pantographic theory ascribes all the adjustment to that mechanism.

The mathematics of Figure 4 then apply. Let b represent the distance from the rotation axis to the intersection of r_s, r_d (Fig. 3). Then, $R=b+r_s \cos \beta$. The length h in Figure 2 elongates in the ratio $h/h_0=R/R_0$, while l contracts in the ratio of the pantograph factor. The letters TD represent this quantity in our FORTRAN computer program. The sum of all the increments l_0 gives the meridian length of a ply, on the drum, while the sum of all the l increments gives the length of that same meridian in the section. Given a section meridian, the pantograph theory predicts a corresponding drum meridian of the length $l_0=\sum l/(TD)$. This result enters into drum-width calculations.

In spiric coordinates the section periphery can take an analytical form, such as a circle or ellipse, or the form of a table of points, and the calculation programs out very nicely.

Cord path equation. Figure 5 shows a photograph of a peeled tire, the cord path, the angle α, and the characteristic differential triangle with sides $Rd\theta$ and $rd\beta$ marked off in white gum strip. By the pantograph theory, with $\cot \alpha = \sqrt{(\csc^2 \alpha - 1)}$,

$$\cot \alpha = \sqrt{\{(B/R)^2 - 1\}},$$

where $B=eR_0/\sin \alpha_0$, $R=b+r \cos \beta$.

The differential triangle of Figure 5 shows that $\cot \alpha=(rd\beta)/(Rd\theta)$ which leads to

$$d\beta = \frac{\sqrt{(B^2 - R^2)}}{r} d\theta$$

as the differential equation of the cord path after expansion.

Expanding the differential form yields $r^2d\beta^2=(B^2-R^2)d\theta^2$, and substituting this into the equation for ds^2 gives,

$$ds^2 = dr^2 + B^2d\theta^2.$$

If $r=a$, then $dr=0$ and $ds=Bd\theta$. Integrating gives $s=eR_0\theta/\sin\alpha_0$. Now the position of the cord on the bead does not change during expansion. For a complete cord, therefore, the length on the drum $=R_0\theta/\sin\alpha_0$, and after expansion into the form of a torus, the cord has stretched in the ratio $e/1$, as postulated. Furthermore, a pie-cut defined by two meridians, with the associated planes passing through the axis of rotation, intercepts equal lengths of cord around the tire. If θ defines the angle of the cut, then the two planes intercept a length $eR_0\theta/\sin\alpha_0$ along any cord which both planes cut.



FIG. 5. Cord path.

The form $ds^2=dr^2+B^2d\theta^2$ allows for a check of the pantograph theory of expansion. Measured lengths on a peeled tire may vary from the pantograph requirement, this latter established by numerical integration. I hope to use this method to determine e along the cord path. This elusive quantity has so far escaped our efforts to handle it analytically, at least in my view.

In current tire practice, straight-cord fabric goes down on the drum, at some angle α_0, and the engineer takes what he gets in the way of a cord path. Given a correct transformation theory, a suitable computer program easily solves the problem, given a desired cord path in the section, what goes down on the drum

to get it? The desired path enters the computer as a table of points, and the computer prints out a table of angles across the width of the building drum. Achieving the path, physically, involves considerably more trouble and expense than calculating it.

Geodesic path. In spiric coordinates, the equation $r=r(\beta)$ defines a surface of revolution. Adapting the theorem of Clairaut [4] to lines on such a surface, a geodesic makes an angle α with the meridian such that $\sin \alpha = c/R$. The pantograph gives, again, $\sin \alpha = R \sin \alpha_0 / R_0 e$. Doing the algebra leads to $\sin \alpha_0 = cR_0 e/R^2$. Determining α_0 as a function of drum meridian length proceeds step by step. This begins by transforming $\sin \alpha$, and a small associated arc, from the section centerline down to the drum centerline. The next increment of section meridian length, with its associated α, goes pantographically down to the drum to become the next increment of drum meridian length, with the proper α_0.

If we require that, at any point, $\sin \alpha = cR$, then we have asked, within the confines of pantographic expansion, for a straight-line path on the drum.

Tire shape. Let a toroidal surface, made of easily deformed material, bear against a flat support so as to develop an appreciable contact patch. The flat surface runs parallel to the rotation axis. Let the toroid have an overall diameter of $2R_u$ when undeformed, and the distance from the rotation axis to the center of the contact patch equal R_0. Briefly, this describes a deflected tire.

Because it combines linear and curvilinear elements in the same continuous range, the surface of a deflected tire does not lend itself to easy analytical expression. However, the following empirical equation appears to represent the centerline of the tread in its course perpendicular to the meridians,

$$R = R_u - (R_u - R_0)e^{-\lambda\theta^3},$$

where R is a radius from the rotation axis to the tread centerline.

Due to the meridian curvature of the tread (Fig. 3) the contact patch has an oval shape. To work this into the equation, let λ vary with the spiric β, in such manner as it must. A study of λ for various kinds of tires may uncover some general pattern in its behavior. If so, a general solution to the load-deflection problem ought soon to follow.

To attempt to inject some analysis into this strongly empirical form, let the center of spiric radii lie at the center of the least-squares circle of the innermost meridian. Since I have not succeeded in finding a derivation of the least-squares circle to list as reference, I will outline it here, in the interests of convenience. A series of points x_i, y_i lie close to a circle. Find the radius r and center p, q of the circle $(x-p)^2+(y-q)^2=r^2$ which minimizes the value of,

$$f(r, p, q) = \sum(r^2 - (x_i - p)^2 - (y_i - q)^2)^2.$$

Setting the partials with respect to r, p, and q equal to zero, and solving the system of equations, yields a solution in terms of the quantities,

$$A = \sum x_i, \qquad B = \sum y_i,$$
$$C = \sum x_i^2, \qquad D = \sum y_i^2,$$
$$E = \sum x_i y_i, \qquad F = \sum (x_i^2 + y_i^2),$$
$$G = \sum x_i(x_i^2 + y_i^2), \qquad H = \sum y_i(x_i^2 + y_i^2),$$

where n = number of points. More algebra gives

$$P = 2nE - 2AB, \qquad S = 2BC - 2AE,$$
$$Q = 2nD - 2B^2, \qquad T = 2BE - 2AD,$$
$$R = nH - BF, \qquad U = BG - AH.$$

Finally,

$$p = \frac{RT - QU}{PT - SQ}, \qquad q = \frac{PU - RS}{PT - SQ}, \qquad r^2 = \frac{F + n(p^2 + q^2) - 2Ap - 2Bq}{n}.$$

With q established as the center of spiric radii (for a symmetric point distribution, $p = 0$), let the tread arc have the form $r = r(\beta)$. Again, the computer does nicely with a table of points. Let the distance of q from the axis of rotation equal the spiric b. Then,

$$R = b + r(\beta) \cos \beta, \qquad R_0 = b + r_0, \qquad R_u = b + r_u(\beta) \cos \beta,$$

which equations define r_0 and r_u. Making the various substitutions in the empirical shape equation yields

$$r(\beta) \cos \beta = r_u(\beta) \cos \beta - (r_u(\beta) \cos \beta - r_0) \exp (\lambda(\beta)\theta^3).$$

A polynomial in θ can replace θ^3 in the exponential. One thinks of this formula in terms of electronic, and not desk, calculations.

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BLUE-EMPTY CHROMATIC GRAPHS

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1. Introduction. Given N points in the plane (no three on a line) and the $\binom{N}{2}$ line segments connecting them in pairs, a chromatic graph is constructed by coloring each segment either red or blue. Each of the $\binom{N}{3}$ subsets of three points will determine a colored triangle. Letting B represent the number of such triangles with all three sides blue and R the number of red triangles, what is the smallest possible value of $B+R$?

The answer was first published in this MONTHLY [1] by A. W. Goodman. Defining an *extremal* chromatic graph as one which, for a given N, has the minimum value of $B+R$, we can state Goodman's result as follows:

THEOREM 1. *In an extremal graph on N points*

$$B + R = 2\binom{u}{3} \text{ if } N = 2u, \qquad \binom{2u}{3} + \binom{2u+1}{3} - u \text{ if } N = 4u+1,$$

$$\binom{2u+1}{3} + \binom{2u+2}{3} - u \text{ if } N = 4u+3.$$

(*Note*: $\binom{n}{k} = 0$ *if* $k > n$, *so that* $B+R=0$ *unless* $N \geqq 6$.)

In addition, Goodman raised the following question: is it possible to construct an extremal graph in which all solid-color triangles are of the same color? Although Goodman easily arrived at an affirmative answer for the case of even N, the question was left open for odd N until a negative answer (for odd $N>7$) was found by P. Erdös and proved in another MONTHLY article [2].

Calling a chromatic graph *blue-empty* if $B=0$ (the case $R=0$ is symmetric), Erdös' conclusion is that no blue-empty graph (for odd $N>7$) is extremal (that is, has the absolute minimum value of $B+R$ given by Theorem 1). This negative result raises the natural question: what is the minimum value of $B+R$ under the restriction $B=0$? In the case of even N (or $N=7$) the minimum is given by Theorem 1; for odd $N>7$, it is necessarily greater than the value of $B+R$ for extremal graphs given by Theorem 1.

This paper offers a simple formula for computing $B+R$ in any chromatic graph (Lemma 1). The formula not only provides a particularly easy proof of Theorem 1 above, but also greatly aids the investigation of *minimal* blue-empty chromatic graphs (graphs which minimize $B+R$, or simply R, under the restriction $B=0$). Theorems 2 and 3 below give, in addition to quantitative results analogous to Theorem 1, a complete and somewhat surprising characterization of the structure of minimal blue-empty chromatic graphs.

2. Some important examples of chromatic graphs.

Two red net graph. Construct a chromatic graph on N indexed points, x_i $(i=1, 2, \cdots, N)$, as follows: draw red lines between points when the sum

of their indices is even and blue lines when it is odd. The points of even index are then pairwise connected by red lines, so that the set of even-indexed points is what we shall call a *red net.* The points of odd index form a disjoint red net which contains one more point than the even red net if N is odd. Since every triangle must have at least one side within one of the nets, the *two red net* graph on N points is blue-empty. If $N=2u$, each red net contains u points and hence $\binom{u}{3}$ red triangles, so that there are $2\binom{u}{3}$ red triangles in the graph. If $N=2u+1$, the number of red triangles in the graph is $\binom{u}{3}+\binom{u+1}{3}$.

Extremal graph. For convenience, assume the N indexed points are the vertices of a regular N-gon. If $N=4u$, $4u+1$, or $4u+2$, red-connect each point to its u nearest neighbors on each side and complete the graph by drawing blue lines. In case $N=4u+3$, red-connect each point to its $2u$ nearest neighbors, as before, but also red-connect points indexed i and j whenever $|i-j|=2u+1$, before completing the graph with blue lines.

It will be shown in the proof of Theorem 1 that the *extremal graph* is, in fact, extremal, as is the *two red net* graph for even N. Theorems 2 and 3 will show that all minimal blue-empty chromatic graphs (for $N>11$) differ from the *two red net* graph by at most minor changes.

3. Fundamental facts about chromatic graphs. Given a chromatic graph on a set of N points, $S=\{x_1, x_2, \cdots, x_N\}$, let I represent the set of points which are red-connected to the point x_i*, and let $\tilde{I}$ represent I with the point x_i added. The complements, $S-I$ and $S-\tilde{I}$, will be denoted by I' and $\tilde{I}'$, respectively. ($\tilde{I}'$ is the set of points blue-connected to x_i.) Given any set T, let (T) denote the number of points in the set, and call the number (I) the *order* of the point x_i (the number of points to which it is red-connected).

LEMMA 1. *In any chromatic graph on N points,*

$$B+R=\binom{N}{3}-\frac{1}{2}\sum_{i=1}^{N}(I)[N-1-(I)].$$

Proof. (I) red lines and $N-1-(I)$ blue lines terminate at the point x_i. There are, therefore, (I) times $N-1-(I)$ distinct *pairings* of a red line and a blue line at x_i, each determining a distinct two-color triangle. But each such two-color triangle has one other "determining" vertex at which a red-blue *pairing* is observed. Hence, the number of two-color triangles is one-half the number of possible *pairings* of blue and red lines at common terminal points—that is, one-half the sum over all points of the product $(I)[N-1-(I)]$. The formula results from the fact that the number of single-color triangles, $B+R$, is equal to the total number of triangles, $\binom{N}{3}$, minus the number of two-color triangles.

It is easy to prove Theorem 1 as a straightforward consequence of Lemma 1,

* This set could be denoted by R_i, using the subscript to link it with the point x_i, but it simplifies the writing to simply capitalize the subscript of a point in order to denote the point set to which it is red-connected, eliminating repetition of the letter R.

since minimizing $B+R$ is equivalent to maximizing the sum of the products of the form $(I)[N-1-(I)]$. Thus, for $N=2u$, each (I) must be u or $u-1$; for $N=4u+1$, each (I) must be $2u$; and for $N=4u+3$, the optimum value of (I) is $2u+1$, but if all points had this order, the number of red lines in the graph would be $\frac{1}{2}(2u+1)(4u+3)$, which is not an integer. The best alternative is to have all points of order $2u+1$ except one, with order $2u$ or $2u+2$. For the above values of (I) the formula of Lemma 1 gives the values of $B+R$ in Theorem 1.

To complete the proof it is necessary to show that graphs with these order numbers can actually be constructed. Consider the *extremal graph* defined in Section 2. For $N=4u$, each point of the graph has order $2u$, which is one of the optimal choices above. For $N=4u+1$, each point has the optimal order $2u$. For $N=4u+2$, each point has order $2u$, which is, once again, an optimal choice. Finally, in constructing the *extremal graph* for $N=4u+3$, each point is at first red-connected to $2u$ other points and then red-connected to one additional point, except for the point indexed $2u+2$, which is further red-connected to both the point indexed 1 and the point indexed $4u+3$, making its order $2u+2$ and thus satisfying the requirement above.

An additional observation of importance is that the *two red net* graph is extremal for $N=2u$, since each point has order $u-1$. It is, therefore, automatically a minimal blue-empty graph.

To investigate blue-empty chromatic graphs we rely heavily upon the following two facts (whose contradictions clearly imply the existence of a blue triangle).

LEMMA 2. *In a blue-empty chromatic graph, if two points x_i and x_j are blue-connected, then every other point is red-connected to at least one of them. That is, $\tilde{I}\cup\tilde{J}=S$.*

LEMMA 3. *For any point x_i in a blue-empty chromatic graph, the set of points blue-connected to x_i, $\tilde{I}'$, is a red net; that is, the points in $\tilde{I}'$ are pairwise red-connected.*

4. The structure of minimal blue-empty chromatic graphs. In Section 2 the two red net graph on N points was defined and shown to be blue-empty. It is easily seen that one blue line between the two red nets may be changed to red without creating any additional red triangles. Additional blue lines between points from different nets may be changed to red without creating any additional red triangles, provided only that no pair of changed lines have a common terminal point. This limitation allows up to u red lines between two u-point red nets, or between a u-point red net and a disjoint $(u+1)$-point red net.

Theorems 2 and 3 state that, for $N>11$, all minimal blue-empty chromatic graphs contain red nets like those of the two red net graph on N points; and, since they cannot contain any *extra* red triangles, these minimal blue-empty graphs must be constructible from the two red net graphs on N points by means of the limited *blue to red* changes described above.

THEOREM 2. *Every minimal blue-empty chromatic graph on $2u$ points ($u>5$) contains $2\binom{u}{3}$ red triangles, contained in two disjoint u-point red nets.*

Proof. The two red net graph on $2u$ points is blue-empty and has $2\binom{u}{3}$ red triangles, so that it is an extremal graph, as was shown after the proof of Theorem 1. Hence, any minimal blue-empty graph on $2u$ points has $2\binom{u}{3}$ red triangles, and since it is therefore an extremal graph, must have only points of order u and $u-1$, as shown in the proof of Theorem 1.

Choosing any point x_i of a minimal blue-empty graph, the set $\tilde{I}'$ of points blue-connected to x_i is a red net by Lemma 3. Take any red net M containing the net $\tilde{I}'$ and not contained in any larger net. (At least one such net must exist.) If it can be shown that M' is also a red net, then the fact that the graph has $2\binom{u}{3}$ red triangles implies that both red nets have u points and the proof is complete.

To show that M' is a red net, suppose that two points, say x_j and x_k, belonging to M' are blue-connected. Since M is not a proper subset of a larger red net, x_j must be blue-connected to some point in M, say x_p, and, applying Lemma 2 (together with the fact that $\tilde{P}\supset M$, since x_p belongs to the red net M), we have

$$\begin{aligned}(\tilde{J}\cap M) \leqq (\tilde{J}\cap\tilde{P}) &= (\tilde{J})+(\tilde{P})-(\tilde{J}\cup\tilde{P})\\ &= (\tilde{J})+(\tilde{P})-(S) \leqq 2(u+1)-2u = 2,\end{aligned}$$

since no point has order greater than u. By the same argument, $(\tilde{K}\cap M)\leqq 2$, and since $\tilde{J}\cup\tilde{K}=S$ by Lemma 2, we have $(M)\leqq(\tilde{J}\cap M)+(\tilde{K}\cap M)\leqq 2+2=4$. But, recalling the assumption that $u>5$, we have the contradiction $(M)\geqq(\tilde{I}')\geqq u-1>4$, since x_i, of order at most u, is blue-connected to at least $u-1$ points. Hence, M' is a red net and the theorem is proved.

THEOREM 3. *Every minimal blue-empty chromatic graph on $2u+1$ points ($u>5$) contains $\binom{u}{3}+\binom{u+1}{3}$ red triangles, contained in two disjoint red nets of u and $u+1$ points.*

Proof. Completing the square in the formula of Lemma 1, we obtain

$$B+R=\binom{2u+1}{3}-\frac{1}{2}\sum_{i=1}^{2u+1}u^2+\frac{1}{2}\sum_{i=1}^{2u+1}[(I)-u]^2,$$

and the problem is reduced to that of minimizing the final term of this expression. In Section 2, the two red net graph on $2u+1$ points was defined and shown to be blue-empty. Points in the $u+1$ point net have order u and those in the u point net have order $u-1$, so that the value of the final term above for the two red net graph is $\frac{1}{2}u$. This term will be larger than $\frac{1}{2}u$ for a chromatic graph in which fewer than $u+1$ points have order u, since more than u points would then have order different from u and hence contribute at least one to the sum. Therefore, any minimal blue-empty graph must contain at least $u+1$ points of order u.

Suppose that there exists a red net of $u+1$ points of order u. Each point in the net, having order u, is red-connected to exactly u other points in the graph,

but must be red-connected to each of the other u points in the red net. Hence, each of the points in the red net must be blue-connected to each of the u points outside the net, which in turn must be pairwise red-connected in order to avoid blue triangles. Thus we have a $u+1$ point red net and a disjoint u point red net, so that the proof is complete for this case.

If, on the other hand, there are no $u+1$ points of order u which are pairwise red-connected, then, since at least $u+1$ points in the graph have order u, there are certainly two points of order u, say x_j and x_k, which are blue-connected. Now $\tilde{J}'$ and $\tilde{K}'$ are red nets of u points by Lemma 3 and are disjoint since $(\tilde{J}' \cap \tilde{K}') = (S) - (\tilde{J} \cup \tilde{K}) = 0$ by Lemma 2. There is, therefore, exactly one point, x_i, which is not contained in either net. Now $\tilde{I}'$ is a red net by Lemma 3. It is easily seen that if all the lines between the red nets $\tilde{J}'$ and $\tilde{K}'$, except for those in $\tilde{I}'$, are changed to blue, the graph will remain blue-empty, for the "subgraph" $\tilde{J}' \cup \tilde{K}'$ would become a *two red net* graph, with, perhaps, additional red lines, and certainly could not contain any blue triangles, while the point x_i could not be a vertex of a blue triangle, so long as $\tilde{I}'$ is a red net.

Changing red lines to blue will certainly not create any new red triangles and hence, the *altered* graph would be a minimal blue-empty graph. If the new graph still contains two disjoint red nets of u and $u+1$ points after (possibly) changing some red lines to blue, then the red nets clearly must belong to the original graph. Thus, we may as well assume that the only red lines in $\tilde{J}' \cup \tilde{K}'$ are those in the red nets $\tilde{I}'$, $\tilde{J}'$, and $\tilde{K}'$ (not necessarily distinct), so that the structure of the graph is completely determined by the values of the parameters j and k, where $j = (\tilde{I}' \cap \tilde{J}')$ and $k = (\tilde{I}' \cap \tilde{K}')$. If either j or k is 0, then x_i is red-connected to all points of one of the nets $\tilde{J}'$ and $\tilde{K}'$ and the proof is complete. Assume, then, that $jk > 0$.

Since the structure of the graph is completely determined by the values of j and k, we can compute the value of the final term in the expression for $B+R$ above in terms of j and k, getting half of

$$[2u - (j + k) - u]^2 + j[(u - 1 + k) - u]^2 + k[(u - 1 + j) - u]^2 = [u - (j + k)]^2 + (j + k) + jk(j + k - 4).$$

Suppose $j+k>4$ for a minimal graph. Then, since we assume $jk>0$, the above expression is strictly greater than $(u-j-k)^2+j+k$, which is always at least u for integral j and k; that is, the final term is necessarily greater than $\frac{1}{2}u$, and the graph cannot be minimal.

If $j+k \leqq 4$ (and $jk>0$) we must have $j+k=2$, 3, or 4, giving respective values for the above expression of $(u-2)^2$, $(u-3)^2+1$, and $(u-4)^2+4$. For $u>5$, these expressions have values greater than u and, hence, the final term in the expression for $B+R$ would exceed $\frac{1}{2}u$, contradicting the assumption that the graph is minimal. We have shown that $jk>0$ is impossible and, hence, the proof is complete.

Theorems 2 and 3 easily give a complete structural characterization of

minimal blue-empty chromatic graphs for $N>11$, as explained at the beginning of Section 4. To completely describe the exceptional cases, $N=6$, 7, 8, 9, 10, and 11, we can apply the methods of Theorems 2 and 3. For odd N, application of the formula used in the proof of Theorem 3 can be made to determine the existence of unique minimal blue-empty graphs, having 4 and 13 red triangles, respectively, for the cases $N=7$ and 9 (as compared with the values 4 and 12 for extremal graphs), and, for the case $N=11$, two exceptional minimal graphs, each having 30 red triangles, the same number as the two red net graph.

A picture of the unique graph for $N=7$ is to be found in [2], while the unique minimal blue-empty graph on nine points is shown in Figure 1. (Red lines are shown in black; blue lines are omitted.) For $N=6$, 8, and 10, the existence of two red nets of $u-1$ points can be established and further investigation results in the observation that, for $N=10$, Theorem 2 continues to hold, while for $N=6$ and 8 we have two exceptional examples and one exceptional example respectively. It should be noted that the appearance of a particular graph may be altered, perhaps, by a permutation of the points, so that *uniqueness* of graphs must refer to uniqueness up to such permutations and the possible geometric variation in position of the points in the plane.

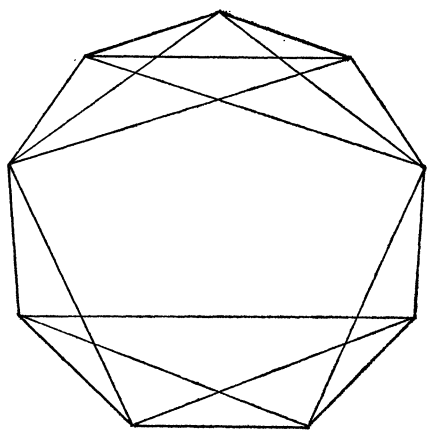


FIG. 1

5. More general problems. Greenwood and Gleason [3] consider chromatic graphs of more than two colors, and deal with sets of k points mutually interconnected by segments of a single color, where k may be larger than 3. As yet, no result as complete as Theorem 1 is known for such general situations. It is easily seen that no formula so simple as that of Lemma 1 can be applied to the investigation of such generalizations, because simple examples reveal that the *order numbers* of the points no longer determine the number of single-color triangles (or *k-nets*). At best, the methods and results of the present paper might provide a good start for more general investigations.

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A LEMMA ON CYCLIC DISLOCATIONS IN DETERMINANTS AND AN APPLICATION IN THE VERIFICATION OF AN IDENTITY

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Introduction. In determining the coefficients in the power series for $\psi\{\phi(x)\}$ where $\phi(x)$ and $\psi(x)$ are themselves power series, it is necessary first to find the coefficients C_{rn} in the expansion

$$\{\phi(x)\}^r = \sum C_{rn}\frac{x^n}{n!}\cdot$$

These are given by

$$C_{rn} = D^n\{\phi(x)\}^r_{x=0} \qquad \left(D^n \equiv \frac{d^n}{dx^n}\right). \tag{A}$$

It is therefore required to find the general form of $D^n(\phi^r)$ in terms of $\phi(x)$ and its derivatives.*

It may be readily verified that

$$\begin{aligned}
\phi^{1-r}D(\phi^r) &= r\phi' \\
\phi^{2-r}D^2(\phi^r) &= r(r-1)\phi'^2 + r\phi\phi'' \\
&= \begin{vmatrix} r\phi' & r\phi'' \\ -\phi & (r-1)\phi' \end{vmatrix} \\
\phi^{3-r}D^3(\phi^r) &= \begin{vmatrix} r\phi' & r\phi'' & r\phi''' \\ -\phi & (r-1)\phi' & (2r-1)\phi'' \\ \cdot & -\phi & (r-2)\phi' \end{vmatrix} \\
\phi^{4-r}D^4(\phi^r) &= \begin{vmatrix} r\phi' & r\phi'' & r\phi''' & r\phi^{(4)} \\ -\phi & (r-1)\phi' & (2r-1)\phi'' & (3r-1)\phi''' \\ \cdot & -\phi & (r-2)\phi' & (3r-3)\phi'' \\ \cdot & \cdot & -\phi & (r-3)\phi' \end{vmatrix}
\end{aligned}$$

* An expansion of $D^n(\phi^r)$ in terms of $D^n(\phi)$, $D^n(\phi^2)$, $D^n(\phi^3)\cdots$ is given in §7.37, p. 160 of Smithsonian Mathematical Formulae and Tables of Elliptic Functions.

It appears that $\phi^{n-r}D^n(\phi^r)$ can be expressed in the form of a determinant

$$(1)\quad \Delta_n(\phi, r) = \begin{vmatrix} r\phi' & r\phi'' & r\phi''' & r\phi^{(4)} & \cdots & r\phi^{(n)} \\ -\phi & (r-1)\phi' & (2r-1)\phi'' & (3r-1)\phi''' & \cdots & \\ \cdot & -\phi & (r-2)\phi' & (3r-3)\phi'' & \cdots & \\ \cdot & \cdot & -\phi & (r-3)\phi' & \cdots & \\ \cdot & \cdot & \cdot & -\phi & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdots & -\phi\ (r-n+1)\phi' \end{vmatrix}$$

where $\phi = \phi(x)$, $\phi^{(n)} = D^n\{\phi(x)\}$ and where the coefficients a_{mn} of ϕ and its derivatives are obtained successively from those in the first row and column by the relation

$$(2)\qquad a_{mn} = a_{m,n-1} + a_{m-1,n-1},$$

the appropriate solution of which is:

$$(3)\qquad a_{mn} = \binom{n-1}{m-1} r - \binom{n-1}{m-2}.$$

The truth of our conjecture will shortly be verified but first we need a lemma.

Let
$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = | C_1 \quad C_2 \quad C_3 |$$

where the C's represent the columns of Δ_3.

Now define δC_1 to be the column obtained from C_1 by dislocating the elements one place in cyclic order, i.e.

$$\delta C_1 = \begin{bmatrix} a_{31} \\ a_{11} \\ a_{21} \end{bmatrix}.$$

Similarly

$$\delta C_2 = \begin{bmatrix} a_{32} \\ a_{12} \\ a_{22} \end{bmatrix}, \qquad \delta C_3 = \begin{bmatrix} a_{33} \\ a_{13} \\ a_{23} \end{bmatrix}.$$

LEMMA: *The sum of the three determinants obtained by dislocating the columns of* Δ_3 *one at a time is zero, viz.*

$$|\delta C_1\ C_2\ C_3| + |C_1\ \delta C_2\ C_3| + |C_1\ C_2\ \delta C_3| = 0.$$

Expanding each of these determinants by the elements of the dislocated column and writing A_{ij} for the cofactor of a_{ij} in Δ_3, we obtain

$$\begin{aligned}|\delta C_1\ \ C_2\ \ C_3| &= a_{31}A_{11} + a_{11}A_{21} + a_{21}A_{31},\\ |C_1\ \delta C_2\ \ C_3| &= a_{32}A_{12} + a_{12}A_{22} + a_{22}A_{32},\\ |C_1\ \ C_2\ \delta C_3| &= a_{33}A_{13} + a_{13}A_{23} + a_{23}A_{33}.\end{aligned}$$

The sum by columns of the three determinants becomes

$$\sum a_{3i}A_{1i} + \sum a_{1i}A_{2i} + \sum a_{2i}A_{3i} \tag{4}$$

which is zero since each of these sums is an expansion of Δ_3 by false cofactors.

If in the process of dislocation the uppermost element is made zero, i.e.

$$\delta C_1 = \begin{bmatrix} \cdot \\ a_{11} \\ a_{21} \end{bmatrix}, \qquad \delta C_2 = \begin{bmatrix} \cdot \\ a_{12} \\ a_{22} \end{bmatrix}, \qquad \delta C_3 = \begin{bmatrix} \cdot \\ a_{13} \\ a_{23} \end{bmatrix},$$

the lemma still holds for, if $a_{3i}=0$ in (4) then each term in the first sum vanishes separately, the other sums being unaffected.

The lemma clearly holds for determinants of any order and for any number of cyclic dislocations, provided only that the number of dislocations is the same for each column. Thus, if

$$\Delta_n = |C_1\ C_2\ C_3\ \cdots C_n|$$

then

$$|\delta C_1\, C_2\, C_3\ \cdots C_n| + |C_1\, \delta C_2\, C_3\ \cdots C_n| + \cdots + |C_1\, C_2\, C_3\ \cdots \delta C_n| = 0.$$

The reduction of $\Delta_n(\phi, r)$. In order to verify our conjecture it will not be necessary to reduce the nth order determinant since the method of reduction follows the same general pattern whatever the order of the determinant. For the sake of brevity, therefore, we will illustrate the general method by systematically reducing the determinant of order four. Expanding Δ_4 by elements of the 4th row,

$$\Delta_4 = (r-3)\phi'\Delta_3 + \phi M_{43} \tag{5}$$

where M_{43} is the minor of the term in the 4th row and 3rd column of Δ_4. Now in the column notation,

$$M_{43} = |C_1\ C_2\ C_4|$$

there being three elements in each column.

$$\begin{aligned}\Delta_3' - M_{43} &= |C_1'\ C_2\ C_3| + |C_1\ C_2'\ C_3| + [|C_1\ C_2\ C_3'| - |C_1\ C_2\ C_4|]\\ &= |C_1' - C_2\ C_2\ C_3| + |C_1\ C_2' - C_3\ C_3| + |C_1\ C_2\ C_3' - C_4|\end{aligned} \tag{6}$$

by elementary operations. But it follows from (2) that $C_2 - C_1' = \delta C_1$, where δC_1 is a cyclic dislocation of C. Similarly $C_3 - C_2' = \delta C_2$ and, provided that the uppermost element of δC_3 is made zero, $C_4 - C_3' = \delta C_3$. But, since the uppermost elements in δC_1 and δC_2 are unconditionally zero, it will be convenient and not incorrect to regard all three dislocated columns as having their uppermost elements made zero. Thus from (6),

$$M_{43} - \Delta_3' = |\delta C_1\ C_2\ C_3| + |C_1\ \delta C_2\ C_3| + |C_1\ C_2\ \delta C_3| = 0$$

by the lemma. Thus $M_{43} = \Delta_3'$. Substituting in (5),

$$\Delta_4 = (r-3)\phi'\Delta_3 + \phi\Delta_3',$$
$$\phi^{r-4}\Delta_4 = (\phi^{r-3}\Delta_3)'.$$

Similarly it can be shown that

$$\phi^{r-3}\Delta_3 = (\phi^{r-2}\Delta_2)',$$
$$\phi^{r-2}\Delta_2 = (\phi^{r-1}\Delta_1)' = (r\phi^{r-1}\phi')' = (\phi^r)''$$
$$\phi^{r-4}\Delta_4 = (\phi^r)^{(4)},$$

that is,

$$\Delta_4(\phi, r) = \phi^{4-r}D^4(\phi^r).$$

The method of reduction used is clearly applicable to the general determinant Δ_n and by this method it can be shown that

$$M_{n+1,n} = \Delta_n',$$
$$\phi^{r-n}\Delta_n = (\phi^{r-n+1}\Delta_{n-1})',$$
$$\Delta_n(\phi, r) = \phi^{n-r}D^n(\phi^r).$$

Thus from equation (A)

$$C_{rn} = \{\phi(0)\}^{r-n}\Delta_n\{\phi(x), r\}_{x=0}.$$

Putting $r = -1$ we find the coefficients in the series expansion of the reciprocal of a power series:

$$[\phi(x)]^{-1} = \sum C_{-1,n}\frac{x^n}{n!}$$

where

$$C_{-1,n} = \frac{\Delta_n\{\phi(x), -1\}_{x=0}}{\{\phi(0)\}^{n+1}} \qquad \phi(0) \neq 0.$$

This formula gives the same result as that given by Whittaker and Watson [1], in terms of a determinant G_n though some manipulation is required in order to verify that the two determinants are related by the identity

$$\Delta_n\{\phi(x), -1\}_{x=0} = (-)^n G_n.$$

The above method can also be used to find alternative forms of the other two expansions given by Whittaker and Watson on the same page, viz.

$$\left[\sum_0^\infty a_n x^n\right]^{1/2}, \qquad \left[\sum_0^n a_r x^r\right]^{-1}.$$

The appearance of determinants in the explicit formulae for the coefficients in a power series is also noticed in the reversion of a power series—see M. Ward [2] and p. 116 of the Smithsonian book of Mathematical Formulae already quoted.

The Legendre polynomials as determinants. Rodrigues' formula for the Legendre polynomials is

$$P_n(x) = \frac{1}{2^n n!} D^n(x^2 - 1)^n = \frac{1}{2^n n!} \Delta_n(\phi, n)$$

where $\phi(x) = x^2 - 1$. Since $\phi' = 2x$, $\phi'' = 2$, $\phi''' = \phi^{(4)} = \phi^{(5)} = \cdots = 0$, we have

$$P_n(x) = \frac{1}{2^n n!} \begin{vmatrix} 2nx & 2n & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 - x^2 & 2(n-1)x & 2(2n-1) & \cdot & \cdot & \cdots & \cdot \\ \cdot & 1 - x^2 & 2(n-2)x & 2(3n-3) & \cdot & \cdots & \cdot \\ \cdot & \cdot & 1 - x^2 & 2(n-3)x & 2(4n-6) & \cdots & \cdot \\ \cdot & \cdot & \cdot & 1 - x^2 & 2(n-4)x & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots 1 - x^2 & 2x \end{vmatrix}$$

Acknowledgment. The author is indebted to the Director, Electrical Research Association, for permission to publish this paper.

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1. E. T. Whittaker and G. N. Watson, Modern Analysis, 4th ed., Cambridge University Press, p. 147, no. 15.

2. M. Ward, The reversion of a power series, Rendiconti del Circolo Matematico di Palermo, vol. 54, 1930, pp. 42–46.

MATHEMATICAL NOTES

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THE GROUP OF THE PYTHAGOREAN NUMBERS

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The complete solution of the equation

$$x^2 + y^2 - z^2 = 0 \tag{1}$$

in integral numbers x, y, z was given by Euclid, Book X, Lemma 1 to Proposition 29. He found the following well-known result: One of the numbers x, y must be even. Let x be even. Then every solution of (1) with $z>0$ can be written in the form

$$x = 2ab, \qquad y = \pm (a^2 - b^2), \qquad z = a^2 + b^2, \tag{2}$$

where a and b are arbitrary integers. Vice versa, every triplet (2) gives us a solution of (1).

The left-hand side of (1) is an indefinite quadratic form with integral coefficients. The pseudo-orthogonal linear substitutions, with integral coefficients of the variables (in this case, x, y, z) which carry the quadratic form into itself are of interest, e.g., in the theory of automorphic functions. An account of the earlier part of the theory can be found in Fricke [1]. Clearly, these linear substitutions form a group the elements of which map integral solutions of (1) onto integral solutions.

In order to formulate our results, we shall use the following notations:

Three integers x, y, z will be called a *proper Pythagorean triplet* if they do not have a common divisor different from ± 1, if x is even, $z>0$, and if (1) is satisfied.

The group of all homogeneous linear transformations of x, y, z with integral coefficients, which leave $x^2+y^2-z^2$ invariant, will be denoted by $\overline{G}$. Its subgroup consisting of those transformations which map proper Pythagorean triplets onto proper Pythagorean triplets and have the determinant ± 1 will be called G.

THEOREM 1. *G acts transitively on the proper Pythagorean triplets. Every triplet can be mapped onto any other by a transformation belonging to G.*

THEOREM 2. *G is isomorphic to the group of all 2×2 matrices with integral coefficients and determinant ± 1.*

THEOREM 3. *The subgroup of substitutions of G which keeps a particular proper Pythagorean triplet fixed is the direct product of an infinite cyclic group and a group of order 2.*

THEOREM 4. *G is of finite index in $\overline{G}$.*

The proof of these results is simple. We observe first that, if x is even, it must be divisible by 4 since y^2-z^2 is divisible by 8 if y and z are odd. Therefore, we may assume that a in (2) is even. Consider the linear substitution c_2

$$a' = \lambda a + \mu b, \qquad b' = \nu a + \rho b \tag{3}$$

where λ, ρ are odd, ν, μ are even, and $\lambda\rho-\nu\mu=1$. By (3), a pair (a, b) is mapped onto a pair (a', b'), where a' is even and b' is odd and where a' and b' are coprime if a and b are coprime.

Now we have

LEMMA 1. *By choosing λ, μ, ν, ρ properly, we can map any coprime pair (a, b) onto the pair $a'=0$, $b'=1$.*

In fact, Euclid's algorithm guarantees the existence of two integers ν^*, ρ^* such that $\nu^*a+\rho^*b=1$. These integers are not uniquely determined; if we put $\nu=\nu^*+kb$, $\rho=\rho^*-ka$, $(k=0, \pm 1, \cdots)$ we also have $\nu a+\rho b=1$, and we can choose k in such a manner that ν is even ($k=0$ if ν^* is even, $k=1$ if ν^* is odd). Automatically, ρ will then be odd. Having determined ν and ρ, we determine λ and μ by $\lambda=b$, $\mu=-a$. Obviously, λ is odd and μ is even, and $\lambda\rho-\nu\mu=1$. This proves Lemma 1.

Now we observe that (2) and (3) define the substitution

$$\begin{aligned} x' &= 2a'b' = (\lambda\rho + \mu\nu)x + (\lambda\nu - \mu\rho)y + (\lambda\nu + \mu\rho)z, \\ y' &= a'^2 - b'^2 = (\lambda\mu - \nu\rho)x + \tfrac{1}{2}(\lambda^2 - \nu^2 - \mu^2 + \rho^2)y + \tfrac{1}{2}(\lambda^2 - \nu^2 + \mu^2 + \rho^2)z, \\ z' &= a'^2 + b'^2 = (\lambda\mu + \nu\rho)x + (\lambda^2 + \nu^2 - \mu^2 - \rho^2)y + \tfrac{1}{2}(\lambda^2 + \nu^2 + \mu^2 + \rho^2)z. \end{aligned} \tag{4}$$

Obviously, (4) is a pseudo-orthogonal transformation that maps proper Pythagorean triplets on proper Pythagorean triplets and carries the left hand side of (1) onto itself. Therefore, (4) is a substitution of G. According to Lemma 1, there exists a substitution (3) which carries a preassigned pair (a, b) (a even) into the pair (0, 1). Therefore, every proper Pythagorean triplet can be carried into one of the triplets (0, 1, 1) or (0, −1, 1). These two triplets can be exchanged by the substitution

$$a' = b, \qquad b' = -a, \tag{5}$$

which induces the substitution $x'=x$, $y'=-y$, $z'=z$. To these substitutions, we may add the substitution

$$a' = -a, \qquad b' = b, \tag{6}$$

which induces the transformation $x'=-x$, $y'=y$, $z'=z$. Together the substitutions (3), (5) and (6) generate the extended modular group c_2', the quadratic representation of which, as described by (4), is at least a part G' of the group G described in Theorem 2, so that obviously G is transitive on the proper Pythagorean triplets. Now, no pseudo-orthogonal substitution outside of G' will map proper Pythagorean triplets onto triplets of the same type so that G' coincides

with G. Consider any substitution of G

$$(7)\quad x' = P_2z + Q_2x + R_2y,\quad y' = P_3z + Q_3x + R_3y,\quad z' = P_1z + Q_1x + R_1y$$

satisfying the pseudo-orthogonality conditions (and the transpose):

$$(8)\qquad \begin{aligned} P_1^2 + Q_1^2 - R_1^2 &= 1, & P_1P_2 - Q_1Q_2 - R_1R_2 &= 0;\\ P_2^2 - Q_2^2 - R_2^2 &= -1, & P_1P_3 - Q_1Q_3 - R_1R_3 &= 0;\\ P_3^2 - Q_3^2 - R_3^2 &= -1, & P_2P_3 - Q_2Q_3 - R_2R_3 &= 0. \end{aligned}$$

Using (2) and analogous formulas for x', y', z', we get from (7)

$$(9)\qquad \begin{aligned} (a' + b')^2 &= \tfrac{1}{2}(P_1 + P_2 + Q_1 + Q_2)(a + b)^2\\ &\quad + \tfrac{1}{2}(P_1 + P_2 - Q_1 - Q_2)(a - b)^2 + \tfrac{1}{2}(R_1 + R_2)\cdot 2(a - b)(a + b),\\ (a' - b')^2 &= \tfrac{1}{2}(P_1 - P_2 + Q_1 - Q_2)(a + b)^2\\ &\quad + \tfrac{1}{2}(P_1 - P_2 - Q_1 + Q_2)(a - b)^2 + \tfrac{1}{2}(R_1 + R_2)\cdot 2(a - b)(a + b), \end{aligned}$$

the coefficients being integers. From (8) we get

$$(10)\qquad \begin{aligned} (P_1 + P_2)^2 - (Q_1 + Q_2)^2 &= (R_1 + R_2)^2,\\ (P_1 - P_2)^2 - (Q_1 - Q_2)^2 &= (R_1 - R_2)^2. \end{aligned}$$

$\frac{1}{2}(R_1+R_2)$ and $\frac{1}{2}(R_1-R_2)$ being integers and relatively prime, we deduce from (10) that

$$(11)\qquad \begin{aligned} \tfrac{1}{2}(P_1 + P_2 + Q_1 + Q_2) &= m^2, & \tfrac{1}{2}(P_1 + P_2 - Q_1 - Q_2) &= n^2,\\ \tfrac{1}{2}(P_1 - P_2 + Q_1 - Q_2) &= r^2 & \tfrac{1}{2}(P_1 - P_2 - Q_1 + Q_2) &= s^2, \end{aligned}$$

where m, n, r, s are integers. Using

$$\begin{aligned} P_1^2 - Q_1^2 - R_1^2 = P_1^2 - P_2^2 - P_3^2 &= -(Q_1^2 - Q_2^2 - Q_3^2)\\ &= -(R_1^2 - R_2^2 - R_3^2) = (ms - nr)^2, \end{aligned}$$

we can solve (11) with respect to $P_1, P_2, \cdots, R_3$. Taking the square root in (9), we get a linear substitution from which we easily deduce (3). Moreover, the only substitutions of x, y, z with integral coefficients and with the properties of keeping the left-hand side of (1) fixed, map the triplet $(0, -1, 1)$ onto itself and are those induced by $a' = \pm a$, $b' = 2ka + b$, where k is an arbitrary integer.

Taking in (1), x, y, z positive and relatively prime, replacing them by $x' = x+\Delta$, $y' = y+\Delta$, $z' = z+\Delta$, and solving for Δ we get another proper solution, x', y', z' deriving from the transformation

$$\begin{pmatrix} -2 & -1 & 2\\ -1 & -2 & 2\\ -2 & -2 & 3 \end{pmatrix}$$

induced by matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

and $z' < z$. Taking x', y', z' positive and performing the same operation we get $z'' < z'$ and so on until we arrive by the method of infinite descent at $z^{(n)} = 1$, $x^{(n)} = 0$, $y^{(n)} = -1$.

This proves all our theorems, except Theorem 4. Now it is easily seen that a substitution of $\overline{G}$ must map a solution of (1) for which x, y, z have as a common divisor only ± 1 onto a solution x', y', z' of (1) with the same property. However, x', y', z' may not form a proper Pythagorean triplet since x' need not be even and z' need not be positive. Therefore, we must extend the group G by ad oining the substitutions

$$x' = y, \quad y' = x, \quad z' = z, \tag{12}$$

$$x' = x, \quad y' = y, \quad z' = -z, \tag{13}$$

in order to obtain $\overline{G}$. It is easily confirmed that G (group of substitutions of x, y, z) forms a normal divisor of index 4 in the group $\overline{G}$ arising from an adjunction of (12) and (13). This also proves Theorem 4.

Reference

1. R. Fricke and F. Klein, Vorlesungen über die Theorie der automorphen Funktionen, vol. 1, Leipzig, 1897.

AN OPEN DISCONTINUOUS FUNCTION

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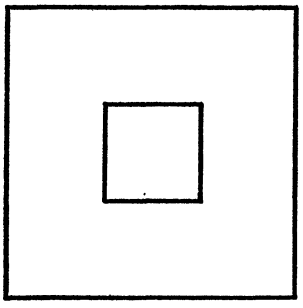
A continuous function can be defined as a function whose inverse takes open sets onto open sets. An open function is a function which directly maps open sets onto open sets. Can a function be open and not continuous? The answer is certainly *yes*, for if we concoct two topologies for a given set, one strictly stronger than the other, then the identity mapping is open and discontinuous. This is the usual answer using strange topologies. We call "Fair Play!", and give a strange answer for the usual topology.

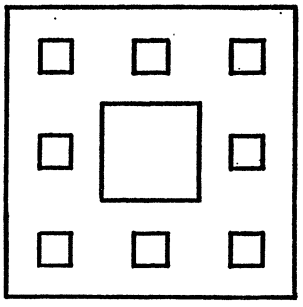
After trying a few examples, one can see that the discontinuities of such a function must be particularly violent and shattering to the nerves. We proceed in fact by the Cantor middle third set on the interval $[0, 1]$. On each of the middle third intervals we define a tangentlike function, asking only that it be strictly monotone and continuous and that its range be the entire y-axis. On the remainder, the Cantor set, we set the function equal to zero.

Due to preservation of unions on direct mapping, it is sufficient to prove openness for intervals. If an interval lies entirely within some middle third interval, the map is clearly an interval due to monotonicity and continuity. If, how-

ever, an interval covers some endpoint of a middle third interval, then it clearly covers entirely some small middle third interval in the neighborhood of this point, and hence the map is the entire y-axis and thus open.

Such a function can also be defined in the plane where we form a middle ninth set from the unit square:





the amount removed having measure

$$\frac{1}{3^2}+\frac{8}{3^4}+\frac{64}{3^6}+\cdots=1.$$

The function here defined on the unit square takes each of the little squares openly on the entire plane, which can be done by bulging a little square slightly, placing it touching the plane and projecting from a point within and on the plane of the little square. The function is defined to map to the origin outside the little squares, and as before is seen to be open.

This last example shows that any axiomatic characterization of analytic functions in terms of metric and topological properties cannot avoid an axiom of continuity in favor of an axiom of openness.

LINEAR TRANSFORMATIONS OF SEQUENCES

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In a recent note, Boyd* proved the following theorem: *If* $a_1s_m+\cdots+a_ns_{m-n+1}$ *tends to a finite limit* s *as* m *tends to infinity, where the* a_i *are real numbers such that* $a_1+\cdots+a_n=1$ *and* $|a_1|>|a_2|+\cdots+|a_n|$, *then* s_m *tends to* s *as* m *tends to infinity.* He further suggested that it should be possible to find a proof of the theorem depending entirely on real-variable arguments.

For simplicity such a proof will be indicated here assuming that n equals 3. It should be reasonably evident that the same type of argument can be carried through for any integral value of n.

By hypothesis, $|a_1s_{p+2}+a_2s_{p+1}+a_3s_p-s|<\epsilon|a_1|$, for $p\geqq N$, and $r\equiv r_2+r_3<1$, where $r_i=|a_i|/|a_1|$.

Since

* A. V. Boyd, Linear transformations of sequences, this MONTHLY, vol. 68, 1961, pp. 262–263.

$$a_1(s_{p+2} - s) = (a_1s_{p+2} + a_2s_{p+1} + a_3s_p - s) - a_2s_{p+1} - a_3s_p + s - a_1s$$
$$= (a_1s_{p+2} + a_2s_{p+1} + a_3s_p - s) + a_2(s - s_{p+1}) + a_3(s - s_p),$$

it follows that

$$|s - s_{p+2}| < \epsilon + r_2|s - s_{p+1}| + r_3|s - s_p|. \tag{1}$$

Assuming that $|s-s_p| < a$, $|s-s_{p+1}| < a$, (1) implies that

$$|s - s_{p+2}| < \epsilon + ar_2 + ar_3 = \epsilon + ar,$$
$$|s - s_{p+3}| < \epsilon + (\epsilon + ar)r_2 + ar_3 = \epsilon(1 + r_2) + a(rr_2 + r_3)$$
$$< \epsilon(1 + r) + ar,$$
$$|s - s_{p+4}| < \epsilon(1 + r_2 + r_3 + rr_2) + a(rr_2 + rr_3)$$
$$< \epsilon(1 + r + r^2) + ar^2,$$
$$|s - s_{p+5}| < \epsilon(1 + r_2 + r_3 + rr_2 + rr_3 + r^2r_2) + a(r^2r_2 + rr_3)$$
$$< \epsilon(1 + r + r^2 + r^3) + ar^2,$$
$$|s - s_{p+6}| < \epsilon(1 + r + r^2 + r^3 + r^4) + ar^3,$$

etc. Since the coefficients of ϵ are bounded by $1/(1-r)$ and the coefficients of a tend to zero, evidently $s_m \to s$.

By using these irregular bounds, we obtain the same types of bounds for larger values of n. For example, if $n=4$, there are 3 terms involving a, 3 involving ar, etc.

A FINITE BOLYAI-LOBACHEVSKY PLANE

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The subject of finite projective geometries has been investigated from various points of view.* In particular, it is a simple matter to construct examples of finite projective geometries in terms of coordinates in a Galois field. From a finite projective plane one obtains a finite affine plane by omitting the points on one line. In an affine plane, there exists through each point X not on a line b exactly *one* line c not meeting b. The question arises at once concerning the existence of *finite* B-L (Bolyai-Lobachevsky) planes, in which for each X not on b there is more than one line through X not meeting b. Proofs have been given of the nonexistence of such planes.† The postulates on which these proofs are based, are in each case stronger than the following set, which we wish to consider here.

* See e.g., Contributions to Geometry, this MONTHLY, vol. 62, 1955, Part II, August-September, and references there.

† B. J. Topel, Bolyai-Lobachevsky planes with finite lines, Rep. Math. Colloquium, Ser. 2, vol. 5–6, Notre Dame, Ind., 1944, pp. 40–42; R. Baer, Polarities in finite projective planes, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 77–93, esp. p. 91; R. Baer, The infinity of generalized hyperbolic planes, Studies and Essays Presented to R. Courant on his Sixtieth Birthday, New York, 1948, pp. 21–27.

A1. The plane $\mathcal{P}$ is a finite collection of elements called points (denoted by capital letters).

A2. There are certain distinguished subsets of the plane $\mathcal{P}$, called lines (denoted by small letters).

A3. There are at least two points on each line.

A4. Two distinct points A and B lie on one and only one line.

A5. The plane $\mathcal{P}$ contains at least four points, no three of which lie on a line.

A6. If a subset of $\mathcal{P}$ contains three points not on a line, and contains all the lines through any pair of its points, then that subset contains all the points of $\mathcal{P}$.

A7. Through each point X not on a line b there pass at least two lines not meeting b.

Using these seven postulates to define a finite B-L plane, we consider the following example of a plane, consisting of the 13 points $A, B, \cdots, M$, arranged on the 26 lines

ABC	ADE	AFH	AGJ	AIL	AKM
BDF	BEI	BGH	BJM	BKL	
CDG	CEJ	CFL	CHK	CIM	
DHI	DJK	DLM	EFK	EGL	EHM
FGM	FIJ	GIK	HJL		

If we define a homogeneous plane to be one such that for each pair of points there is a collineation carrying the first into the second, then the preceding example is not homogeneous, since there is no collineation carrying A into C.* It would be interesting to know methods for constructing additional examples of finite B-L planes (in particular, homogeneous planes), and to learn the limitations on the number n of points on a line and the number ν of lines on a point. It is easily verified that if the number n is a power of a prime, then ν is congruent to 0 or 1 (mod n). The total number of points in the plane is $p=1+\nu(n-1)$, and the number λ of lines is $\lambda=\nu^2-(\nu^2-\nu)/n$.

The following evidences of the lack of homogeneity in the example given above have been observed. There is just one triple of lines through the point A, namely, ADE, AFH, AGJ, which is disjoint from a pair of nonintersecting lines, namely, BKL and CIM. A similar statement is true for the points B, F, and H. Through the point C, on the other hand, the triple ABC, CDG, CIM, is disjoint from the pair EFK and HJL, while the triple CDG, CEJ, CIM, is disjoint from the pair AFH and BKL. A similar statement is true for the points D and G. However, there exists no triple of lines through E, I, J, K, L, or M which is disjoint from a nonintersecting pair of lines.

Another aspect of the lack of homogeneity is that through each of the points C, D and G there are six triples of lines which meet every line, while through

* The example is homogeneous in the sense of Topel, since each line contains the same number of points.

each of the remaining points there are only four such triples. Also the two disjoint quadruples *AKM*, *BEI*, *CDG*, *HJL*, and *AGJ*, *BEI*, *CHK*, *DLM* both omit the point *F*, while there exist no disjoint quadruples which omit *C* or *E*.

At least four finite $B-L$ planes, which are homogeneous in the sense used here, have been constructed by Professors Karl Menger, Gordon Pall and Abe Sklar. One of these is not homogeneous in the sense of Topel.

CONVOLUTION PRODUCTS AND QUOTIENTS AND ALGEBRAIC DERIVATIVES OF SEQUENCES

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T. A. Newton [1] illustrates the use of Laplace's generating functions in finding, operationally, the recursion relation for the coefficients in the power series solution of a differential equation. There is an algebraic side to this which does not appear when one uses Laplace's generating functions with their continuous variable, t, and power series with the related questions of convergence. Newton [1] takes as an example the differential equation

$$(1 + t^2)v'' + 2tv' - 2v = 0 \tag{1}$$

and obtains from it the relation

$$(n + 1)b_{n+2} + (n - 1)b_n = 0. \tag{2}$$

In this note it will be shown how (1) and (2) may be replaced respectively by

$$(1 + \tau^2)D^2\xi + 2\tau D\xi - 2\xi = 0 \tag{3}$$

and

$$\{(\nu + 1)\xi_{\nu+2} + (\nu - 1)\xi_\nu\} = 0 \tag{4}$$

where ξ is a variable on sequences in both (3) and (4), where the roots in ξ of both (3) and (4) are sequences, where (3) and (4) define the same two-parameter family of sequences, and where all the operations indicated in (3) and (4) are algebraic, involving no questions of convergence. In (3), τ is not a variable but a special sequence to be introduced below, $D\xi$ is an *algebraic derivative* to be defined below, and $2\tau D\xi$, for example, is a *convolution product* of the sequences 2τ and $D\xi$.

The word *sequence* will be used here, not for a succession of quantities, but for a sign such as 2τ, $\{\nu^2\}$, $\{1, 2, 3, 4, \cdots\}$, or ξ which denotes an assignment of quantities to the nonnegative integers. In (4), $(\nu-1)\xi_\nu$ is not a convolution product of two sequences but becomes an ordinary product of quantities as soon as the variables ν and ξ are fixed or replaced by appropriate values; $(\nu-1)$ and ξ_ν are not sequences whereas $\{\nu-1\}$ and $\{\xi_\nu\}$ are sequences, the braces serving to bind out the variable ν. The more complete notation $\{\nu-1\}_\nu$ will not be necessary in this note since ν will be the only variable to be bound out in this way.

A convolution calculus for sequences (for the record) was presented by Jozef Eliáš [2], who begins—unfortunately, it seems to this author—with a definition of convolution product as follows:

$$\begin{aligned}(5)\qquad &\{\xi_0, \xi_1, \xi_2, \xi_3, \cdots\} * \{\eta_0, \eta_1, \eta_2, \eta_3 \cdots\} \\ &= \{0, \xi_0\eta_0, \xi_0\eta_1 + \xi_1\eta_0, \xi_0\eta_2 + \xi_1\eta_1 + \xi_2\eta_0, \cdots\}\end{aligned}$$

where Eliáš requires the lead term of the product to vanish. The author has developed an operational calculus for sequences, to be outlined in this note, based on the more familiar convolution product mentioned, for example, in Newton's note [1], and also in Feller [3], as follows:

$$\begin{aligned}&\{\xi_0, \xi_1, \xi_2, \xi_3, \cdots\}\{\eta_0, \eta_1, \eta_2, \eta_3, \cdots\} \\ (6)\qquad &= \left\{\xi_0\eta_0, \xi_0\eta_1 + \xi_1\eta_0, \xi_0\eta_2 + \xi_1\eta_1 + \xi_2\eta_0, \cdots, \sum_{n=0}^{\nu} \xi_n\eta_{\nu-n}, \cdots\right\}.\end{aligned}$$

These calculi are patterned after Mikusiński's operational calculus [4] which is based on the convolution integral of two functions. Mikusiński defines an *algebraic derivative* ([4], p. 261), and it is the analogue of this algebraic derivative as applied to sequences that permits the purely algebraic side of Newton's program in [1] to be brought into focus.

Four important examples of convolution products are

$$(7)\qquad \{c, 0, 0, \cdots\}\{\xi_0, \xi_1, \xi_2, \cdots\} = \{c\xi_0, c\xi_1, c\xi_2, \cdots\};$$

$$(8)\qquad \{0, 0, 0, \cdots\}\{\xi_0, \xi_1, \xi_2, \cdots\} = \{0, 0, 0, \cdots\};$$

$$(9)\qquad \{1, 0, 0, \cdots\}\{\xi_0, \xi_1, \xi_2, \cdots\} = \{\xi_0, \xi_1, \xi_2, \cdots\};$$

$$(10)\qquad \{0, 1, 0, \cdots\}\{\xi_0, \xi_1, \xi_2, \cdots\} = \{0, \xi_0, \xi_1, \xi_2, \cdots\}.$$

It follows from (7) that numbers may be identified with and used as *numerical sequences*, i.e. sequences all of whose terms are zero except possibly the lead term. Equations (8) and (9) are special cases of (7) which reveal the zero and unit elements in this convolution algebra. Equation (10) shows that the sequence $\{0, 1, 0, \cdots\}$ serves as a translation operator which shifts the terms of a sequence one place to the right introducing a zero in the lead position. This sequence will be denoted by τ. That is, $\tau = \{0, 1, 0, \cdots\}$ and it becomes the basic building block for the operational forms of sequences. Other translation operators are positive integral powers of τ. Thus

$$\tau^n = \{\underbrace{0, \cdots, 0}_{n \text{ zeros}}, 1, 0, \cdots\}$$

and

$$(11)\qquad \tau^n\{\xi_0, \xi_1, \xi_2, \cdots\} = \{\underbrace{0, \cdots, 0}_{n \text{ zeros}}, \xi_0, \xi_1, \xi_2, \cdots\}.$$

Equation (11) shows that

$$\tau^n\{f(\nu)\} = \{f(\nu - n)\} \tag{12}$$

provided that $f(-1)=f(-2)=f(-3)= \cdots =f(-n)=0$. For example

$$\tau^2\{(\nu+1)(\nu+2)\} = \{(\nu-1)\nu\} = \{\nu^{(2)}\},$$
$$\tau^2\{2, 6, 12, 20, \cdots\} = \{0, 0, 2, 6, 12, 20, \cdots\},$$

and more generally,

$$\begin{aligned}\tau^n\{(\nu+1)(\nu+2)\cdots(\nu+m)g(\nu)\} \\ = \{(\nu+1-n)(\nu+2-n)\cdots(\nu+m-n)g(\nu-n)\},\end{aligned} \tag{13}$$

for $m \geqq n$.

The relationship between the convolution product of Eliáš (5) and the convolution product of this note may now be stated as $\xi * \eta = \tau\xi\eta$, where the asterisk distinguishes the Eliáš product.

The coordinate-type convergence of Cooke [5] justifies the following equation:

$$\{\xi_0, \xi_1, \xi_2, \cdots\} = \xi_0 + \xi_1\tau + \xi_2\tau^2 + \cdots$$

where the partial sums formed from the right hand side form a succession of sequences which is *always* convergent, converging to the sequence on the left hand side.

It may be shown with no great trouble that the class of sequences under ordinary addition and convolution multiplication forms a commutative ring with a unit element and with no divisors of zero—i.e., an integral domain. As a result, the *convolution quotient*, ξ/η, where $\eta \neq 0$, of two sequences may be defined as the unique root in x of the equation:

$$\eta x = \xi, \qquad \eta \neq 0, \tag{14}$$

As a first illustration of such convolution quotients, equation (11) shows that a sequence whose first n or more terms are zero may be divided by τ^n and the resulting convolution quotient exists as a sequence. For example:

$$\xi = \frac{\{0, \xi_0, \xi_1, \xi_2, \cdots\}}{\tau} = \frac{\{0, 0, \xi_0, \xi_1, \xi_2, \cdots\}}{\tau^2} = \frac{\{0, 0, 0, \xi_0, \xi_1, \xi_2, \cdots\}}{\tau^3},$$

$$\frac{\{f(\nu)\}}{\tau^n} = \{f(\nu+n)\}, \qquad \textit{provided } f(0) = f(1) = \cdots = f(n-1) = 0,$$

$$\frac{\{\nu^{(3)}\}}{\tau^2} = \frac{\{\nu(\nu-1)(\nu-2)\}}{\tau^2} = \{(\nu+2)(\nu+1)\nu\},$$

$$\begin{aligned}\frac{\{\nu^{(m)}g(\nu)\}}{\tau^n} &= \frac{\{\nu(\nu-1)(\nu-2)\cdots(\nu-m+1)g(\nu)\}}{\tau^n} \\ &= \{(\nu+n)(\nu-1+n)\cdots(\nu-m+1+n)g(\nu+n)\}.\end{aligned}$$

for $m \geqq n$.

Equation (14) is not always solvable in x in the domain of sequences (e.g. $\tau x = 1$ is not) and this unsolvability may be remedied by making the usual extension of an integral domain to a field of ordered pairs, but this is not necessary for the purposes of this note.

The existence of convolution quotients as sequences permits the construction of *operational forms* of sequences in terms of τ. Using ν as a variable on the non-negative integers and c as an arbitrary number we have

$$\begin{aligned}(1 - c\tau)\{c^\nu\} &= [\{1, 0, \cdots\} - \{0, c, 0, \cdots\}]\{c^\nu\} \\ &= \{1, -c, 0, \cdots\}\{1, c, c^2, \cdots\} \\ &= \{1, 0, \cdots\} = 1.\end{aligned}$$

This yields the operational form of $\{c^\nu\}$:

$$\{c^\nu\} = \frac{1}{1 - c\tau} \tag{15}$$

and, as a special case,

$$\frac{1}{1 - \tau} = \{1, 1, \cdots\} = \text{summing operator.} \tag{16}$$

Notice that it requires no inverse operator, G^{-1}, as in Newton [1], to express $\{c^\nu\}$ in terms of $1/(1-c\tau)$. From the equation

$$\tau\{\xi_{\nu+1}\} = \{0, \xi_1, \xi_2, \cdots\} = \xi - \xi_0$$

we obtain the operational form of $E\xi$ or $\{\xi_{\nu+1}\}$:

$$E\xi = \{\xi_{\nu+1}\} = \frac{\xi - \xi_0}{\tau}. \tag{17}$$

Replacing ξ by $E\xi$ successively in (17) gives:

$$E^2\xi = \{\xi_{\nu+2}\} = \frac{\xi - \xi_0 - \xi_1\tau}{\tau^2}, \tag{18}$$

$$\vdots$$

$$E^m\xi = \{\xi_{\nu+m}\} = \frac{\xi - \xi_0 - \xi_1\tau - \xi_2\tau^2 - \cdots - \xi_{m-1}\tau^{m-1}}{\tau^m}. \tag{19}$$

Equations (17), (18), and (19) are analogous to equations like $\mathfrak{L}\{f'(t)\} = s\mathfrak{L}\{f(t)\} - f(0)$ in Laplace Transform theory, whose use in solving differential equations permits the automatic imposition of terminal conditions bypassing arbitrary constants; such equations in any operational calculus or transformation calculus might be called *terminal-condition-equations*. Equations (17), (18), and (19) permit the automatic imposition of terminal conditions when this convolution calculus for sequences is applied to the solution of recursion relations.

Equation (19) corresponds to (4) in Newton's note [1], and to equation (3) of [6, p. 26]. The right hand side of (19), however, is not a formula in a variable, τ but is an operational form in terms of the sequence τ, and it is not merely a correspondent of the sequence $E^m\xi$, but *equals* $E^m\xi$.

The algebraic derivative $D\{\xi_\nu\}$ of a sequence is defined as follows:

$$D\{\xi_\nu\} = \{(\nu+1)\xi_{\nu+1}\} = \{\xi_1, 2\xi_2, 3\xi_3, \cdots\}. \tag{20}$$

It follows with no great trouble that:

$$\begin{aligned} &\tau D\xi = \{\nu\xi_\nu\}, \qquad D\tau^n = n\tau^{n-1}, \qquad D[f(\tau)]^n = n[f(\tau)]^{n-1}Df(\tau), \\ &D(\xi_0 + \xi_1\tau + \xi_2\tau^2 + \cdots) = \xi_1 + 2\xi_2\tau + \cdots, \\ &D(a\xi + b\eta) = aD\xi + bD\eta, \qquad a, b \text{ numbers, } \xi, \eta \text{ sequences}, \\ &D(\xi\eta) = \xi D\eta + \eta D\xi, \qquad D\xi/\eta = \frac{\eta D\xi - \xi D\eta}{\eta^2}. \end{aligned} \tag{21}$$

The algebraic derivative may be used to obtain new operational forms. From (15) we obtain

$$D\frac{1}{1-c\tau} = D\{c^\nu\}, \qquad \frac{c}{(1-c\tau)^2} = \{(\nu+1)c^{\nu+1}\} = \{c, 2c^2, 3c^3, \cdots\}.$$

Again, starting with (16) and using (21), we obtain

$$\{1, 1, 1, \cdots\} = \frac{1}{1-\tau}, \qquad \{\nu\} = \frac{\tau}{(1-\tau)^2},$$

$$\{\nu^2\} = \frac{\tau+\tau^2}{(1-\tau)^3}, \qquad \{\nu^3\} = \frac{\tau+4\tau^2+\tau^3}{(1-\tau)^4}.$$

It is now possible to state the relationship between equations (1) and (3). As a generalization of (1), consider a linear nth order differential equation in which the coefficients are expandable in Maclaurin series:

$$\left(\sum_{k=0}^{\infty} a_{0k}t^k\right)v^{(n)} + \left(\sum_{k=0}^{\infty} a_{1k}t^k\right)v^{(n-1)} + \cdots + \left(\sum_{k=0}^{\infty} a_{nk}t^k\right)v = 0 \tag{22}$$

where v is a function of the continuous variable t, and $v^{(m)} = (d^m/dt^m)v$. Then (22) is converted to the corresponding *algebraic derivative* equation (23) below, by replacing t by τ, replacing v by ξ and $v^{(m)}$ by $D^m\xi$, $m = 1, 2, \cdots, n$, and interpreting the resulting terms as convolution products of sequences. This gives us

$$\left(\sum_{k=0}^{\infty} a_{0k}\tau^k\right)D^n\xi + \left(\sum_{k=0}^{\infty} a_{1k}\tau^k\right)D^{n-1}\xi + \cdots + \left(\sum_{k=0}^{\infty} a_{nk}\tau^k\right)\xi = 0. \tag{23}$$

In Newton's notation [1], an application of the operator G^{-1} to (22) produces (23) where

$$G^{-1}(uv) = (G^{-1}u)(G^{-1}v), \qquad u, v \text{ functions of } t,$$
$$G^{-1}(v^{(m)}) = D^m G^{-1}(v), \qquad G^{-1}(t^m) = \tau^m.$$

It remains to see how (23) is converted to an equivalent recursion relation and in particular how (3) is converted into (4). This is easy for the special case to which attention will now be restricted in which the coefficient of $D^m\xi$ in (23) is a polynomial of degree m or less. It follows from (20) that

$$\begin{aligned} D^2\xi &= \{(\nu+1)(\nu+2)\xi_{\nu+2}\} = \{1\cdot 2\xi_2, 2\cdot 3\xi_3, 3\cdot 4\xi_4, \cdots\}, \\ &\vdots \\ D^m\xi &= \{(\nu+1)(\nu+2)\cdots(\nu+m)\xi_{\nu+m}\}. \end{aligned} \tag{24}$$

And now, using (13), any one of the latter equations may be multiplied through by a power of τ not exceeding in degree the order of the derivative involved to yield:

$$\begin{aligned} \tau D\xi &= \{\nu\xi_\nu\}, \\ \tau^2 D^2\xi &= \{\nu(\nu-1)\xi_\nu\}, \\ &\vdots \\ \tau^m D^m\xi &= \{\nu^{(m)}\xi_\nu\}, \qquad \nu^{(m)} = \nu(\nu-1)(\nu-2)\cdots(\nu-m+1), \end{aligned} \tag{25}$$

$$\begin{aligned} \tau^n D^m\xi &= \{(\nu+1-n)(\nu+2-n)\cdots(\nu+m-n)\xi_{\nu+m-n}\} \\ &= \{(\nu+m-n)^{(m)}\xi_{\nu+m-n}\}, \qquad n \leqq m. \end{aligned} \tag{26}$$

Equation (24) corresponds to equation (6) in Newton [1] and (24) and (25) correspond to equations found in [6, p. 28]. These equations permit the quick conversion of a linear algebraic-derivative equation with polynomial coefficients —the degree of the polynomial not exceeding the order of the derivative it multiplies—to an equivalent recursion relation. For example, (3) is converted to (4) using (24), (25) and (26) as follows:

$$(1+\tau^2)D^2\xi + 2\tau D\xi - 2\xi = 0, \tag{3}$$
$$D^2\xi + \tau^2 D^2\xi + 2\tau D\xi - 2\xi = 0,$$
$$\{(\nu+1)(\nu+2)\xi_{\nu+2}\} + \{\nu(\nu-1)\xi_\nu\} + 2\{\nu\xi_\nu\} - 2\{\xi_\nu\} = 0,$$
$$\{(\nu+1)(\nu+2)\xi_{\nu+2} + (\nu^2+\nu-2)\xi_\nu\} = 0,$$
$$\{(\nu+2)[(\nu+1)\xi_{\nu+2} + (\nu-1)\xi_\nu]\} = 0,$$
$$\{(\nu+1)\xi_{\nu+2} + (\nu-1)\xi_\nu\} = 0, \quad \text{since } \nu \text{ varies on nonnegative integers.} \tag{4}$$

Then $\xi_2-\xi_0=0$, $2\xi_3+0\xi_1=0$, $3\xi_4+\xi_2=0$, $4\xi_5+2\xi_3=0$, $5\xi_6+3\xi_4=0$, $6\xi_7+4\xi_5=0$, so that ξ_0 and ξ_1 are arbitrary and

$$\xi = \xi_0 + \xi_1\tau + \xi_0\left(\tau^2 - \frac{\tau^4}{3} + \frac{\tau^5}{5} - \frac{\tau^8}{7} + \cdots\right), \tag{27}$$

$$\xi = \xi_0\{1, 0, 1, 0, -\tfrac{1}{3}, 0, \tfrac{1}{5}, 0, \cdots\} + \xi_1\{0, 1, 0, 0, 0, \cdots\}. \tag{28}$$

Now one might say that (27) solves (3) and that (28) solves (4); however both (27) and (28) solve both (3) and (4), and (3) and (4) appear as alternate and equivalent ways of defining a two parameter family of sequences.

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THE RAPID FACTORISATION TO HIGH ACCURACY OF QUARTIC EXPRESSIONS

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Consider the quartic expression $x^4+\alpha x^3+\beta x^2+\gamma x+\delta$ resolved into quadratic factors $(x^2+ax+b)(x^2+cx+d)$ so that

$$a + c = \alpha, \tag{1}$$

$$b + ac + d = \beta, \tag{2}$$

$$bc + ad = \gamma, \tag{3}$$

$$bd = \delta. \tag{4}$$

The solution of these equations to high accuracy is required and a method of iteration followed by improvement will be employed.

1. Iteration. Following on an initial choice of a (or c) or b (or d) an iterative process for the solution of equations (1) to (4) may be devised, and both of these have been given by Porter and Mack in [1]. The second alternative, namely iteration on b (or d) is preferable however, as suitable limits on the initial choice have been determined by the above-mentioned authors. These are

(a) If δ is positive (but not equal to γ^2/α^2) then b lies between 0 and $+\sqrt{\delta}$;

(b) If δ is negative, let $H=\frac{1}{4}\alpha^2+\frac{1}{4}(\gamma^2/\delta)-\beta$. Then if H is positive b lies between 0 and $+\sqrt{|\delta|}$, if H is negative b lies between 0 and $-\sqrt{|\delta|}$, and if $H=0$, $b=\sqrt{8}$;

(c) If δ is positive and equal to γ^2/α^2 then $b=\alpha=\sqrt{\delta}$.

It is to be noted that these are not necessarily the only possible values of b. It should also be noted that case (c) is exceptional.

Equations (1) to (4) are rearranged as

$$d = \delta/b, \tag{5}$$

$$a = (\alpha b - \gamma)/(b - d), \tag{6}$$

$$c = \alpha - a, \tag{7}$$

$$g = b + d + ac - \beta, \tag{8}$$

and the process consists of computing in turn d, a, and c from these equations corresponding to chosen values of b (within the range indicated by (a) or (b) above), and testing for $g=0$ in (8).

The convergence of the method may be improved by choosing successive values of b on the basis of the already computed values of g regarded as a function of b, i.e. by interpolating inversely for $g=0$. It is, however, necessary to ensure that the values of b are sufficiently near to the desired interpolate for polynomial representation of the inverse function to be adequate, i.e. for the process to converge.

A practical way of carrying out the inverse interpolation is to incorporate in the scheme Aitken's method of interpolation by a sequence of linear cross-means [2]. This may be done quite simply and with little extra writing by incorporating the Aitken scheme alongside the column for b and by working it right to left, rather than left to right as is usual. Furthermore, as it is required to interpolate for $g=0$, the usual column headed "parts" is not required, the column of values of g being used instead. The requirement, mentioned in the last paragraph, for the inverse interpolation to converge is easily checked by observing that the linear cross-means themselves converge from the start. Should this not happen in the early stages then it indicates that some relatively poor values of b have been taken, so that only the two rows with the smallest values (in modulus) of g should be retained and used to start a fresh Aitken array.

Consider the expression $x^4-3x^3+7x^2-9x-11$, $\alpha=-3$, $\beta=7$, $\gamma=-9$, $\delta=-11$. δ is negative; therefore, using condition (*b*) H is determined and seen to be negative, whereupon $0>b>-\sqrt{11}$. The following scheme is constructed

Aitken's method			b	d	$b-d$	a	c	g
			-1	11	-12	-1	-2	5
		-1.7692	-2	5.5	-7.5	-2	-1	-1.5
	-1.6950	-1.7218	-1.8	6.1111	-7.9111	-1.8202	-1.1798	-0.5414
-1.6942	-1.6942	-1.6944	-1.695	6.4897	-8.1847	-1.7209	-1.2791	-0.0041
			-1.6942	6.4927	-8.1869	-1.7201	-1.2799	$+0.0001$

Notes: The initial values of b lie roughly in the center of the permissible range. The first (linear) interpolate, -1.7692, is almost certainly not accurate and so $b=-1.8$ is used in the next row. The quadratic interpolate, $b=-1.6950$,

is derived from the pairs of values of b and g on the first three rows. The completion of the fourth row enables the values of -1.6944, -1.6942, -1.6942 to be computed in the Aitken array and, to four decimal places, the calculation is finished. As usual in Aitken's scheme, figures common to a column of iterates may be omitted from succeeding columns. The convergence of the cross-means is established from the start and so no difficulties arise in the inverse interpolation.

2. Improvement. The method of Porter and Mack is a first order iterative process, even when Aitken's scheme is incorporated. For high accuracy it is desirable to use a second order process so that the required number of figures are obtained without an excessive number of stages. One such process is the generalisation, given originally by Bairstow [3], of the Newton-Raphson method to quadratic factors, and this may be applied to improve *any* approximation. In the present case, however, the coefficients as given by Porter and Mack's method are known to be correct to a certain number of significant figures, and this fact enables us to derive a special simplified procedure for carrying out the Bairstow process, leading to double accuracy. Suppose the values a', b', c' and d' of the parameters have been computed as in Section 1, and that the required accurate values are given by $a=a'+\Delta a$, $b=b'+\Delta b$, $c=c'+\Delta c$, and $d=d'+\Delta d$, where the increments may, of course, be either positive or negative. Substitution into equations (1) to (4) gives, on retaining first order small quantities only,

$$\Delta a + \Delta c = \alpha - (a' + c'), \tag{9}$$

$$c'\Delta a + \Delta b + a'\Delta c + \Delta d = \beta - (b' + d' + a'c'), \tag{10}$$

$$d'\Delta a + c'\Delta b + b'\Delta c + a'\Delta d = \gamma - (b'c' + a'd'), \tag{11}$$

$$d'\Delta b + b'\Delta d = \delta - b'd', \tag{12}$$

and these are linear in the increments. Consider the right hand sides of the equations. The first is exactly zero since c' was found by subtraction of a' from α. The second equals $-g$, and the remaining two depend only on the rounding off errors in the calculation of d' and a'. It is, however, known that one stage of this second order improvement will yield double the number of significant figures, and accordingly, the last line of the iterative scheme is recalculated to twice the accuracy. These new values of a', b', c', and d' now ensure that the right hand sides of equations (11) and (12), as well as of equation (9), are zero to the required accuracy. The equations are now readily solvable with solution

$$\Delta a = -\Delta c, \qquad \Delta d = -(d'/b')\Delta b,$$

$$\Delta b\{1 - (d'/b')\} + \Delta c(a' - c') = -g,$$

$$\Delta c = -\Delta b(c'b' - a'd')/\{b'(b' - d')\}.$$

The algebraic solution of the last two equations is trivial, leading to explicit expressions for the increments, but the computation is reduced if the formulae are kept in the above form. The coefficients of the increments in these expressions need only be worked to roughly one more significant figure than is known for g.

It is useful to note that $b'-d'$, occurring in the last formula, is already available in the repeated row and that the quantity d'/b' occurs twice in the coefficients of the increments.

Reworking the last row of the example to 8 decimal places gives

b	d	$b-d$	a	c	g
−1.6942	6.49273993	−8.18693993	−1.72012988	−1.27987012	0.00008277

The equations for the increments are accordingly

$$\Delta a = -\Delta c, \qquad \Delta d = 3.8323\Delta b,$$
$$4.8323\Delta b - 0.4403\Delta c = -0.00008277,$$
$$\Delta c = -0.9615\Delta b,$$

with solution

$$\Delta b = -0.00001575, \qquad \Delta d = -0.00006035, \qquad \Delta c = 0.00001514.$$

Hence $a=-1.72014502$, $b=-1.69421575$, $c=-1.27985498$, $d=6.49267957$ and the product of the quadratic factors is

$$x^4 - 3x^3 + 6.99999999x^2 - 8.99999996x - 10.99999999.$$

With practice, using a suitable desk calculator, the iteration to about four significant figures can be carried out in fifteen to twenty minutes, and the improvement to double accuracy in about a further five minutes.

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CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

ON THE DIFFERENTIABILITY OF A CERTAIN WELL-KNOWN FUNCTION*

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The function given by $f(x)=0$, if x is irrational or $x=0$, and $1/q$, if $x=p/q$, $(p, q)=1$ is a well-known example of a function which is continuous on a dense set (the irrationals and zero) and also discontinuous on a dense set (the nonzero rationals). Another question which may be asked is: "Where is $f(x)$ differentiable?" Since $f(x)=f(x+1)$ if $x\neq 0$ and $x\neq -1$, it suffices to consider the interval $0\leqq x\leqq 1$. Furthermore, since f is not continuous on the set of nonzero rationals, the only possible points of differentiability are the irrationals and zero. The object of this note is to give an elementary proof that the function is not differentiable at these points.

If $x=0$ and $h\neq 0$, $[f(x+h)-f(x)]/h=f(h)/h$. Let $\{h_i\}$ be a sequence of irrationals having zero as limit. Then $\lim_{i\to\infty} f(h_i)/h_i = 0$. Now let $h_i = 1/i$, $i=1, 2, \cdots$. Then $f(h_i)/h_i=(1/i)/(1/i)=1$. Therefore $\lim_{h\to 0} f(h)/h$ does not exist and f is nondifferentiable at $x=0$.

If x is an irrational number $0<x<1$ and if $h\neq 0$, $[f(x+h)-f(x)]/h = [f(x+h)]/h$. If $\{h_i\}$ is a sequence of real numbers having zero as limit such that $x+h_i$ is irrational for each i, then $\lim_{i\to\infty} [f(x+h_i)]/h_i=0$. Let the decimal representation of x be $.a_1a_2\cdots a_n\cdots$. Choose $h_i=.a_1a_2\cdots a_i-x$. Since $x\neq 0$, $a_i\neq 0$ for some i. Let N be the least integer such that $a_N\neq 0$. Then $f(x+h_i) = f(.a_1a_2\cdots a_i)\geqq 10^{-i}$ for all $i\geqq N$, and $|h_i|\leqq 10^{-i}$. Hence, $|[f(x+h_i)]/h_i|\geqq 1$ for all $i\geqq N$. Therefore $\lim_{h\to 0}[f(x+h)]/h$ does not exist and f is not differentiable.

ANOTHER CHARACTERIZATION OF CIRCLES

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Circles are typical examples of Jordan curves and there have already been many characterizations of circles. This little note picks out a property, formally replacing *a path* by *a circle* in the statement of the Jordan curve theorem, which, as we shall see, also characterizes a Jordan curve to be a circle.

THEOREM. *A Jordan curve is a circle if and only if through any two points lying both in its interior or both in its exterior, there is a circle passing through these two points and not intersecting the curve.*

* Sometimes called the ruler function.

Proof. The *only if* part is almost obvious if we map the curve and the given two interior (or exterior) points to X-axis and two points of upper plane respectively by means of a suitable Möbius Transformation.

For the *if* part, let C be such a Jordan curve and D its diameter. There exist four sequences of points, $\{P_n\}$, $\{Q_n\}$, $\{\overline{P}_n\}$, and $\{\overline{Q}_n\}$ with the properties:

(i) P_n, Q_n are interior points and $\overline{P}_n$, $\overline{Q}_n$ are exterior points,

(ii) $\rho(P_n, Q_n) > D - 1/n$,

(iii) $\overline{P}_n$, P_n, Q_n, $\overline{Q}_n$ lie on a straight line and with the order $\overline{P}_n$, P_n, Q_n, $\overline{Q}_n$.

(iv) $\rho(\overline{P}_n, P_n) = \rho(Q_n, \overline{Q}_n) = 2/n$.

There correspond, by assumption, two sequences of circles C_n and $\overline{C}_n$ where C_n passes through P_n, Q_n and lies wholly in the interior of C, while $\overline{C}_n$ passes through $\overline{P}_n$, $\overline{Q}_n$ and lies wholly in the exterior of C.

By the Weierstrass Theorem, there is an increasing set of indices k_n such that both $\lim_{n\to\infty} P_{k_n}$ and $\lim_{n\to\infty} Q_{k_n}$ exist. Then, if we let

$$\lim_{n\to\infty} P_{k_n} = \lim_{n\to\infty} \overline{P}_{k_n} = P, \qquad \lim_{n\to\infty} Q_{k_n} = \lim_{n\to\infty} \overline{Q}_{k_n} = Q$$

we necessarily have

$$\lim_{n\to\infty} C_{k_n} = \lim_{n\to\infty} \overline{C}_{k_n} = C,$$

the common limit circle with segment $\overline{PQ}$ as its diameter.

COROLLARY. *Circle-preserving mappings are continuous.*

Editor's note for *Another Characterization of Circles*, by Wu-Teh Hsiang.

This paper, written by an eleventh-grade student in Taipei, Formosa, appears to have unusual potential interest for readers in this country. Because of the nature of the material and its authorship it is printed as it was received, without editing.

QUASI-TRIGONOMETRY*

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The student in his initial introduction to trigonometry is quite happy to accept the trigonometric functions defined with reference to a system of rectangular coordinates. Later when he encounters hyperbolic functions he is often curious as to a possible connection with the circular functions.

In this note it is our purpose to consider a trigonometry based, not on the right angle, but on the angle $\frac{1}{3}\pi$ and later on an arbitrary angle λ. In the case of the arbitrary angle λ, we shall define quasi-elliptic, -parabolic, and -hyperbolic trigonometries. Phillips [1] has proposed a project of similar nature and has worked out rudimentary definitions and tables. However, it is our desire to

* Prepared in a National Science Foundation Undergraduate Research Participation Program at Colorado State University by Mr. Strand under the directionof Professor Stein.

forego these basic aspects in lieu of investigating more general relations and parallelisms to conventional trigonometry.

1. Definitions. We define our quasi-functions* with the aid of the oblique coordinate system shown in Figure 1 where $0 \leqq \lambda < 2\pi$. The definitions, which are analogous to those of conventional trigonometric functions, use n as the distance from the origin to the point $P(x, y)$ with x as the abscissa and y as the ordinate. Thus, the functions are defined as

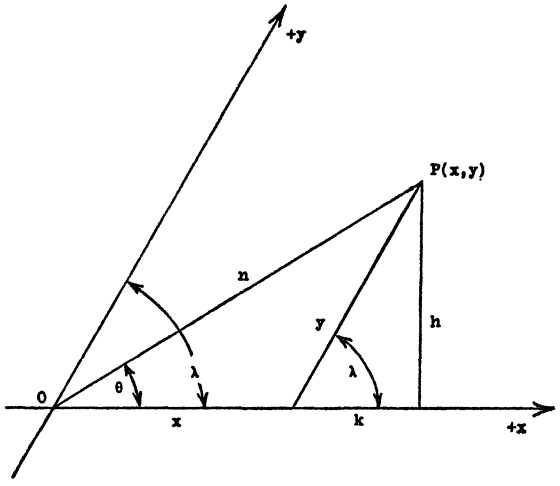


FIG. 1

$$\text{(1)} \qquad \begin{aligned} \operatorname{nis}\theta &= \frac{y}{n}, & \operatorname{conis}\theta &= \frac{x}{n}, \\ \operatorname{nat}\theta &= \frac{y}{x}, & \operatorname{conat}\theta &= \frac{x}{y}, \\ \operatorname{ces}\theta &= \frac{n}{x}, \quad \text{and} & \operatorname{coces}\theta &= \frac{n}{y}. \end{aligned}$$

2. Basic identities. To establish the basic identities we consider the case where $\lambda = \frac{1}{3}\pi$. We must augment the defining triangle in Figure 1 with h, the perpendicular distance from the x-axis to the point $P(x, y)$, and k, the distance from the abscissa to the intersection of h and the x-axis. Our identities are derived from the conventional trigonometric identities. Therefore, we need to express our quasi-functions in terms of $\sin\theta$ and $\cos\theta$.

* Our terminology is essentially the same as that proposed by Phillips in the original presentation of his paper at the AAPT meeting, Iowa City, 1952.

If $h \neq 0$, we may write

$$\text{nis } \theta = \frac{y}{n} = \frac{y}{h} \cdot \frac{h}{n} = \csc \lambda \sin \theta, \tag{2}$$

or

$$\sin \theta = \tfrac{1}{2}\sqrt{3} \text{ nis } \theta. \tag{2'}$$

But this result is also true even if $h = 0$. Similarly, if $y \neq 0$, we may write

$$\cos \theta = \frac{x+k}{n} = \frac{x}{n} + \frac{k}{n} = \frac{x}{n} + \frac{k}{y} \cdot \frac{y}{n} = \text{conis } \theta + \cos \lambda \text{ nis } \theta, \tag{3}$$

or

$$\cos \theta = \text{conis } \theta + \tfrac{1}{2} \text{ nis } \theta. \tag{3'}$$

But this result likewise is true if $y = 0$.

By squaring both sides of (2') and (3'), and adding corresponding members we obtain

$$1 = \sin^2 \theta + \cos^2 \theta = \tfrac{3}{4} \text{nis}^2 \theta + \text{conis}^2 \theta + \text{nis } \theta \text{ conis } \theta + \tfrac{1}{4} \text{nis}^2 \theta,$$

or

$$\text{nis}^2 \theta + \text{conis}^2 \theta = 1 - \text{nis } \theta \text{ conis } \theta. \tag{4}$$

The reader can easily verify that

$$\text{nat } \theta = \frac{\text{nis } \theta}{\text{conis } \theta}, \qquad \text{ces } \theta = \frac{1}{\text{conis } \theta}, \text{ etc.,}$$

from (1). By dividing (4) by $\text{nis}^2 \theta$ we obtain

$$1 + \text{conat}^2 \theta = \text{coces}^2 \theta - \text{conat } \theta. \tag{5}$$

By dividing (4) by $\text{conis}^2 \theta$ we obtain

$$\text{nat}^2 \theta + 1 = \text{ces}^2 \theta - \text{nat } \theta. \tag{6}$$

3. Composite angle formulae. The formulae for the functions of composite angles are derived from those of conventional trigonometry. Thus, by substituting from (2') and (3') into

$$\sin(\theta + \Phi) = \sin \theta \cos \Phi + \cos \theta \sin \Phi$$

we obtain, after simplification,

$$\text{nis}(\theta + \Phi) = \text{nis } \theta \text{ conis } \Phi + \text{conis } \theta \text{ nis } \Phi + \text{nis } \theta \text{ nis } \Phi. \tag{7}$$

In a similar manner we obtain

$$\text{nis}(\theta - \Phi) = \text{nis } \theta \text{ conis } \Phi - \text{conis } \theta \text{ nis } \Phi, \tag{8}$$

$$\text{conis}(\theta + \Phi) = \text{conis}\,\theta\,\text{conis}\,\Phi - \text{nis}\,\theta\,\text{nis}\,\Phi, \tag{9}$$

$$\text{conis}(\theta - \Phi) = \text{conis}\,\theta\,\text{conis}\,\Phi + \text{nis}\,\theta\,\text{nis}\,\Phi + \text{conis}\,\theta\,\text{nis}\,\Phi. \tag{10}$$

By letting $\theta=\Phi$ in (7) and (9), we obtain the double angle formulae

$$\text{nis}\,2\theta = 2\,\text{nis}\,\theta\,\text{conis}\,\theta + \text{nis}^2\,\theta, \tag{11}$$

$$\text{conis}\,2\theta = \text{conis}^2\,\theta - \text{nis}^2\,\theta. \tag{12}$$

The reader can easily investigate other ramifications of the quasi-functions such as functions of negative angles and reduction formulae. However, it is our interest to proceed at this time to the general case.

4. An arbitrary angle. For an arbitrary angle λ, $0 \leqq \lambda < 2\pi$, between the oblique coordinates, the functions are defined the same as in (1). However, it is easily shown that (4) becomes

$$1 = \text{conis}^2\,\theta + (2\cos\lambda)\,\text{conis}\,\theta\,\text{nis}\,\theta + \text{nis}^2\,\theta. \tag{13}$$

In (13) we make the substitutions $X=\text{conis}\,\theta$ and $Y=\text{nis}\,\theta$, where X and Y are the rectangular coordinates of the point P. Thus, (13) becomes

$$1 = X^2 + (2\cos\lambda)XY + Y^2. \tag{14}$$

We shall use (14) to define various types of quasi-trigonometries. If (14) represents an ellipse, we shall call our quasi-functions elliptic in nature, and the resulting trigonometry will be called quasi-elliptic trigonometry. Similarly, if (14) represents a parabola or a hyperbola we shall call our functions, respectively, quasi-parabolic and -hyperbolic in nature.

It is obvious from (2), (3), and (14) that when $\lambda=\frac{1}{2}\pi$ we have at once conventional trigonometry.

The discriminant of (14) is $D=4\cos^2\lambda-4$. Thus, in order for (14) to be a parabola, we must have $D=0$, or $\cos^2\lambda=1$, which implies that $\lambda=0$ or $\lambda=\pi$. For each of these values of λ, (14) reduces to two parallel lines which can be considered as a degenerate parabola.

To have an ellipse we must have $D<0$, or $\cos^2\lambda<1$, or, using the positive square root only, $\cos\lambda<1$, which implies that $0<\lambda\leqq\frac{1}{2}\pi$.

In the case of the hyperbola we need $D>0$, or $\cos^2\lambda>1$, again using only the positive square root; this leads us to the conclusion that λ is imaginary.

In the latter case the most propitious results will be obtained if we regard λ of the form $\lambda=i\Phi$, where Φ is real. Similarly, it is most favorable to define new quasi-hyperbolic functions in terms of the argument $i\theta$ where θ is real.

Thus using (2) and (3) we have

$$\text{hnis}\,\theta = \text{nis}\,i\theta = \csc\lambda\sin i\theta = \text{csch}\,\Phi\sinh\theta. \tag{15}$$

Likewise we have

$$\text{hconis}\,\theta = \text{conis}\,i\theta = \cos i\theta - \cos\lambda\,\text{nis}\,i\theta = \cosh\theta - \coth\Phi\sinh\theta. \tag{16}$$

By solving (15) and (16) for cosh θ and sinh θ, we obtain our basic quasi-hyperbolic identity

(17) $\cosh^2\theta - \sinh^2\theta = 1 = \text{hconis}^2\,\theta + (2\cosh\Phi)\,\text{hnis}\,\theta\,\text{hconis}\,\theta + \text{hnis}^2\,\theta.$

We note the similarity between this equation and (13). Further identities involving quasi-hyperbolic functions can be found by proceeding in the manner used in Section 2. Although the similarity between (13) and (17) will cause the corresponding identities to be very much alike, we must remember the different nature of (17). In (17) the arguments are imaginary, whereas in (13) we were concerned with real arguments.

5. Conclusions. Conventional trigonometry can be looked on as a special case of what we choose to call *quasi-trigonometry*. If the angle λ in Figure 1 is $\frac{1}{2}\pi$, conventional trigonometry results. By allowing λ in (14) to assume other real values, we obtain what we might call elliptic functions and parabolic functions in analogy to the circular functions of conventional trigonometry. These functions we prefer to call quasi-elliptic and quasi-parabolic functions. By allowing λ to assume imaginary values, we obtain what we call quasi-hyperbolic functions. These show some connection between the circular and hyperbolic functions with which the student is already familiar.

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POSTULATES FOR THE POSITIVE INTEGERS

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Among the postulates given in [1] for the system J^+ of all positive integers are the following.

(i) *It is closed under uniquely defined binary operations of addition and multiplication, which are associative, commutative, and distributive.*

(iii) *Furthermore, the following cancellation laws hold in* J^+:

(17) *If* $m + x = n + x$, *then* $m = n$.

(18) *If* $mx = nx$, *then* $m = n$.

(iv) *Again, for any two elements* m *and* n *of* J^+ *exactly one of the following alternatives holds:* $m=n$, *or* $m+x=n$ *has a solution* x *in* J^+, *or* $m=n+y$ *has a solution* y *in* J^+.

Further we are told in [1], "Note in particular that the cancellation law for addition is now assumed as a postulate, since it can no longer be deduced from the possibility of subtraction, as it was in Section 2 of Chapter I."

Below we derive postulate (iii) from postulates (i) and (iv).

Proof of (17). Suppose that $m+x=n+x$. If there is y in J^+ such that $m+y=n$ then, by (i), $(m+x)+y=n+x$. But $m+x=n+x$. Thus, by (iv), we have a contradiction. Hence there is no y in J^+ such that $m+y=n$. In a similar way we can show that there is no y in J^+ such that $m=n+y$. Hence $m=n$, by (iv).

Proof of (18). Suppose that $mx=nx$. If there is y in J^+ such that $m+y=n$ then, by (i), $mx+yx=nx$. But $mx=nx$. Also yx is in J^+, by (i). Thus, by (iv), we have a contradiction.

The rest of the proof is as in that of (17).

This concludes the derivation of postulate (iii).

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MATHEMATICAL EDUCATION NOTES

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THE TEACHING OF MATHEMATICS FOR MANAGEMENT CAREERS

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When nonmathematicians discover the value of mathematics in their own specialty, they often request that the classical curriculum be curtailed to make room for *ad hoc* combinations of special mathematical techniques to meet their particular need.

No one would dare suggest such a thing at the universities of Bologna, Salamanca or Paris, where mathematics is taught as the Queen of Science.

Sometime during the twelfth century, however, France cut down on foreign aid, and many English students at the University of Paris had to go home. They gathered at Oxford and formed a corporation to continue their studies. Part of the cultural loss that must accompany such a change-over was the demoting of mathematics from the Queen to the handmaid of science.

This Cinderella-like humility still afflicts some English-speaking mathematicians. Let me quote from the April 1957 issue of the American Mathematical Monthly. The article is, *If this be Treason . . .* by R. P. Boas, Jr., of Northwestern University.

"If I had to name one trait that more than any other is characteristic of professional mathematicians," he writes, "I should say that it is their willingness, even eagerness, to admit that they are wrong. A sure way to make an impression on the mathematical community is to come forward and declare, 'You are doing such-and-such all wrong and you should do it *this* way.' Then everybody says,

'Yes, how clever you are,' and adopts your method. . . .

"It would be pleasant to teach only the new and exciting kinds of mathematics; it would be comforting to teach only the really useful kinds. The traditional topics are some of the topics that once were either new and exciting, or useful. They have persisted partly by mere inertia—and that is bad—but partly because they still serve a real purpose, even if it is not their ostensible purpose. Let us keep this in mind when we are revising the curriculum."

The mathematics of finance is a refreshing exception to this revisionist approach. There is a story about a naughty but charming little girl who wanted to get into heaven. She ran around the golden walls tossing an apple in the air until it fell inside. Then she asked permission to go in and get her apple. Once she was inside she was so nice to everybody that they let her stay.

The mathematics of finance threw the apple of Arabic numerals and algebra into the university campus where it has grown into a magnificent tree. But the little girl, herself, continued to be naughty. She exists for a number of conflicting purposes. One, is to tell management what is happening. Two, is to tell the banker that the business is doing very well. Three, is to tell the tax collector that the business is not doing well at all. Four, is to apportion investment capital in harmony with One, Two, and Three, above.

I believe this problem would have been minimized if the "mathematics of finance" had been taught *as mathematics*, and the presentation of the resulting information had been taught separately.

In our society where the poorest citizen can be a stockholder in the largest corporation, the mathematical techniques used in finance are of interest to everyone, as are the mathematical techniques for sound strategy, scientific planning, and for decision-making under uncertainty.

With added examples and exercises provided by the instructor, the present texts on *finite mathematics* could be used to offer modern mathematics to those business administration, engineering, and physical science students who want to prepare for management roles in their chosen field.

The basic techniques are: mathematical statistics, including decision theory and Monte Carlo methods; probability, logic, set theory, and queuing theory; matrix theory and linear algebra.

These subjects are now offered as *mathematics for students of the social and biological sciences.* Why not offer a general program of modern mathematics, taught *as mathematics?* Its principal supporters would be science, engineering, and business administration undergraduates, but every student on the campus could be urged to participate.

In this way liberal arts students would know about the existence of objective techniques for planning and making decisions. The resulting increase in the number of students taking mathematics might well support an even broader program. After all, St. Patrick and St. Augustine were liberal arts majors, yet the first converted a whole island with a trivial example from the theory of sets, and the second published the first paper on transfinite numbers.

THE FIRST TWO INTERNATIONAL MATHEMATICAL OLYMPIADS FOR STUDENTS OF COMMUNIST COUNTRIES*

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Mass problem-solving contests known as *mathematical olympiads* are held yearly throughout the Soviet Union and in the other Communist countries [1, 2, 3, 4, 10]. These contests have proved to be of great value, especially in raising the level of mathematical culture of large numbers of secondary school students and in discovering talented youngsters in this discipline.

In 1959 the Rumanian Society of Mathematical and Physical Sciences (SSMF) took the initiative in organizing the first such mathematical competition on an international scale. The Society was supported in this undertaking by the Ministry of Education of the Rumanian People's Republic.

Invitations to participate in the International Mathematical Olympiad were sent to the mathematical societies and the ministries of education of all European Communist countries. Each country was invited to send a delegation of 8 recent graduates from its secondary schools accompanied by a mathematics educator as its leader. The leaders were requested to prepare problems for submission to the organizational committee of the olympiad.

The First International Mathematical Olympiad [6, 7, 11] for students of the Communist countries took place in Orasul Stalin in Rumania during the period July 23–July 31, 1959. All European Communist countries except Albania were represented. With the exception of the Soviet Union, which sent only 4 students, all delegations consisted of 8 members each. Eight girls were among the 52 participants.

The delegations were composed of students who rated highest in their country's mathematical olympiad of 1959. As no mathematical olympiad had been held in East Germany up to that time, its delegation was composed of high school graduates who had demonstrated excellent performance on their final examinations.

The leaders of the delegations formed the Committee of the Olympiad, chaired by Professor G. D. Simionescu of Bucharest. From material submitted by the delegation leaders, the Committee selected the problems for the competition, which were then translated into seven languages. Since the problems were of varying degrees of difficulty, each problem was assigned a maximum point value which could be awarded for its perfect solution.

The full content of the two written contests, conducted on July 24 and 25, six problems in all, is given below. The participants were given three hours each day. The country which submitted the problem and the maximum score for its solution appear in parentheses.

* This is part of a Survey of Recent East European Literature in Elementary, Secondary, and College Mathematics, a project conducted by A. L. Putnam and I. Wirszup, Department of Mathematics, The University of Chicago, under a grant from the National Science Foundation.

First day of the Olympiad (Arithmetic, algebra, trigonometry)

1. Prove that the fraction $(21n+4)/(14n+3)$ is irreducible for every natural number n. (Poland, 5 points)

2. For what real values of x is

$$\text{a. } \sqrt{\{x+\sqrt{(2x-1)}\}}+\sqrt{\{x-\sqrt{(2x-1)}\}}=\sqrt{2}$$

$$\text{b. } \sqrt{\{x+\sqrt{(2x-1)}\}}+\sqrt{\{x-\sqrt{(2x-1)}\}}=1$$

$$\text{c. } \sqrt{\{x+\sqrt{(2x-1)}\}}+\sqrt{\{x-\sqrt{(2x-1)}\}}=2$$

where only nonnegative values are admitted for the square roots? (Rumania, 8 points)

3. Let x denote an angle (i.e., a real number); let a, b, c denote three arbitrary real numbers. Consider the quadratic equation in $\cos x$

$$a\cos^2 x+b\cos x+c=0.$$

Using the numbers a, b, and c, form a quadratic equation in $\cos 2x$ whose roots correspond to the roots of the original equation. Compare the given and formed equations for $a=4$, $b=2$, $c=-1$. (Hungary, 7 points)

Second day of the Olympiad (Geometry)

1. Construct a right triangle, its hypotenuse c being given, if it is known that the median drawn to the hypotenuse is the geometric mean of the two remaining sides of the triangle. (Hungary, 5 points)

2. An arbitrary point M is selected in the interior of the segment $\overline{AB}$. The squares $AMCD$ and $MBEF$ are constructed on the same side of $\overline{AB}$ with the segments $\overline{AM}$ and $\overline{MB}$ as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at the point M and also at some other point N. Let N' denote the point of intersection of the straight lines AF and BC.

a) Prove that the points N and N' coincide.

b) Prove that the straight lines MN pass through some fixed point S independent of the choice of the point M. (By PQ is meant the straight line passing through the points P and Q, while $\overline{PQ}$ denotes the segment from P to Q.)

c) Find the locus of the midpoints of the segments $\overline{PQ}$ as the point M varies between A and B. (Rumania, 8 points)

3. Two given planes P and Q intersect along the straight line p. The point A is given in the plane P, and the point C in the plane Q; none of these points lies on the straight line p.

Find an isosceles trapezoid $ABCD$ (having AB parallel to CD) in which a circle can be inscribed, with vertices B and D lying in the planes P and Q respectively. (Czechoslovakia, 7 points)

The solutions were read and graded by the Committee of the Olympiad. The total possible score for the correct solution of all problems was 40 points. One student from Czechoslovakia achieved this highest score. A student from Rumania took second place, and a student from Hungary third place.

No rules had been established beforehand for determining the relative rank of the participating delegations as units. Nevertheless, an unofficial final standing was drawn up, in which Rumania and Hungary were awarded first and second places, respectively. In order to include the Soviet delegation, which consisted of only 4 participants, the number of points its members scored (111) was doubled. As a result, the Soviet Union took third place, followed by Czechoslovakia, Bulgaria, Poland, and East Germany.

A rather detailed summary of the results of the First Olympiad is given in the following table.

Country	Total points achieved (possible 320)	Prizes			Honorable Mention (from 33 to 25 points)
		I 37 or more points	II 36 points	III 35 or 34 points	
Rumania	249	1	2	2	1
Hungary	233	1	1	2	1
Soviet Union	222 (111×2)	—	—	1	2
Czechoslovakia	192	1	—	—	4
Bulgaria	131	—	—	—	1
Poland	122	—	—	—	1
East Germany	40	—	—	—	—

The Second International Mathematical Olympiad [8, 11] also took place in Rumania, in the mountain resort village of Sinaia, July 18–July 25, 1960. This time, however, only five countries sent delegations; each consisted of 8 students accompanied by two leaders. Neither the Soviet Union nor Poland were represented at this olympiad.

Four problems instead of three were given in the competition on geometry. The students were allowed three hours on the first day and four hours on the second.

The problems used in the Second International Olympiad, with the name of the country which submitted each, are given below. The reports of the olympiad do not indicate the point value assigned to each problem.

First day of the Olympiad (Arithmetic, algebra, trigonometry)

1. Find all three-digit numbers, each of which upon division by 11 gives a quotient equal to the sum of the squares of the numerical values of the digits of the dividend. (Bulgaria)

2. Solve the inequality

$$\frac{4x^2}{\{1-\sqrt{(1+2x)}\}^2} < 2x + 9. \qquad \text{(Hungary)}$$

3. Given the right triangle ABC with hypotenuse BC subdivided into n congruent parts, where n is an odd number. Let α denote the angle whose vertex is point A and which is subtended by the congruent segment containing the midpoint of the hypotenuse of the given triangle. Denote the lengths of the altitude and of the hypotenuse by h and a respectively. Prove that

$$\tan \alpha = \frac{4nh}{(n^2-1)a}. \qquad \text{(Rumania)}$$

Second day of the Olympiad (Geometry)

1. Construct the triangle ABC, given the lengths h_a and h_b of its altitudes drawn from the vertices A and B, respectively, and the length m_a of the median drawn from the vertex A. (Hungary)

2. Given the cube $ABCDA'B'C'D'$.

a) Determine the locus of the midpoints of the segments $\overline{xy}$, where x is an arbitrary point of the segment $\overline{AC}$, and y an arbitrary point of the segment $\overline{B'D'}$.

b) Determine the locus of the points z, lying in the interior of the segments $\overline{xy}$ (defined in part a of this problem) and satisfying the relation

$$|\overline{zy}| = 2|\overline{zx}|. \qquad \text{(Czechoslovakia)}$$

3. Given a right circular cone and the sphere inscribed in it. About this sphere is circumscribed a right circular cylinder, one of whose bases lies in the plane of the base of the given cone. Let v_1 denote the volume of the cone, and v_2 the volume of the cylinder.

a) Prove that the equality $v_1 = v_2$ cannot take place.

b) Indicate the minimum value k, for which the equality $v_1 = kv_2$ holds. For this case construct the angle at the vertex of an axial section of the cone. (Bulgaria)

4. Given an isosceles trapezoid with bases a, b and altitude h.

a) By construction, find a point P lying on the axis of symmetry of the trapezoid such that each of the nonparallel sides of the trapezoid subtends a right angle with vertex P.

b) Find an expression in terms of a, b, and h for the distance between the point P and one of the bases of the trapezoid.

c) Determine under what conditions the construction of the point P is possible. (Consider all cases which may take place.) (East Germany)

The results of the Second Olympiad are summarized in the table below.

Country	Total points achieved	Prizes			Honorable Mention (32–29 points)
		I more than 40 points	II 40–37 points	III 36–33 points	
Hungary	248	2	2	—	1
Czechoslovakia	257	1	1	2	2
Rumania	248	1	1	1	1
Bulgaria	175	—	—	1	2
East Germany	38	—	—	—	1

The international olympiads made it possible to compare the achievements of the best students of one country with those of other countries. In addition, the delegation leaders had an opportunity to exchange ideas concerning the work of the olympiad committees in the various countries, and more generally to discuss questions of the teaching of mathematics and of the school curriculum.

Just as in other national or local olympiads for students of the higher grades, the problems used in the international olympiads were based on the material of the secondary school mathematics courses which are planned along similar lines in all Communist countries. The solutions require of the participant not only a mastery of all branches of elementary mathematics, but also ability and considerable training in independent mathematical thinking and a wide experience with similar types of intriguing and often tricky problems.

Although no general conclusions should be drawn from the somewhat surprising results of the first two international olympiads, they do offer indications of the high level of mathematical training of some secondary school students in Rumania, Hungary and Czechoslovakia. In Rumania, for example, there is a long tradition of organizing mathematical contests. During the prewar years the number of participants was rather small. Since that time the contests have expanded considerably in size and importance. The 1959 Rumanian Mathematical Olympiad attracted more than 40,000 pupils of grades 5 through 11.

East Germany, having had no experience in contests of this type, finished far behind the other countries in the competitions. As a matter of fact, no German student received more than 10 points in the first olympiad.

The German delegation to the First International Olympiad was assigned the task of gathering information and ideas concerning the organization, content and conduct of mathematical olympiads in the other countries. It made a thor-

ough study, especially of the Rumanian olympiads. The resulting report by the leader of the German delegation [6] suggested long-range measures for raising student interest in mathematics and for expanding extracurricular activities in East Germany. In particular, some procedures were proposed leading to the organization of mathematical contests based on the model of other Communist countries.

The recommendations made by the leader of the East German delegation to the First International Mathematical Olympiad have found an echo in a resolution passed by the Politbureau of the Central Committee of the Communist Party of East Germany (SED) on May 17, 1960. The resolution, entitled *On the Improvement and Further Development of Polytechnical Training in Secondary Schools*, explicitly stressed the importance of mathematical extracurricular activities and the organization of mathematical olympiads [5].

As a result, mathematical contests were soon organized in various parts of East Germany. Among these the First Olympiad of Young Mathematicians in Leipzig [5] took place on June 13, 1960. The First East Berlin Mathematical Olympiad was organized in 1961 by the Ministry of National Education, with the participation of representatives of Humboldt University. According to a report [9], 7240 pupils of grades 7 through 12 took part in it. This number represented 32% of the 7th through 12th grade pupils in the East Berlin schools.

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PROFESSIONAL CONTENT IN SOVIET TEACHER-TRAINING CURRICULA IN MATHEMATICS

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In an earlier paper, Clarence B. Lindquist outlined the major features of one teacher-training program in mathematics currently in operation in Soviet pedagogical institutes [1]. Similar outlines have appeared in works by Medlin and others [2]. Lindquist pointed out that the Ministry of Higher and Secondary Special Education exercises supervisory control over teacher education in all fields. Uniformity of programs of instruction throughout the Soviet Union is achieved through promulgation of "official" study plans and syllabi by the various Republics' ministries of education.

In view of the uniformity of teacher-training programs achieved in the Soviet Union by these means, detailed analyses of curricula in specific areas are possible. The purpose of this paper is to provide such an analysis in the area of mathematics with particular attention given professional subject matter.

Two official curricula are available to the Soviet pedagogical student preparing to teach mathematics—the Mathematics-Technical Drawing curriculum and the Mathematics-Physics curriculum. Both programs are of five year duration. The following mathematics and professional courses are found in both curricula.*

Mathematics	Total Class Hours	Professional	Total Class Hours
Mathematical Analysis	408	Pedagogy	120
Analytic Geometry	172	History of Pedagogy	72
Projective and Descriptive Geometry	110	Psychology	84
Foundations of Geometry	64	School Hygiene	36
Higher Algebra	192	Elementary Mathematics	400
Foundations of Arithmetic	36	Methods of Teaching Mathematics	194
Real Variable	50	Special Practical Training in Mathematics	92
Complex Variable	54	Visual Aids	36
Seminar in Mathematics	84	Special Preparation	48
Theory of Numbers	48	Practice Teaching	16 weeks
Elective in Mathematics	54		

The Mathematics-Technical Drawing curriculum includes courses in differential geometry and the history of mathematics available to the Mathematics-Physics

* Slight variations occur in the class hours allotted individual courses in the Mathematics-Physics and Mathematics-Technical Drawing curricula. The class hours given here are those for the Mathematics-Physics curriculum.

major as electives only. Courses in physics, technical drawing, and general education (political science, foreign language, physical education, etc.) required in these curricula are listed in the translations of Soviet syllabi given by Lindquist, Medlin, *et al.*, and, hence, will not be restated here.

The reader will note the inclusion under *Professional Courses* of the courses *Elementary Mathematics* and *Special Practical Training in Mathematics*. Spanning *seven* semesters and occupying 400 class hours, the course *Elementary Mathematics* is one of the most extensive and important courses in both curricula. Only one other course, *Mathematical Analysis*, is allotted more time (408 hours over a four-semester period). Basically, the *Elementary Mathematics* program is a sophisticated recapitulation of the most important topics of the Soviet secondary school mathematics program. In addition, the course syllabus includes topics of contemporary interest designed to broaden and deepen the teacher's mathematical outlook.

The official government syllabus for the course *Elementary Mathematics* lists the following major topics [3].

First semester—*The arithmetic of rational numbers*: whole numbers, divisibility of whole numbers, rational numbers, finite continued fractions and indeterminate equations.

Second semester—*Theory and practice of calculation*: precise computation, elements of approximate computation, aids to approximate computation (slide rule, tables), graphical solution of equations and numerical methods of isolating roots.

Third semester—*Geometry*: introduction, plane figures and their characteristics, similar figures, geometric values (measures), spatial figures and their characteristics, geometric values (volume and surface area).

Fourth semester—*Geometric constructions*: introduction, constructions with ruler and compass, basic geometric loci of points in a plane and the method of geometric loci in the solution of construction problems, geometric transformations of the plane and the method of transformations in the solution of construction problems, algebraic methods for the solution of construction problems, solvability of construction problems with ruler and compass, loci in space.

Fifth semester—*Algebra*: elementary methods for the solution of algebraic equations in one unknown, combinatorial theory, binomial and polynomial theorems, polynomials in several variables, nonlinear systems of algebraic equations in several unknowns, inequalities.

Sixth semester—*Irrational and transcendental equations*: irrational algebraic equations in the real domain, exponential and logarithmic equations in the real domain, trigonometric equations in the real domain.

Seventh semester—*Trigonometry* (plane and the elements of spherical): trigonometric functions of a real argument, inverse trigonometric functions, analytic representation of trigonometric functions, applications of trigonometry to geometry, elements of spherical geometry and trigonometry.

Comparison of the preceding topics with the Soviet secondary school mathematics syllabus indicates substantial agreement [4]. Certain topics (continued fractions; constructability; spherical trigonometry; etc.), though not found in the secondary school program, provide the prospective teacher with valuable background and enrichment material. An earlier paper by Wirszup discusses these Soviet enrichment materials in some detail [5].

Many of the texts used in the seven semester *Elementary Mathematics* sequence are modern in approach and content—far more modern than Soviet secondary school texts. Most have been prepared especially for the sequence by professors in pedagogical institutes or, in some cases, by research mathematicians. Included in the long list of approved texts for the *Elementary Mathematics* sequence are the following:

(A) V. M. Bradis, *Teoreticheskaia arifmetika* (Theoretical arithmetic). Moscow: Uchpedgiz, 1954.
(B) Ia. S. Bezikovich, *Priblizhennye vychisleniia* (Approximate computation). GTTI, 1953.
(C) B. V. Kutuzov, *Geometria* (Geometry). Moscow: Uchpedgiz, 1954.
(D) S. I. Novoselov, *Spetzial'nyi kurs elementarnoi algebry* (Special course in elementary algebra). Moscow: Soviet Science Publishing House, 1958.
(E) S. I. Novoselov, *Spetzial'nyi kurs trigonometrii* (Special course in trigonometry). Moscow: Soviet Science Publishing House, 1957.

The Kutuzov text *Geometry* has been translated into English and published by the School Mathematics Study Group for use by American teachers and students (*Studies in Mathematics*, Volume IV). The two Novoselov texts have been translated by various groups and individuals and are awaiting publication. These, in particular, are of interest because of the skill with which Novoselov displays the role of modern mathematical concepts at the elementary level.

Of the remaining professional courses in the Soviet curriculum, only the methods course, the practical training course, and the course entitled "Special Preparation" are of interest to the mathematician.

The course *Methods of Teaching Mathematics* follows and slightly overlaps the *Elementary Mathematics* course. The *Elementary Mathematics* course is intended to provide a foundation upon which the methods course can build. The methods course in mathematics at pedagogical institutes consists of 194 class hours of lectures, laboratory activities, and "practical" activities in secondary schools, spread over a four-semester period. The lecture course consists of a series of lectures on topics of primary importance to the Soviet mathematics teacher and, in particular, to the new teacher. Official texts and syllabi, lesson plans, visual aids in mathematics, evaluation procedures, extra-class activities, instructional techniques, and the Soviet philosophy of instruction all are touched upon in the lecture series. Also included are mathematical or quasi-mathematical topics of particular importance to the teacher—induction and deduction, axioms

and theorems, and a lengthy collection of "tips" for teaching specific topics included in the official secondary school syllabus.

Laboratory activities include the preparation of specimen documents (lesson plans, calendar plans, reports, tests, etc.), familiarization activities with official Soviet school syllabi and texts, exercises to develop familiarity with the literature in the field, the preparation and use of models and visual aids, and the preparation of reports and notebooks relating to the preceding activities.

Practical activities to be carried out in secondary schools include the observation of approximately six classes at different levels and a general familiarization with school activities. Observations are made in groups rather than individually and are followed by a conference with the teacher observed.

Examinations and term papers are regular parts of the methods course. Government certification examinations taken at the end of the 5-year training period may include "The Teaching of Mathematics" as a major area if elected by the candidate.

Two Soviet methods texts are frequently met:

V. M. Bradis, *Metodika prepodavaniia matematiki v srednei shkole* (Methods of teaching mathematics in the secondary school). Moscow: Uchpedgiz, 1954.

and

S. E. Liapin (Ed.), *Metodika prepodavaniia matematiki* (Methods of teaching mathematics). Moscow: Uchpedgiz, 1952.

It is the writers' understanding that the Bradis text is the current "official standard."

Practical Training in Mathematics is a 92-hour course spanning three semesters. Actually, it is comprised of three separate practical training courses—one in visual aid (devices and models) preparation, one in field measurement, and a third in techniques of calculation. The first phase of the practical training program consists of 20 hours of instruction and workshop practice in the construction of three or four mathematical models using cardboard, wood, glass, sheet metal, wire, or thread. One important aim of this portion of the practical training program is to acquaint future mathematics teachers with tools and materials used in the construction of visual aids. The thirty odd possible model projects listed in the practical training syllabus (according to materials used) include [6]:

(A) *Cardboard*—Models illustrating area formulas for the triangle, parallelogram, and trapezoid; models of cubes separated into two prisms, into three pyramids, and into six pyramids.

(B) *Wood and plywood*—Hinged models of angles and quadrilaterals; various geometric figures and solids; surveyor's instruments.

(C) *Wire and sheet metal*—Models illustrating central and axial symmetry; solid geometry model kit.

(D) *Glass*—Polyhedra; inscribed and circumscribed solids.

The standard text for this section of the practical training course is:

P. Ia. Dorf, *Nagliadnye posobiia po matematike* (Visual aids for mathematics). Moscow: Uchpedgiz, 1955.

The second phase of the practical training program deals with field measurement. The purpose of this phase is to acquaint the future teacher with simple geodetic instruments of the type used in secondary schools for demonstration purposes. Teachers also are familiarized with more complicated professional instruments. Preliminary work with the instruments often is done in the gymnasium or on the institute grounds. Field exercises include [7]:

(A) Surveying with the cross-staff.
(B) Angle measurement with the cross-staff.
(C) Surveying with the plane-table and approximation with the naked eye.
(D) Levels and leveling procedures.
(E) The solution of problems on determining inaccessible distances and heights by various means.

The computational portion of the practical training course includes lessons and exercises dealing with nomography; methods for solving systems of equations in four to six unknowns; graphical approximation of the derivative of a function and of the definite integral of a function over a real interval; the use of tables and interpolation theory; and the method of least squares [8].

In addition to the methods course, practical training, and the course in elementary mathematics, future Soviet mathematics teachers must enroll in a professional course entitled *Special Preparation.* All Soviet mathematics teachers are expected to be able to supervise at least two types of extra-class activity—usually including a mathematics club or circle. The *Special Preparation* course apparently is designed to provide some formal instruction along these lines. An elective dealing specifically with extra-class work in mathematics also is available.

The remaining courses in the professional portion of the pedagogical curriculum in mathematics (and a related discipline) are not sufficiently different from American university courses in psychology and education to warrant discussion here. The mathematical portion of the total curriculum, however, deserves consideration. Although the mathematics courses taught in Soviet pedagogical institutes appear to be of sound mathematical character, they are, in a sense, "professionalized" courses. The texts used in most courses appear excellent, but most are aimed directly at the prospective secondary school mathematics teacher. Topics of interest and relevance for secondary school teachers are given special attention. Current textbooks examined by these writers appear to be modern in approach and rich in mathematical content. Wirszup's complimentary remarks concerning the nature, content, and quality of the Soviet teacher-training texts in use prior to 1956 seem equally valid today [9].

A brief comparison between the mathematical portions of the Soviet teacher-training program and the recommendations for the training of teachers of mathematics recently published by the Mathematical Association of America may prove informative. The third of the five levels of competence established by the MAA (level III—Teachers of High School Mathematics—"these teachers are qualified to teach a modern high school mathematics sequence in grades nine through twelve") requires as a minimum the following [10]:

(A) Three courses in analysis
(B) Two courses in abstract algebra
(C) Two courses in geometry beyond analytic geometry
(D) Two courses in probability and statistics
(E) Two upper-class electives; e.g., introduction to real variables, number theory, etc. . . .

(One of these courses should contain an introduction to the language of logic and sets.)

Approximate translation of Soviet "class hours" into "semester-hour courses" was accomplished by considering 16 class hours equivalent to 1 semester hour. MAA "courses" are understood to be three semester-hour presentations [11].

Despite imprecisions of translation, it is apparent from the following tabulation that the Soviet curriculum meets and exceeds the MAA's level III requirements in all areas except statistics.

MAA Level III		Soviet	
Analysis	3 courses	Analysis	5 five semester-hour courses
Abstract Algebra	2 courses	"Higher" Algebra	4 three semester-hour courses
Geometry (beyond analytics)	2 courses	Projective and Descriptive Geometry Foundations of Geometry	2 five semester-hour courses
Probability and Statistics	2 courses	Theory of Probability	1 three semester-hour course (elective only)
Upper level (electives)	2 courses	Real Variable	1 three semester-hour course
		Complex Variable	1 three semester-hour course
		Theory of Numbers	1 three semester-hour course
		Foundations of Arithmetic	1 two semester-hour course
		Seminar	1 five semester-hour course

Despite the fact that the Soviet program appears to exceed the minimum MAA level III requirements by a substantial margin, it is doubtful that it approximates MAA level IV preparation (Master's degree *in mathematics*). The semi-professional character of the mathematics courses included in the Soviet program causes it to fall short of fifth-year level preparation in mathematics at first-rate Soviet or American universities. Nonetheless, the Soviet program is a sound one from both the mathematical and professional points of view.

There can be little doubt that Soviet mathematics teachers completing the present curricula are prepared to discharge their duties effectively.

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7. *Ibid.*, p. 10.

8. *Ibid.*, pp. 12–13.

9. I. Wirszup, *op. cit.*, pp. 521–522.

10. Recommendations for the training of teachers of mathematics, this MONTHLY, vol. 67, 1960, p. 987.

11. *Ibid.*, p. 984.

MATHEMATICAL EDUCATION IN SOVIET ENGINEERING SCHOOLS

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Introduction. The Trade Union of Educational and Scientific Workers of the USSR served as host in 1960 to a comparative education seminar and field study. The Phi Delta Kappa's commission on International Education joined with the Comparative Education Society to co-sponsor this undertaking.

This program was directed by the Ministries of Education in each of the Soviet Republics visited. The emphasis was upon the changes which were being introduced in the USSR as a result of important reforms taking place at the present time.

Provisions were made for personal contacts and meetings with educators and students. There were opportunities to visit classes and discuss mutual problems both with and without interpreters.

During the five week survey fifty American delegates inspected schools in Moscow, Tbilisi and Kharkov of European Russia, as well as Alma Ata, Stalinabad, Tashkent and Samarkand in Asia. Visits to teaching and research institutions were supplemented by trips to libraries, bookstores, museums, factories and collective farms.

The purpose of this report is to describe mathematics education in Soviet engineering schools. Specifically, an attempt is made to present the mathematics

program, then make some observations on its implementation, and some comments on students and faculty.

The Program. The Mathematics Program as submitted to directors of higher technical educational institutions by the Ministry of Higher Education in 1959–60 consists of a minimum of 360 contact hours. This minimum is distributed in the following way:

1. Analytic Geometry, vector algebra, determinants	72 hrs.
2. Introduction to analysis	40
3. Differential calculus of functions of one variable	68
4. Integral calculus of functions of one variable	64
5. Differential calculus of functions of several variables	20
6. Integral calculus of functions of several variables	24
7. Series	36
8. Differential equations	36
	360 hrs.

This minimum program is generally supplemented by additional mathematics to meet the needs of individual schools.

The course in mathematics begins with a general survey of the nature, aims, problems and methods of mathematics, its meaning for practical science and its value to science in general.

The first part of the program is not taught as an independent course in analytic geometry, but constitutes a geometric introduction into mathematical analysis. Sequence in the teaching of the parts of the program is not prescribed. For example, the first and second parts can be taught together, one supplementing the other.

All mathematics teaching is directed to the mastery of general principles as a means for solving theoretical and practical problems in engineering. At every point in the learning of higher mathematics, special attention is given to computational methods, utilizing computers which are available in higher technical institutes. In general, engineering education is offered only at the higher technical educational institutions, not at the universities.

The Soviet educational system is highly standardized. Curricula at all levels of education are uniform throughout the USSR. However, instruction in the Soviet Republics of central Asia is at a significantly lower grade than that in the Moscow area.

Almost all of the educational institutions concerned with engineering training are under the general direction of the Soviet Ministry of Higher Education. This Ministry is responsible for the general planning of the higher education system as well as the supervision of the curriculum of each specialty. However, institutions of higher learning are administered in part by those union republics in which they are located.

Students. Students of the Soviet Union have a high regard for college education. Higher Technical Educational Institutions admit students from the ten-year schools, and the best 5% from the technikums. It is Soviet policy to give

preference for admission to those applicants who have two years of industrial experience, the aim being, that eventually 80% of the admissions would be in this class.

Each applicant must pass 5 admission examinations: in mathematics, physics, chemistry, a foreign language and Russian. The rules allow the student to apply for only one field of specialization in one institute in a given year. If he fails to secure admission, he may not apply for another year. An average of one in four who seek entrance, gain admission to engineering institutions.

Lack of money does not stand in the way of higher education for anyone. From 80 to 90% of the engineering students are given grants based on need and aptitude. The amounts of the grants are sufficient to take care of food and lodgings. At present about one third of the engineering students are girls. Engineering students and practicing engineers are exempt from military service. Since a great deal of importance is placed on a higher education for future prestige and economic status, attrition is low, averaging only about 10%.

Faculty. Faculties enjoy prestige second to no other professional group in the Soviet Union. The requirements for teaching positions are associated with degrees. In general, the position of professor requires the doctor's degree, that of the docent, a kandidat's degree and a diploma is sufficient for the position of an instructor. A diploma is awarded on completion of the undergraduate 5 or $5\frac{1}{2}$ year engineering course. Of the faculty, approximately 5% are professors, 45% are docents and 50% are instructors.

The teaching of mathematics is done by professors in lectures to large groups during approximately half of the contact hours, the rest of the time the students attend recitation periods which are usually conducted by instructors and involve discussion of assigned problems.

In the Soviet Union there is no tenure as we know it. Appointments are for a 5-year period. However, if performance has been satisfactory, reappointments are usual. Retirement is not compulsory at any age if a teacher is able to perform his duties. When a faculty member is 60 (women 55) he may retire at 40% of his salary. The average annual income of a Soviet professor exceeds the income of a skilled worker by about five times. Although a professor is supposed to work six hours a day, six days a week, it is difficult to find out how this time is allotted between teaching, research and other duties.

From discussions with faculty members, it seemed that they were better informed on educational developments in the U.S.A. than the Americans are of similar developments in the Soviet Union. This is due, in part, to their effective translating services. Also many of the Soviet educators are able to read English technical journals.

Reforms. At present there is a gradual reorganization of engineering education. It is planned to improve the mathematical preparation of students through the raising of the quality of instruction, closer ties with professional departments and greater emphasis on applied mathematics. The new reforms call for more socially useful and productive work. This change is being introduced to develop

a wholesome attitude toward labor and the need of creating a greater supply of skilled manpower.

There is some confusion as to how these reforms are to be implemented; however, Soviet educators were agreed that the indicated changes will introduce greater variety and bring Soviet schools closer to life in Communist society. The reforms suggest that the Soviets aim to prepare specialists to serve the needs of the State to a greater extent than ever before. These needs are determined by industrial plans and by a political system dedicated to materialism.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1501. *Proposed by Brother Alfred, St. Mary's College, California*

There are 15 men on the board of directors of a large company and 20 committees associated with the board. It is required to set up these committees so that: (1) Each board member shall belong to four committees; (2) Each committee shall have three board members; (3) No two committees shall have more than one board member in common.

E 1502. *Proposed by H. S. Shapiro, New York University*

Let

$$\frac{1 + a_1x + a_2x^2 + \cdots}{1 + b_1x + b_2x^2 + \cdots} = 1 + c_1x + c_2x^2 + \cdots,$$

where $0 \leqq a_i \leqq 1$, $0 \leqq b_i \leqq 1$. Prove that $|c_n|$ does not exceed the nth Fibonacci number.

E 1503. *Proposed by Richard Sinkhorn, University of Wisconsin*

Evaluate $\lim_{x \to 0} [(\pi/2) \csc (\pi x/2) - \Gamma(x)]$, where $\Gamma(x)$ is the gamma function.

E 1504. *Proposed by R. C. Lyness, Great Singleton, England*

Find positive integers x, y, z, a, b, c, d such that: (1) $x^n + y^n + z^n = a^n + b^n + c^n + d^n$ for $n = 1, 2, 3$, (2) $z > y > x$, and (3) $z \neq y + x$.

E 1505. *Proposed by R. L. Caskey, Oklahoma State University*

Construct a triangle given, in position, a vertex, the midpoint of the side opposite that vertex, and the nine-point center. Describe the region in which the nine-point center must be located so that the problem will have a solution when the other two points are arbitrarily placed.

SOLUTIONS

A Number and Its Reverse

E 1453 (*Corrected*) [1961, 572]. *Proposed by José Gallego-Diaz, University of Puerto Rico*

Let A be the sum of the digits of a natural number N, let $B=A+N$, let A' be the sum of the digits of the number B, and let $C=B+A'$. Find N if the digits of C are those of N in reverse order.

(Dedicated to the memory of Victor Thébault.)

Solution by D. L. Silverman, Beverly Hills, Calif. If N has k digits, then C cannot exceed N by more than $18k$. On the other hand, if C is the "reverse" of N, it can be shown by induction that C exceeds N by at least $9(10^{(k-2)/2})$ if k is even and by at least $99(10^{(k-3)/2})$ if k is odd. This quickly implies that $k \leqq 2$, and since N is obviously not a one-digit number, we have

$$N = 10x + y, \qquad 1 \leqq x < y \leqq 9,$$

$$B = 11x + 2y,$$

$$C = 11x + 2y + p + (11x + 2y - 10p) = 10y + x,$$

where p is the greatest integer contained in $(11x+2y)/10$. That is,

$$7x - 2y = 3p,$$

giving two solutions, $x=1$, $y=2$ and $x=6$, $y=9$. N is therefore 12 or 69.

Also solved by Louisa R. Alger, J. W. Baldwin, Merrill Barnebey, E. E. Bosman, Brother Alfred, W. E. Buker, D. I. A. Cohen, R. W. Derby, C. L. Dotton, E. S. Eby, J. W. Ellis, Jane Evans, Anton Glaser, N. V. Glick, Michael Goldberg, Robert Goldberg, J. B. Herreshoff, Alice E. A. Hunt, D. C. B. Marsh, L. V. Mead, Otto Mond, Walter Penney, J. L. Pietenpol, David Sachs, Anthony Sapienza and Ray Sommers (jointly), A. E. Schoenberger, R. T. Shannon, J. L. Shores, S. J. Sidney, Banson Sing, E. L. Spitznagel, Jr., R. P. Tapscott, Guy Torchinelli, C. C. Yalavigi, and the proposer.

A System of Matrix Equations

E 1471 [1961, 573]. *Proposed by J. L. Brenner, Stanford Research Institute, Menlo Park, California*

If A, B are invertible matrices of the same dimension, it is not always possible to solve $XY=A$, $YX=B$ for X, Y; A and B must be similar, since $X^{-1}AX=B$. Under what conditions on invertible A, B, C can one solve $XY=A$, $YZ=B$, $ZX=C$ for X, Y, Z?

Solution by the proposer. Let W be a square root of CBA. (Every invertible matrix has a square root.) Then $Z=WA^{-1}$, $Y=BAW^{-1}$, $X=AWA^{-1}B^{-1}$ solve the problem. That is, there are no further necessary conditions.

It is clear how the problem can be generalized.

Also solved by Leonard Carlitz, E. S. Eby, David Friedman, Michael Goldberg, S. H. Greene, C. R. MacCluer, D. C. B. Marsh, F. D. Parker, David Sachs, S. J. Sidney, and W. C. Waterhouse.

Editorial Note. If the system of matrix equations has a solution X, Y, Z, we have (as shown by Friedman and Parker) $ACB = XYZXYZXX^{-1} = XBACX^{-1}$ and $BAC = YZXYZXYY^{-1} = YCBAY^{-1}$, whence ACB, CBA, BAC are similar to each other. Again (as shown by Eby, MacCluer, Marsh, Sidney, and Waterhouse), $AC=XYZX=XBX$. In view of the above solution we therefore have: If A, B, C are nonsingular matrices of the same dimension, then (1) ACB, CBA, BAC are similar, (2) there exists a matrix T such that $AC=TBT$. These constitute two interesting facts to establish without a prior knowledge of the above solution to E 1471.

Adeals and Ideals

E 1472 [1961, 573]. *Proposed by C. H. Cunkle, Utah State University, and W. H. Lesser, Franklin and Marshall College*

In a recent text, a ring R is defined as an additive Abelian group, closed and associative under multiplication and for which the distributive laws hold, and a subset M of R is defined to be an ideal, provided M is closed under addition from within ($x\in M$ and $y\in M$ implies $x+y\in M$) and closed under multiplication from within and without ($m\in M$ and $r\in R$ implies $mr\in M$ and $rm\in M$). Show that with this definition an ideal M need not be a subring of R.

I. *Solution by Underwood Dudley, University of Michigan.* Let R be the additive group of integers with the trivial multiplication $ab=0$ for all a, b in R. Then the set of all nonnegative integers is an ideal but not a subring.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* Let R be the set of all 2×2 matrices, $\left(\begin{smallmatrix} 0 & j \\ 0 & 0 \end{smallmatrix}\right)$, where j ranges over the ring of integers; let M be the subset obtained from nonnegative j. All postulates on R and M are satisfied, but M is not a ring.

III. *Solution by W. C. Waterhouse, Harvard University.* Let R be the ring of all real polynomials in x whose constant terms vanish. Let $M\subset R$ be defined by the condition that the coefficient of x be nonnegative. M then satisfies the given definition but is not a subring, as $x\in M$, $-x\notin M$.

Also solved by D. W. Bailey, J. L. Brown, Jr., D. I. A. Cohen, J. R. Durbin, R. L. Farrell, David Friedman, Marcelle Friedman, L. D. Goldstone, G. A. Heuer, J. F. Hurley, W. G. Leavitt, Gerald Leibowitz, Jiang Luh, C. R. MacCluer, T. G. McLaughlin, J. S. Monroe, Jr., D. A. Moran, Richard Platek, D. A. Robinson, David Sachs, S. J. Sidney, J. L. Sieber, D. L. Silverman, E. L. Spitznagel, Jr., Evelyn A. Strawbridge, Guy Torchinelli, E. W. Wallace, R. J. Whitley, A. B. Willcox, and the proposers. Late solutions by R. J. Cormier and J. C. Ferrar.

Editorial Note. If the first condition on M should be replaced by "M is closed under subtraction from within," then M would be a subring of R.

Solution III is interesting in that if furnishes an example where R is a ring without proper divisors of zero.

In a recent undergraduate course in abstract algebra, Willcox defined a subset M of a ring R to be an *Adeal* if it satisfies the conditions of the problem, and an *ideal* is then an Adeal which is also a subring. He then asked the class to find sufficient conditions on R to insure that every Adeal be an ideal. One such (trivial) condition is that R possess an identity element. The weakest condition found is that for each $a \in R$ there exist a nonzero integer n such that $na \in aR+Ra$, where na has the usual meaning. It is an interesting problem to try to find still weaker conditions.

Square Triangular Numbers

E 1473 [1961, 573]. *Proposed by J. L. Pietenpol, Columbia University*

Show that there are infinitely many square triangular numbers.

I. *Solution by A. V. Sylwester, U. S. Naval Ordnance Lab., Corona, Calif.* If $N[i]=i(i+1)/2$, the ith triangular number is a square, then

$$N[4i(i+1)] = 4N[i](2i+1)^2$$

is also a square. Since the first triangular number is a square, there exist an infinite number of square triangular numbers.

II. *Solution by Erwin Just, Bronx Community College, Bronx, N. Y.* It is known that the Pell equation $p^2-2q^2=1$ has an infinite number of positive integral solutions, which implies that there are an infinite number of perfect squares of the form $(p^2-1)/2$. Therefore there exist infinitely many square triangular numbers of the form $p^2(p^2-1)/2$.

III. *Solution by R. M. Warten, Madison, Wisconsin.* The following theorem was stated and proved by me in the article "An elementary method for finding all numbers both triangular and square," *Pi Mu Epsilon Journal*, vol. 2, no. 8, Spring 1958, pp. 364–366:

THEOREM. *Define the sequence* $\{K_n\}$ *of positive integers as follows*: (1) $K_1 = a_1^2 b_1^2$ *with* $a_1=b_1=1$, (2) $K_n=a_n^2b_n^2$ *for* $n=2, 3, \cdots$, *where* $a_n=a_{n-1}+b_{n-1}$, $b_n=2a_{n-1}+b_{n-1}$. *Then* $\{K_n\}$ *contains all and only those positive integers which are both triangular and square.*

For references to some less elementary methods, see the same article.

Also solved by Leon Bankoff, William Becker, W. Bluger, D. A. Breault, Brother R. L. Fitz and Brother T. C. Wesselkamper (jointly), Brother Joseph Heisler, J. L. Brown, Jr., Leonard Carlitz, D. I. A. Cohen, R. J. Cormier, M. S. Demos, G. C. Dodds, Underwood Dudley, S. C. Eisenstat, David Forslund, David Friedman, H. M. Gehman, N. V. Glick and Richard Katz (jointly), Michael Goldberg, L. D. Goldstone, Jay Gottesfeld, S. H. Greene, Cornelius Groenewoud, Fritz Herzog, Vern Hoggatt, J. A. H. Hunter, Gerald Janusz, G. W. Kessler, J. D. E. Konhauser, Robert McGuigan, P. B. Manchester, D. C. B. Marsh, D. A. Moran, R. J. Oberg, C. C. Oursler, F. D. Parker, R. V. Parker, Walter Penney, George Piranian, Peter Rosenthal, David Sachs, Waclaw Sierpinski, S. J. Sidney, D. L. Silverman, Sister Miriam Francis, Philip Smedley, Eric Sturley, Paul Stygar, Dmitri Thoro, Guy Torchinelli, Kermit Uggen, W. C. Waterhouse, L. A. Zalcman, David Zeitlin, and the proposer. Late solutions by R. H. Awglin, J. B. Muskat, Stanton Philipp and E. P. Starke.

Editorial Note. The result of this problem is not new and many references can be given to earlier proofs, such as (1) L. E. Dickson, *A History of the Theory of Numbers* (Chelsea, 1952), vol. 2, p. 10, notes that the problem was solved by L. Euler on August 10, 1730; also see p. 16 and p. 27, (2) W. Sierpinski, *Teoria Liczb*, 3rd ed. (Warsaw, 1950, Monografie Matematyczne, t. 19), p. 517, (3) Problem E 954 this MONTHLY [1951, 568], where it is shown that the nth triangular square number is

$$[(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2]/32,$$

(4) *Scripta Mathematica*, 20 (1954), p. 213, Note 386, (5) M. N. Khatri, "Triangular numbers which are also squares," *Mathematics Student*, 27 (1959), pp. 55–56, (6) W. Sierpinski, "Sur les nombres triangulaires carrés," *Publication de la Faculté d'Électrotechnique de l'Université à Belgrade*, no. 65 (1961), pp. 1–4, (7) W. Sierpinski, "Sur les nombres triangulaires carrés," *Bull. Soc. Royale Sciences Liège*, 30e année (1961), pp. 189–194. In the last two references, Sierpinski proves if $t_x = x(x+1)/2 = y^2$ is a square triangular number, then the next square triangular number is $t_{3x+4y+1} = (2x+3y+1)^2$. Starting with $x=1$, one obtains all the square triangular numbers, of which the first six are $t_1 = 1$, $t_8 = (6)^2$, $t_{49} = (35)^2$, $t_{288} = (204)^2$, $t_{1681} = (1189)^2$, $t_{9800} = (6930)^2$. In reference (6) above Sierpinski also proves a result due to D. Blanusa, that if t_{x_n} is the nth term in the sequence of square triangular numbers, then

$$x_n = [(\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n]^2/4, \quad y_n = [(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n]/4\sqrt{2}.$$

K. Zarankiewicz has asked if there are infinitely many right triangles each of whose three sides is a triangular number. This is still an open question, and to date only one such triangle is known —the one with sides $t_{132} = 8778$, $t_{143} = 10296$, $t_{164} = 13530$. Sierpinski, in references (6) and (7) above, has shown that there are infinitely many right triangles whose legs are consecutive triangular numbers and whose hypotenuse is a whole number. The fact that $t_5^2 + t_8^2 = (39)^2$ and $t_7^2 + t_9^2 = (53)^2$ shows that there are other right triangles with triangular numbers as legs and with a whole number for hypotenuse.

F. D. Parker has shown that if $(r-2)/2$ is not a square, then there are an infinite number of r-gonal square numbers.

The Equation $m^{n^m} = n^{m^n}$

E 1474 [1961, 573]. *Proposed by Alvin Hausner, City College of New York*

Show that the equation $m^{n^m} = n^{m^n}$ has no solutions in positive integers with $m \neq n$.

Solution by Erwin Just, Bronx Community College, Bronx, N. Y. Without loss of generality assume $n > m \geqq 1$. If $m = 1$, or if $m = 2$ and $n = 3$ or 4, we readily find that $n^{m^n} > m^{n^m}$. Suppose $m = 2$ and $n > 4$. Since $f(x) = x^{1/x}$ is a decreasing function for $x > e$, $4^{1/4} > n^{1/n}$, or $4^n > n^4$, or $2^n > n^2$, or $n^{2^n} > 2^{n^2}$, and $n^{m^n} > m^{n^m}$. Finally, suppose $n > m \geqq 3$. Then $m^{1/m} > n^{1/n}$, or $m^n > n^m$, and again $n^{m^n} > m^{n^m}$. It follows that the equation $m^{n^m} = n^{m^n}$ has no solution in positive integers if $m \neq n$.

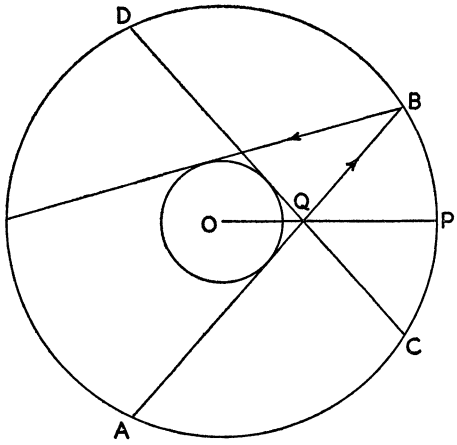
Also solved by William Becker, Robert Bowen and Chuck Carniglia and Alan Felzer (jointly), Brother R. L. Fitz, J. L. Brown, Jr., Leonard Carlitz, Carl Cohen, D. I. A. Cohen, D. T. Cottingham, M. S. Demos, Jane Di Paola, G. C. Dodds, Underwood Dudley, J. R. Durbin, C. E. Franti, N. V. Glick, Michael Goldberg, Gerald Janusz, D. C. B. Marsh, R. J. Oberg, Allen Rubenstein, David Sachs, A. E. Schoenberger, R. T. Shannon, Marlow Sholander, S. J. Sidney, D. L. Silverman, Eric Sturley, Guy Torchinelli, Kermit Uggen, W. C. Waterhouse, L. A. Zalcman, David Zeitlin, and the proposer. Late solutions by Ronald Alter, J. C. Ferrar and Stanton Philipp.

The Cylindrical Mirror

E 1475 [1961, 573]. *Proposed by R. H. C. Newton, Berkhamsted, England*

Let OP be a radius of a right section of a perfectly reflecting circular cylinder. Let Q be any point on OP, and let a ray leave Q and reflect from the inner surface of the cylinder. Does it ever return to Q, and if so, under what conditions?

Solution by N. V. Glick, North American Aviation, Inc. We assume the ray to be in the plane of the section, and refer to the accompanying figure. Project ray QB back to A and construct a circle with O as center and tangent to AB. It follows by elementary geometry that all the rays must be tangent to this



circle. Now through Q only two tangents to the circle can be drawn, namely AB and CD. The ray, to get back to Q, must get either to A or to C. Denote angle AOB by α and angle AOC by β. A necessary and sufficient condition that the ray get to A is that there exist positive integers m and n such that $m\alpha = n(2\pi)$, or that α and π be commensurable. A necessary and sufficient condition that the ray get to C is that there exist positive integers m and n such that $m\alpha = n(2\pi) + \beta$.

Also solved by Leon Bankoff, Michael Goldberg, S. H. Greene, L. V. Mead, David Sachs, Paul Stygar, N. M. Wickstrand and the proposer.

Some of the solutions submitted were incomplete. The problem is related to the famous problem of Alhazen. Bankoff called attention to Victor Thébault, "Geodesics," *Scripta Mathematica*, June-Sept. 1955, p. 148, and to Prob. 168, *Mathematical Visitor*, Jan. 1887, p. 171.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5007. *Proposed by A. V. Boyd and D. A. Higgs, University of the Witwatersrand, Johannesburg, South Africa*

Sum: $\operatorname{sech} \frac{1}{2}\pi - \frac{1}{3} \operatorname{sech} \frac{3}{2}\pi + \frac{1}{5} \operatorname{sech} \frac{5}{2}\pi - \cdots$.

5008. *Proposed by Hans Zassenhaus, University of Notre Dame*

1. The words $W(A, B, C)$ in 3 letters A, B, C and of finite length form a semigroup with the empty word as identity element if juxtaposition is used as the rule of multiplication; it remains a semigroup S if every word $W_1W_2W_2W_3$ with repetition of a nonempty subword W_2 is replaced by W_1W_3.

Show that the mapping $W(A, B, C) \to W(ABCBA, BCACB, CABAC)$ is an isomorphism of this semigroup onto a proper subsemigroup.

2. Find a word of shortest length in S that is not a proper subword of any other word of S.

5009. *Proposed by Albert Wilansky, Lehigh University*

Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[0, 1]$ such that $f_n \to 0$ uniformly but f_n' does not tend uniformly to 0. Show that there exists a function g which is not continuously differentiable on $[0, 1]$ and which is uniformly approximable by a finite linear combination of the f_n. (As an example, $f_n(x) = (\sin n\pi x)/n$. In this case g may be any continuous function.)

5010. *Proposed by Hans Schneider, University of Wisconsin*

For a permutation σ of $(1, \cdots, m)$ let $s(k)$ be the number of cycles of length k in a factorization of σ into cyclic permutations, and set $s^*(k) = \sum_{k|d} s(d)$. Let P_σ be the permutation matrix associated with σ: $P_\sigma(i, j) = 1$ or 0 according as $\sigma(i) = j$ or $\sigma(i) \neq j$. For a permutation τ of $(1, \cdots, n)$ define $t(k)$, $t^*(k)$ and P_τ similarly. Show that the m by n matrix (with complex elements) of maximum rank r satisfying $P_\sigma A = AP_\tau$ has $r = \sum_k \phi(k)w^*(k)$, where ϕ is Euler's function and $w^*(k) = \min \{s^*(k), t^*(k)\}$. If $m = n$, deduce that there exists a nonsingular A if and only if $s(k) = t(k)$ for all k, and that, in this case, there exists a permutation matrix π such that $A\pi$ commutes with P_σ if and only if $P_\sigma A = AP_\tau$. (These results include Lemmas 1 and 2 of R. Brauer, *Connection between ordinary and modular characters of groups of finite order*, Ann. Math., 42, 1941, 926–935).

5011. *Proposed by Julius Lieblein, David Taylor Model Basin, Washington, D. C.*

For a fixed $k>0$, find all continuous functions, $f(t)$, such that $I=\int_x^{kx} f(t)dt$ is independent of x, $x>0$.

5012. *Proposed by M. F. Ruchte and B. T. Sims, Iowa State University*

Recall that a domain of integrity is an associative ring which contains no proper divisors of zero, and that a division ring is a domain of integrity whose nonzero elements form a multiplicative group. Prove (or disprove) the following converse of a theorem of Ore:

Let S be a domain of integrity with identity 1. If S is imbedded in a division ring F, show that the intersection of every two nonzero right (left) ideals is nonzero.

SOLUTIONS

Uspensky's Derivative Formula

3534 [1932, 116; 1961, 814]. *Proposed by J. V. Uspensky.* Show that

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t-1}{t^r-1}\right)\right]_{t=1} = [1-r][1-2r]\cdots[1-(n-1)r]r^{-n}.$$

Comment by L. Carlitz, Duke University. Pinzka is quite correct in his comment. However, the theorem, though spoiled by a simple misprint, is too neat to be relegated to limbo. We shall prove, for arbitrary r, that

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t-1}{t^r-1}\right)^n\right]_{t=1} = [1-r][1-2r]\cdots[1-(n-1)r]r^{-n}.$$

By Lagrange's formula (Pólya-Szegö, *Aufgaben und Lehrsätze*, v. 1, p. 125), if $w=z/\phi(z)$, where $\phi(0)\neq 0$, then

$$z = \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}(\phi(x))^n}{dx^{n-1}}\right]_{x=0}. \tag{1}$$

Take $\phi(z)=z/w$, $w=(z+1)^r-1$. Then

$$\begin{aligned} z+1 = (w+1)^{1/r} &= \sum_{n=0}^{\infty} \frac{\frac{1}{r}\left(\frac{1}{r}-1\right)\cdots\left(\frac{1}{r}-n+1\right)}{n!} w^n \\ &= \sum_{n=0}^{\infty} \frac{(1-r)(1-2r)\cdots[1-(n-1)r]}{n!}\frac{w^n}{r^n}. \end{aligned} \tag{2}$$

Comparing (1) and (2) we get

$$\left[\frac{d^{n-1}(\phi(x))^n}{dx^{n-1}}\right]_{x=0} = \left[\frac{d^{n-1}}{dt^{n-1}}(\phi(t-1))^n\right]_{t=1}$$
$$= \left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t-1}{t^r-1}\right)^n\right]_{t=1}$$
$$= \frac{(1-r)(1-2r)\cdots[1-(n-1)r]}{r^n}.$$

Eight Points in Space

4306 [1948, 431]. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Let there be given eight arbitrary points in space. Prove that one can always select three of them which do not form an acute-angled triangle.

Comment by D. A. Moran, University of Illinois. The proof has been given in a much stronger form, namely that six arbitrary points in space possess the stated property. See H. T. Croft, *On 6-point configurations in 3-space*, J. London Math. Soc., v. 36, part 3, July 1961.

Diagonals of a Polygon

4444 [1951, 422]. *Proposed by J. H. Braun, Illinois Institute of Technology*

Prove that no three diagonals of a regular polygon of odd order are concurrent at any point other than the vertices.

Comment by D. A. Moran, University of Illinois. The stated result has been proved for regular polygons of prime order. See H. T. Croft and M. Fowler, *On a problem of Steinhaus about polygons*, Proc. Cambridge Phil. Soc., v. 57, part 3, July 1961, p. 686.

Harmonic Function

4940 [1960, 1034]. *Proposed by M. S. Klamkin, AVCO Research, and D. J. Newman, Yeshiva University*

Problem no. 151 in the "Scottish" book of problems due to Wavre, poses the question of the existence of a harmonic function defined in a region containing a cube in its interior such that it vanishes on all its edges. Show that such a function exists for any number of dimensions.

Solution by Fred Suvorov, Princeton University. Consider

$$h(x_1, x_2, \cdots, x_n) = (\sinh (n-1)^{\frac{1}{2}}x_1)\cdot(\sin x_2)\cdots(\sin x_n).$$

h is harmonic in n-space and vanishes on the n-cube of side π. (And, of course, many other places.)

Also solved by the proposer.

Bateman's Polynomial

4941 [1961, 66]. *Proposed by Leonard Carlitz, Duke University*

Bateman's polynomial $F_n(z)$ is defined by

$$F_n(z) = {}_3F_2\left[\begin{matrix}-n, & n+1, & \frac{1}{2}(1+z);\\ 1, & 1 & \end{matrix}\right]$$

in the usual notation for generalized hypergeometric functions. Now if $p=2n+1$ is prime, show that

$$F_n(0) = \begin{cases} 0 \pmod p & (n \text{ odd}) \\ 4a^2 \pmod p & (n \text{ even}), \end{cases}$$

where a^2 is the odd square in the decomposition $p=a^2+b^2$.

This result is analogous to that of problem 4628 [1956, 348].

Solution by the proposer. Since $n \equiv -\frac{1}{2} \pmod p$,

$$F_n(0) = {}_3F_2\left[\begin{matrix}-n, & n+1, & \frac{1}{2};\\ 1, & 1 & \end{matrix}\right] \equiv {}_3F_2\left[\begin{matrix}-n, & -n, & -n;\\ 1, & 1 & \end{matrix}\right]$$

$$\equiv \sum_{r=0}^{n} (-1)^r \binom{n}{r}^3 \pmod p.$$

But it is known that

$$\sum_{r=0}^{n} (-1)^r \binom{r}{n}^3 = \begin{cases} 0 & (n \text{ odd}) \\ (-1)^m \dfrac{(3m)!}{(m!)^3} & (n = 2m); \end{cases}$$

moreover

$$(-1)^m \frac{(3m)!}{(m!)^3} = (-1)^m \binom{2m}{m} \frac{(2m+1)\cdots(3m)}{m!} \equiv \binom{2m}{m}^2 \pmod p.$$

Now, by a theorem of Gauss,

$$2^{-2m}\binom{2m}{m} \equiv 2a \pmod{p = 4m+1},$$

where a is the unique odd integer determined by $p=a^2+b^2$, $a \equiv b+1 \pmod 4$. Then $F_{2m}(0) \equiv \binom{2m}{m}^2 \equiv 4a^2 \pmod p$.

Property of a Convex Set Symmetric about the Origin

4943 [1961, 67]. *Proposed by D. J. Newman, Yeshiva University*

Let A and B be two convex plane sets, both symmetric about the origin. Suppose that $z \in A$, $\zeta \in A$ implies that either $z+\zeta \in B$ or $z-\zeta \in B$. Show that this persists for n vectors, *i.e.*, $z_i \in A$, $i=1, 2, \cdots, n$ implies $z_1 \pm z_2 \pm \cdots \pm z_n \in B$ for some choice of $\pm$ signs.

Solution by K. A. Post, Technical University, Eindhoven, Netherlands. The problem is equivalent to the following theorem: Given a convex plane set A, symmetric about the origin. Then $x, y, z \in A$ implies: at least one of the six vectors $x \pm y$, $x \pm z$, $y \pm z \in A$.

If two vectors are linearly dependent, the solution is trivial. So suppose $\lambda x + \mu y + \nu z = 0$, $\lambda\mu\nu \neq 0$, and let $|\nu| = \text{Min}(|\lambda|, |\mu|, |\nu|)$. So we have $z = \rho x + \sigma y$, $|\rho| \geqq 1$, $|\sigma| \geqq 1$. Since A is symmetric, we may assume $\rho \geqq 1$, $\sigma \geqq 1$, by using $-x$ instead of x, $-y$ instead of y. Now it is easily seen that $x+y$ belongs to the convex hull of the vectors x, y, and z and therefore $x+y \in A$.

Remarks: (1) The requirement of convexity for B is superfluous. (2) The problem and solution can be generalized to n dimensions: *Given a convex set A in n-dimensional space, symmetric about the origin. Then $x_1, x_2, \cdots, x_{n+1} \in A$ implies that at least one n-combination of x_i with $+$ and $-$ signs $\in A$.*

Also solved by Robert Breusch, H. E. Bray, and the proposer.

Partitions of n

4944 [1961, 67]. *Proposed by Morris Newman, National Bureau of Standards*

Prove that $p(n)$, the total number of partitions of n, is odd for infinitely many values of n and even for infinitely many.

Solution by J. H. van Lint, Technical University, Eindhoven, Netherlands. We use the well known formula (cf. Hardy and Wright, *Theory of Numbers*)

$$\left\{1 + \sum_{n=1}^{\infty} p(n)x^n\right\} \cdot \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k+1)/2} = 1,$$

i.e.

$$\left\{1 + \sum_{n=1}^{\infty} p(n)x^n\right\} \cdot (1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots) = 1.$$

Suppose m is the largest number for which $p(m)$ is even (or odd). Take $n = m + \frac{1}{2}(m+1)(3m+2)$. By the above formula:

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots + (-1)^{m+1}p(m) = 0.$$

In both cases the number of odd terms in the left member of this equation would be odd, namely $2m+1$ or 1 respectively. This is a contradiction. Hence the assertion is proved.

Also solved by the proposer.

Editorial Note. A. Mąkowski notes that O. Kolberg has proved the present theorem in *Mathematica Scandinavica* 7, 1959, 377–378.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

A Second Course in Statistics. By Robert Loveday. Cambridge University Press, 1961. xi+155 pp. $1.85.

This book is a sequel to the author's *A First Course in Statistics.* The author describes the difference between the two books as follows: "The keyword of *A First Course in Statistics* is *observation*. . . . In *A Second Course in Statistics* the important idea is *probability*." It is an index of the condensed style of presentation that this "important idea" is the subject of a chapter containing just two pages of text (plus three pages of technical examples and exercises).

The student will find in this book a good selection of basic statistical techniques and terminology, compressed into a remarkably small space. At high school level, such an exposure to statistical method might well prove beneficial, even though the underlying logic is not fully understood.

However, the author's hope that the book may be useful for university students seems rather too optimistic. While agreeing that "rigorous mathematical treatment" is not to be expected in a book at this level, soundness in logic and clarity in explanation are always desirable.

Defects in this respect are particularly noticeable on pages 77 and 109, where significance tests are used as bases for statements of inverse probability. Also confusing is the statement (p. 114) that Spearman's rank correlation is "nothing more than a quickly calculated approximation to r_{xy}." The possibility of applying significance tests using Spearman's rank correlation (as described, for example, in M. G. Kendall's *Rank Correlation Methods*) is ignored.

These defects, and others of a less glaring nature, reduce the value of the book for the more mature student. However, the book does provide the rather younger student with some idea of modern statistical philosophy, and this may justify its existence.

N. L. JOHNSON
Case Institute of Technology

Probability: A First Course. By Frederick Mosteller, Robert E. K. Rourke and George B. Thomas, Jr. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961. xv+319 pp. $5.00.

Both this volume and *Probability with Statistical Applications* by the same authors (see below) have emerged from the text, *Introductory Probability and Statistical Inference*, prepared for the Commission on Mathematics, College Entrance Examination Board. Much of the material in this text was used as the basis of the nationally-televised NBC Continental Classroom course in *Probability and Statistics* presented early in 1961.

This text is probably best suited for a one-semester course in Probability given in a mathematics or statistics department but primarily for nonmathematics majors. For an understanding of the material covered, a second course in high-school algebra is required, but no knowledge of calculus is assumed.

The first four chapters are concerned with finite sample spaces and include the usual introductory topics (e.g., permutations, combinations, mutually exclusive events, independent events, conditional probability, random drawings, and Bayes' theorem). Chapter 5 introduces the notion of a random variable in a careful manner not generally found in an elementary book. This chapter also includes discussion of mean, variance, and a readable development of Chebyshev's theorem. Chapter 6 discusses the binomial distribution and Chapter 7 ("Some Statistical Applications of Probability"), the last chapter, includes brief discussions of such statistical topics as estimation, testing of hypotheses, confidence limits, and ends with a most interesting example of Bayesian inference. Also included in the text are two appendices. One deals with sets and the other discusses summation notation. At the end of the book there is a comprehensive set of tables, the outstanding feature of which is the inclusion of a table of binomial probabilities (both individual and cumulative).

The book is carefully written with precise statements of definitions and theorems. The authors have chosen to define sample variance with divisor n rather than $(n-1)$ but have remarked that both definitions are common and have explained the bias so introduced. Each new topic is introduced by a numerical example before being treated in complete generality. In addition to many worked examples, the text includes an abundance of well-selected exercises. However, it is unfortunate that no answers are given for the exercises. For a first edition, the text is remarkably free of typographical errors.

There is little that the reviewer has found to criticize. However, perhaps some use of tree diagrams would have clarified the discussion of elementary probability laws related to mutually exclusive events and conditional probability. Also, on several occasions there seems to be repetition in the worked examples; two examples are identical except for the numbers involved.

The reviewer recommends that this text be given careful consideration for a one-semester course in probability at the freshman level. There are few books on the market which are as carefully planned and as clearly and concisely written as this one.

HARVEY J. ARNOLD
Wesleyan University

Probability with Statistical Applications. By Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas, Jr. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961. xv+478 pp. $6.50.

Much of this volume is identical to *Probability: A First Course* by the same authors. There are three new chapters and an addition to a fourth. Most of the new material is statistical in nature. To avoid repetition, the reader is referred

to the preceding review.

The additional chapters include discussions of joint distributions, continuous distributions, central limit theorem for the binomial, sampling theory (with and without replacement), and least squares. There is also a third appendix, dealing with a theorem on independence. Also included are answers to the even-numbered exercises.

This reviewer found the discussion of least squares, although somewhat clumsy in the absence of calculus, extremely clear and well written. The treatment presented is unusual and refreshing. Most texts not using calculus avoid any discussion of the minimizing procedure.

This text is definitely not another cookbook of techniques of statistics, but rather an elementary mathematical treatment of probability with some statistical applications. It does not mention such topics as the Poisson distribution, small sample theory and the "t" distribution, chi-square tests and the analysis of variance. This text is well suited for a one-semester course for nonmathematics majors as a background for further work in statistical applications. There are few good books now on the market which serve this purpose and this reviewer considers this one excellent.

HARVEY J. ARNOLD
Wesleyan University

Introduction to Vector Analysis. By Harry F. Davis. Boston, Allyn and Bacon, 1961. xii+359 pp. $7.95.

The first four chapters of this book contain the material suggested by the title, namely elementary vector algebra and analysis in Euclidean three space up to the integral theorems of Gauss and Stokes. The presentation here is not particularly unusual; in fact, the author describes it as being "in certain respects old-fashioned." The author does not strive for a high level of rigor but seems intent on developing the reader's intuition. For example, the principal theorems are first given over-simplified "proofs" (clearly labeled as such) which are not unlike those found in many physics books in which usable tools are needed in a hurry. Following each such "proof" is a proof—usually a proof of the result with additional simplifying assumptions, e.g. Gauss' Theorem is proved for spherical regions.

It is Chapter 5 which makes this book different from many others with similar titles. The section titles, listed here in order, indicate the subjects touched on in this chapter: Introduction, Cartesian Spaces, Vector Spaces, Matrices, Linear Transformations, Linear Functionals, Grassman Products, Scalar Products and Vector Products, Linear Transformations (Continued), Change of Basis, The Tensor Viewpoint, Special Classes of Transformations, Scalar Fields and the Gradient, Vector Fields, Historical Notes. This is a formidable list of topics to appear in a vector analysis course after the major work of the course is done. The author attempts to keep the discussion from getting out of hand by treating in detail only the three dimensional case in

many instances. Despite this the reviewer feels that the student would master very little of this material and probably would have a false sense of accomplishment from learning the new vocabulary if he did not suffer from loss of interest and frustration because of the nearly impossible task confronting him.

The exposition of this much material in a 93-page chapter is necessarily hurried. In some places this causes significant omissions. For example, the text of the section in which linear dependence is defined contains no example of a linearly dependent set of vectors. In the section on change of basis, one is never quite sure whether the symbol "$\overset{e}{1}$" stands for the first vector in the old basis, its components (1, 0, 0) in the old basis, or its components in the new basis.

The text seems to contain no serious errors or misprints. Although there are numerous exercises there is a shortage of problems to challenge the capable student. Answers and/or hints are given for most of the exercises. There is a very complete index.

H. E. CHRESTENSON
Reed College

From an Ivory Tower. By Bernard A. Hausmann, S.J. The Bruce Publishing Co., Milwaukee, Wisconsin, 1960. vii+122 pp. $3.50.

This is not, as is claimed on the dust jacket, "A giant step toward a 'meeting of the minds' of mathematicians and philosophers," (nor, I think, was it seriously intended as such), but it is a book which may arouse some interest in the foundations and philosophy of mathematics among students, and among philosophers who may not have thought about these matters. It deals on an elementary level with a variety of topics, including Euclidean and non-Euclidean geometry, groups, the number system, infinite cardinals, and Boolean algebra. Unfortunately, the author's terminology is unconventional, and the writing shows signs of having been rushed.

The most serious mathematical blunder is in the definition of the product of two Dedekind cuts (p. 71), under which the product of two cuts is not a cut at all. Nearly as bad, are, "The set of all positive even integers is equal to the set of all positive even and odd integers," (Theorem 3, p. 83), "The set of all real and imaginary numbers is called the set of all complex numbers," (p. 74, after the introduction of "imaginary" numbers as ordered pairs), and "What is common to a and not-a is nothing, i.e., the null class," (p. 93). Section II of chapter VI is called "Relative numbers," but these are never introduced or explained; the section introduces "rational integers" (integers) as ordered pairs of "natural integers." And functional notation is introduced, without a word of explanation, in the chapter on Boolean algebra.

Perhaps the difficulty is that expressed in the following manner on page 90: "Moreover symbols are so much more concise than words, and enable the mind to grasp a thought so much more rapidly and accurately." The book abounds with statements like, "The reason why this will not be done is the difficulty of the task," (p. 52), "Let x be real and defined by the relation $0 \leqq x \leqq 1$," (p. 85),

"Euclid's proof of many theorems is false . . . the lack of consequence is due . . . " (p. 54), and "If their arguments do not conclude, this chapter would . . . " (p. 76).

Father Hausmann's philosophy of mathematics can be quarrelled with: he believes in real definitions (p. 7), crucial experiments (p. 34), and a technical distinction between postulates and axioms (p. 94); he believes that cardinal numbers are properties of sets (that property that all equi-numerous sets have in common is their number) (p. 86), that symbolic logic is preferred in mathematics to Aristotelian logic for reasons of convenience (p. 91) (although Aristotle had, so far as I know, no logic of relations at all), and that "In any situation where the null class cannot occur, Aristotelian logic is perfectly adequate" (p. 112).

No defense is offered for these beliefs, and I can disagree with all of them. But I can only approve of Father Hausmann's desire to narrow the gulf between Scholasticism and modern developments in logic and the foundations of mathematics, and I can only applaud his willingness to narrow that gulf from both sides. I wish he had done it better. Perhaps he should not have made an effort "to avoid the technicalities and elisions which would please mathematicians" (p. vi).

HENRY E. KYBURG, JR.
The Rockefeller Institute

BRIEF MENTION

Elementary Differential Equations (Second Edition). By W. T. Martin and Eric Reissner. Addison-Wesley, Reading, Mass., 1961. ix+331 pp. $6.75.

The changes from the first edition, which was reviewed in this MONTHLY, v. 65 (1958). pp. 457–458, include more exercises, additional work on linear motion with variable mass and Fourier series, new sections on plane motion and differential operator methods, and various minor improvements.

The Main Stream of Mathematics. By Edna E. Kramer. Fawcett World Library, New York, 1961. 352 pp. 75 cents.

The original, hard-cover, edition was reviewed in this MONTHLY, v. 58 (1951), pp. 501–502. Dover Publications reprints (paper).

Vector Analysis. By Edwin Bidwell Wilson. 1960, xviii+436 pp. $2.00. Founded upon the lectures of J. Willard Gibbs.

New Thinking in School Mathematics. Office for Scientific and Technical Personnel of the Organisation for European Economic Co-operation, Paris, 1961. 246 pp. (paper).

This book consists of two parts:

(1) A report of a two-week seminar held at Royaumont, France in late 1959 on "New Thinking in Mathematical Education." The U. S. representatives were M. H. Stone, A. W. Tucker, E. G. Begle, R. E. K. Rourke, and H. F. Fehr.

(2) A summary of a survey of practices and trends in school mathematics in each of the O.E.E.C. member countries, in Canada, and in the U. S.

As long as the supply lasts, complimentary copies of this book may be obtained by writing to the Buffalo office of the Mathematical Association of America.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professors Lenora D. Briggs and E. M. Hughes, Nebraska State Teachers College, Chadron, represented the Association at the inauguration of Dr. F. Clark Elkins as President of Nebraska State Teachers College, Chadron, on November 6, 1961.

Professor O. H. Hamilton, Oklahoma State University, represented the Association at the inauguration of Dr. James R. Scales as President of Oklahoma Baptist University on December 10, 1961.

Professor D. R. Ryan, Gonzaga University, represented the Association at the inauguration of The Very Reverend John P. Leary as President of Gonzaga University on October 28, 1961.

Adelphi College: Dr. J. K. Thurber, New York University, has been appointed Assistant Professor; Assistant Professor Donald Solitar has been promoted to Associate Professor.

Colorado State University: Drs. D. L. Bentley, Stanford University, and Kenzo Seo, Purdue University, have been appointed Assistant Professors; Dr. E. E. Remmenga is on sabbatical leave at the Waite Agricultural Research Institute, Adelaide, Australia.

Los Angeles State College: Messrs. H. R. Gillette, University of California, Berkeley, and S. S. Gendelman, and Drs. J. H. DeHardt, Siegfried Krapf, and K. H. Roth have been appointed Assistant Professors; Assistant Professor R. J. Diamond has been promoted to Associate Professor.

Mary Washington College: Mr. L. J. Jones, Miss Lois J. Reid, and Miss Mary E. Montgomery have been appointed Instructors.

Miami University: Miss Michelle Harrell, and Messrs. E. B. Games and L. F. Volk have been appointed Instructors; Professor W. L. Strother is on leave of absence for the academic year at the University of Florida; Professor C. E. Capel has been appointed Acting Chairman of the Department of Mathematics.

New York University: Drs. Ram Gnanadesikan, Bell Telephone Laboratories, and Frank Scalora, International Business Machines, have been appointed Adjunct Associate Professors; Drs. Gilbert Baumslag, C. S. Gardner, Frank Karal, Anneli Lax, Robert Lewis, J. G. Stampfli, and N. S. Rosenfeld have been appointed Assistant Professors; Messrs. M. J. Meisner, New York University, and Arthur Schlissel, Hunter College, have been appointed Instructors; Professors Aryeh Dvoretsky, Hebrew University, Jerusalem, Israel, Beno Eckmann, Swiss Federal Institute of Technology, Zurich, Switzerland, and B. H. Neumann, University of Manchester, England, have been appointed Visiting Professors; Associate Professor H. B. Keller has been promoted to Professor.

North Dakota State University: Dr. C. G. Schilling, Gannon College, has been appointed Associate Professor; Mr. J. B. Sperry, Shawnee Mission North High School, Merriam, Kansas, has been appointed Instructor; Messrs. G. W. Nelson and Richard Shermoen have been promoted to Assistant Professors.

Oberlin College: Miss Joanne Phillips, Brown University, has been appointed Instructor; Associate Professor Samuel Goldberg has been promoted to Professor; Assistant Professor E. T. Wong has been promoted to Associate Professor.

Pennsylvania State University: Professor H. B. Ribeiro, University of Nebraska, and Dr. S. G. Mrowka, University of Michigan, have been appointed Associate Professors;

Drs. Bruce Barnes, Jet Propulsion Laboratory, Pasadena, California, R. A. Moore, Indiana University, Donald Rung, University of Notre Dame, S. N. Singh, University of California, Berkeley, and H. R. Stevens, Hamburg University, Germany, have been appointed Assistant Professors; Dr. Bruno Mueller, University of Mainz, Germany, and Mr. H. S. Hahn, University of Illinois, have been appointed Visiting Assistant Professors; Professor Friedrich Kasch, University of Heidelberg, Germany, has been appointed Distinguished Visiting Professor; Associate Professors Lowell Schoenfeld and Josephine Mitchell have been promoted to Professors; Assistant Professor S. F. Mack has been promoted to Associate Professor.

Southern Illinois University: Dr. C. E. Langenhop, Mathematica, Princeton, New Jersey, has been appointed Professor; Assistant Professor F. G. Asenjo, Georgetown University, and Drs. L. D. Gates, Jr., Babcock and Wilcox Company, Lynchburg, Virginia, and T. H. Starks, E. I. du Pont de Nemours, Wilmington, Delaware, have been appointed Associate Professors; Dr. P. E. Long, Oklahoma State University, has been appointed Assistant Professor; Mr. J. S. Brown has been promoted to Instructor.

University of Buffalo: Mr. W. E. Price, Mrs. Dolores C. Schmidt, and Miss Mary E. Graves have been appointed Instructors.

University of California, Los Angeles: Drs. J. W. Smith and Y. H. Clifton, Massachusetts Institute of Technology, have been appointed Assistant Professors; Associate Professor Alfred Horn has been promoted to Professor; Assistant Professors C. C. Chang and P. C. Curtis, Jr. have been promoted to Associate Professors; Professor P. H. Daus retired July 1, 1961 with the title Professor Emeritus.

University of Cincinnati: Assistant Professor Frank Wagner, Marquette University, and Dr. K. C. Ha, University of North Carolina, have been appointed Assistant Professors; Assistant Professors Jean D. Gibbons, Mercer University, and T. L. Hicks, Ozarks Junior College, and Messrs. Y. G. Tsuei, Colorado State University, H. T. Anderle, and John Niehaus have been appointed Instructors; Mr. K. R. Meyer has been appointed Acting Instructor; Associate Professors W. E. Restemeyer, E. F. White, and J. L. Baker, Jr. have been promoted to Professors; Associate Professor H. D. Lipsich has been promoted to Professor and appointed Head of the Department of Mathematics; Professor Arno Jaeger has been appointed Director of Graduate Studies.

University of Delaware: Dr. R. W. Kennard, E. I. du Pont de Nemours, Wilmington, Delaware, has been appointed Associate Professor; Assistant Professors C. C. Braunschweiger and E. J. Pellicciaro have been promoted to Associate Professors.

University of Miami: Dr. Edwin Duda, University of Virginia, has been appointed Assistant Professor; Messrs. R. W. Fitzgerald and D. P. Stadtlander have been appointed Instructors; Professor E. J. Moulton retired June 1961.

University of Minnesota-Institute of Technology: Visiting Associate Professor I. S. Gal has been appointed Associate Professor; Drs. Alfred Aeppli, Cornell University, and Howard Jenkins, Stanford University, have been appointed Assistant Professors; Dr. Lisl N. Gal has been appointed Research Associate; Professor Max Koecher, University of Münster, Germany, has been appointed Visiting Professor; Associate Professors Edgar Reich, Helmut Röhrl, M. L. Stein and H. F. Weinberger have been promoted to Professors.

University of Montreal: Dr. Anatole Joffe, McGill University, has been appointed Assistant Professor; Messrs. Bernard Courteau, Jean Menard, and Marc Venne have been appointed Senior Instructors; Assistant Professor Roland Guy has been promoted to Associate Professor; Professor Alexis Zinger is on leave for the academic year at the University of Oxford, England; Professor Maurice L'Abbe represented Canada at the second meeting of the Groupement de Mathematiciens d'Expression Latine held at Florence and Bologna, Italy, from September 26 to October 3, 1961.

University of Southwestern Louisiana: Drs. D. R. Andrew, University of Pittsburgh,

and J. W. Kenelly, Jr., University of Florida, have been appointed Assistant Professors; Assistant Professor O. J. Huval has been promoted to Associate Professor.

University of Western Ontario: Drs. C. N. Lee, University of Michigan, and M. M. Ali have been appointed Assistant Professors; Miss Anne E. Bode, Huron College, has been appointed Instructor; Assistant Professor G. W. Goes, DePaul University, has been appointed N. R. C. Postdoctoral Fellow; Associate Professor W. H. Wehlau has been promoted to Professor; Assistant Professors G. E. Cross and James Sanders have been promoted to Associate Professors; Professor Arthur Woods and Mr. W. H. Adamson retired July 1, 1961.

University of Wisconsin: Associate Professor C. H. Wilcox, California Institute of Technology, has been appointed Professor; Professor J. A. Nohel, Georgia Institute of Technology, has been appointed Associate Professor; Dr. L. S. Levy, University of Illinois, has been appointed Assistant Professor; Drs. F. J. Kosier and F. J. Forelli, University of California, Berkeley, and C. H. Edwards, University of Tennessee, have been appointed Instructors; Associate Professor C. W. Curtis has been promoted to Professor; Assistant Professors Anatole Beck, Fred Brauer, Joshua Chover, E. A. Robinson, and Hans Schneider have been promoted to Associate Professors; Mr. G. O. Losey has been promoted to Assistant Professor.

University of Wyoming: Mr. L. E. Shader and Miss Nancy C. Baggs have been appointed Instructors; Assistant Professor Greta Neubauer retired June 30, 1961.

Washington State University: Drs. W. M. Cunnea, University of California, Berekley, and H. C. Wiser, University of Utah, have been appointed Assistant Professors; Messrs. H. B. Emerson, Washington State University, C. J. Potratz, Pacific Lutheran College, J. C. Pelzl, Mankato State College, and H. F. Rahmlow, University of the Pacific, have been appointed Instructors; Assistant Professors W. E. Barnes and C. T. Long have been promoted to Associate Professors; Associate Professor O. W. Rechard has been promoted to Professor.

Mr. William Adams, Rensselaer Polytechnic Institute, has been appointed Instructor at Siena College.

Associate Professor O. R. Ainsworth, University of Alabama, has been promoted to Professor.

Dr. Furio Alberti, University of Illinois, Chicago Undergraduate Division, has been promoted to Assistant Professor.

Professor Lillian K. Bradley, Prairie View Agricultural and Mechanical College, has been appointed Associate Professor at Texas Southern University.

Dr. C. K. Bradshaw, San Jose State College, has been promoted to Assistant Professor.

Brother Andrew, St. Anthony's High School, Smithtown, New York, has been appointed Chairman of the Department of Mathematics at St. Francis Preparatory School, Brooklyn, New York.

Brother L. John Saino, St. Mel High School, Chicago, Illinois, has been appointed Acting Chairman of the Department of Mathematics at Lewis College.

Associate Professor H. N. Carter, University of Tulsa, has been promoted to Professor.

Associate Professor R. L. Chittim, Bowdoin College, is on leave at the University of London, England, on a National Science Foundation Faculty Fellowship.

Dr. D. R. Clutterham, The Martin Company, Orlando, Florida, has been appointed Manager of the newly organized department, Computers and Digital Information.

Mr. A. A. Gibbons, William Jewell College, has accepted a position as Chemical Sales Representative with the Pittsburgh Plate Glass Company, Dallas, Texas.

Assistant Professor J. D. Harrell, Jr., Agricultural and Technical College of North Carolina, has been appointed Assistant Professor at North Carolina College at Durham.

Mrs. Annabelle Henry, Ohio State University, has been appointed Instructor at the University of Akron.

Professor Emeritus T. H. Hildebrandt, University of Michigan, has been appointed Visiting Professor at Western Michigan University.

Dr. William Huebsch, Institute for Advanced Study, has been appointed Associate Professor at Western Reserve University.

Dr. R. E. Hughs, Lehigh University, has been promoted to Assistant Professor.

Mr. C. C. Kilby, Jr., System Development Corporation, has been appointed Assistant Professor at Central Connecticut State College.

Mrs. Gladdis E. Loehr, Arizona State University, has accepted a position as Mathematician at the Pacific Missile Range, U. S. Naval Missile Center, Point Mugu, California.

Assistant Professor M. A. McKiernan, Illinois Institute of Technology, has been appointed Associate Professor at the University of Waterloo, Ontario, Canada.

Assistant Professor Norbert Martin, College of Steubenville, has been promoted to Associate Professor.

Dr. A. M. Mood, General Analysis Corporation, Santa Monica, California, has accepted a position as Vice-President of CEIR, Beverly Hills, California.

Assistant Professor M. G. Ossesia, Duquesne University, has been appointed Professor at Slippery Rock State College.

Mr. William Outerson, Purdue University, has been appointed Instructor at the University of Connecticut.

Professor P. S. Pretz, St. Benedict's College, retired June 1, 1961 with the title Professor Emeritus.

Associate Professor B. E. Rhoades, Lafayette College, has been named a Jones Faculty Lecturer for the year 1961–62. The Jones Faculty Lecture Series was established at Lafayette College under the auspices of a fund for the reward of superior teaching.

Associate Professor J. A. Schumaker, Montclair State College, has been appointed Professor and Chairman of the Department of Mathematics at Rockford College.

Mr. C. C. Thompson, Yale University, has been appointed Associate Professor at Hollins College.

Mr. L. J. Weill, Louisiana State University, has been appointed Instructor at the University of Alabama.

Assistant Professor P. W. Zehna, Colorado State College, has accepted a position as Statistician with CEIR, Palo Alto, California.

Mr. Neill McShane, Yale University, died on March 11, 1961. He was a member of the Association for six years.

The notice of the death of Professor E. S. Ashcraft (this MONTHLY, vol. 68, 1961, p. 688) was in error. The correct notice follows:

Professor Emeritus T. B. Ashcraft, Colby College, died December 17, 1960. He was a charter member of the Association.

GRADUATE FELLOWSHIPS IN APPLIED MATHEMATICS

To afford opportunity for graduate study directed mainly toward the fields of applied mathematics and the related fields of mathematical analysis, the Mathematics Research Center, U. S. Army, offers a number of fellowships to candidates with distinguished academic records. A candidate must have a bachelor's degree in mathematics, physics, or engineering, and he must obtain admission to the Graduate School of the University of Wisconsin. He must have his proposed course of study approved by the Director of the Mathematics Research Center. He must be a United States citizen.

A fellowship holder is expected to devote his full time to study that leads to a master's

degree in applied mathematics, analysis, or statistics at the University of Wisconsin. He will receive full University residence credit for his work. Further information and blanks for application may be obtained by writing Professor R. E. Langer, Director, Mathematics Research Center, U. S. Army, The University of Wisconsin, Madison 6, Wisconsin.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

AWARD FOR DISTINGUISHED SERVICE TO MATHEMATICS

The Board of Governors of the Mathematical Association of America at its meeting on August 28, 1961, in Stillwater, Oklahoma, has voted to name Dr. Mina S. Rees (Mrs. L. Brahdy), Dean of Graduate Studies of the City University of New York, as the first recipient of the Award for Distinguished Service to Mathematics.

A certificate and monetary award in the amount of five hundred dollars was presented to Dean Rees at the time of the January 1962 meeting of the Association in Cincinnati, Ohio.

In establishing the conditions of this recognition it was specified that the award was to be made "for outstanding service to mathematics, other than mathematical research" and that "the contribution should be such as to influence significantly the field of mathematics or mathematical education on a national scale" (not necessarily in the United States).

The following citation was prepared by Professor Wallace Givens, Chairman of the Committee on the Award for Distinguished Service to Mathematics, with the assistance of information contributed by Dr. F. J. Weyl and President John J. Meng of Hunter College.

Citation of Mina S. Rees (Mrs. L. Brahdy).

The distinguished career of Dean Mina S. Rees in administration of government and academic enterprises which is here recognized was based on a sound mathematical education at Hunter College, where she graduated summa cum laude, at Columbia University and at the University of Chicago which awarded her the degree of Doctor of Philosophy in 1931. After teaching mathematics at Hunter College as Instructor, Assistant Professor and Associate Professor during the years 1926–43, she entered government service in 1943 as Technical Aide and Executive Assistant to the Chief of the Applied Mathematics Panel in the Office of Scientific Research and Development. In 1946 she was named Head of the Mathematics Branch of the Office of Naval Research and later (1949–53) continued her service in the ONR as Director of the Mathematical Sciences Division and (1952–53) as Deputy Science Director. During these years in Washington, Mina Rees firmly built into the permanent structure and policies of the ONR the principle that the full scope of mathematics should form part of the total scientific effort properly supported by government sponsored research programs.

In recognition of her service during the war years, Dean Rees was awarded the President's Certificate of Merit as well as the King's Medal for Service in the Cause of Freedom from Britain.

Of her work in establishing the current pattern of research in mathematics, a Washington colleague has expressed his appreciation in the following terms:

> "Through her personal imagination, initiative, and judgement, as well as her indefatigable leadership in rousing this country's mathematical community to state its case and turn its effort to the development of ways for the effective use of the re-

search support funds which thus became available to it in increasing amounts, she has contributed perhaps more than any other single person to the scope and wealth of present day mathematical research activity in the United States which has been made possible by the judiciously channelled massive injection of Federal funds."

In 1953, Mina Rees returned to Hunter College to resume her long interrupted work there in the position of Professor of Mathematics and Dean of Faculty. In July 1961 her appointment as Dean of Graduate Studies in the newly established City University of New York was announced. In this position she heads a graduate council which coordinates existing graduate work in the four senior colleges of the University and which supervises the organization of new doctoral programs.

Among her publications which have contributed to the vision of the national need for development of mathematics are the following:

The mathematics program of the Office of Naval Research. Bull. Am. Math. Soc., vol. 54, 1948.
Professional opportunities in mathematics. (With H. W. Brinkman, Z. I. Mosesson, S. A. Schelkunoff and S. S. Wilks), this MONTHLY, vol. 58, 1951.
Modern mathematics and the gifted student, The Mathematics Teacher, vol. 46, 1953.
The mathematician in government establishments. Proc. of a Conf. on Training in Appl. Math., Am. Math. Soc., 1953.
Mathematics and federal support. Science, vol. 119, 1954.
New frontiers for mathematicians. Pi Mu Epsilon Journal, vol. 2, 1955.
Mathematicians in the market place, this MONTHLY, vol. 65, 1958.
Support of higher education by the Federal Government, this MONTHLY, vol. 68, 1961.

On the need for personnel in the field of digital computation and on the growth of this activity, her articles include:

The federal computing machine program. Science, vol. 112, 1950.
Digital computers—their nature and use. American Scientist, vol. 40, 1952.
Computers. The Scientific Monthly, vol. 79, 1954.
Digital computers, this MONTHLY, vol. 62, 1955.
The impact of the computer, The Mathematics Teacher, vol. 51, 1958.

More technical publications include:

Division algebras associated with an equation whose group has four generators, Am. Jour. Math., vol. 54, 1932.
On the solution of nonlinear hyperbolic differential equations by finite differences. (With R. Courant and E. Isaacson.) Comm. on Pure and Appl. Math., vol. 5, 1952.

She also contributed by her editorship or partial authorship of classified volumes published in 1947 by the Applied Mathematics Panel of the National Defense Research Council (Three Volume Summary Technical Report and volume 2 of Theory of Air Warfare).

On the international scene, Dean Rees has served as Honorary Advisor to the British Universities Summer Schools since 1957 and in 1948 published an article on "Applied Mathematics in Western Europe" in the Monthly Research Report of the Office of Naval Research.

The extent of the contributions to the common welfare and in particular to the mathematical and scientific development of the nation for which Dean Mina Rees is here honored are indicated by the following list of positions she has held.

Public Service:

Advisory Committee on Math., National Bureau of Standards Member, 1954–58, Chairman, 1954–57.

Advisory Panel for Mathematics, National Science Foundation Member, 1955–58.
General Sciences Advisory Panel, Department of Defense Member, 1958–61.
Comm. on Survey of Research in Math. in U.S.A., National Research Council Member and Chairman of Subcommittee, 1954–56.
Advisory Board, Computation and Exterior Ballistics Laboratory, U. S. Naval Proving Ground, Dahlgren Member, 1958–61.
Steering Committee, Regional Orientation Conferences in Math. Educ. Member, 1960–61.
Comm. of Mathematicians Advisory to Continental Classroom Vice Chairman, 1959–61.
1959 Summer Writing Group, School Mathematics Study Group Member.
Regional Committee, Woodrow Wilson Fellowship Program Member, 1957–61.

Service in Scientific and Educational Groups:

American Mathematical Society: Trustee, 1955–59.
Mathematical Association of America: Nominating Committee for 1959, Chairman; Committee on Advisement and Personnel, Member, since 1961.
Mathematical Association of America, N. Y. Metropolitan Section: Vice Chairman, 1955; Chairman, 1956.
American Association for the Advancement of Science: Vice President and Chairman, Sect. A, 1953–54; Member, Board of Directors, 1957–60.
Mathematics Division, National Research Council: Member, 1953–56; Executive Committee, 1954–56; Committee on Applications of Mathematics, 1953–59; Committee on Regional Development, 1955–57; Nominating Committee, 1960.
Society for Industrial and Applied Mathematics: Member of Council, 1955–58; Committee on Visiting Lecturers, 1959–60; Representative on AAAS Council, 1958–61.
Policy Committee for Math.: Member, 1952–53.
Conference Board of the Mathematical Sciences: Member, since 1959; Vice Chairman, since 1961.
New York Academy of Sciences: Member of Council, 1957–60.
Executive Committee, American Conference of Academic Deans, since 1960.
Consultant: Institute of International Education; Courant Institute of Mathematical Sciences, N.Y.U.; Bureau of Census; National Security Agency; Office of Naval Research.

COMBINED MEMBERSHIP LIST

The 1961–1962 edition of the Combined Membership List of AMS, MAA, and SIAM was mailed to members of the Association in December. Any member who failed to receive a copy may obtain one by writing to the Buffalo office of the Association.

Each member should consult his entry in both the alphabetical list and the geographical list to make sure that the entries are correct. Necessary changes should be reported to the Buffalo office. Department chairmen are also requested to consult departmental entries.

It is expected that copy for the next List will be sent to the printer on October 1. Corrections should be received by the Buffalo office no later than September 25.

HARRY M. GEHMAN, *Executive Director*

THE OCTOBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the Mathematical Association of America was held on Friday, October 20, 1961, at Indiana State College, Terre Haute, in conjunction with the annual meeting of the Indiana Academy of Science. Fifty-four persons attended, including 36 members of the Association. Professor Merrill Shanks of Purdue University presided at the morning session, which was given over to a panel discussion on the subject, *Aspects of the curriculum revolution in mathematics*. The topics discussed were:

The objectives of the SMSG curricula, by Professor Robert Troyer, Indiana University.

The high school teacher and SMSG, by Mr. Allan Weinheimer, North Central High School, Indianapolis, Indiana.

The small college and the advanced freshman, by Professor Paul Mielke, Wabash College.

A new methods course, by Professor Joseph Kennedy, Indiana State College.

The new Indiana certification requirements, by Professor P. D. Edwards, Ball State Teachers College.

Professor John Yarnelle, Hanover College, presided at the afternoon session, which included a business meeting and an invited hour address by Professor Harry Pollard of Purdue University. Professor Pollard's subject was *The revival of celestial mechanics.*

PAUL MIELKE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, May 5, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, U. S. Naval Weapons Laboratory, Dahlgren, Virginia, April 28, 1962.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA, College of St. Teresa, Winona, May 12, 1962.

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NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, November 24, 1962.

NORTHERN CALIFORNIA

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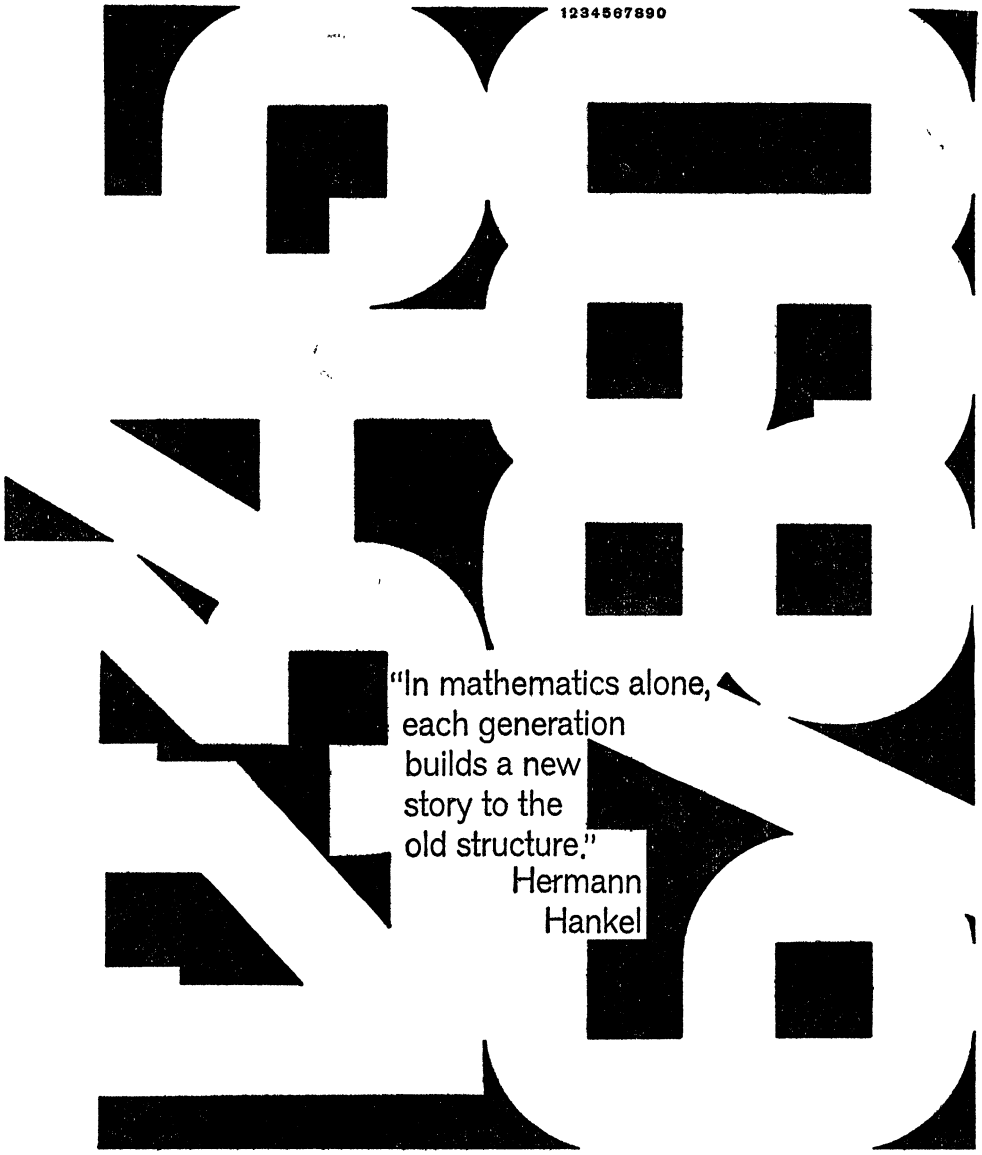
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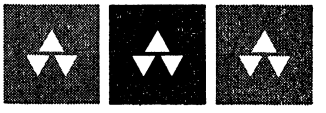
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ON THE MATHEMATICS CURRICULUM OF THE HIGH SCHOOL

The following memorandum was composed by several of the undersigned and sent to 75 mathematicians in the United States and Canada. No attempt was made to amass a large number of signatures by canvassing the entire mathematical community. Rather, the objective was to obtain a modest number from men with mathematical competence, background, and experience and from various geographical locations. A few of the undersigned, whose support is indeed welcomed, volunteered their names when they learned about the memorandum from a colleague.

The mathematicians of this country now have a more favorable climate in which to develop and gain acceptance of improvements in mathematics education. Indeed a number of groups have recognized the opportunity and are working hard and with the best of intentions to utilize it.

It would, however, be a tragedy if the curriculum reform should be misdirected and the golden opportunity wasted. There are, unfortunately, factors and forces in the current scene which may lead us astray. Mathematicians, reacting to the dominance of education by professional educators who may have stressed pedagogy at the expense of content, may now stress content at the expense of pedagogy and be equally ineffective. Mathematicians may unconsciously assume that all young people should like what present day mathematicians like or that the only students worth cultivating are those who might become professional mathematicians. The need to learn much more mathematics today than in the past may cause us to seek shortcuts which, however, could do more harm than good.

In view of the possible pitfalls it may be helpful to formulate what appear to us to be fundamental principles and practical guidelines.

1. For whom. The mathematics curriculum of the high school should provide for the needs of all students: it should contribute to the cultural background of the general student and offer professional preparation to the future users of mathematics, that is, engineers and scientists, taking into account both the physical sciences which are the basis of our technological civilization, and the social sciences which may need progressively more mathematics in the future. While providing for the other students, the curriculum can also offer the most essential materials to the future mathematicians. Yet to offer such subjects to all students as could interest only the small minority of prospective mathematicians is wasteful and amounts to ignoring the needs of the scientific community and of society as a whole.

2. Knowing is doing. In mathematics, knowledge of any value is never possession of information, but "know-how." To know mathematics means to be able to do mathematics: to use mathematical language with some fluency, to do problems, to criticize arguments, to find proofs and, what may be the most

important activity, to recognize a mathematical concept in, or to extract it from, a given concrete situation.

Therefore, to introduce new concepts without a sufficient background of concrete facts, to introduce unifying concepts where there is no experience to unify, or to harp on the introduced concepts without concrete applications which would challenge the students, is worse than useless: premature formalization may lead to sterility; premature introduction of abstractions meets resistance especially from critical minds who, before accepting an abstraction, wish to know why it is relevant and how it could be used.

3. Mathematics and science. In its cultural significance as well as in its practical use, mathematics is linked to the other sciences and the other sciences are linked to mathematics, which is their language and their essential instrument. Mathematics separated from the other sciences loses one of its most important sources of interest and motivation.

4. Inductive approach and formal proofs. Mathematical thinking is not just deductive reasoning; it does not consist merely in formal proofs. The mental processes which suggest what to prove and how to prove it are as much a part of mathematical thinking as the proof that eventually results from them. Extracting the appropriate concept from a concrete situation, generalizing from observed cases, inductive arguments, arguments by analogy, and intuitive grounds for an emerging conjecture are mathematical modes of thinking. Indeed, without some experience with such "informal" thought processes the student cannot understand the true role of formal, rigorous proof which was so well described by Hadamard: "The object of mathematical rigor is to sanction and legitimatize the conquests of intuition, and there never was any other object for it."

There are several levels of rigor. The student should learn to appreciate, to find and to criticize proofs on the level corresponding to his experience and background. If pushed prematurely to a too formal level he may get discouraged and disgusted. Moreover the feeling for rigor can be much better learned from examples wherein the proof settles genuine difficulties than from hair-splitting or endless harping on trivialities.

5. Genetic method. "It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state" wrote James Clerk Maxwell. There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea. This may suggest a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race—retrace its great lines, of course, and not the thousand errors of detail.

This genetic principle may safeguard us from a common confusion: If A is

logically prior to B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history. On the whole, we may expect greater success by following suggestions from the genetic principle than from the purely formal approach to mathematics.

6. "Traditional" mathematics. The teaching of mathematics in the elementary and secondary schools lags far behind present day requirements and highly needs essential improvement: we emphatically subscribe to this almost universally accepted opinion. Yet the often heard assertion that the subject matter taught in the secondary schools is obsolete should be closely scrutinized and should not be taken simply at face value. Elementary algebra, plane and solid geometry, trigonometry, analytic geometry and the calculus are still fundamental, as they were fifty or a hundred years ago: future users of mathematics must learn all these subjects whether they are preparing to become mathematicians, physical scientists, social scientists or engineers, and all these subjects can offer cultural values to the general students. The traditional high school curriculum comprises all these subjects, except calculus, to some extent; to drop any one of them would be disastrous.

What is bad in the present high school curriculum is not so much the subject-matter presented as the isolation of mathematics from other domains of knowledge and inquiry, especially from the physical sciences, and the isolation of the various subjects offered from each other; even the techniques and theorems within the same subject appear as isolated, disconnected tricks to the student, who is left in the dark about the origin and the purpose of the manipulations and facts that he is supposed to learn by rote. And so, unfortunately, it often happens that the material offered appears as useless and boring, except perhaps, to the few prospective mathematicians who may persist despite the curriculum.

7. "Modern" mathematics. In view of the lack of connection between the various parts of the present curricula, the groups working on new curricula may be well advised in seeking to introduce unifying general concepts. We think, too that judicious use of sets and of the language and concepts of *abstract* algebra may bring more coherence and unity into the high school curriculum. Yet, the spirit of modern mathematics cannot be taught by merely repeating its terminology. Consistently with our principles, we wish that the introduction of new terms and concepts should be preceded by sufficient *concrete* preparation and followed by genuine, challenging applications and not by thin and pointless material: one must motivate and apply a new concept if one wishes to convince an intelligent youngster that the concept warrants attention.

We cannot enter here into detailed analysis of the proposed new curricula, but we cannot leave unsaid that, in judging them on the basis of the guidelines stated above (Sections 1–5), we find points with which we cannot agree.

Of course, not all mathematicians have the same taste. Mathematics has

many aspects. It can be regarded as an instrument to understand the world around us: mathematics presumably possessed this value for Archimedes and Newton. Mathematics can also be regarded as a game with arbitrary rules where the principal consideration is to stick to the rules of the game: some such view may be considered suitable for certain problems of foundations. There are several other aspects of mathematics, and a professional mathematician may favor any one. Yet when it comes to teaching, the choice is not a mere matter of taste. We may expect that an intelligent youngster would want to explore the world around him, but we cannot expect him to learn arbitrary rules: why just these and not others?

At any rate, we fervently wish much success to the workers on the new curricula. We wish especially that the new curricula should reflect more the connection between mathematics and science and carefully heed the distinction between matters logically prior and matters which should have priority in teaching. Only in this way can we hope that the basic values of mathematics, its real meaning, purpose, and usefulness will be made accessible to all students, including of course, the prospective mathematicians. The recently expressed "wide spread concern about a trend to excessive emphasis on abstraction in the teaching of mathematics to engineers" points in the same direction.*

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THE BRING-JERRARD EQUATION AND WEIERSTRASS ELLIPTIC FUNCTIONS

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We assume that the reader is acquainted with the basic properties of elliptic functions which are of the Weierstrass $\wp$-type. We will quickly summarize what we need concerning the $\wp$-function and refer the reader to [2, pp. 146–188 and pp. 227–228] for a complete discussion.

E1. If ω_1 and ω_2 are complex numbers with a nonreal ratio, then a Weierstrass $\wp$-function with periods ω_1 and ω_2 exists. We denote this function by $\wp(z; \omega_1, \omega_2)$.

E2. $\wp(z; \omega_1, \omega_2)$ has so-called invariants $g_2(\omega_1, \omega_2)$ and $g_3(\omega_1, \omega_2)$ which enter into the differential equation satisfied by $\wp$, namely: $[\wp'(z)]^2 = 4\wp^3(z) - g_2\wp(z) - g_3$.

E3. If g_2 and g_3 are arbitrarily assigned complex numbers meeting the condition $g_2^3 - 27g_3^2 \neq 0$, then a $\wp$-function with invariants g_2 and g_3 exists. We denote this function by $\wp(z \mid g_2, g_3)$. In particular, if g_2 is chosen to be 0 and g_3 is taken to be any complex $a \neq 0$, then the function $\wp(z \mid 0, a)$ exists. It is this latter type of $\wp$-function which will be of service to us in this work.

E4. Any $\wp$-function is of order 2 since it has but one pole of order 2 in each period-parallelogram. Thus $\wp$ takes each complex value (including ∞) exactly twice in any period-parallelogram. We will assume that ω_2/ω_1 has a positive imaginary part and will work throughout with the basic period-parallelogram P with vertices at 0, ω_1, $\omega_1+\omega_2$ and ω_2. More definitely, P consists of the points of

the straight line segments 0 to ω_1 and 0 to ω_2, excluding ω_1 and ω_2, as well as the interior of the parallelogram having these segments as adjacent sides.

E5. We will denote the half-periods $\frac{1}{2}\omega_1$, $\frac{1}{2}(\omega_1+\omega_2)$ and $\frac{1}{2}\omega_2$ by Ω_1, Ω_2 and Ω_3, respectively. We will also let Ω stand for any Ω_j $(j=1, 2, 3)$, indifferently. It is known that $\wp$ takes its values exactly twice at the Ω's so that $\wp'(\Omega_j)=0$ $(j=1, 2, 3)$. The Ω'*s* constitute the only solutions *of* $\wp'(z)=0$ in P.

E6. Concerning the distribution of values of $\wp$ in P, we can be more specific than in E4 and E5. Suppose z^* is in P and $z^*\neq 0, \Omega$. Where is the second z in P, say z^{**}, such that $\wp(z^{**})=\wp(z^*)$? The answer is that z^{**} and z^* are symmetric with respect to the proper Ω. Only one Ω is suitable as a center of symmetry if z^{**} is to lie in P when z^* is in P. It is clear that if z^* lies on a side of P, then z^{**} is on the same side and if z^* is an interior point of P, then z^{**} is likewise interior to P. When we speak of points z^*, z^{**} in the above relation, it is understood that z^* (and hence z^{**}) is not an Ω. It may be shown that $\wp'(z^*)=-\wp'(z^{**})$.

We shall refer to the above list of properties of $\wp$ when necessary.

We now discuss the necessary algebraic aspects of this paper. As is well-known, the equation $x^5+c_1x^4+c_2x^3+c_3x^2+c_4x+c_5=0$, with arbitrary complex coefficients c_j, can be transformed to the *Bring-Jerrard* type $y^5+Ay+B=0$ by a Tschirnhaus transformation [1, pp. 212–214]. Assuming $A, B\neq 0$ and setting $y=(B/A)x$, we get the equation $x^5+ax+a=0$ where $a=A^5/B^4\neq 0$. We will work throughout with this latter equation. It is important for us to know for what a the equation $x^5+ax+a=0$ has multiple roots. Now, $x^5+ax+a=0$ has multiple roots if and only if the polynomials x^5+ax+a and $5x^4+a$ (the derivative of x^5+ax+a) have common zeros. This happens only when $a=-5^5/4^4$ as is easily shown. Therefore *the equation* $x^5+ax+a=0$ *has five simple roots for each complex* $a\neq -5^5/4^4$. The particular equation $x^5-(5^5/4^4)x-5^5/4^4=0$ has $-\frac{5}{4}$ as a double root, a simple positive root and a pair of conjugate imaginary roots as is not difficult to verify.

It is given as an exercise in [3, p. 184] to prove that the elliptic function $\wp(z\,|\,g_2, g_3)\wp'(z)+\wp^2(z)-1$ has five zeros $z_1, z_2, \cdots, z_5$ in P and the numbers $\wp(z_j)$ satisfy the quintic equation $4x^5-x^4-g_2x^3+(2-g_3)x^2-1=0$. This can be done by using the differential equation for $\wp$ mentioned in E2. It is our purpose to extend the result of this exercise to all quintic equations via the Bring-Jerrard type. *We will pay particular attention to the multiplicities of the roots.*

Let us define, for each complex $a\neq 0$, the function

$$f_a(z) = \wp(z)\wp'(z) + i\sqrt{a}\wp(z) + 2i\sqrt{a}$$

where $\wp(z)=\wp(z\,|\,0, a)$, $i=\sqrt{-1}$, and $\sqrt{a}$ is a fixed root of a. Observe that $f_a(z)$ is an elliptic function of order 5 since both $\wp$ and $\wp'$ have only one pole of orders 2 and 3, respectively, in P. Thus there are precisely five values $z_1, z_2, \cdots, z_5$ in P, possibly with multiplicities, such that $f_a(z_j)=0$ $(j=1, 2, \cdots, 5)$.

LEMMA 1. *The numbers* $\wp(z_j\,|\,0, a)$ *are roots of the equation* $x^5+ax+a=0$.

Proof. Since $f_a(z_j)=0$ $(j=1, 2, \cdots, 5)$ we see that

$$\wp(z_j)\wp'(z_j) = -i\sqrt{a}\wp(z_j) - 2i\sqrt{a}.$$

Squaring both sides of the above equation and using the differential equation for $\wp$ in E2, we find

$$\wp^2(z_j)[4\wp^3(z_j) - a] = -a\wp^2(z_j) - 4a\wp(z_j) - 4a.$$

Transposing the right-hand member, cancelling the terms $\pm a\wp^2(z_j)$ and dividing by 4, we get

$$\wp^5(z_j) + a\wp(z_j) + a = 0.$$

Thus the numbers $\wp(z_j)$ are roots of $x^5+ax+a=0$.

LEMMA 2. *$f_a(z)$ has five simple zeros $z_1, \cdots, z_5$ in P for $a \neq -5^5/4^4$.*

Proof. Assume this is not the case and that at z_1, say, $f_a(z_1)=f_a'(z_1)=0$. Then, since $\wp''(z|0, a)=6\wp^2(z)$, we have

$$f_a(z_1) = \wp(z_1)\wp'(z_1) + i\sqrt{a}\wp(z_1) + 2i\sqrt{a} = 0$$

and

$$f_a'(z_1) = 10\wp^3(z_1) + i\sqrt{a}\wp'(z_1) - a = 0.$$

Solving for $\wp'(z_1)$ from the first of these equations and substituting in the second equation, we get, after some algebraic simplification: $\wp^4(z_1)=-\frac{1}{5}a$. But, by Lemma 1, this implies that one of the four values of $(-\frac{1}{5}a)^{\frac{1}{4}}$ is a root of $x^5+ax+a=0$, making $a=-5^5/4^4$. Since $a\neq -5^5/4^4$ by hypothesis, $f_a(z)$ vanishes only once at each z_j and $z_1, \cdots, z_5$ are all mutually different.

LEMMA 3. *If $a_0=-5^5/4^4$, then $f_{a_0}(z)$ has a double zero at a z_1 in P with $\wp(z_1|0, a_0)=-\frac{5}{4}$. Further, $f_{a_0}(z)$ has three additional simple zeros z_3, z_4, z_5 in P.*

Proof. Consider the function $\wp(z|0, a_0)$. We know, by E4, that there exists a z_1 in P such that $\wp(z_1|0, a_0)=-\frac{5}{4}$. From the differential equation of E2, we have: $[\wp'(z_1|0, a_0)]^2=4\wp^3(z_1)-a_0$. Solving for $\wp'(z_1)$, we get $\wp'(z_1)=\pm\frac{15}{16}\sqrt{5}$. The reason for the double sign in $\wp'(z_1)$ is that $-\frac{5}{4}$ is assumed twice in P in accordance with E4 (see also E6). Substituting these numerical values of $\wp(z_1)$, $\wp'(z_1)$ into $f_{a_0}(z)$, $f_{a_0}'(z)$ shows $f_{a_0}(z_1)=f_{a_0}'(z_1)=0$ while $f_{a_0}''(z_1)\neq 0$. We omit the details of this computation except for noting that $\wp'(z_1)$ must be taken as $-\frac{15}{16}\sqrt{5}$ if $\sqrt{a_0}$ in $f_{a_0}(z)$ is taken to be $\frac{25}{16}i\sqrt{5}$, while $\wp'(z_1)=+\frac{15}{16}\sqrt{5}$ if $\sqrt{a_0}=-\frac{25}{16}i\sqrt{5}$. Thus, a choice of $\sqrt{a_0}$ fixes z_1 uniquely between the two z's for which $\wp(z)=-\frac{5}{4}$.

Let r_3, r_4, r_5 denote the three simple roots of $x^5+a_0x+a_0=0$ $(r_1=r_2=-\frac{5}{4})$. Just as we found for the double root $-\frac{5}{4}$, there exist numbers z_3, z_4, z_5 in P such that $\wp(z_j)=r_j$ $(j=3, 4, 5)$. Use of the differential equation gives $\wp'(z_j)=\pm\sqrt{(4r_j^3-a_0)}$ for $j=3, 4, 5$. The appearance of the double sign was already

explained. Substitution of r_j, $\pm\sqrt{(4r_j^3-a_0)}$ for $\wp$, $\wp'$ in $f_{a_0}(z)$ gives $f_{a_0}(z_j)=0$ $(j=3, 4, 5)$ because $r_j^5+a_0r_j+a_0=0$. Again, we suppress the details of the easy computation except for noting, as before, that the signs of the $\wp'(z_j)$, and hence the z_j themselves, are uniquely determined by which of the two possible $\sqrt{a_0}$ is used in defining $f_{a_0}(z)$. Since no two of r_3, r_4, r_5 are the same, the numbers z_3, z_4, z_5 are different and it is clear that the z_j $(j=3, 4, 5)$ must be simple zeros of $f_{a_0}(z)$ since $z_1(=z_2)$ is a double zero and $f_{a_0}(z)$ has but five zeros in P.

LEMMA 4. *If z_1 and z_2 are distinct zeros of $f_a(z)$, then $\wp(z_1|\,0, a)\neq\wp(z_2)$.*

Proof. This was already seen to be the case for $a=a_0$ in Lemma 3. We now show that this holds in general. Assume Lemma 4 is false, i.e., let $\wp(z_1)=\wp(z_2)$ with $z_1\neq z_2$. Then by E6, we find $\wp'(z_2)=-\wp'(z_1)$ and hence:

$$\begin{aligned}&\wp(z_2)\wp'(z_2)+i\sqrt{a}\wp(z_2)+2i\sqrt{a}\\ =\;&-\wp(z_1)\wp'(z_1)+i\sqrt{a}\wp(z_1)+2i\sqrt{a}=0,\\ &\wp(z_1)\wp'(z_1)+i\sqrt{a}\wp(z_1)+2i\sqrt{a}=0.\end{aligned}$$

Subtracting, we get $2\wp(z_1)\wp'(z_1)=0$. Now $\wp(z_1)\neq 0$ since $a\neq 0$, so that $\wp'(z_1)=0$. Therefore z_1 is one of the three values Ω by E5. But this means that $\wp$ assumes a value three times in P: twice at z_1 which is one of the Ω's and again at z_2 with $z_2\neq z_1$. We thus have a contradiction since $\wp$ assumes its values exactly twice in P. Hence different z's among $z_1, \cdots, z_5$ provide different values of $\wp$.

Putting Lemmas 1 through 4 together gives the following

THEOREM. *For each complex number $a\neq 0$, the numbers $\wp(z_j|\,0, a)$ are the roots, with proper multiplicities, of the equation $x^5+ax+a=0$. Here, z_j $(j=1, 2, \cdots, 5)$ are the zeros of the elliptic function $f_a(z)$ in a period-parallelogram.*

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ARITHMETICAL FUNCTIONS ASSOCIATED WITH THE UNITARY DIVISORS OF A GENERALISED INTEGER

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1. Introduction. In a previous paper [1], I have defined *generalised integers* as follows. Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers (generalised primes) such that $1<p_1<p_2<\cdots$. Form the set $\{l\}$ of all possible p-products, i.e., products $p_1^{v_1}p_2^{v_2}\cdots$ where $v_1, v_2, \cdots$ are integers $\geqq 0$

of which all but a finite number are 0. Call these numbers *generalized integers* and suppose that no two generalised integers are equal if their v's are different. Then arrange $\{l\}$ as an increasing sequence: $1=l_1<l_2<\cdots$.

Eckford Cohen, [2], has defined the *unitary divisors* of a positive integer n as follows. A divisor $d>0$ of the positive integer n will be called *unitary* if $d\delta=n$ and $(d, \delta)=1$. The aim of this paper is to define arithmetical functions associated with the unitary divisors of a generalised integer and to show some of their properties.

2. Unitary divisors. A divisor d, $(d\in\{l\})$, of the generalised integer l will be called *unitary* if $d\delta=l$ and $(d, \delta)=1$. For generalised integers l, m, denote by $(l, m)_*$ the greatest divisor of l which is a unitary divisor of m; in the case $(l, m)_*=1$, l will be said to be *semiprime* to m.

Let $\{l_n\}$ be the set of generalised integer $l_1, \cdots, l_n$. Let (l_n) be the set of generalised integers semiprime to l_n and contained in $\{l_n\}$.

It will be noted that any generalised integers $l\leqq l_n$ can be represented *uniquely* in the form $l=dx$, where d is a unitary divisor of l_n and x is contained in (l_n/d); in fact, $d=(l, l_n)_*$. This observation can be restated as the enumerative principle:

THEOREM 1. *The generalised integers dx, where d ranges over the unitary divisors of l_n, and for each d, x ranges over the set (l_n/d), constitute the set $\{l_n\}$.*

3. The unitary totient function $\phi^*(l_n)$. Define $\phi^*(l_n)$ to be the number of generalised integers contained in $\{l_n\}$ which are semiprime to l_n.

THEOREM 2. $\sum_{d\delta=l_n,(d,\delta)=1}\phi^*(d)=n$, *i.e., the summation extends over all unitary divisors d of l_n.*

Proof. Consider $l_m\leqq l_n$. Either $(l_m, l_n)_*=1$, or there is a divisor of l_m, say d, which is also a unitary divisor of l_n, so that $(l_m, l_n)_*=d$.

In this case $(l_m, l_n/d)_*=1$ and there are $\phi^*(l_n/d)$ generalised integers with this property. Hence $\sum_d\phi^*(l_n/d)=n$ from Theorem 1. As d ranges over the unitary divisors of l_n, so does l_n/d. Hence $\sum_d\phi^*(d)=n$.

Note $\phi^*(p^n)=[p^n]-1$, where, by definition $[l_n]=n$. Also neither $[l_n]$ nor $\phi^*(l_n)$ are multiplicative.

4. Multiplicative functions. Suppose $\theta(l_n)$ is multiplicative and $l_n=p_1^{v_1}p_2^{v_2}\cdots p_k^{v_k}$.

THEOREM 3. $$\sum_{\substack{d\delta=l_n\\(d,\delta)=1}}\theta(d)=(1+\theta(p_1^{v_1}))(1+\theta(p_2^{v_2}))\cdots(1+\theta(p_k^{v_k})).$$

Proof. When $l_n=1$, assume r.h.s. $=1$. The r.h.s. is the sum of products of the form $\theta(p_i^{v_i})\theta(p_j^{v_j})\cdots(p_r^{v_r})=\theta(p_i^{v_i}p_j^{v_j}\cdots p_r^{v_r})$ in which all possible unitary divisors are contained and no unitary divisor occurs more than once. This is what occurs on the l.h.s.

Note. Putting $\theta(l_n)=1$ and $\theta(l_n)=l_n$ in Theorem 3 gives the number and sum of the unitary divisors of l_n.

THEOREM 4. *If $g(l_n)$ and $h(l_n)$ are multiplicative functions, then the unitary convolution*

$$f(l_n) = \sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} g(d)h(\delta)$$

of $g(l_n)$ and $h(l_n)$ is also multiplicative.

Proof. If $g(1)=h(1)=1$, then $f(1)=1$. Suppose $l_n=l_pl_q$, $(l_p, l_q)=1$. If $l_n=d\delta$, then d and δ have unique factorisations, $d=d_1d_2$, $\delta=\delta_1\delta_2$ such that $l_p=d_1\delta_1$, $l_q=d_2\delta_2$. Moreover $(d, \delta)=1\Leftrightarrow(d_1, \delta_1)=(d_2, \delta_2)=1$. Hence

$$\begin{aligned} f(l_n) &= \sum_{\substack{d_1\delta_1=l_p \\ d_2\delta_2=lq \\ (d_1,\delta_1)=(d_2,\delta_2)=1}} g(d_1d_2)h(\delta_1\delta_2) \\ &= \sum_{\substack{d_1\delta_1=l_p \\ (d_1,\delta_1)=1}} g(d_1)h(\delta_1) \sum_{\substack{d_2\delta_2=lq \\ (d_2,\delta_2)=1}} g(d_2)h(\delta_2) \\ &= f(l_p)\cdot f(l_q). \end{aligned}$$

5. The unitary Möbius function $\mu^*(l_n)$. Let $w(n)$ denote the number of distinct prime divisors of l_n, $w(1)=0$. Define $\mu^*(l_n)=(-1)^{w(n)}$. Then $\mu^*(l_n)$ is multiplicative.

THEOREM 5.

$$\sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \mu^*(d) = 0 \quad \text{when} \quad l_n \neq 1.$$

$$\sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \mu^*(d) = 1 \quad \text{when} \quad l_n = 1.$$

$$\sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \frac{\mu^*(d)}{d} = \left(1-\frac{1}{p_1^{v_1}}\right)\left(1-\frac{1}{p_2^{v_2}}\right)\cdots\left(1-\frac{1}{p_k^{v_k}}\right) \quad \text{when} \quad l_n \neq 1.$$

$$\sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \frac{\mu^*(d)}{d} = 1 \quad \text{when} \quad l_n = 1.$$

These statements follow immediately from Theorem 3 by putting $\theta(l_n)=\mu^*(l_n)$ and $\theta(l_n)=\mu^*(l_n)/l_n$ respectively.

THEOREM 6. *A unitary inversion formula.*

$$\text{If } G(l_n) = \sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} F(d)$$

then

$$F(l_n) = \sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \mu^*(d)G(\delta).$$

Proof.

$$\sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} F(d) = \sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \sum_{\substack{DE=d \\ (D,E)=1}} \mu^*(D)G(E) = \sum_{E/l_n} G(E) \sum_{\substack{D\delta=l_n/E \\ (DE,\delta)=1 \\ (D,E)=1}} \mu^*(D)$$

$$= \sum_{\substack{E/l_n \\ (E,l_n/E)=1}} G(E) \sum_{\substack{D\delta=l_n/E \\ (D,\delta)=1}} \mu^*(D)$$

$$= G(l_n),$$

from Theorem 5. Applying this theorem to Theorem 2 shows that

$$\phi^*(l_n) = \sum_{\substack{d\delta=l_n \\ (d,\delta)=1}} \mu^*(d)[\delta].$$

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LOCALLY RECURRENT FUNCTIONS

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1. Introduction. We shall say that a function $f(x)$ is locally recurrent at x_0 if every deleted neighborhood of x_0, $N(x_0)$, contains an element x such that $f(x)=f(x_0)$. Thus the Dirichlet function, 1 when x is rational and 0 when x is irrational, is locally recurrent everywhere. The function $f(x)=x \sin 1/x$ is locally recurrent at the origin if $f(0)=0$. In this paper we raise the following questions:

(A) Does there exist an everywhere locally recurrent function which is nonconstant and continuous?

(B) Does there exist a function, nonconstant and continuous, locally recurrent everywhere over (say) right neighborhoods? (The distinction from case (A) is that there the recurrent property may be found on either the left of the point in question or on the right but not necessarily on both, whereas here we ask for the recurrent property always at a point to the right of the point in question.)

(C) Does there exist a function, nonconstant and continuous, locally recurrent almost everywhere over right neighborhoods?

The answers to (A) and (C) are affirmative, but the answer to (B) is no! At first glance this behavior is unexpected—one would conjecture that (A) and (B) would have the same answer. The failure of (B), however, could be anticipated from a now celebrated result originally established by Besicovitch [1]. He exhibited a continuous function with no unilateral derivative. Here we mean that the upper derivative on the right (possibly $+\infty$) and the lower derivative on the right (possibly $-\infty$) are different and dually for the left derivatives. Several mathematicians have questioned the validity of this complicated example, but there is no point in entering such a controversy since A. P. Morse [3] has also produced an example. If the answer to (B) were otherwise, then it would be reasonably easy to secure Besicovitch-type examples whereas this is apparently a deep question, because the local recurrent behavior would imply that the derivative, if it existed, would have to be zero, and this would be impossible. (B) also answers a conjecture due to G. Aumann (private communication). He believes there is no continuous function with the property that the upper right derivative is always ∞ and the lower left derivative is $-\infty$ and dually. If there were such a function, it would have to be locally recurrent both in right neighborhoods and in left neighborhoods since infinitely often in $N(x_0)$ we would have $f(x')>f(x_0)$, $f(x'')<f(x_0)$, $x'>x_0$, $x''>x_0$ and dually for $x'<x_0$, $x''<x_0$.

The author [2] gave an example of a continuous function which, as it turns out, is locally recurrent almost everywhere, although at the time, the purpose was merely to describe a continuous function without a derivative *avoiding the use of any limiting operations* other than those necessary in constructing the real line. One could also prove, though it was not done there, that this function has almost everywhere an upper derivative $+\infty$ and a lower derivative $-\infty$ unilaterally, but on the exceptional set a unilateral derivative of $+\infty$ or $-\infty$ exists. While the ideas we use here stem from this earlier paper, we require a somewhat stronger type of theorem so that all ideas will be developed *ab initio*.

2. A class of continuous functions. On the unit interval I we can represent any number x in the form

$$x = .x_1x_2x_3 \cdots$$

where each x_k is one of the digits 0, 1, 2, $\cdots$, $b-1$ with b the base. We set

$$f(x) = .u_1u_2u_3 \cdots$$

where u_k is some function of $x_1, \cdots, x_k$:

$$u_k = h_k(x_1, x_2, \cdots, x_k),$$

and we shall say that u_k is determined antecedently, i.e., it depends only on those x_k's up to and including the kth place. We shall assign the value 0 or the value 1

to u_k in all cases so that $f(x)$ is represented in base 2. Conventionally we agree that $u_1=1$. We construct the u_i's as follows:

$$\begin{aligned} u_k &= u_{k-1} \quad \text{if} \quad x_k = x_{k-1} = 0 \\ &\qquad\quad \text{or} \quad x_k = x_{k-1} = b-1 \\ u_k &\neq u_{k-1} \quad \text{if} \quad x_k = b-1, \quad x_{k-1} \neq b-1 \\ &\qquad\quad \text{or} \quad x_k = 0, \quad x_{k-1} \neq 0. \end{aligned}$$

In all other cases u_k is arbitrary except for the restriction imposed above by antecedence. Some such restriction is essential, however, since otherwise a rule could be formulated so that the value of x_k with k large could affect the value of u_j with j small and destroy continuity.

Two things are of especial interest in this construction: (i) the sharp variation from rigid specification in the four instances stated above in the determination of the value of u_k to the almost complete freedom in the other cases, and (ii) the uniqueness of the functional values. Actually (i) and (ii) are, in a sense, related. By (ii) we of course mean that $f(x)$ is independent of the notation employed so that in base 3, .2000 $\cdots$ and .1222 $\cdots$ have the same functional correspondent, for example, although they are both different representations of $\frac{2}{3}$. At the same time $x=.200000001\cdots$ and $x'=.122222221\cdots$ are *close* together, but are distinct numbers so that the restriction in our definition ensures that $f(x)$ and $f(x')$ are *close* together.

THEOREM. *The function $f(x)$ described above is continuous.*

Proof. Suppose x and x' agree for the first k places. Then $f(x)$ and $f(x')$ agree for the first k places so that $|f(x)-f(x')|<2^{-k+1}$. (The fact that from the $k+1$ place onward $f(x)$ might consist only of zeros and $f(x')$ only of ones or vice versa prevents a stronger inequality.) But x and x' can also be close together in the following way: x and x' agree for the first p places with $x_{p+1}=x'_{p+1}+1$, $x_j=0$, $x'_j=b-1$ for $p+1<j<k+2$. By antecedence $u_j=u'_j$ for the first p places, and clearly $x_{p+1}\neq 0$, $x'_{p+1}\neq b-1$. Hence we have (the initial three dots indicating agreement for the first p places):

$$f(x) = \cdots 1000\cdots \quad \text{or} \quad \cdots 1000\cdots,$$

$$f(x') = \cdots 1000\cdots \quad \text{or} \quad \cdots 0111\cdots$$

or their *complements* obtained by interchanging zeros and ones. In any case $|f(x)-f(x')|<2^{-k+1}$. A precisely analogous case arises when $x_{p+1}=x'_{p+1}-1$. By choosing k sufficiently large, continuity is ensured.

3. Construction of a locally recurrent function. We proceed to invent a function satisfying the conditions listed in (A) making use of the continuity theorem previously established. We shall require several different number bases, in particular 2, 3, and 16.

We begin by expressing x in base 3 and initially will consider only right neighborhoods. We therefore agree never to represent x in a form requiring an unbroken series of twos (e.g., we write .2000 $\cdots$ instead of .1222 $\cdots$). We use the following rules to construct a particular function with functional values notated in base 2:

$$\begin{aligned} u_{k-1} = u_k \quad &\text{if} \quad x_{k-1} = x_k = 0 \\ &\text{or} \quad x_{k-1} = x_k = 2 \\ &\text{or} \quad x_{k-1} = 0, \quad x_k = 1 \\ &\text{or} \quad x_{k-1} = 2, \quad x_k = 1, \\ u_1 = 1. & \end{aligned}$$

In all other cases $u_{k-1} \neq u_k$. Since these rules satisfy the conditions of our continuity theorem, our function $u = f(x)$ is continuous. We can express these same rules in tabular form where X means $u_{k-1} \neq u_k$, and a blank space means $u_{k-1} = u_k$:

x_{k-1}/x_k	0	1	2
0			X
1	X	X	X
2	X		

For example, if

$$x = .010211200 \cdots,$$

$$f(x) = .110110100 \cdots.$$

In the subsequent discussion x will always denote a fixed point in (0, 1) with associated functional value u while $x' > x$ will denote a point (variable) in the right neighborhood of x. As the size of the neighborhood is reduced, x' will necessarily lie closer to x. The associated functional value of x' is denoted by u'. For convenience we agree that x and x' agree for the first $2k+1$ places of their ternary expansions. We now construct a table listing possible combinations of x_{2k+1} and x_{2k+2} with the u_{2k+1} and u_{2k+2} values associated with these places and likewise for x' and u'. For simplicity we shall frequently omit subscripts and write (say) u' to mean (say) u'_{2k+1}. To avoid complexity we agree that $u_{2k+1} = 0$. The reverse case is immediate by *complementation* interchanging zeros and ones. The last tabular entry is exceptional in that we are given three entries beginning with x_{2k} and will require additional comment.

x	u	x'	u'	
00	00	02	01	$\surd$
10	01	12	01	
20	01	22	00	$\surd$
01	00	02	01	$\surd$
11	01	12	01	
21	00	22	00	
2 12	0 01	2 22	0 00	$\surd$

Observe that 02 and 22 do not appear. The first case is impossible to handle by this treatment and is discussed below. The second case need not be considered since not all pairs in the expansion of x can have the form 22 from some point onwards. For the last case we can restrict attention to 2 12 since the cases 0 12 and 1 12 could be handled by allowing the first discrepancy to occur at the *preceding* even place which ends in either 0 or 1 and is already provided for in the earlier tabular entries. Similarly the cases 0 02 and 1 02 present no difficulty leaving only the case 2 02 as the sole source of trouble.

The checked entries in the table show discrepancies in the corresponding u and u' values so that the recurrent property is momentarily surely absent. This disagreement will be taken care of later by a second transformation h. In all of these cases, however, we have either 00 versus 01 or vice versa for u and u'. Passing to the complements, these u and u' values become 11 and 10 or vice versa.

The special case where only the combinations 02 and 22 appear in the expansion of x from some point onwards is treated by considering x'' where $x'' < x$ so that the neighboring point is in a left neighborhood of x. Here we replace 2 02 with 2 00 so that the associated u and u'' values are 0 10 and 0 11 or 1 01 and 1 00 so that the discrepancy is of the same kind as previously observed.

In making the second transformation h so that the recurrent property holds i.e.

$$h[f(x)] = h[f(x')] \text{ (or possibly } h[f(x'')]),$$

we first represent these functional values in base 16 in the usual way, i.e. we partition the binary expansion of $f(x)$ into groups of 4:

$$f(x) = .u_1u_2u_3u_4 \mid u_5u_6u_7u_8 \mid \cdots$$

and replace each group of 4 by its base 16 representative so that the group 0111 would be replaced by 7, for example. Since this procedure is merely a change in notation, it has no effect on continuity.

Making this change, suppose

$$f(x) = .y_1y_2y_3 \cdots$$

where each y_i is one of the numbers 0, 1, $\cdots$, 15. Let the transformation h yield real numbers notated in base 2:

$$h[f(x)] = .v_1v_2v_3 \cdots, \quad v_k = 0 \text{ or } 1, \quad v_1 = 1,$$

where we set $v_{k-1} = v_k$ whenever

(a) $y_{k-1} = 0, \quad y_k = 0, 1, 2, 3, 4, 5, 6, 7,$

(b) $y_{k-1} = 15, \quad y_k = 8, 9, 10, 11, 12, 13, 14, 15,$

(c) $y_{k-1} \neq 0, \quad y_{k-1} \neq 15, \quad y_k = 2, 3, 6, 7, 8, 9, 12, 13.$

In all other cases $v_{k-1} \neq v_k$.

Clearly h is continuous by our theorem of Section 2. We now show that additionally

$$h[f(x)] = h[f(x')], \qquad x' > x$$

or

$$h[f(x)] = h[f(x'')], \qquad x'' < x \text{ in the special case.}$$

If $f(x)=f(x')$, this is trivially true. If $f(x)\neq f(x')$ with the first discrepancy in their binary expansions occurring at the $4k$th place, then we could have either

$$|--00| \quad \text{versus} \quad |--01|$$

or

$$|--10| \quad \text{versus} \quad |--11|$$

where the dashes indicate agreement between corresponding u and u' values. The partitioning into a group of four which we used is convenient because of the use of base 16 to follow. We have of course displayed only that group of four at which the first discrepancy occurs. In base 16 we could have, for example, $|0000|=0=y_k$ versus $|0001|=1=y_k'$ or $|0010|=2=y_k$ versus $|0011|=3=y_k'$ or the reverse possibility interchanging y_k and y_k'. If we inspect all of the eight possibilities, we secure only the pairs listed below for y_k and y_k' or vice versa:

y_k	0	2	4	6	8	10	12	14	y_k'
y_k'	1	3	5	7	9	11	13	15	y_k

We assert that under h, $v_k=v_k'$. For, consider any permissible pair such as 4 versus 5. Observe that if $y_{k-1}=y_{k-1}'=0$, then if $y_k=4$ or 5, $v_{k-1}=v_k$ while $v_{k-1}'=v_k'$ by rule (a) above. Since $v_{k-1}=v_{k-1}'$, $v_k=v_k'$. Again if $y_{k-1}=y_{k-1}'=15$, for our pair $v_{k-1}\neq v_k$ and $v_{k-1}'\neq v_k'$ by (b). By (c) $v_{k-1}\neq v_k$ and $v_{k-1}'\neq v_k'$ for the remaining possibilities. More generally, for any pair in our little table, either both numbers occur in the group $0, \cdots, 7$ under (a) or both are excluded so that $v_k=v_k'$, and a similar state exists under rules (b) and (c) so that we have agreement at least through the first k v values of $h[f(x)]$ and $h[f(x')]$ (or x'').

The second possibility is that the initial discrepancy between $f(x)$ and $f(x')$ occurs at the $4k+2$ place. (Recall that we originally arranged matters so that the first difference occurred at an even place.) We then have the possibilities:

$$\text{(i)} \quad |00??| \quad \text{versus} \quad |01??|,$$
$$\text{(ii)} \quad |10??| \quad \text{versus} \quad |11??|.$$

In (i) y_{k+1} could be either 0, 1, 2, 3 with y_{k+1}' one of 4, 5, 6, 7 or vice versa. In (ii) similarly we have the possibilities 8, 9, 10, 11 versus 12, 13, 14, 15. In case (i) 0, 1, 4, 5 are all included in rule (a), all excluded in (b), and all excluded in (c). Thus by a proper choice of the ?? entries in x' (or x''), we can secure $v_{k+1}=v_{k+1}'$. With the remaining possibilities under (i), 2, 3, 6, 7 we find these are all

included in (a), excluded in (b), and included in (c) so that again there exists a choice so that $v_{k+1}=v'_{k+1}$. The (ii) case is similar. Thus these discrepancies in u and u' do not affect the final function values through the point where the first disagreement in their expansions occurs.

The next point is to observe that the point in the expansion of x (fixed) and $x'>x$ (or $x''<x$) where we first encounter a difference between x and x' (or x'') is at our disposal. Thus we can make $x'-x$ as close to zero as we please, (or $x-x''$ alternatively). Thus, given any neighborhood of x, we can find an x' (we will omit further reference to x'') with the property that x' lies in the neighborhood and $v_{2k}=v'_{2k}$ with $x_j=x'_j$, $1\leqq j<2k$, $x_{2k}<x'_{2k}$. Furthermore, we can so choose the number x' that recurrence takes place. For $h[f(x)]$ consists of some arrangement of zeros and ones. If we find $y_{2k}\neq y'_{2k}$, we choose y'_{2k+1} so that $v_{2k+1}=v'_{2k+1}$. This can always be done since we can choose y'_{2k+1} so that $v'_{2k}=v'_{2k+1}$ or so that $v'_{2k}\neq v'_{2k+1}$, and we make the proper choice. Now translate y'_{2k+1} into base 2 obtaining in our notation $u'_{4k+1}u'_{4k+2}u'_{4k+3}u'_{4k+4}$. Then choose $x'_{4k+1}, \cdots, x'_{4k+4}$ so that under f we obtain the u' entries of our last sentence. Treating every subsequent v'_k in this way, we can assert that

$$h[f(x)] = h[f(x')]$$

for an x' constructed by this procedure. The axiom of choice is implicitly assumed.

Since f and h are both continuous, $h[f(x)]$ is continuous. It is obvious that the function is never constant since we could terminate x' with zeros from some point onwards whereupon $h[f(x')]$ would terminate with zeros. We could also alter x' so that $f(x')$ from some point onwards has the form $|1010|1010|\cdots$ so that $h[f(x')]$ would terminate in a series of alternate zeros and ones from some point onwards. We have thus established the validity of statement A.

We have also established the assertion in C since the only values of x not in the right neighborhood must consist exclusively of zeros and twos from some point onwards. But it is well known that the measure of any set of points in (0, 1) which fails to contain a specific digit in its expansion is zero. Likewise if the first k digits are fixed and thereafter a fixed digit does not appear the measure of such sets is also zero. Since measure is a subadditive function, the measure of the union of such sets is zero.

R. P. Dilworth (private communication) has provided the author with a proof that B is false which is as ingenious as it is simple. Dilworth's argument runs as follows:

Let S be the set of all x such that $g(x)=g(x_0)$, $x_0\in I$. By continuity S is closed. Consequently S^c is open and on the real line can be expressed as a union of disjoint open intervals (a_n, b_n). In particular $g(a_n)=g(x_0)$. But local recurrence on the right implies that $g(x)=g(x_0)$ for some $x\in(a_n, b_n)$ contrary to the meaning of S^c. Hence $S^c=\phi$ so that continuity and local right recurrence imply that g is a constant. This, then, establishes that B is false.

It is interesting to observe that $h[f(x)]$ is symmetric about $x=\frac{1}{2}$. This means that we could have replaced questions (B) and (C) referring to right neighborhoods by left neighborhoods. Finally the function can be extended to the real line by setting $h[f(x)]=h[f(x+k)]$ where k is any integer.

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MATHEMATICAL NOTES

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BUNCHES OF CONES

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If four right circular cones in 3-space have a common vertex and are mutually externally tangent, then their semivertical angles $\alpha_1, \cdots, \alpha_4$ are connected by the relation

$$\left\{\sum_{i=1}^{4} \cot \alpha_i\right\}^2 = 4 + 2\sum_{i=1}^{4} \cot^2 \alpha_i. \tag{1}$$

To prove this result, suppose that the first three cones are fixed, their semivertical angles being α_i and their axes having direction cosines l_i, m_i, n_i $(i=1,2,3)$. Then the corresponding quantities for the fourth cone must satisfy the equations $l_4 l_i + m_4 m_i + n_4 n_i = \cos(\alpha_4+\alpha_i)$, where $i=1, 2, 3$.

These equations are homogeneous and linear in the five quantities l_4, m_4, n_4, $\cos \alpha_4$, and $\sin \alpha_4$. Solving them for l_4, m_4, n_4 and substituting the solution in the identity $l_4^2+m_4^2+n_4^2=\cos^2 \alpha_4+\sin^2 \alpha_4$, we obtain the equation determining α_4 in

the form $a\cos^2\alpha_4+b\cos\alpha_4\sin\alpha_4+c\sin^2\alpha_4=0$. Hence the required relation is algebraic (indeed, quadratic) in $\cot\alpha_4$, and it is also symmetric in $\cot\alpha_1, \cdots, \cot\alpha_4$.

Denote by $p_1, \cdots, p_4$ the elementary symmetric functions in $\cot\alpha_1, \cdots, \cot\alpha_4$. Then we have shown that the required relation can be expressed as a polynomial relation in $p_1, \cdots, p_4$, and we next consider the special case $\alpha_1=\alpha_2=\alpha_3=\alpha$, $\alpha_4=\beta$.

In this case the axes of the first three cones are the edges of an equiangular trihedron of angle 2α whose circumscribing right circular cone has semivertical angle $\alpha+\beta$. The cosine formula of spherical trigonometry then gives $\cos 2\alpha=\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)\cos(2\pi/3)$, which is equivalent to

$$3\cot^2\alpha+6\cot\alpha\cot\beta-\cot^2\beta=4, \tag{2}$$

and this equation is quadratic in $\cot\alpha$. In this special case, we also have

$$p_1=3\cot\alpha+\cot\beta, \qquad p_2=3\cot^2\alpha+3\cot\alpha\cot\beta, \tag{3}$$

and the other p_i are of higher degree in $\cot\alpha$. Hence the required general relation, which reduces to (2) in the special case under discussion, is independent of p_i for $i>2$, and so it is obtained by the elimination of $\cot\alpha$ and $\cot\beta$ from (2) and (3), yielding $p_1^2-4p_2=-4$. Finally, giving p_1 and p_2 their general values, we obtain (1) as required.

This method can be used to obtain a similar equation for the case when the cones do not touch, but are such that each pair is inclined at the same nonzero angle θ. The problem can also be generalized to n-dimensional space, and (1) is the particular case $n=3$, $\theta=\pi$ of the general formula

$$\left\{\sum_{i=1}^{n+1}\cot\alpha_i\right\}^2=(n+\sec\theta)\left\{1-\cos\theta+\sum_{i=1}^{n+1}\cot^2\alpha_i\right\}, \tag{4}$$

which is proved elsewhere [3] in a different way. This investigation originated with a study of the problem solved by Soddy [1] in 1936, for which a proof can be found in Section 1.5 of [2], and which has been generalized to n-dimensions by Gosset [5] and others. Equation (4) above permits us to make further generalizations to hypercircles lying on the surface of a hypersphere and to nontangent hypercircles. The original *flat* version can be recovered by writing a_i/r for α_i and letting r tend to infinity.

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A GEOMETRIC FACET OF THE EULERIAN CONSTANT

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Let γ denote the Eulerian constant, and $k(=1.461\cdots)$ the zero of the psi-function: $\psi(x)=-\gamma+\sum_{n=0}^{\infty}\{(n+1)^{-1}-(n+x)^{-1}\}$. Then

$$\begin{aligned}\gamma &= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+k}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k-1}n^2+\left(\frac{k+1}{k-1}\right)n+\frac{k}{k-1}\right)^{-1} \qquad (1)\\ &= \sum_{n=0}^{\infty}\frac{stu}{s(t+u)n^2+2\{stu(s+t+u)\}^{1/2}n+tu},\end{aligned}$$

where

$$(2)\qquad s=\frac{k-1}{k},\qquad t=\frac{2-2\{k(2-k)\}^{1/2}}{k-1},\qquad u=\frac{2+2\{k(2-k)\}^{1/2}}{k-1}.$$

Let s_n denote the fraction in the last expression of (1), i.e.

$$(3)\qquad \gamma=\sum_{n=0}^{\infty}s_n.$$

We note that (3) has the following interesting geometric interpretation:

THEOREM. *If T and U are coplanar tangent circles with no interior points in common, with radii t and u respectively (cf.* (2)*), then the Eulerian constant is the sum of the radii of the infinite "chain" of circles successively inscribed into the "curvilinear angle" (T, U), the initial member of the chain having radius s* (cf. (2)). *This circle-chain representation for γ (as given by* (1)*) is "termwise faithful" in that the rth circle of the chain will have a radius equal to the rth term of the series* (1).

Proof. Let S be a circle with radius s, which is coplanar with, tangent to, and has no interior point in common with T and U, respectively. Let h denote the distance from the center of S to the line passing through the centers of T and U; then,

$$(4)\qquad h=\frac{2\{stu(s+t+u)\}^{1/2}}{t+u}.$$

Let us now execute a geometric inversion which leaves S invariant, and whose center is at the point P of tangency between T and U; then the radius r of the circle each of whose points remain fixed under this inversion satisfies the relation

$$r^2 = 4stu/(t + u). \tag{5}$$

Let j denote the distance from P to the inverse of U; then

$$j = 2st/(t + u). \tag{6}$$

Now, let S_1 denote the smallest circle tangent to S, T, U; let S_n, for each integer $n>1$, denote the smallest circle tangent to S_{n-1}, T, U. Let one of the common tangents to S_n and its inverse, which passes through P, touch them in the points A and B respectively. Let p_n denote the distance from P to the center of the inverse of S_n. Then the radius of S_n ($n \geqq 0$, $S_0 \equiv S$) is given by

$$\begin{aligned} \frac{s \cdot \overline{PA}}{\overline{PB}} &= \frac{s \cdot \overline{PA} \cdot \overline{PB}}{\overline{PB}^2} = \frac{sr^2}{p_n^2 - s^2} \\ &= \frac{sr^2}{(s - j)^2 + (h + 2sn)^2 - s^2}. \end{aligned} \tag{7}$$

Using (4), (5), (6) in the last expression of (7) we see that the radius of S_n is s_n as defined above.

Remark (by the referee): The theorem can also be established using Soddy's Theorem and a mathematical induction.

A NOTE ON CIRCULAR PROBABILITY PROBLEMS

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The purpose of this note is to point out that probabilities for the "arc length problem" which is considered in [2] may be read directly from existing tables [3] and from a graph [6, p. 654].

Two closely related problems are generated if a circle C_1 of radius R is dropped upon a fixed circle C_2 of radius D. These involve (a) the probability that C_1 covers a randomly selected point on the circumference of C_2 and (b) the probability that C_1 covers a randomly selected point within C_2. The first of these two is the arc length problem. The second, which could be called the area problem, is considered in [1] where extensive tables may be found.

In [1] the following function is defined:

$$p(R, r) = \int_A \int \frac{1}{2\pi} e^{-(x^2+y^2)/2} dy dx = \int_0^R e^{-(r^2+t^2)/2} I_0(rt) t dt$$

where A is the region within $(x-x_0)^2+(y-y_0)^2=R^2$, and $x_0^2+y_0^2=r^2$. This integral gives the probability that the circle C_1 captures the point (x_0, y_0) if the center of C_1 is aimed at the origin with aiming errors being circularly normally distributed. Very extensive tables of $q(R, r)=1-p(R, r)$ have been prepared by Marcum [3].

Let $P(R, D)$ denote the probability that C_1 covers a randomly selected point

on the circumference of C_2. If the point (x_0, y_0) is chosen at random on this circumference, then $p(R, r)$ becomes $p(R, D)$. Also if $\phi = \tan^{-1}(y_0/x_0)$, then $h(\phi) = 1/2\pi$, $0 \leqq \phi < 2\pi$ is the probability density function describing the behavior of (x_0, y_0). Now $P(R, D)$ is

$$Pr\begin{bmatrix}\text{circle } C_1 \text{ captures} \\ \text{the point } (x_0, y_0)\end{bmatrix}\begin{bmatrix}\text{position of the ran-} \\ \text{dom point is } (x_0, y_0)\end{bmatrix} \cdot Pr\begin{bmatrix}\text{position of random} \\ \text{point is } (x_0, y_0)\end{bmatrix}$$

integrated over all values of (x_0, y_0). Hence,

$$P(R, D) = \int_0^{2\pi} p(R, D)\frac{1}{2\pi}\,d\phi = p(R, D)$$

so that this probability may be read directly from existing tables.

The tables of Marcum are difficult to obtain. However, Solomon's graph, which is readily available, yields answers sufficiently accurate for most applied problems. To use his Figure 1 relabel the horizontal axis D, the vertical axis R, and read the answer directly. Apparently Solomon used another set of tables [4] prepared by the Rand Corporation to draw his graph but these tables are also difficult to obtain.

The method used in [2] leads to

$$P(R, D) = \frac{1}{2}\left[1 - e^{-(R^2+D^2)/2}I_0(RD) - \frac{D^2 - R^2}{D^2 + R^2}\int_0^{(R^2+D^2)/2} e^{-u}I_0\left(\frac{2RD}{R^2 + D^2}u\right)du\right].$$

Since we also have

$$P(R, D) = \int_0^R e^{-(D^2+t^2)/2}I_0(Dt)tdt,$$

an identity relationship which is very difficult to prove directly has been established as a by-product. Incidently this involves the troublesome integral

$$\int_0^x e^{-u}I_0(ku)du, \qquad 0 < k < 1$$

which turns up occasionally in the literature. A partial table is found in [5].

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ON R-COHERENCE CLASSES AND PARTITIONS

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1. Introduction. In a recent note [1], F. Harary showed that a parity relation partitions its field distinctly. This strengthened the well-known proposition concerning equivalence relations. The purpose of this note is to generalize these results further. We employ Wallace's principle stated below, which was shown by Wallace [3] to be equivalent to the axiom of choice.

A *relation* R over a non-empty set X is a subset of the Cartesian product $X \times X$. The set X is called the *field* of the relation. The notation xRy [x non Ry] is used to indicate $(x, y) \in R$ [(x, y) non$\in R$]. If $x \in X$, then the set $R(x)$ is defined as follows: $R(x) = \{y \mid y \in X \text{ and } yRx\}$.

Following Rado and Reichelderfer [2], a subset C of X is called *R-coherent* if and only if for every pair of distinct elements $x, y \in C$, x and y are *R-comparable*; that is, either xRy or yRx. A set $C \subset X$ is called an *R-coherence class* if and only if C is a maximal R-coherent subset of X; that is, C is an R-coherent subset of X, and C is not a proper subset of any R-coherent set in X. In particular, we note that the degenerate subsets of X are R-coherent.

Generalizing [1], a relation R over X *semipartitions its field distinctly* if there exists a partition of X into subsets such that for any pair of distinct elements $x, y \in X$, x and y are R-comparable if and only if x and y lie in the same subset.

We now state Wallace's principle.

(W) Any R-coherent subset of X is contained in a maximal R-coherent subset of X.

2. R-coherence classes. Theorem 1, below, is an immediate consequent of (W).

THEOREM 1. *If R is an arbitrary relation over X, the family F of R-coherence classes covers X.*

Proof. This follows from the above and the fact that $\{x\}$, for every $x \in X$, is an R-coherent set.

We refer to the following property of a relation R over X as property (T).

(T) For $x, y, z \in X$, if x and y are R-comparable, and if y and z are R-comparable, then x and z are R-comparable.

A relation possessing property (T) might appropriately be called a *semitransitive* relation.†

THEOREM 2. *If R is a relation over X and R has property* (T), *then R semipartitions X distinctly.*

Proof. By Theorem 1, the family F of R-coherence classes covers X. Let F_1 and F_2 be any two distinct elements of F. Assume $F_1 \cap F_2 \neq \phi$. Let $y \in F_1 \cap F_2$.

* Now at the Goodyear Aircraft Corporation, Akron, Ohio.

† The author is indebted to the referee for this and other valuable suggestions.

Let x and z be any pair of distinct elements of $F_1 \cup F_2$, and suppose (without any loss of generality) that $x \neq y$. Since F_1 and F_2 are R-coherent sets, x and y are R-comparable. Either $y = z$, or $y \neq z$.

If $y = z$, then, in view of the preceding statements, x and z are R-comparable. If $y \neq z$, then y, z, being distinct elements of an R-coherent set, are R-comparable. By the above and property (T), x and z are R-comparable. Therefore, $F_1 \cup F_2$ is an R-coherent set that properly contains F_1 and F_2. This is a contradiction, since F_1 and F_2 are maximal R-coherent subsets of X. Thus F is a partition of X.

Since the elements of F are the maximal R-coherent subsets of X, R semipartitions X distinctly.

3. A special case. According to Harary [1], a relation R over X is a *parity relation* if and only if R has the following properties:

(1) R is *irreflexive* (x non Rx for every $x \in X$);

(2) R is *symmetric* (xRy implies yRx for $x, y \in X$);

(3) R is *distinctly transitive* (for any distinct $x, y, z \in X$, xRy and yRz imply xRz).

It is evident that a parity relation R over X has property (T). Therefore, in view of Theorem 2, a parity relation semipartitions its field distinctly. But by definition, a parity relation is symmetric. It should be noted that if R is a symmetric relation over X, property (T) and distinct transitivity are logically equivalent. Simple examples show this is not true in general.

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2. T. Rado and P. Reichelderfer, On cyclic transitivity, Fundamenta Mathematicae, vol. 34, 1947, pp. 14–29.

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REMARK ON THE PRECEDING NOTE

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In the preceding note, Bednarek [1] obtains an interesting generalization of my note [2]. We show that these two results are equivalent by deducing the theorem of [1] from that of [2], and make a related remark.

As in [1], let R be a relation whose field is X. For any subset C of X, the *subrelation* $\langle C \rangle$ *generated by* C is that relation having C as its field and containing all those ordered pairs $(x, y) \in R$ such that both x and y are in C. A relation R is *complete* if for every two distinct elements $x, y \in X$, xRy or yRx, i.e., x and y are R-comparable. Thus in [1], a subset C of X is called R-coherent if the relation $\langle C \rangle$ is complete. Probably the most important theorem on complete relations is:

REDEI'S THEOREM [3]. *Every complete relation R whose field X is finite contains a directed path $(x_1, x_2), (x_2, x_3), \cdots, (x_{n-1}, x_n)$ in which all n points (or elements) of X appear.*

Recall from [2] the definitions of a *weakly connected* digraph (directed graph) and of a *weak component* of a digraph. Of course these concepts apply to any relation, irreflexive or not. In these terms, a relation R is *semitransitive* if whenever two distinct points are in the same weak component, they are R-comparable.

THEOREM [1]. *A semitransitive relation R semipartitions its field X distinctly.*

THEOREM [2]. *A parity relation (irreflexive, symmetric, and distinctly transitive) partitions its field distinctly.*

Proof of Theorem [1] from Theorem [2]. Let R be a semitransitive relation whose field is X. Let R^s be the *symmetrized relation* of R, i.e., $R^s = \{(x, y)$: x and y are R-comparable$\}$.

Let $\overline{R}$ be the maximal irreflexive subrelation of R^s. Then $\overline{R}$ is a parity relation with the same field X. By Theorem [2], $\overline{R}$ partitions its field X distinctly. Hence R semipartitions X distinctly, proving Theorem [1].

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1. A. R. Bednarek, On R-coherence classes and partitions, this MONTHLY, vol. 69, 1962, preceding note.

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AN IDENTITY RELATED TO THE DEDEKIND-VON STERNECK FUNCTION

ECKFORD COHEN, University of Tennessee

For integers n and r, $r>0$, let $\phi(r)$ and $\mu(r)$ denote the Euler and Möbius functions, respectively, and let (n, r) denote the greatest common divisor of n and r. In addition, let $\theta(n, r)$ represent the Nagell totient, that is, the number of positive integers $a \leqq r$ such that $(a, r) = (n-a, r) = 1$. In this note we prove the curious identity,

$$\sum_{d\delta=r} \frac{\theta(\delta, d)\mu^2(\delta)}{\phi(d)\phi(\delta)} = 1. \tag{1}$$

In fact, let $\Phi(n, r)$ denote the Dedekind-von Sterneck function [2, Sec. 3], defined by

$$\Phi(n, r) = \frac{\phi(r)\mu(m)}{\phi(m)}, \qquad \left(m = \frac{r}{(n, r)}\right); \tag{2}$$

we shall deduce the representation,

$$\Phi(n, r) = \phi(r) \sum_{d\delta=(n,r)} \frac{\theta\left(\frac{r}{d}, d\right)\mu\left(\frac{r}{d}\right)\mu(\delta)}{\phi(d)\phi\left(\frac{r}{d}\right)}, \tag{3}$$

from which (1) results, on placing $n=0$. For a discussion of (3), the reader is referred to the comments at the end.

Our proof is based on the following inversion principle. Let $f(n, r)$ be an even function (mod r), that is, $f(n, r)=f((n, r), r)$ for all integers n; then with $r=r_1r_2$ [4],

$$g(r_1, r_2) = \sum_{d|r_1} f\left(\frac{r_1}{d}, r\right) \rightleftarrows f(n, r) = \sum_{d\delta=(n,r)} g\left(d, \frac{r}{d}\right)\mu(\delta). \tag{4}$$

We shall also make use of the following relation [1, Corollary 21],

$$\sum_{\substack{d|r \\ (d,n)=1}} \frac{\mu(d)}{\phi(d)} = \frac{\theta(n, r)}{\phi(r)}, \tag{5}$$

the summation extending over the divisors of r prime to n.

Proof of (3). On the basis of (5), the definition of $\mu(r)$, and the multiplicativity of $\mu(r)$ and $\phi(r)$, one obtains with $r=r_1r_2$,

$$\sum_{d|r_1} \Phi\left(\frac{r_1}{d}, r\right) = \phi(r) \sum_{d|r_1} \frac{\mu(dr_2)}{\phi(dr_2)} = \frac{\phi(r)\mu(r_2)}{\phi(r_2)} \sum_{\substack{d|r_1 \\ (d,r_2)=1}} \frac{\mu(d)}{\phi(d)}$$

$$= \frac{\phi(r)\mu(r_2)}{\phi(r_2)}\left(\frac{\theta(r_2, r_1)}{\phi(r_1)}\right).$$

Application of (4) with $f(n, r)=\Phi(n, r)$ leads immediately to (3).

Comments. The representation (3) of $\Phi(n, r)$ is *unique* in the sense of (4). This representation may be viewed as the dual of Hölder's formula [2, (21) and (24)],

$$\Phi(n, r) = \sum_{d|(n,r)} d\mu\left(\frac{r}{d}\right), \tag{6}$$

since the latter identity arises in an analogous manner as a consequence of the inversion relation dual to (4); in particular, see [3] and [5, Sec. 4].

The analogy between (3) and (6) becomes even clearer, if one notes, on the basis of [2, (13), (34), $k=s=1$], that (6) has the equivalent formulation

$$(7) \qquad \Phi(n,r) = \phi(r) \sum_{d \mid (n,r)} \frac{\phi^*\left(\frac{r}{d}, d\right)\mu\left(\frac{r}{d}\right)}{\phi(d)\phi\left(\frac{r}{d}\right)},$$

where $\phi^*(n,r)$ denotes the number of positive integers $a \leqq r$ such that $((a, n-a), r) = 1$.

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4. ——— An arithmetical inversion principle, Bull. Amer. Math. Soc., vol. 65, 1959, pp. 335–336.

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CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

A NOTE ON ALTERNATING SERIES

PHILIP CALABRESE, Illinois Institute of Technology

It is not often that a classroom assignment develops into something worth writing about. Most of the time it just develops into something worth putting off until after dinner. So when Professor Curme, my calculus teacher, asked the class to determine how many terms of the alternating harmonic series one would have to add in order that the sum be within .00005 of the limit, my reaction was, "Okay, I'll do it after my accounting homework."

Well, true to the mathematics majors' code, I looked for an easy way to do

it, and at the next session gave my answer of 10,000 terms. Unfortunately, Professor Curme, substantiated by Love and Rainville's calculus book,* said my answer was one half the number of terms needed to insure the necessary accuracy. Well, I went home, thought about it some more, and wrote up this proof. I hope there aren't too many holes in it!

Given a convergent alternating series, $\sum_{n=1}^{\infty}(-1)^{n+1}U_n$, that satisfies the added condition $(U_n - U_{n+1}) > (U_{n+1} - U_{n+2})$, one can estimate the limit of the series within any accuracy ϵ by adding up the first n terms, where $U_n \leqq 2\epsilon$. That is, the sum of the first n terms of the series is within an ϵ of the limit if $U_n \leqq 2\epsilon$. This is equivalent to saying that $|S_n - L| \leqq \epsilon$ if $U_n \leqq 2\epsilon$, where S_n is the sum of n terms, and L the limit of the series.

Proof. For simplicity, we introduce the following notation. Let $S_n = \sum_{i=1}^{n}(-1)^{i+1}U_i$, $a = U_{n+1} - U_{n+2}$, $b = U_{n+3} - U_{n+4}, \cdots$, $a' = U_n - U_{n+1}$, $b' = U_{n+2} - U_{n+3}, \cdots$. We can write the alternating series as follows:

$$S_n + (-1)^n(a + b + \cdots),$$

or

$$S_{n-1} + (-1)^{n+1}(a' + b' + \cdots).$$

By our assumption, $a < a'$, $b < b', \cdots$. Therefore

$$(a + b + \cdots) < (a' + b' + \cdots).$$

But both groupings yield the same limit L. That is,

$$S_n + (-1)^n(a + b + \cdots) = L,$$
$$S_{n-1} + (-1)^{n+1}(a' + b' + \cdots) = L,$$

and

$$|S_n - L| = (a + b + \cdots),$$
$$|S_{n-1} - L| = (a' + b' + \cdots).$$

Therefore

$$|S_n - L| < |S_{n-1} - L|.$$

Now, since in every alternating series the limit L must lie between any two successive sums S_{n-1} and S_n, and since $U_n = |S_n - S_{n-1}|$, it follows that $U_n = |S_n - L| + |S_{n-1} - L|$.

Now if we find a term for which $2\epsilon \geqq U_n$, $U_n = |S_n - L| + |S_{n-1} - L| \leqq 2\epsilon$. But $|S_n - L| < |S_{n-1} - L|$. Therefore $|S_n - L| + |S_n - L| < 2\epsilon$, and finally, $|S_n - L| < \epsilon$. Also, since $|S_{n+1} - L| < |S_n - L|$, we have $|S_{n+1} - L| < \epsilon$. So, all the sums following S_n are within an ϵ of the limit.

Furthermore, if one can find a term U_n which is exactly equal to 2ϵ, S_n is

* Love and Rainville, Differential and Integral Calculus, 5th ed., Macmillan, p. 369, No. 23.

the first sum that meets the necessary conditions. For $U_n = |S_n - L| + |S_{n-1} - L| = 2\epsilon$. But $|S_n - L| < |S_{n-1} - L|$. Therefore $|S_n - L| < \epsilon < |S_{n-1} - L|$.

For an example, let us consider the well-known alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$. The condition $a < a'$ is easily verified. Let us now take $\epsilon = .00005 = 1/20{,}000$. Then using the relation $U_n \leqq 2\epsilon = 2(1/20{,}000)$ we get $1/n \leqq 1/10{,}000$ or $n \geqq 10{,}000$. Therefore, $S_{10{,}000}$ is within .00005 of the limit of the series. It is also the first sum within that accuracy because we found a U_n exactly equal to 2ϵ.

In conclusion, I would like to thank Professor G. L. Curme for encouraging me to write this paper, and for helping to direct me into mathematics.

SOME TRIGONOMETRIC PRODUCTS

R. C. MULLIN, University of Waterloo

Products of the form $\prod_{k=1}^{n-1} f(k\pi/n)$, where f is any one of the six basic trigonometric functions, may easily be formed once $\prod_{k=1}^{n-1} \sin(k\pi/n)$ and $\prod_{k=1}^{n-1} \cos(k\pi/n)$ are known. These two latter products are readily found by elementary means.

We begin by noting that $e^{i\theta}$ and $e^{-i\theta}$ are the zeros of $z^2 - 2z\cos\theta + 1$. Also, the polynomial $\sum_{k=0}^{n-1} z^{2k} = (z^{2n} - 1)/(z^2 - 1)$ has $2n-2$ zeros, and these are obviously $e^{ik\pi/n}[k = \pm 1, \pm 2, \cdots, \pm(n-1)]$. Hence we obtain the identity

$$(*) \qquad \prod_{k=1}^{n-1} [z^2 - 2z\cos(k\pi/n) + 1] = \sum_{k=0}^{n-1} z^{2k}.$$

By placing $z = \mp 1$, we at once obtain

$$\prod_{k=1}^{n-1} [1 \pm \cos(k\pi/n)] = n/2^{n-1};$$

multiplying, we find

$$\prod_{k=1}^{n-1} [1 - \cos^2(k\pi/n)] = (n/2^{n-1})^2.$$

Hence

$$\prod_{k=1}^{n-1} \sin(k\pi/n) = n/2^{n-1},$$

since all factors $\sin(k\pi/n)$ are positive.

Now put $z = -i$ in identity (*) to give

$$\prod_{k=1}^{n-1} 2i\cos(k\pi/n) = \tfrac{1}{2}[1 - (-1)^n].$$

If n is even, $\prod_{k=1}^{n-1} \cos(k\pi/n) = 0$, a fact otherwise obvious, since $\cos \frac{1}{2}\pi = 0$. If n is odd, it immediately follows that

$$\prod_{k=1}^{n-1} \cos(k\pi/n) = (-1)^{\frac{1}{2}(n-1)}/2^{n-1},$$

and we can at once deduce that, for n odd,

$$\prod_{k=1}^{n-1} \tan(k\pi/n) = (-1)^{\frac{1}{2}(n-1)}n.$$

ON NONNEGATIVE POLYNOMIALS

LOUIS BRICKMAN,* Yale University, and LEON STEINBERG, Remington Rand Univac

1. Introduction. It is rather well known and easy to prove that a polynomial which is nonnegative on the whole real line can be written as the sum of the squares of two real polynomials. Here we discuss the analogous question for polynomials nonnegative in an interval, finite or half infinite. All polynomials mentioned are understood to have real coefficients.

Beginning with the interval $[0, \infty)$, we quote a result appearing in [3] and [5; 5].

THEOREM 1. *Let the polynomial $f(x)$ be nonnegative for nonnegative x. Then there exist polynomials $p(x)$, $q(x)$, $r(x)$, $s(x)$ such that*

$$f(x) = p^2(x) + q^2(x) + [r^2(x) + s^2(x)]x. \tag{1}$$

In this note we improve Theorem 1 and obtain

THEOREM 2. *Let the polynomial $f(x)$ be nonnegative for nonnegative x. Then there exist polynomials $p(x)$ and $q(x)$ such that*

$$f(x) = p^2(x) + q^2(x)x. \tag{2}$$

The method of proof is quite elementary. Moreover the technique can be applied to obtain a simple proof of the following somewhat deeper result of F. Lukács [5; 4] and [2; 35].

THEOREM 3. *Let the polynomial $f(x)$ be nonnegative in the finite interval $[a, b]$. Then $f(x)$ can be written*

$$f(x) = p^2(x)(x - a) + q^2(x)(b - x) \tag{3}$$

if the degree of $f(x)$ is odd, and

$$f(x) = p^2(x) + q^2(x)(x - a)(b - x) \tag{4}$$

if the degree of $f(x)$ is even. $p(x)$ and $q(x)$ can be chosen so that the degree of each term of (3) *and* (4) *does not exceed the degree of $f(x)$.*

* The first author was supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command under Contract No. AF49(638)224.

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A SET OF MATRICES WHOSE INVERSES HAVE FOR ELEMENTS THE ELEMENTS OF THE ORIGINAL MATRIX DIVIDED BY THE ORDER OF THE MATRIX

JOHN P. HOYT, U. S. Naval Academy

Let r be a primitive nth root of 1, i.e. an nth root which is not also an mth root for some proper divisor m of n. Then the n by n matrix $A_n=(a_{ij})$, where $a_{ij}=r^{ij}$, $(i, j=1, \cdots, n)$, has for its inverse $B_n=(b_{ij})/n$, where $b_{ij}=r^{i(n-j)}$, $(i, j=1, \cdots, n)$.

Proof. $A_nB_n=(c_{ik})$, where

$$c_{ik}=\frac{1}{n}\sum_{j=1}^{n}a_{ij}b_{jk}, \qquad (i, k=1, \cdots, n)$$

$$=\frac{1}{n}\sum r^{ij}r^{j(n-k)}=\frac{1}{n}\sum r^{j(n+i-k)}.$$

(1) For $k=i$, $c_{ik}=c_{ii}=(1/n)\sum_{j=1}^{n}r^{jn}=1$, since $r^n=1$.

(2) For $k\neq i$, $s=(n+i-k)$ is not divisible by n since $-(n-1)\leqq i-k\leqq(n-1)$ and $i-k\neq 0$. Thus $p=r^s\neq 1$ because of our choice of r as a primitive nth root of 1.

Then

$$c_{ik}=\frac{1}{n}\sum_{j=1}^{n}r^{js}=\frac{1}{n}\sum_{j=1}^{n}p^j=0$$

since

$$0=p^n-1=(p-1)\sum_{j=0}^{n-1}p^j=(p-1)\sum_{j=1}^{n}p^j \quad \text{and} \quad p-1\neq 0.$$

We have shown that $C_n=A_nB_n$ is the identity matrix of order n. Hence B_n is the inverse of A_n.

We note in the definition of B_n that $b_{ij}=r^{i(n-j)}=a_{i\,n-j}$ for all i and $j=1, \cdots, n-1$, while $b_{in}=1$. Thus the columns of B_n are merely those of A_n permuted and multiplied by $(1/n)$. Specifically, aside from the factor $(1/n)$, the first $n-1$ columns of B_n are the first $n-1$ columns of A_n in reverse order, while the nth column of B_n is the same as the nth column of A_n, namely, a column of 1's.

Several examples follow:

For $n=2$, $r=-1$,

$$A_2=\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \qquad B_2=\tfrac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}.$$

For $n=3$, let $r=\cos 120°+i \sin 120°$ where $i=\sqrt{-1}$ or let $r=\cos 120° -i \sin 120°$. Then

$$A_3 = \begin{pmatrix} r & r^2 & 1 \\ r^2 & r^4 & 1 \\ r^3 & r^6 & 1 \end{pmatrix} = \begin{pmatrix} r & r^2 & 1 \\ r^2 & r & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad B_3 = \tfrac{1}{3}\begin{pmatrix} r^2 & r & 1 \\ r & r^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For $n=4$, let $r=i=\sqrt{-1}$. Then

$$A_4 = \begin{pmatrix} i & i^2 & i^3 & 1 \\ i^2 & i^4 & i^6 & 1 \\ i^3 & i^6 & i^9 & 1 \\ i^4 & i^8 & i^{12} & 1 \end{pmatrix} = \begin{pmatrix} i & -1 & -i & 1 \\ -1 & 1 & -1 & 1 \\ -i & -1 & i & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$B_4 = \tfrac{1}{4}\begin{pmatrix} -i & -1 & i & 1 \\ -1 & 1 & -1 & 1 \\ i & -1 & -i & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

THE EQUATION OF THE CONIC

ROSE LARIVIERE and GRACE M. NOLAN, University of Illinois, Chicago

Eliminating the xy term from the general equation of the conic by the usual rotation of coordinate axes does not satisfy the students who have been taught to generalize. They prefer a well-known application of quadratic equation theory which does not limit the coefficient of xy to any specific value.

We assume that the coefficient B' of the xy term, after a rotation of axes through an angle θ, has been selected or is otherwise known. The given equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad A \geqq 0, \tag{1}$$

and the required equation, after the rotation to transform Bxy to $B'x'y'$, is to be written

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F = 0. \tag{2}$$

Then $A'+C'=A+C$ and $A'C'=-\frac{1}{4}(B^2-4AC-B'^2)$. To obtain A' and C' we have therefore to solve the characteristic quadratic given the sum and the product of its roots. An ambiguity remains, however, since we must determine which root is A' or which is C'.

[In what follows c will represent $\cos\theta$, and s will represent $\sin\theta$.]

To distinguish between A' and C' we note that

$$A' = Ac^2 + Bsc + Cs^2,$$

$$A' - A = (C - A)s^2 + Bsc = \frac{s}{2c}[2(C - A)sc + 2Bc^2]$$

$$= \frac{s}{2c}[2(C - A)sc + B(c^2 - s^2) + B(c^2 + s^2)] = \frac{s}{2c}[B' + B],$$

therefore

(3) $A' - A = \frac{1}{2}(B' + B)\tan\theta$. Similarly $C' - C = -\frac{1}{2}(B' + B)\tan\theta$.

If we limit ourselves to a rotation through a positive acute angle, which is always sufficient, $\tan\theta > 0$, and A' will be greater than A if $(B' + B) > 0$ or less than A if $(B' + B) < 0$. [These conclusions will be reversed whenever $\tan\theta < 0$.] Consequently the sign of $B' + B$ determines our choice of A' and C'.

For the computation of the remaining coefficients D' and E', the angle of rotation θ is required. But B, B', A and A' are now known so that equation (3) yields $\tan\theta$ specifically, from which $\sin\theta$ and $\cos\theta$ can be derived. D' and E' are then readily found by using the customary rotation transformation matrix.

Thus equation (2) can be determined with a minimum of calculation though it retains an arbitrary $x'y'$ term.

MATHEMATICAL EDUCATION NOTES

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All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

COOPERATIVE MATHEMATICS TESTS

A PROGRESS REPORT

SHELDON S. MYERS, Educational Testing Service, Princeton

In the spring of 1958, initial planning was started for a new series of end-of-course mathematics tests for the Cooperative Test Division of Educational Testing Service. The ferment and change occurring in mathematics curricula around the country lent urgency to the plans, but at the same time made the task much more difficult.

In the fall of 1958, the older end-of-course Cooperative Mathematics Tests were each reviewed by at least five reviewers. The results of this survey provided

one of the bases for the design and specifications of a new series of mathematics tests. By the early part of 1959, an advisory committee of ten members had been established to help in the development of test specifications. Some members of the committee had expressed apprehension at developing new achievement tests in mathematics at a time when so much development of new mathematics curricula was taking place. It was felt, however, that since the entire test development process required about four years, work on the new tests should proceed, even though compromises would be necessary now and revisions of the new tests needed several years later.

By the summer of 1959, the specifications for 11 new tests (Arithmetic, Structure of the Number System, Algebra I, Algebra II, Algebra III, Geometry I, Geometry II, Trigonometry, Analytic Geometry, Calculus I and Calculus II) were developed with the indispensable assistance of the advisory committee. Consideration of three other tests—Probability and Statistical Inference, Elementary Analysis, and Finite Mathematics was postponed until a later date.

Since the Arithmetic Tests involved three parallel forms, and the others, two parallel forms each, a total of 23 new forms was planned. Two pretests for each of these forms were scheduled, making a total of 46 new pretests to be written by 46 item writers, for a total of over 1850 questions. These writers began work during the summer of 1959 after an elaborate kit of instructions, sample items, and test specifications had been sent to each. From November, 1959, to March, 1960, the Mathematics Section of Test Development Division at ETS selected and reviewed 1850 items for 46 new pretests which were tried in a selected national population of schools in May, 1960. Each of the pretests was reviewed by four competent mathematics teachers.

In our zealous efforts to modernize and up-date the content of these end-of-course tests, we learned to our dismay that the pretests were too difficult for our sample of schools, with mean scores averaging about 35 per cent of the total possible scores. After much careful discussion and planning, it was decided to prepare, as a first step, final forms only in Arithmetic, Algebra I, Algebra II, Geometry I, Geometry II. The purpose of this was to see whether it was possible to select appropriate items from the pretests and revise them so that forms appropriate in content and difficulty could be constructed for pretesting and norming in May, 1961.

After many revisions, the Algebra and Geometry Tests were given a quick, experimental try-out in two large high schools in metropolitan Philadelphia, in January, 1961. This try-out revealed that the forms still required drastic revision to make them appropriate in difficulty. After another thorough overhauling which involved the writing of many new, more appropriate questions, Arithmetic, Algebra, and Geometry were pretested during May, 1961. These tests were administered to students in three selected school populations—suburban, urban, and rural. The results justified our labors, and the Cooperative Test Division is now proceeding with the publication of the following tests: Arithmetic, Forms A, B, and C; Algebra I, Forms A and B; Algebra II, Forms

A and B; and Geometry, Forms A and B (the original two levels for Geometry are represented by Parts I and II in the final forms). These will be available to schools in the spring of 1962, accompanied by preliminary interpretive data based on performance by selected reference groups.

Meanwhile, work is under way to subject Algebra III, Analytic Geometry, Calculus, and Structure of the Number System to similar extensive revisions in preparation for further tryouts in May, 1962. At the same time, Arithmetic, Algebra I, Algebra II, Geometry, and Trigonometry will be subjected to norming on large samples of students in appropriate reference groups. Results of this norming administration will form the basis for the Manual for Interpreting Scores for these tests, providing more comprehensive data to supersede the tentative information described above. The other tests will be normed in 1962.

WOMEN IN MATHEMATICAL CAREERS*

Women earned one third of all baccalaureates, one third of all master's degrees, but only one tenth of all doctorates in 1958. Education was the most popular major for women at all levels, whereas mathematical subjects ranked about eleventh in a field of fifteen majors. More women obtained the bachelor's and master's degrees in mathematics than in the physical sciences, but more women (69) obtained the doctorate in the physical sciences than in mathematics (15).

In 1957, 1166 women were graduated with an undergraduate major in mathematics, which was nearly 1% of the total number of women graduates. Of this group, 8% continued their education and 86% became employed. In the employed group, 42% were employed as mathematicians or statisticians, 42% were teachers, 3% were research workers, and 16% were in various other occupations.

In the 1956–58 National Register of Scientific and Technical Personnel, 1277 women were listed in mathematical occupations, which was nearly 11%, of the total mathematical listings. In this group, 53% were employed by colleges and universities, 29% by private industry, 13% by governmental agencies, and 5% in other occupations. The Ph.D. degree was held by 19%, the master's by 49%, the bachelor's by 30%, and about 1% held less than a bachelor's degree. The median annual salary for this group was $6026.

According to a study by the Educational Testing Service, less than 50% of the upper one third of high ability women high school graduates continue their education, whereas over 60% of the men of the same ability level do so. This is one indication of the loss of potential talent for science and mathematics among women.

* Adapted by D. J. Dessart of the State University College, Oneonta, New York, from National Science Foundation, *Women in Scientific Careers*, Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C.

NEW APPOINTMENTS IN MATHEMATICS IN THE OFFICE OF EDUCATION

The United States Commissioner of Education, Sterling M. McMurrin, has recently announced the appointment of several specialists to fill positions in Instruction, Organization, and Services Branch of the Office of Education.

Dr. Edwina Deans, formerly elementary supervisor in Arlington County, Virginia, and one of the nominees for vice president of the National Council of Teachers of Mathematics in the 1962 election, representing the elementary level, will serve as elementary mathematics specialist. In this position, Dr. Deans will survey trends in public school systems throughout the country relative to mathematics programs, and act as adviser to state and local programs in setting up and evaluating experimental mathematics projects.

Dr. Lauren Woodby, formerly Head of the Department of Mathematics at Central Michigan University, Mount Pleasant, Michigan, has been appointed secondary school mathematics specialist. In his position in secondary mathematics he will have essentially the same kind of responsibilities for secondary schools that have been assigned to Dr. Deans for the elementary level. Dr. Woodby will work with Dr. Kenneth E. Brown of the Office of Education.

Dr. J. Ned Bryan will head up a special project aimed at developing techniques of discovery and encouragement of superior abilities among American youth, from kindergarten through graduate school. Before going to the Office of Education, Dr. Bryan was director of the North Central Association Project on Guidance and Motivation of Superior and Talented Students supported by the Carnegie Corporation of New York. Earlier he was director of the NEA Academically Talented Pupil Project. Dr. Bryan has been a staff member of the University of West Virginia and of Rutgers University.

PILOT STUDY ON THE USE OF SPECIAL TEACHERS IN MATHEMATICS IN GRADES 4, 5, AND 6

The Science Teaching Improvement Program of the American Association for the Advancement of Science is sponsoring a pilot study this year on the Use of Special Teachers of mathematics in grades 4, 5, and 6. Schools participating are in the Washington, Baltimore, and Wilmington, Delaware area. The teachers are using the sample textbooks of the School Mathematics Study Group for grades 4, 5, and 6.

The study is considered a follow-up of the study on the Use of Special Teachers in Science and Mathematics which the Science Teaching Improvement Program has conducted during the two academic years, 1959–61. The first study on the Use of Special Teachers, in which the school systems of Cedar Rapids, Iowa, Lansing, Michigan, Washington, D. C., and Woodford County, Kentucky cooperated, was completed at the end of the school year, 1961. Final reports of the study were issued early this year.

This year's study also is considered as a pilot program for a more extensive program, which may be started in 1962–63, in which would be involved not

only special teachers using the new materials in mathematics, but also new materials in science at the elementary school level. In the first study on the use of special teachers, the course of study for each school was the regular course of study for that school system. None of the new materials in mathematics or science were used. It is believed that the use of special teachers may be especially helpful in a period in which new materials will be tried out in the classrooms. Hence, one of the purposes of the study is to compare the effectiveness of teachers, who can devote full time to the teaching of mathematics, with general classroom teachers in the use of the new materials in mathematics.

It is also part of the plan to experiment with inservice programs for teachers using the new sample courses. Professor John A. Brown of the University of Delaware, and Mildred Cole, lecturer at the University of Maryland, are in charge of the inservice work with teachers in the pilot study. Both Dr. Brown and Miss Cole have been active participants in the work of the School Mathematics Study Group, and have done extensive inservice education work with elementary teachers.

School systems participating, 1961–62, in the pilot study are Montgomery County, Maryland; the Baltimore City Schools; the public schools of Newark, Delaware; the Lora Little School, the Stanton Elementary School, and the Tatnall School in Wilmington, Delaware. The Tatnall School is a private school, the others are all public schools. The most extensive use of the materials is in Newark, Delaware where some 17 teachers are involved in grades 4, 5, and 6.

The purpose of the report to be issued at the end of the year will be to assist the school systems in making decisions for future use of new materials in mathematics, and also to serve as a basis for developing a more extensive project in both science and mathematics for the next two years.

While it was intended in planning the pilot study that all teachers involved be full-time teachers of mathematics in grades 4, 5, and 6, in working out plans with the school systems, several other plans of administration were also developed. The Baltimore city schools, for example, have an experimental program in team teaching and in the use of resource teachers. Some of the special teachers in Baltimore, therefore, are working part time as special teachers, and part time with the team teacher and resource teacher experiments.

FEDERAL AID TO EDUCATION*

Although the 1st Session of the 87th Congress adjourned without completing action on general aid or aid to higher education, an impressive record of enactments in specialized areas of education was made, including appropriations for: the Peace Corps; training teachers of the mentally retarded; the Office of Vocational Rehabilitation; an experimental program in research and demonstration

* Abstracted from *Washington Outlook on Education* of the National Education Association, October 6, 1961.

projects to combat juvenile delinquency; an extension of the National Defense Education Act of 1958, which includes special benefits for the teaching of science, mathematics and modern foreign languages, and under which some course content work in English may be started; payments to school districts in federally impacted areas; the extensive education program for the National Science Foundation; Agricultural Research Service of the Department of Agriculture for research activities; the Center for Cultural Interchange between East and West; and college housing.

EARNED DEGREES IN HIGHER EDUCATION

A brief summary of earned degrees in higher education appeared in the November 4, 1961 issue of *School and Society*. Written by Walter Crosby Eells, the article is based upon an analysis of data published in the two latest editions of *American Universities and Colleges* (Washington: American Council on Education, 1959, 1960) and supplemented by reports of the United States Office of Education for the four latest years available: "Earned Degrees Conferred by Higher Education Institutions" (Washington: U. S. Office of Education, Department of Health, Education, and Welfare).

More than 7,500,000 earned degrees of baccalaureate or higher level have been conferred by 290 of the major institutions of higher education in the United States since their organization date and through 1958–59. Eleven of the institutions have awarded more than 100,000 degrees. The first four ranking institutions by number of degrees are: the University of California, Columbia University, New York University, and the University of Michigan.

REGIONAL TEPS CONFERENCES 1962

The theme for the regional TEPS conferences held in January, 1962, was *New Horizons: Moving from Ideas to Action*. Don Davies, new Executive Secretary of the TEPS Commission of the National Education Association, indicated that the central purpose of the conferences was to stimulate mobilization of the teaching profession to assume responsibility for achieving and maintaining high standards of preparation and practice for its members. Major proposals of the New Horizons Task Force provided a starting point for the discussions.

Among groups which nominated participants for the regional conferences were the Conference Board of the Mathematical Sciences, the National Council of Teachers of Mathematics, and the American Association for the Advancement of Science.

The new Executive Secretary, Don Davies, formerly a member of the Department of Education of the University of Minnesota, replaced Dr. T. M. Stinnett as Executive Secretary in the summer. Dr. Stinnett has become Assistant Executive Secretary for Professional Development and Welfare of the National Education Association.

CONANT TO STUDY TEACHER EDUCATION

James B. Conant has launched a study of the education of American teachers with the support of the Carnegie Corporation of New York. This study follows closely on the earlier Conant studies which have considered junior and senior high school education and have resulted in publications which have attracted wide attention.

In the first year of the study the plan is to limit inquiries to the preparation of elementary and secondary school teachers in the academic subjects. Consideration of the preparation of other teachers and of special service personnel, as well as matters of certification and accreditation, will be delayed until the second year. Because of time limitations, the staff will be able to study only a small sample of the various patterns of the 1100 colleges and universities which prepare teachers.

Among staff members who will assist Dr. Conant are: John I. Goodlad, Professor, School of Education, UCLA; Jeremiah S. Finch, Professor of English and former Dean of the College, Princeton University; William H. Cartwright, Professor and Chairman, Department of Education, Duke University; Robert F. Carbone, former instructor, Graduate School of Education, University of Chicago; and E. Alden Dunham, a member of Dr. Conant's staff for the past three years.

THE CENTER FOR PROGRAMED INSTRUCTION*

The Center for Programed Instruction is a nonprofit, educational organization, incorporated under the laws of the State of New York with the approval of the Commission on Education. Due to the rapid expansion of the field of programmed instruction and the increasing demands upon the Collegiate School Automated Teaching Project for information and guidance, the Center for Programed Instruction has been established to extend the activities of the Collegiate School Project by effectively translating research finding into practical classroom application.

The staff of the Center is composed of teachers and administrators who do programming and workshop instruction. A group of experienced research personnel from the fields of education and psychology are also included. P. Kenneth Komoski, President of the Center, has been a teacher and administrator at the Collegiate School, New York City, for the last eight years. Since 1959 he has directed the Collegiate School's work in programming and the use of programmed instruction. This work at Collegiate, done under grants from the Fund for the Advancement of Education and the Carnegie Corporation of New York, has formed the basis for the foundation of The Center for Programed Instruction. The Center, through an additional grant from the Carnegie Corporation, is extending and continuing work with many schools throughout the country in

* From a report of The Center for Programed Instruction, Inc., 365 West End Ave., N. Y., N. Y.

research and development of programmed instruction. Perhaps the most significant aspect of the work of the Center is the fact that a research project, initiated at the school level, is committed to a radical examination of both the learning and teaching processes.

Description of materials in mathematics which are being programmed, or for which the program is complete, are as follows:

Pre-Primary and Primary Grades. A *preverbal* program in number concepts and symbolism is being written and tried in cooperation with the Center at the Dalton School, New York City. It treats number concepts, the decimal system, addition and subtraction. Special visual and tactile aids have been developed by the authors to complement the program. It is being tested and evaluated in 1961–62.

Junior High School. Junior high school mathematics utilizing the Maryland Mathematics Project ideas and the ideas of the School Mathematics Study Group at Yale University is being programmed at the Center. Experimental units in *Introductory Probability*, *Prime Numbers and Factoring*, and *Nonmetric Geometry* are available.

Senior High School. A program in sets, relations, and functions has been completed and revised. The material covered is essentially that recommended by the College Entrance Examination Board Commission on Mathematics. The program was originally written for high school students, but trials with eighth graders and individual students in the second, fifth, and sixth grades have been successful. The range of time to complete the program was between four and seven hours. This program, presented in programmed text form, is available on request.

College Preparatory Mathematics. Efforts to program these years of high school college preparatory mathematics reflecting modern curriculum revisions are being continued at the Center. Experimental programmed units covering the following topics are available in research packages accompanied by a detailed instruction manual: numbers and numerals; directed numbers and real numbers; sets, relations and functions (now available in revised form); mappings (many to one, one to one); binary operations (addition, multiplication, etc.); linear equations; and other topics.

Mathematics for Science. Programmed topics to complement work in the physical sciences are being prepared. This material will include units in *Logarithms and Scientific Notation* and *Use of the Slide Rule*, which are now available for experimental purposes.

Similar work on programming has been completed for a program on high school chemistry, reflecting recent approaches to the teaching of chemistry such as the CBAP and CHEM studies. Some of the units created by the Physical Science Study Committee are being incorporated in the program. More

difficult topics in general science for use in grades four through nine are also being programmed. Typical of such units are *Use of the Microscope* and *Atomic Structure*. These units are being tested in 1961–62.

PUBLICATIONS OF INTEREST

A report of the three TEPS conferences, (Bowling Green, 1958, Kansas, 1959, and San Diego, 1960) has appeared as a book by G. K. Hodenfield and T. M. Stinnett. *The Education of Teachers*, Prentice-Hall, Inc., $0.55 paperback, $3.95 hard cover. Mr. Hodenfield is Education Reporter for the Associated Press and, along with other newsmen, covered day-by-day progress of all three conferences. The book not only reflects the damage inflicted on schools by the persistent feud between academicians and educationists, but also records what was on the minds of the thousands of participants at the three conferences.

In the wide publicity now being given to mathematics, especially the new course materials in mathematics, one article of interest reviewed the work in geometry for primary graders reporting the work of Suppes and Hawley. This article appeared in *Parade* for October 29, 1961. *Parade* is a supplement to many Sunday newspapers. The publishers of *Parade* indicate that their magazine reaches ten million families.

An interesting, unusual review of activities in the field of teacher education in mathematics by Lawrence T. Casto of the University of Kansas, appears in the October, 1961 issue of the *Bulletin of the Kansas Association of Teachers of Mathematics*. All of the major studies have been included.

An article entitled "Professional Mathematicians Work in Industry and Government" appeared in the *Monthly Labor Review* for September 1961. Characteristics of employees and of their positions are tabulated based upon a survey conducted by the Bureau of Labor Statistics and sponsored by the Association's Committee on Non-Teaching Mathematical Employment. Copies of the article (Reprint No. 2374) may be obtained without charge from Bureau of Labor Statistics, 341 Ninth Avenue, Room 1025, New York 1, New York.

A very useful article entitled "A Bibliography for Careers in Mathematics," by Nura D. Turner, appeared in *Science Education* for October, 1961 and also in the November issue of the *Mathematics Teacher*. The bibliography is annotated.

ELEMENTARY PROBLEMS AND SOLUTIONS

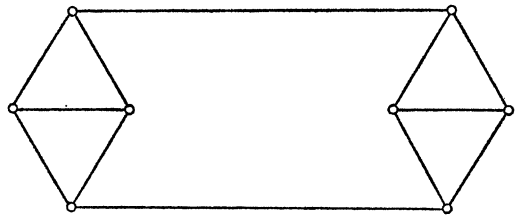
EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1506. *Proposed by L. R. Ford, Jr., C-E-I-R, Inc., Beverley Hills, Calif.*

The vertices and edges of a certain polyhedron, from which the inside and the faces have been removed, may be imbedded in the plane, allowing stretching. Sketch the original polyhedron if the plane imbedding is as shown.



E 1507. *Proposed by D. J. Newman, Yeshiva University*

If a square is partitioned into n convex polygons, show that the total number of edges present is at most $3n+1$.

E 1508. *Proposed by Leonard Carlitz, Duke University*

David Allison has proved (this MONTHLY [1961, 272]) that the identity in n

$$\left(\sum_{r=1}^{n} r^k\right)^p = \left(\sum_{r=1}^{n} r^m\right)^q, \qquad p > q,$$

holds only when $p=3$, $q=1$, $k=1$, $m=3$. Show that

$$\left(\sum_{r=1}^{n} (-1)^r r^k\right)^p = \left(\sum_{r=1}^{n} (-1)^r r^m\right)^q, \qquad p > q,$$

is never satisfied for $k \geqq 1$, $m \geqq 1$.

E 1509. *Proposed by A. G. Konheim, IBM, Yorktown Heights, New York*

Sum for all real $t \in (-1, 1)$ the series

$$1+\frac{2}{3}t^2+\frac{2}{3}\frac{4}{5}t^4+\frac{2}{3}\frac{4}{5}\frac{6}{7}t^6+\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}t^8+\cdots.$$

E 1510. *Proposed by A. J. Goldman, National Bureau of Standards*

If A and B are sets of complex numbers, let $B-1=\{b-1: b\in B\}$, $a(B-1)=\{a(b-1): b\in B\}$, and $A*B=\cap\{a(B-1): a\in A\}$. Suppose $A=\{z: |z-1|\leqq k\}$, where $k>1$, and that $B=\{z: |z|\geqq 1\}$. Find $A*B$.

SOLUTIONS

A Property of Isosceles Triangles

E 1476 [1961, 667]. *Proposed by M. T. Salhab, Illinois Institute of Technology*

In triangle ABC, $AB=AC$, D is the midpoint of BC, E is the foot of the perpendicular from D on AC, and F is the midpoint of DE. Prove that AF is perpendicular to BE.

I. *Solution by Leon Bankoff, Los Angeles, California.* Let H be the foot of the perpendicular from B on AC. Then BE and AF are corresponding medians of the similar right triangles CBH and DAE, the corresponding sides of which are mutually perpendicular. Hence AF is perpendicular to BE.

II. *Solution by D. C. Stevens, New York University.* Let A be (0, 0), B be $(4a, 4b)$, and C be $(4c, 0)$. Then $a^2+b^2=c^2$, D is $(2a+2c, 2b)$, E is $(2a+2c, 0)$, and F is $(2a+2c, b)$. One now easily finds that $m_{AF}m_{BE}=-1$.

III. *Solution by Stanton Philipp, Seal Beach, Calif.* Using elementary vector algebra we have $AF\cdot BE=(AE+EF)\cdot(BD+DE)=AE\cdot BD+EF\cdot BD+EF\cdot DE=(AD+DE)\cdot BD+EF\cdot BD+EF\cdot DE=DE\cdot BD+EF\cdot BD+EF\cdot DE=DE\cdot DC-(DE\cdot DC)/2-(DE\cdot DE)/2=(DE\cdot DC)/2-(DE\cdot DE)/2=DE\cdot(DC-DE)/2=(DE\cdot EC)/2=0$.

Also solved by A. N. Aheart, D. W. Bailey, Merrill Barnebey, Mark Benard, R. E. Blewster, Jr., J. L. Botsford, R. E. Bowen, D. A. Breault, Brother Alfred, Brother T. Brendan, Brother T. C. Wesselkamper, W. E. Buker, R. I. Canfield, F. G. Carty, Gil Chisholm, Jr., D. I. A. and S. P. Cohen (jointly), N. A. Court, A. E. Crofts, Jr., D. M. Danvers, J. F. Dillon, H. P. Dodge, Underwood Dudley, Jane Evans, G. E. Floyd, Herta T. Freitag, Stuart Friedman, Philip Fung, W. M. Gentleman, A. F. Gilman III, N. V. Glick, Michael Goldberg, R. P. Goldberg, L. D. Goldstone, Richard Goodman, S. H. Greene, Bernard Greenspan, Cornelius Groenewoud, G. W. Grotts, Weiner Held, Frank Herlihy, Erwin Just and Norman Schaumberger (jointly), Roman Kaluzniacki, Geoffrey Kandall, B. R. Kay, M. R. Kirch, David Klapphulz, J. D. E. Konhauser, J. F. Leetch, F. Leuenberger, Esther A. Linfield, Frank McGee, P. B. Manchester, Wallace Manheimer, D. C. B. Marsh, Beckham Martin, Emerson Martin, Jr., W. F. Mayhew, M. V. Mielke, C. N. Mills, Roderick Montgomery, J. R. Naylor, P. R. Nolan, G. Normandin, R. J. Oberg, Woody Pang, C. E. Parker, F. D. Parker, W. R. Ransom, N. A. Robins, A. P. Rollett, J. J. Segedy, E. M. Scheuer, S. D. Shore, J. L. Sieber, D. L. Silverman, Barny Simon, Michael Skalsky, W. C. Song, Kirk Stewart, Wilmont Toalson, J. W. Toole, Guy Torchinelli, Rita Wagner, James Walton, W. C. Waterhouse, Charles Wexler, Hazel S. Wilson, Dale Woods, Larry Zalcman, David Zeitlin, and the proposer. Late solutions by Peter Balint, Romae J. Cormier, Andrew DiGirolamo, Ragnar Dybvik, Mike Madden, J. B. Muskat, C. S. Ogilvy, C. F. Pinzka, L. A. Ringenberg, Richard Shramko, E. L. Spitznagel, Jr., A. A. Warntz, Roscoe Woods, and Bell Young.

Re $f(n)=n^2-n+41$

E 1477 [1961, 667]. *Proposed by Sidney Kravitz, Dover, New Jersey*

It is known that $f(n)=n^2-n+41$ yields prime numbers for $n=1, \cdots, 40$. Prove that (a) $f(n)$ is never divisible by a positive integer <41, (b) $f(n)$ is never a perfect square except for $n=41$, (c) for each n there exists an m such that $f(m)=f(n)f(n+1)$, (d) $f(1722)$ is the smallest f with four, not necessarily distinct, prime factors.

Solution by D. C. Stevens, New York University. More generally, suppose that $f(n)=n^2-n+p$ yields prime numbers for $n=1, \cdots, p-1$. (a) If q, some prime less than p, divides $f(n)$ for some n, then q divides $f(n')$ where $n' \equiv n \pmod{q}$ and $0<n' \leqq q$. But $f(n')$ is prime by assumption, so $f(n)$ is never divisible by a number $<p$. (b) If $f(n)=a^2$, the above implies that $a \geqq p$. $f(p)=p^2$. Since f is strictly increasing, and $f(a)<a^2$, $f(a+1)>a^2$ for $a>p$, $f(n)=a^2$ is impossible for $a>p$. (c) $f(n)f(n+1)=f(n^2+p)$. (d) The smallest number with four factors $\geqq p$ is p^4. But $f(p^2)<p^4$ and $f(p^2+1)>p^4$. The next possibility is $p^3(p+2)$. The assumptions imply $p+2$ is a prime. $f(p(p+1))=p^3(p+2)$.

The above is valid for $p=3$, 5, 11, 17, 41, and possibly other values.

Also solved by J. W. Baldwin, M. T. L. Bizley, Brother Alfred, A. L. Buchman, F. G. Carty, D. I. A. Cohen, D. Drasin and J. Yeager (jointly), Underwood Dudley, Philip Fung, N. V. Glick, Michael Goldberg, S. H. Greene, J. E. Homer, Jr., Erwin Just and Norman Schaumberger (jointly), P. B. Manchester, Wallace Manheimer, D. C. B. Marsh, M. V. Mielke, P. R. Nolan, Hugh Noland, R. J. Oberg, Stanton Philipp, N. A. Robins, I. D. Ruggles, D. L. Silverman, The Cincinnatians, Guy Torchinelli, J. E. Vinson, W. C. Waterhouse, David Zeitlin, and the proposer. Late solutions by Ronald Alter, D. A. Breault, Gus DiAntonio, A. J. Kokar, D. B. Lloyd, R. H. C. Newton, and Philip Rose.

The proposer pointed out that the smallest f with three factors occurs at $n=421$ (Problem 395, *Mathematics Magazine*, Nov. 1959), and the smallest f with six factors occurs at $n=139{,}564$ (*Recreational Mathematics Magazine*, June 1961, p. 50).

Digital Roots of Perfect Numbers

E 1478 [1961, 667]. *Proposed by Jonathan Sondow, University of Wisconsin*

Prove that the digital root of every even perfect number except 6 is 1.

I. *Solution by Erwin Just and Norman Schaumberger, Bronx Community College.* An even perfect number must have the form $2^{n-1}(2^n-1)$, where 2^n-1 is a prime, which implies n is odd for $n>2$. Now $2^{n-1}(2^n-1)-1=2^{2n-1}-2^{n-1}-1 \equiv (2^n+1)(2^{n-1}-1)$, which is divisible by 9 since each of the last two factors is divisible by 3.

II. *Solution by Wallace Manheimer, Franklin K. Lane High School, Brooklyn, New York.* Let $N=2^{k-1}(2^k-1)$. For $k>2$ and even, 2^k-1 is composite, and N is not a perfect number. For odd k, the digital roots of 2^k-1 form the recurring sequence 1, 7, 4, $\cdots$. The corresponding roots of 2^{k-1} are 1, 4, 7, $\cdots$, and $N \equiv 1 \pmod 9$.

III. *Remarks by David Zeitlin, Remington Rand Univac.* Since the digital root of a number (base 10) is the remainder when divided by 9, we have the following comments from L. E. Dickson, *History of the Theory of Numbers*, vol. 1, New York, 1952:

p. 9. "Tartaglia (1506–1559) observed that any perfect number except 6 yields the remainder 1 when divided by 9."

p. 25. "H. Novarese (1887) proved that every perfect number of Euclid's type ends in 6 or 28, and that each one >6 is of the form $9k+1$."

Also solved by R. E. Bowen, D. A. Breault, Brother Alfred, J. L. Brown, Jr., F. G. Carty, D. I. A. Cohen, Underwood Dudley, N. V. Glick, Michael Goldberg, R. P. Goldberg, Sidney Kravitz, Warren McCausland, Andrzej Makowski, D. C. B. Marsh, R. J. Oberg, F. D. Parker, D. L. Silverman, D. C. Stevens, Guy Torchinelli, W. C. Waterhouse, and the proposer. Late solutions by J. F. Dillon and C. F. Pinzka.

Several solvers pointed out that this problem appears, with solution, in Maxey Brooke, "On the digital roots of perfect numbers," *Mathematics Magazine*, vol. 34, no. 2 (Nov.–Dec. 1960), p. 100.

Restricted Choices

E 1479 [1961, 668]. *Proposed by Morton Abramson, McGill University*

Find the number of ways of choosing k elements from n elements $x_1, \cdots, x_n$ so that no three consecutive elements appear in any choice.

Solution by the proposer. We wish to find the number of distinguishable arrays of $n-k$ dashes and k dots along a straight line with the restriction of no three consecutive dots appearing. To do this we array only the $n-k$ dashes along a straight line and consider the $n-k+1$ spaces or slots, those between the dashes, the space before the first dash, and the space after the last dash. What then is the number of ways of inserting the k dots into the spaces with at most 2 dots being inserted into any one space? There are

$$\binom{n-k+1}{i}\binom{n-k+1-i}{k-2i} = \binom{n-k+1}{k-i}\binom{k-i}{i}$$

such distinguishable insertions with exactly i spaces each containing 2 dots. It follows that our required number is

$$\sum_{i=0}^{[k/2]} \binom{n-k+1}{k-i}\binom{k-i}{i}.$$

Also solved by Brother T. C. Wesselkamper, Virginia Christian, S. J. Einhorn, Seymour Geisser, W. M. Gentleman, S. H. Greene, J. F. Leetch, Nathan Mantel, D. C. B. Marsh, Michael Skalsky, D. C. Stevens, Warren Teitelman, and W. C. Waterhouse. Late solution by J. D. Nulton.

Many of these solutions disagreed with the solution published above.

Probability of a Polygon

E 1480 [1961, 668]. *Proposed by Leo Flatto and A. G. Konheim, IBM, Yorktown Heights, New York*

Let the line segment $[0, 1]$ be divided into $n+1$ segments by n points

$P_1, \cdots, P_n$. Assume the $\{P_i\}$ are independent random variables each uniformly distributed on $[0, 1]$. What is the probability p that the $n+1$ segments can be joined to form an $(n+1)$-sided polygon?

Solution by G. A. Heuer, Concordia College, Moorhead, Minnesota. The required property fails if and only if there is a closed subinterval of length $1/2$ containing none of the P_i. If this subinterval is $[0, 1/2]$, the corresponding probability is $1/2^n$. If this subinterval lies between the two smallest P_i, the probability is similarly $1/2^n$. There are $n+1$ disjoint and equally likely such cases, giving a complementary probability of $(n+1)/2^n$. Hence $p = 1-(n+1)/2^n$.

Also solved by Seymour Geisser, N. V. Glick, Michael Goldberg, D. C. B. Marsh, Michael Skalsky, Kirk Stewart, and the proposers.

Geisser pointed out that the problem is a special case of theoretical exercise 7.2 on p. 307 of *Modern Probability Theory and Its Applications*, by E. Parzen.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5013. *Proposed by L. E. Ward, Jr., U. S. Ordnance Test Station, China Lake, California.*

Let R denote the set of real numbers, Z the set of complex numbers. Are R and Z isomorphic when considered as additive groups?

5014. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

It is well known that an equilateral triangle cannot be imbedded in a square lattice. However, it can be done in a cubic lattice. Can this be extended, i.e., can any regular polygon be imbedded in a cubic lattice of high enough dimension?

5015. *Proposed by Basil Gordon and E. G. Strauss, University of California at Los Angeles*

Suppose $f(x)$ is continuous on $[a, b]$ and n times differentiable on (a, b). Suppose $a = x_0 < x_1 < \cdots < x_n = b$. Show that

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ f(x_0) & f(x_1) & \cdots & f(x_n) \end{vmatrix} = \frac{1}{n!} f^{(n)}(\xi) \prod_{i>j} (x_i - x_j)$$

where $a<\xi<b$.

5016. *Proposed by J. S. Frame, Michigan State University*

An infinite sequence of even polynomials $P_n(x)$ is defined by the differential equation $P_n''(x) = -P_{n-1}(x)$, together with the boundary conditions $P_0(x) = 1$, $P_n(1) = P_n(-1) = 0$ for $n>0$. Prove that

$$\int_{-1}^{1} P_{n-1}(x) P_{m-n}(x) dx = 4^m(4^m - 1) \cdot 2B_m/(2m)!,$$

where B_m denotes the mth Bernoulli number, given by the expansion

$$1 - \frac{t}{2} \cot \frac{t}{2} = \sum B_m t^{2m}/(2m)!.$$

5017. *Proposed by Leonard Carlitz, Duke University*

Let $w=f(z)$ be an analytic function of the complex variable $z=x+iy$ that is an algebraic function of the two real variables x, y. Show that w is an algebraic function of z.

5018. *Proposed by D. J. Newman, Yeshiva University*

Show that a rectangle can be covered by two smaller rectangles similar to itself if and only if it is not a square.

SOLUTIONS

Idempotent Element

3133 [1925, 261]. *Proposed by A. A. Bennett*

Show that in every set containing but a finite number of elements and admitting an associative rule of multiplication, there must be an idempotent element; that is, one which is equal to its square. Show that this does not continue to hold if the set contains an infinite number of elements.

Solution by T. M. K. Davison, University of Toronto. Let a be an arbitrary element of a given finite set S. Now S contains $a^2 \neq a$ (else there is nothing to prove). Hence consider the set $A = \{a^n \mid n = 1, 2, \cdots\}$. This set is finite since it is contained in S. Therefore there exist positive integers r, s such that $a^r = a^s$, $r>s\geqq 1$. For, if not, the powers of a are all distinct and hence generate an infinite set.

We can find a q such that $s+q\equiv 0 \pmod{r-s}$ where $0\leqq q<r-s$. That is, $s+q=k(r-s)$ for an integer $k>0$ for $s+q>0$, $r-s>0$. Now from $a^r=a^s$ it follows by induction that, for every $n\geqq 1$, $a^{nr-(n-1)s}=a^r=a^s$. Multiplying both sides by a^p, where $0\leqq p<r-s$, we have, for every $n\geqq 1$,

$$(*) \qquad a^{nr-(n-1)s+p} = a^{n(r-s)+s+p} = a^{s+p}.$$

Consider $a^{k(r-s)}$ where $k(r-s)=s+q$. Now

$$a^{k(r-s)}\cdot a^{k(r-s)} = a^{k(r-s)}\cdot a^{s+q} = a^{k(r-s)+s+q} = a^{s+q} = a^{k(r-s)}$$

by an application of (*). Hence $a^{k(r-s)}$ is an idempotent element of S.

The infinite set $\{2^n \mid n=1, 2, \cdots\}$ does not possess an idempotent element.

Polynomials Related to the Bernoulli Polynomials

4945 [1961, 67]. *Proposed by N. R. Riesenberg, Brooklyn College*

If $F_n(x)$ is the coefficient of z^n in the expansion of $2hze^{xz}/(e^{hz}-e^{-hz})$ in ascending powers of z so that $F_0(x)=1$, $F_1(x)=x$, $F_2(x)=(3x^2-h^2)/6$, etc., show that:

(1) $F_n(x)$ is a homogeneous polynomial of degree n in x and h.

(2) $dF_n(x)/dx=F_{n-1}(x)$, $n\geqq 1$.

(3) $\int_{-h}^{h} F_n(x)dx=0$, $n\geqq 1$.

(4) If $y=a_0F_0(x)+a_1F_1(x)+a_2F_2(x)+\cdots$, where a_i are real constants, then the mean value of d^ry/dx^r in the interval $-h\leqq x\leqq h$ is a_r.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since $r/(e^r-1)=\sum_{k=0}^{\infty} B_kr^k/k!$, we find that

$$F_n(x) = \sum_{k=0}^{\infty} \frac{B_k(2h)^k(x+h)^{n-k}}{k!(n-k)!},$$

where B_k are Bernoulli numbers. (1) is obvious. Now (2) follows from

$$dF_n(x)/dx = \sum_{k=0}^{n-1} \frac{B_k(2h)^k(x+h)^{n-k-1}}{k!(n-k-1)!} = F_{n-1}(x).$$

To prove (3) we note that $\sum_{k=0}^{n} \binom{n+1}{k} B_k=0$, $n\geqq 1$, and therefore

$$\int_{-h}^{h} F_n(x)dx = (2h)^{n+1} \sum_{k=0}^{n} \frac{B_k}{k!(n-k+1)!} = 0.$$

Since $y(x)=\sum_{k=0}^{\infty} a_kF_k(x)$, then $y^{(r)}(x)=\sum_{k=r}^{\infty} a_kF_{k-r}(x)$, and since $\int_{-h}^{h} F_0(x)dx=2h$,

$$(2h)^{-1}\int_{-h}^{h} y^{(r)}(x)dx = \sum_{k=r}^{\infty} a_k(2h)^{-1}\int_{-h}^{h} F_{k-r}(x)dx = a_r$$

by (3). This proves (4).

Also solved by A. E. Danese, Jane Evans, S. H. Greene, W. M. H. Kantor, A. G. Konheim, and Vaclav Vitek.

Editorial Note. W. C. Waterhouse finds the problem posed in Whittaker and Watson, *Modern Analysis*, p. 336, ex. 49.

A Summation

4946 [1961, 67]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Let $S_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+1/n$. Sum $\sum_{n=1}^{\infty} S_n/n!$

I. *Solution by W. H. M. Kantor, Undergraduate, Brooklyn College.* Consider the more general problem of evaluating $g(x) = \sum_{n=1}^{\infty} a_n S_n x^n$, having given $f(x) = \sum_{n=1}^{\infty} a_n x^n$. By the ratio test, they have the same radius of convergence. Since

$$S_n = \int_0^1 \sum_{i=1}^{n} y^{i-1} dy = \int_0^1 \frac{1-y^n}{1-y}\, dy,$$

it follows that if x is within the interval of convergence then

$$g(x) = \sum_{n=1}^{\infty} a_n x^n \int_0^1 \frac{1-y^n}{1-y}\, dy = \int_0^1 \left\{ \frac{1}{1-y} \sum_{n=1}^{\infty} (a_n x^n - a_n x^n y^n) \right\} dy$$

$$= \int_0^1 \frac{f(x) - f(xy)}{1-y}\, dy.$$

For the particular case, $f(x) = e^x$, $x = 1$, we have

$$g(1) = \int_0^1 \frac{e - e^y}{1-y}\, dy = e \int_0^1 \frac{1-e^{-y}}{y}\, dy.$$

The value of the integral on the right is $0.79660\cdots$ (Franklin, *Treatise on Advanced Calculus*, p. 570), so that the required sum is $2.16538\cdots$.

II. Solution by J. W. Wrench, Jr., The David Taylor Model Basin, Washington, D. C. Consider the more general sum $y = \sum_{n=1}^{\infty} S_n x^n/n!$. The function y is found to satisfy the differential equation $xy' - xy = e^x - 1$. The solution, satisfying the condition $y \to 0$ when $x \to 0$, is

$$y = e^x[\gamma + \ln x - Ei(-x)],$$

where γ is Euler's constant and $Ei(-x) = -\int_x^{\infty} e^{-t} t^{-1} dt$, $x > 0$, is the exponential integral for a negative real argument.

When $x = 1$, we find for the sum of the proposed series the expression $e \cdot [\gamma - Ei(-1)]$, which is numerically equal to 2.16538 22153 26936 35942, to 20 decimal places.

III. *Comment by J. H. van Lint, Technical University, Eindhoven, Netherlands.* The solution can be found in Erdelyi-Magnus-Oberhettinger-Tricomi: *Higher Transcendental Functions*, part 2, p. 143, formula (5).

Also solved by W. N. Anderson, Jr., Merrill Barnebey, W. J. Blundon, Edward Bregstone,

J. A. Brown, W. G. Brown, R. G. Buschman, P. R. Chernoff, G. H. de Visme, Ragnar Dybvik, J. A. Faucher, S. H. Greene, J. W. Haake, A. G. Konheim, D. S. Newman, Hugh Noland, R. C. Read, W. C. Shaw, F. C. Smith, O. E. Stanaitis, W. C. Waterhouse, David Zeitlin, and the proposer.

Bounded Operator on Hilbert Space

4947 [1961, 181]. *Proposed by D. J. Newman, Yeshiva University*

Let $a_{mn}=\exp(-m-ne^{-2m})$. Prove that the infinite matrix $A=(a_{mn})$ represents a bounded operator on Hilbert space. That is, it takes vectors $(x_1, x_2, \cdots)$ for which $\sum x_n^2<\infty$ into vectors $(y_1, y_2, \cdots)$ for which $\sum y_n^2<\infty$.

Solution by the Proposer. What we prove is that A^TA is bounded; this will suffice since $xA^TAx^T=(Ax)^2$, and so the finiteness of $(Ax)^2$ is implied.

Now, A^TA has elements

$$b_{mn}=\sum_k e^{-(k+me^{-2k})}e^{-(k+ne^{-2k})}=\sum_k e^{-2k}e^{-(m+n)e^{-2k}}$$

$$<2\int_0^\infty e^{-2u}e^{-(m+n)e^{-2u}}du=\int_0^1 e^{-(m+n)t}dt<\int_0^\infty e^{-(m+n)t}dt=\frac{1}{m+n}.$$

However, the Hilbert matrix $(1/(m+n))$ is known to be bounded. The result therefore follows.

Also solved by Fred Suvarov.

Sphere in Hilbert Space

4948 [1961, 181]. *Proposed by Lawrence Glasser, Carnegie Institute of Technology*

Show that the volume of any sphere in Hilbert space is zero.

Solution by D. J. Newman, Yeshiva University. Consider the more general problem: Suppose $f(x)$ is an even function, increasing from 0 to ∞ on the positive axis. Let V_n be the volume of the region in n-space given by

$$f(x_1)+f(x_2)+\cdots+f(x_n)\leqq 1,$$

then $V_n\to 0$ as $n\to\infty$.

Let $V_n(t)$ denote the volume of the region $f(x_1)+\cdots+f(x_n)\leqq t$, $f(x_i)\leqq 1$ for all $i=1, 2, \cdots, n$. Now pick a number u, $u>0$, such that $f(u)\geqq 1$, and then pick a number v, $v>0$, such that $\int_{-u}^{u}e^{-vf(x)}dx\leqq\frac{1}{2}$. We now have

$$\frac{1}{2^n}=\left(\int_{-u}^{u}e^{-vf(x)}dx\right)^n=\int_{-u}^{u}\cdots\int_{-u}^{u}e^{-v(f(x_1)+\cdots+f(x_n))}dx_1\cdots dx_n$$

$$\geqq\int_{f(x_i)\leqq 1}\cdots\int e^{-v(f(x_1)+\cdots+f(x_n))}dx_1\cdots dx_n=\int_0^\infty e^{-vt}dV_n(t)$$

$$\geqq\int_0^1 e^{-vt}dV_n(t)\geqq e^{-v}V_n(1)=e^{-v}V_n.$$

Hence $V_n \leqq e^v/2^n$, which implies the result. For the given problem, we merely take $f(x) = x^2$.

Comment by John V. Ryff, Stanford University. Presumably the spirit of the problem is to determine where a measure, or at least a finitely additive nonnegative set function, which is translation invariant, may be imposed upon a Hilbert space assigning nonzero measure to some sphere. Further one assumes that if S is any measurable set and α is any nonnegative real number, then $m(\alpha S) = \alpha m S$ where m is the measure.

With this meaning the problem has been solved by Charles Loewner. See *Grundzüge Einer Inhaltslehre im Hilbertschen Raume*, Annals of Mathematics, 40 (1939), in which it is shown that any sphere must have zero (or infinite) measure.

Also solved by Jose Gallego-Diaz, Fred Suvarov, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Analytic Geometry and Calculus. By Frank L. Juszli. Prentice Hall, Englewood Cliffs, New Jersey, 1961. 348 pp. $6.75.

This is a cookbook text on the standard subjects of analytic geometry and calculus. It is unsuitable for any class of college level students since, among other things, it lacks an adequate description of functions, limits and continuity, it omits entirely the mean value theorem for derivatives, and it hints only vaguely at the fundamental theorem of calculus.

Some claims on the dust cover are revealing: "The author's approach is especially useful to two groups: (1) the lay reader who wishes to increase his understanding of the developments in science and engineering through a more complete knowledge of mathematics and (2) the reader currently working in a technical or industrial field who needs to refresh his mathematical background to keep pace with the growth of his job *Analytic Geometry and Calculus* maintains a graphical approach throughout . . . the author emphasizes techniques and interpretations rather than abstract theory. Calculus is focused in its importance as a *tool*" These claims are more or less realized, but the exposition (especially of integration) is so poor that this book would most likely bewilder its intended audience.

CHARLES J. HIMMELBERG
University of Kansas

Advanced Calculus, An Introduction to Analysis. By Watson Fulks. Wiley, New York, 1961. xv+521 pp. $9.50.

This book presents many topics traditionally associated with the title, including properties of continuous and finitely differentiable functions, implicit functions, the Riemann integral, and certain infinite processes. Vector notation and methods are emphasized. Courses called advanced calculus fall somewhere in a wide spectrum and opinions differ concerning both the level and coverage. Exposition in the present book appears to be careful but the point of view is closer to current practice in elementary courses than to that in function theory or real analysis. It contains much useful material. It may or may not be the suitable text for a particular situation.

The student is unlikely to discover the nature or relevance of Dedekind real numbers from the sketch pp. 3–14 unless the teacher elaborates considerably. The statement p. 14 that each nonempty set (of reals) which is bounded above has a supremum is labeled and subsequently referred to in crucial spots as a fundamental property rather than as a theorem. Complex numbers are introduced p. 403, not constructively, but by appropriation of a root of -1.

Definitions and theorems tend generally to rule out important pathological cases and infinite limits. For example, extrema of a function f at boundary points of the domain are not included. There is no mention of integrability of a function f which fails to be defined and continuous on only a finite subset of the interval, a fact needed in the discussion of line integrals. The term *curve* defined p. 172 as the range of a vector-function has only tenuous connection with curvilinear integrals which follow, and a similar remark applies to surface on p. 307.

GEORGE M. EWING
University of Oklahoma

Analytic Geometry and Calculus. By L. J. Adams and Paul A. White. Oxford, New York, 1961. x+932 pp. $9.75.

This text assumes basic training in algebra and trigonometry and covers the usual topics in a beginning course in the subject and the following which are not so commonly included: Newton's method, line integrals, Green's theorem, and some differential equations.

After an initial chapter on the analytic geometry of the line, the calculus is introduced. Definitions of limits are stated intuitively and the fundamental theorems about them stated but not proved. Applications to engineering and physical sciences are stressed and the exercises are well-chosen and numerous. The distribution of topics is such that those in analytic geometry may be omitted for classes in calculus alone.

The treatment of maxima and minima is careful. The reviewer noted with pleasure the classical method for finding a conic through five points. Within the bounds set by the authors the treatment of integration is clearly developed.

As is usual with many textbooks, the treatment of directed distances is less than adequate and the use of the same symbol for a line segment and its length is latter-day heresy. There are a few exceptions to the authors' statement "if the proof of a theorem is not given, this fact is stated clearly," in the sense that important results are stated informally at times without indication that proof is necessary. Two such examples are the fundamental theorem on the decomposition into partial fractions and the fact that a curve crosses its tangent at a point of inflection.

But every book has its defects and there seem to be few here. This should be a very teachable text, enjoyed by students and teachers alike.

BURTON W. JONES
The University of Colorado

Elements of Mathematical Statistics. By Howard W. Alexander. Wiley, New York, 1961. xi+367 pp. $7.95.

This book is intended as a text for a first course in mathematical statistics for students who have had a year of calculus but not necessarily any previous work in statistics or probability theory. The material covered is indicated by the chapter headings, which are: 1. Discrete Distributions; 2. Continuous Distributions; 3. Sampling Distributions I; 4. Statistical Inference; 5. Sampling Distributions II; 6. Regression and Correlation; 7. Analysis of Variance. There is an appendix on matrices, a brief bibliography, and a set of the usual tables.

The first three chapters deal with sufficient descriptive statistics, probability and sampling theory to provide a basis for statistical inference (Chapter 4) which is the main theme of the book. Chapter 5 supplements Chapter 3, and Chapters 6 and 7 deal with methodology.

The book is clearly written and logically arranged. An ample number of problems are given at the end of each section with answers to the odd numbered ones. The reviewer feels that this book will prove to be quite adequate as a text in a first course in mathematical stistics.

PAUL M. HUMMEL
University of Alabama

Games of Strategy: Theory and Application. By Melvin Dresher. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961. xii+186 pp. $6.75.

This book attempts to treat formally, but in an elementary way, games of strategy. The author succeeds in this and in creating a highly motivated, well-illustrated, and very readable book.

Chapters I, II and III are typical: The concept of a matrix game is introduced and illustrated by example. The minimax theorem is stated and a constructive proof is given. Observations are made on this theorem and applied to several particular games. Chapter IV deals very briefly with extensive games.

Many ways of solving games are presented: examining submatrices, successive approximations, etc.

Continuous games and their solutions are considered and the required background in Stieltjes integrals is contained in the text. Convex games are defined, developed and illustrated, and an amazing variety of dueling problems is considered, as well as a tactical air war game. Finally, infinite games with separable payoff functions are considered.

The book would be excellent as an introduction to games of strategy, and is well within the grasp of a college junior or senior.

MORTON DAVIS
Princeton University

Calculus with Analytic Geometry. By Donald E. Richmond. Addison-Wesley, Reading, Mass., 1959. xv+458 pp. $8.75.

The first five chapters are the author's *Introductory Calculus* (Reviewed in this MONTHLY, vol. 67, 1960, pp. 490–491). The different approach is itself a recommendation.

For those under pressure to teach some calculus early, Chapter 5, Complex Numbers and Trigonometry, contains a development of the identities, graphs, and approximations, with only some prior knowledge of triangles needed. Both differentiation and area appear within the first third of the book and linear differential equations at about the two-thirds mark. For those cramped for time, here is an excellent selection of topics in a space 300 to 400 pages shorter than usual and so arranged that the later chapters are mutually independent.

The problems are adequate in number, but not abundant. Analytic Geometry is buried; rotation and translation are problems in Chapter 9 (on Vectors) and conics are studied in Chapter 10 (on The Inverse Square Law). Some elegant developments aid in keeping the text short. For a first course there seems to be little of importance missing while included are uncommon topics such as brief discussions of Laplace transforms, Rutherford's scattering law, and the separation of variables for partial differential equations. Strong emphasis is placed on error estimates for approximating polynomials.

One detracting feature is the application in at least four places of the theorem on bounded, monotonic sequences, the explicit statement of which I was unable to find, even though sequences and least upper bounds are introduced early. Among others are the early distinction between f and $f(x)$ soon lost [p. 45] to "the function $f(x)\cdots$," the statement [p. 208] "If, however, $c<0$ and $x<0$, $x+\sqrt{(x^2+c)}$ is negative . . . ," and the notation [p. 189] "$d/dx\int_a^x f(x)dx$."

R. G. BUSCHMAN
Oregon State University

Introduction to Complex Analysis. By Zeev Nehari. Allyn and Bacon, Inc., 1961. ix+258 pp. $7.50.

Professor Nehari has succeeded in writing a splendid text in complex variable theory that is suitable for a "one-semester course for seniors and first-year graduate students in mathematics, engineering, and the sciences."

His approach is straightforward. The complex numbers appear as combinations of reals, $a+bi$, where i is a solution (assumed to exist) of the equation $x^2+1=0$. (The pair-of-reals description of the complex numbers occurs only in passing.) The first three chapters, entitled *Complex Variables*, *Analytic Functions*, and *Complex Integration* give the basic principles of the theory, culminating in the Cauchy Integral Theorem.

The longest chapter in the book is the fourth, *Applications of Cauchy's Theorem*. Here, the Cauchy integral formula and the Taylor and Laurent expansions are introduced. There is also a discussion of isolated singularities, of rational functions and their partial fraction decomposition, the residue theorem and some of its consequences—evaluation of real integrals, argument principle, Rouché's theorem, maximum principle and the Poisson integral. This chapter will be a mine of information for the student.

Three shorter chapters on conformal mapping, physical applications, and additional topics conclude the book. These latter topics include analytic continuation, the gamma and zeta functions, the symmetry principle, and the construction of the harmonic conjugate by means of the conjugate coordinates z and $\bar{z}$ introduced in the first chapter.

The only misprint of any consequence noted by the reviewer is the omission of a factor n^{-1} in a summation representing the logarithm on page 103. Also, Exercise 2 on page 33 duplicates Exercise 3 on page 20. It seems somewhat regrettable that Professor Nehari has omitted any references or suggestions for further reading. Indeed, he does not even mention his own book, *Conformal Mapping*, that could be read with profit by anyone who has studied the text under review.

All in all, however, it is a pleasure to acknowledge the lucid style of this book. Its coherent structure and commendable selection of problems will make it a fine introduction to a respected, venerable, and useful subject.

ERNEST C. SCHLESINGER
Wesleyan University

BRIEF MENTION

The Scientific Papers of J. Willard Gibbs.

Volume 1—Thermodynamics, Dover, New York, 1961. xxvi+434 pp. $2.00.
Volume 2—Dynamics, Vector Analysis and Multiple Algebra, Electromagnetic Theory of Light, Miscellaneous Papers, Dover, New York, 1961. viii+284 pp. $2.00.

Elementary Principles in Statistical Mechanics. By J. Willard Gibbs, Dover, New York, 1960, xviii +207 pp. $1.45.

Synopses for Modern Secondary School Mathematics.

School Mathematics in OEEC Countries: Summaries.

These reports grew out of the Royaumont seminar. Both publications are available free on request to O.E.E.C. Mission, Publications Office, Suite 1223, 1346 Connecticut Avenue, N.W., Washington 6, D. C.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Western Michigan University: Dr. J. W. Petro, State University of Iowa, has been appointed Assistant Professor; Associate Professor J. C. McCully is on leave on an N. S. F. Fellowship for the academic year at Harvard University.

Mr. D. R. Beuerman, University of Buffalo, has accepted a position as Associate Mathematician with the Texaco Research Center, Beacon, New York.

Mr. D. L. Bohmont, Colorado State University, has been appointed Instructor at the University of Vermont.

Mr. A. C. Calianese, Fort Meade, Maryland, has been appointed Instructor at St. Peter's College.

Mr. R. K. Clark, University of California, Berkeley, has accepted a position as Mathematician with the Matson Navigation Company, San Francisco, California.

Associate Professor Mamie M. Davis, Wesleyan College, has been appointed Associate Professor at Greensboro College.

Mr. P. A. Davis, University of British Columbia, has accepted a position as Research Scientist with the Lamont Geological Observatory of Columbia University.

Mr. Patrick Di Caprio, St. Bonaventure University, has accepted a position as Agent with the Metropolitan Life Insurance Company, Schenectady, New York.

Mr. E. L. Eagle, Bendix Aviation Corporation, Ann Arbor, Michigan, has accepted a position as Design Engineer with the Martin Company, Denver, Colorado.

Mr. Gene Evans, Ardmore, Oklahoma, has been appointed Instructor at Abilene Christian College.

Mr. H. W. Gould, West Virginia University, has been promoted to Assistant Professor.

Dr. B. R. Hare, University of Florida, has been appointed Instructor at Duke University.

Dr. D. W. Hight, Oklahoma State University, has been appointed Assistant Professor at Arkansas State College.

Mr. S. A. Hoffman, Burroughs Corporation, Paoli, Pennsylvania, has accepted a position with Kettelle and Wagner, Paoli, Pennsylvania.

Mr. R. B. Howe, Mississippi State University, has been appointed Instructor at Louisiana Polytechnic Institute.

Assistant Professor Hyman Kamel, Rensselaer Polytechnic Institute, has been appointed Associate Professor at Howard University.

Mr. Eric Kilpatrick, University of Kentucky, has accepted a position as Registrar Assistant at the Veteran's Administration Hospital, Lexington, Kentucky.

Mrs. Roberta Mielke, University of Minnesota, has been appointed Instructor at Macalester College.

Miss Vivian Morgan, Rocketdyne, Canoga Park, California, has accepted a position as Programmer with the Lawrence Radiation Laboratory, Berkeley, California.

Mr. E. A. Newburg, Combustion Engineering, Windsor, Connecticut, has accepted a position with the Travelers Research Center, Hartford, Connecticut.

Mr. W. A. Rhea, Oklahoma State University, has been appointed Assistant Professor and Chairman of the Department of Mathematics at William Penn College.

Mr. P. M. Rivot, Brooklyn College, has accepted a position as Associate Engineer with the Sperry Gyroscope Company, Great Neck, New York.

Miss Jean E. Sammet, International Business Machines, Cambridge, Massachusetts, has accepted a position with the Sylvania Electric Products, Needham, Massachusetts.

Mr. R. E. Shermoen, University of Illinois, has been appointed Instructor at the University of New Mexico.

Assistant Professor R. L. Stanley, Washington State College, has been appointed Associate Professor at Portland State College.

Professor Emeritus E. P. Starke, Rutgers, The State University, has been appointed Professor at Bloomfield College.

Mr. Seymour Steinberg, Atlanta University, has been appointed Instructor at Monmouth College.

Associate Professor R. F. Steward, Western Carolina College, has been promoted to Professor and appointed Head of the Department of Mathematics.

Dr. Beauregard Stubblefield, International Electric Corporation, Paramus, New Jersey, has been appointed Associate Professor at Michigan State University Oakland.

Mr. S. J. Turner, Hercules Powder Company, Cumberland, Maryland, has accepted a position with Litton Industries, Maryland.

Mr. E. J. Vought, Pomona College, has been appointed Assistant Professor at California State Polytechnic College.

Mr. M. H. West, Convair, Forth Worth, Texas, has accepted a position as Research Engineer with General Dynamics/Astronautics, San Diego, California.

Associate Professor E. D. Chestnutt, Virginia Polytechnic Institute, died February 15, 1961.

Visiting Professor F. R. Georgia, Clarkson College of Technology, died October 2, 1961.

Mrs. Nellie P. Miser, (Mrs. W. L.), Ada, Ohio, died November 28, 1961. She had been a member of the Association for thirty-four years.

Professor R. E. Root, United States Naval Postgraduate School, died September 23, 1961. He had been a member of the Association for forty-five years.

Mr. J. C. Sowell, Mississippi Southern College, died February 13, 1961.

Associate Professor O. E. Strong, Indiana State Teachers College, died July 20, 1961. He had been a member of the Association for eleven years.

Mr. L. A. Wold, University of Washington, died May 28, 1961.

NEW YORK STATE MATHEMATICS TEACHERS MEET

The twelfth annual meeting of the Association of Mathematics Teachers of New York State will be held at Hotel Syracuse and Technical High School, Syracuse, New York on April 27 and 28, 1962. Approximately forty meetings will be conducted to study issues related to the teaching of mathematics at all levels from elementary school to college. The General Session on Saturday morning will be addressed by Dr. Mina Rees, Dean of Graduate Studies, The City University of New York. Dean Rees will inform the members of the work of the Conference Board of the Mathematical Sciences on which she serves as Vice-Chairman. There will be three major section meetings on Friday evening for elementary teachers, secondary teachers, and college teachers, respectively. The Elementary Section will hear Dr. Vincent Glennon, Professor of Education, Syracuse University, on the topic "Where is the Middle of the Road in Elementary School Mathematics?" A panel of secondary school teachers will consider "UICSM, SMSG, and the Regents Syllabus." The College Section will be addressed by Dr. Robert Wisner of Michigan State University Oakland. Dr. Wisner is Executive Director of the Committee on the Undergraduate Program of the Mathematical Association of America. He will discuss recommendations made by this committee regarding the training of teachers of mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting at the University of British Columbia at Vancouver, British Columbia, Canada, on August 28, 29 and 30, 1962. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registering either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $1 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms. These forms must be completed and returned to Providence not later than August 3, 1962, in order to be included free of charge in the listings at the Summer Meeting in Vancouver. Forms which arrive after this closing date, but before August 10, will be included in the register at the meeting for a late registration fee of $3.00, and will also be included in the printed listings, but not until ten days after the meeting. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

RECIPROCITY AGREEMENT WITH THE SOCIÉTÉ MATHÉMATIQUE DE BELGIQUE

At its meeting on January 24, 1962, the Board of Governors authorized the first reciprocity agreement under the general arrangements approved by the Board at its meeting on August 28, 1961. Under these provisions, the Association has entered into a reciprocity agreement with the Société Mathématique de Belgique by which members of the Société living outside the United States and Canada may become members of the MAA without paying an initiation fee. On the other hand, members of the MAA living in the United States or Canada who become members of the Société pay only one-half of the regular dues for members living abroad. The regular dues of the Société are $4 for 1962; therefore members of the MAA living in the United States or Canada may become members of the Société by paying $2.

Those members of the Association wishing to take advantage of this arrangement should write to Professor G. Hirsch, Société Mathématique de Belgique, 317 Avenue Charles Woeste, Brussels, Belgium.

HENRY L. ALDER, *Secretary*

OCTOBER MEETING OF THE OKLAHOMA SECTION

The Oklahoma Section of the Mathematical Association of America met in conjunction with the Oklahoma Council of Teachers of Mathematics on Friday, October 27, 1961 at Classen High School, Oklahoma City. There were 83 members of the Association in attendance.

In the business meeting, the following officers were elected: Chairman, Professor Harold Huneke, University of Oklahoma; Vice-Chairman, Professor Katherine C. Mires, Northestern State College; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The program was as follows:

1. *The new look in mathematics*, by Dr. Julius Hlavaty, DeWitt Clinton High School, New York, New York.

2. *Teaching mathematics in the sixties*, by Dr. Hlavaty.

3. *Using computers in today's mathematics curriculum*, by Professor R. V. Andree, University of Oklahoma.

R. V. ANDREE, *Secretary*

THE NOVEMBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the Mathematical Association of America was held on November 4, 1961, at Moorhead State College, Moorhead, Minnesota. Professor Marion Smith, Moorhead State College, presided at the morning session and the Section Chairman, Professor Charles Hatfield, University of North Dakota, presided at the afternoon session. There were 71 persons registered for the meeting, of whom 56 were members of the Association.

At the business meeting, Professor John Hafstrom, University of Minnesota at Duluth, made available for distribution some mimeographed booklets which had been compiled by his committee on undergraduate curricula, at the request of the Minnesota Section. These booklets give the catalog descriptions of mathematics departmental offerings for most of the colleges represented in the Minnesota Section. Sister Seraphim M. Gibbons, College of St. Catherine, gave a brief description of plans for conducting the 1962 High School Mathematics Contest and announced that the papers would be machine-graded this year.

By invitation of the Section, Professor R. H. Cameron, University of Minnesota, gave the main address of the morning, entitled, *An integral calculus for a function space.*

A feature of the afternoon program was a talk by Professor J. B. Serrin, University of Minnesota, entitled, *Applied mathematics in the college curriculum.* Serrin stressed the need for a balance between application and abstraction in the undergraduate mathematical offerings. His talk was followed by a general discussion, in which it became evident that there was quite general agreement with the policies he had proposed.

The following papers were presented:

1. *A participant's comments on the Conference on Undergraduate Research in Mathematics at Carleton*, by Professor Robert MacDowell, Antioch College (currently Visiting Professor at Carleton College).

The Conference on Undergraduate Research in Mathematics, held at Carleton College, June 19–23, 1961, was reviewed. Various definitions of *undergraduate research* were offered, and some arguments favoring a program involving undergraduates in research were presented.

2. *A paradox in set theory??*, by Professor J. E. Hafstrom, University of Minnesota at Duluth.

If $S = a_1, a_2, a_3, \cdots$ is a countably infinite set, does there exist a strictly increasing and uncountable chain of subsets of S? One argument is given with the conclusion that no uncountable chain exists. Another argument shows that such a chain does exist. Question: Is there a paradox here or, if not, which of the arguments is invalid?

3. *Report on the International Symposium on Nonlinear Vibrations in Kiev, 1961*, by Professor W. S. Loud, University of Minnesota.

A description of the activities of the Symposium was given, followed by some brief remarks on the speaker's impressions of life in Kiev.

4. *Iterates of a real-valued function*, by Professor G. U. Brauer, University of Minnesota.

Let n denote a positive integer, and let $f(x)$ be a real-valued continuous function; $f^n(x)$ denotes the function obtained by iterating f n times. It is shown that if the derivative of f is bounded by 1, then f satisfies $f^n(x) \geq 0$ for all x if and only if: (i) $f(x) \geq x + \lambda_0$, $f^2(x) \geq x + 2\lambda_0$ for $x < -\delta_0$, where λ_0 and δ_0 are positive numbers: (ii) $0 \leq f(x) \leq a$ on $(-\delta_0, a)$ for some nonnegative number a; (iii) $f(x) \geq 0$ for $x > 0$ or f is bounded above for $x < 0$; and (iv) $f(x) \leq x - \lambda_0$ for $x > a_1$, where a_1 is the smallest nonnegative point where the graph of f crosses the x-axis.

5. *An integral calculus for a function space*, by Professor R. H. Cameron, University of Minnesota. (By invitation)

This paper discusses various ways of setting up an integral for functionals $F(x)$ on the space $C[a, b]$ of continuous functions $x(t)$ on the interval $a \leq t \leq b$. The method used is to first set up an integral on the space of finite dimensional functionals, $F(x) = f[x(t_1), \cdots, x(t_n)]$ by defining $\int_{C[a,b]} F(x)dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1, \cdots, u_n) g_n(u_1, \cdots, u_n, t_1, \cdots, t_n) du_1 \cdots du_n$ and then extending the definition by a limit process. Thus an integral is defined for finite dimensional functionals and this integral is shown to be additive, homogeneous, and monotonic. However, it cannot be extended in the natural way to a larger class of functionals.

6. *A simple method for generating rational approximations*, by Mr. D. C. Olivier, Carleton College.

A simple method is described for approximating a real-valued function by a rational function over a finite interval. The approximating function is obtained by requiring the fit to be exact at points which are the roots of the Chebyshev polynomial of appropriate degree orthogonalized on the interval of the approximation.

7. *Recreational mathematics in college teaching*, by Professor Merrill Barnebey, University of North Dakota.

The speaker gave a brief mention of places in the college curricula where the recreational aspects of the subject can be emphasized, as well as making some light-hearted remarks relating to logic and to the history of mathematics.

8. *Generalized synthetic division*, by Professor F. J. Arena, North Dakota State University.

The speaker reviews synthetic division and shows how to modify the process when the divisor is of the form $ax + b$, where neither a nor b is zero, and a is not 1. He then shows how the process can be extended to the case where the divisor is a polynomial of higher degree.

MURRAY BRADEN, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Ursinus College, Collegeville, Pennsylvania, on November 25, 1961. The Chairman, Professor S. S. McNeary, Drexel Institute of Technology, presided at the meeting. The meeting was attended by 89 persons, including 71 members of the Association.

At the business meeting, the following officers were elected: Chairman, Professor D. W. Western, Franklin and Marshall College; Secretary-Treasurer, Professor V. V. Latshaw, Lehigh University; Third Member of the Executive Committee, Professor A. E. Filano, West Chester State College.

The following papers were presented at the meeting:

1. *Gaussian polynomials and pentagonal numbers*, by Professor Hans Rademacher, University of Pennsylvania.

The Gaussian polynomials have some properties analogous to those of the binomial coefficients. Gauss used them for a factorization of certain sums of roots of unity. I. Shur indicated that they can be used for the proof of Euler's identity $\prod_{m=1}^{\infty}(1-x^m)=\sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda}x^{\lambda(3\lambda-1)/2}$, where the exponents $\frac{1}{2}\lambda(3\lambda-1)$ are the pentagonal numbers.

2. *ALGOL 60*, by Mr. A. G. Grace, Jr., Radio Corporation of America, Cherry Hill, New Jersey.

Following an endorsement of Professor Forsythe's ideas with respect to the importance of the computer in the undergraduate curriculum (this MONTHLY, October 1959), ALGOL 60 is introduced as a language which can be easily learned and effectively used by the student in communicating with both man and machine. Algorithms for several elementary computing procedures are used to illustrate the simplicity of the basic language. The power of ALGOL 60 and its implicit complexity are indicated by a syntactical analysis of the *for* statement. The dearth of compilers and the lack of suitable educational materials are thought to be vanishing problems.

3. *Recommendations of the Panel on Physical Science and Engineering, Committee on the Undergraduate Program in Mathematics*, by Dr. H. O. Pollak, Bell Telephone Laboratories, Murray Hill, New Jersey.

The above panel has prepared recommendations for the mathematical education of engineers which examine both the courses which should be available and the spirit in which they should be taught. It is recommended that two years of calculus with analytic geometry, plus a semester each of linear algebra and of probability and statistics be required of all engineers. A group of further courses should be available, especially for those going on to graduate work. All courses should contain a mixture of motivation, intuition, mathematical sophistication and mathematical rigor.

F. L. DENNIS, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Annual Fall Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Catholic University, Washington, D. C., on Saturday, December 2, 1961. Professor W. K. Morrill, Chairman of the Section, presided. There were 142 persons present, including 99 members of the Association.

The following papers were presented:

1. *Infinite periodic sequences of* 1*'s and* -1*'s*, by Professor J. P. Hoyt, U. S. Naval Academy.

The function $(-1)^{n-1}$, $(n=1, 2, \cdots)$ *generates* an infinite periodic sequence of 1's and -1's with period 1, -1. We attempt to find a function that will *generate* an infinite periodic sequence of 1's and -1's of arbitrary period. If we limit ourselves to $(-1)^{f(n)}$ where $f(n)$ is a polynomial, we find the period must have a length which is a power of 2. If we do not limit ourselves to the above-mentioned form, we can easily construct a function which will generate an infinite periodic sequence of 1's and -1's of arbitrary period. In the course of the solution, we are led to a set of matrices with interesting inverses.

2. *Algebra for freshmen*, by Professor Jean-Pierre Meyer, Johns Hopkins University.

This paper describes an experimental course in abstract algebra for freshmen which was introduced into the Johns Hopkins curriculum last year. It discusses the educational philosophy and contents of this course.

3. *On the mean value theorem of the differential calculus*, by Dr. J. B. Diaz, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

In a recent paper, Professor A. K. Aziz and the speaker obtained a mean value theorem for vector valued functions of a real variable which plays a role analogous to that of Lagrange's mean

value theorem in the differential calculus of real valued functions. This report contains an analysis of various methods for proving these theorems.

4. *Methods of message compression*, by Mr. W. H. Land, Jr., IBM Communications Center, Federal Systems Division, Rockville, Maryland.

This paper discusses how compression of a binary message $X(L)$ may be employed to utilize communication channels and memory storage facilities more efficiently. The role of compression in the general communications system is presented. In addition, compression concepts are applied in the more specific case of removing correlations which exist in the message sequence of a Digital TV system. The information destroying and information preserving transformations are discussed. In particular the operations of smoothing and vertical-and-horizontal differencing are introduced using a digitized intensity function $Z_D = F_D(X, Y)$ of picture coordinates to illustrate these concepts.

5. *Introducing Frederick College*, by Professor J. M. Long, Frederick College.

6. *Some nomograms in conventional dynamic astronomy*, by Mr. C. R. White, Aberdeen Proving Ground, Maryland.

Nomographic design equations are derived for any planet or asteroid in elliptic motion about the sun, as viewed from earth at the same instants of time or at selected increments of such times.

7. *Some aspects of a calculation of* π *to* 100,000 *decimals*, by Dr. Daniel Shanks and Dr. J. W. Wrench, Jr., Department of the Navy, David Taylor Model Basin, Washington, D. C.

On July 29, 1961 the authors computed π to over 100,000 decimals on an electronic computer, using two independent formulas due respectively to Störmer and Gauss. This computation requires billions of arithmetic operations. To minimize these the formulas mentioned were adopted, and several programming devices were developed. An alternative strategy, one which has never been used, and is based on a formula of Ramanujan, was also discussed. It was shown that a computation to a million decimals is hardly possible at this time. A brief discussion was given concerning the distribution of the digits in this approximation.

8. *Semantic information*, by Mr. C. J. Maloney, U. S. Army Chemical Corps, Fort Detrick, Frederick, Maryland.

Semantic and signal information theory have in common the mathematical model consisting of an n-dimensional euclidean space. The former is of value in maximizing the transmission of information over a channel of limited capacity which may be subject to interference. For this purpose text is encoded in groups of successive symbols into shorter codes which exploit the frequency of occurrence of the original symbols and the statistical property of the interference. Two major distinctions between the communication and the information retrieval contexts lie in the fact that in the latter (1) probability plays no part and (2) shortened codes are only achieved at the cost of loss of semantic information.

9. *Developing city-wide mathematics examinations*, by Miss Carol V. McCamman, Calvin Coolidge High School, Washington, D. C.

A 1956 discussion of mathematics standards in Washington, D. C., showed wide disagreement between junior and senior high school teachers, but almost complete agreement that there should be uniformity of standards in algebra and the sequential mathematics in senior high school. Thus, a committee of teachers developed an elementary algebra test used experimentally in 1957. A new form of this test has been made each year since, and used city-wide. Two years ago an intermediate algebra test was also used, and past spring there were city-wide final examinations in elementary and intermediate algebra and plane geometry.

10. *Report on the Meeting of the Mathematical Association of America at Stillwater, Oklahoma, August 1961*, by Dr. C. H. Frick, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

HERTA T. FREITAG, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, May 5, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, U. S. Naval Weapons Laboratory, Dahlgren, Virginia, April 28, 1962.

METROPOLITAN NEW YORK, University Heights Campus, New York University, May 12, 1962.

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA, College of St. Teresa, Winona, May 12, 1962.

MISSOURI, Missouri School of Mines, Rolla, April 28, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, November 24, 1962.

NORTHERN CALIFORNIA

OHIO, Muskingum College, New Concord, May 5, 1962.

OKLAHOMA, Northwestern State College, Alva, April 13–14, 1962.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.

ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, May 4–5, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

SOUTHERN CALIFORNIA

SOUTHWESTERN, University of New Mexico, Albuquerque, April 27–28, 1962

TEXAS, Rice University, Houston, April 6–7, 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, May 5, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

THE MATHEMATICAL ASSOCIATION OF AMERICA

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The Association is a national organization of persons interested in mathematics at the college level. It was organized at Columbus, Ohio, in December 1915 with 1045 individual charter members and was incorporated in the State of Illinois on September 8, 1920. Its present membership is over 12,000, including more than 500 members residing in foreign countries.

Any person interested in the field of mathematics is eligible for election to membership. Annual dues of $5.00 includes a subscription to the American Mathematical Monthly. Members are also entitled to reduced rates for purchases of the Carus Mathematical Monographs and the MAA Studies, and for subscriptions to Mathematics Magazine.

Further information about the Association, its publications and its activities may be obtained by writing to:

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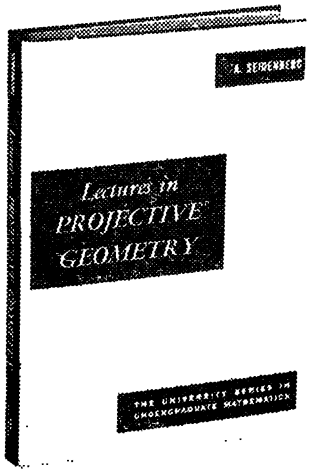
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IMPLICATIONS FOR THE COLLEGES OF THE NEW SCHOOL PROGRAMS

PAUL C. ROSENBLOOM, University of Minnesota and
Minnesota State Department of Education

Mathematicians are generally becoming aware of the existence of new programs in high school mathematics, such as those of the School Mathematics Study Group (SMSG), the University of Illinois Committee on School Mathematics, the Ball State College Project, and the Commission on Mathematics of the College Entrance Examination Board. The Ball State curriculum is already out in the form of commercial textbooks, and a number of leading publishers are also publishing textbooks along the same lines. Last year about five per cent of the secondary schools in the country already had at least one class using the new curricular materials. As textbooks become available for the new types of courses, these courses will spread even more rapidly.

For some years to come, therefore, we will be in a transitional period during which we will have a growing proportion of entering freshmen who will have had a new kind of high school background. We may expect that within a few years they will be in the large majority. Colleges must begin preparing now for the influx of such students.

The differences between these various programs may seem large to those who are deeply involved in any one of them. To one who has no vested interest in them, these differences are relatively minor in comparison with their common features. These common features have been very well summarized in a report prepared for the California State Department of Education by Professor W. H. Meyer of the University of Chicago [1].

The first general aim of all these programs is to make a course in calculus, supplemented by analytic geometry, the standard beginning course for college freshmen. This course will ordinarily require a foundation of four years of mathematics in high school. We must encourage students to take four years of mathematics in high school by publicizing more widely the increased mathematical background demanded in almost all professions.

The first consequence is that within a few years the colleges will be able to reduce drastically their courses in college algebra, trigonometry, and remedial high school mathematics. An accelerated qualifying course in algebra and trigonometry may occasionally be necessary for students who had less than four years in high school, but even this concession should be discouraged when the qualifying work can be obtained outside the college. Not only *can* the colleges cut down on high school courses, but they *must*, so as to enable their faculty to give the courses they should be giving at the college level. Colleges must set time limits on offering these courses for credit and a further time limit beyond which they will not offer these courses at all. This time table must be announced publicly, and must be communicated to the high schools feeding into any given college. As a by-product, this will give these high schools an incentive to improve

their own programs, since they will then be given full responsibility for teaching high school mathematics.

For the immediate future, colleges should begin turning these courses over to high school teachers, and as long as they must offer high school subjects, they should use the new curricular materials. In many places one can begin by having teachers from nearby high schools teach these subjects part time at the college, perhaps with supervision during the first year, and perhaps at such salaries that they could serve as rewards for outstanding high school teachers. A college may start with its own campus school, or with teachers who are used for supervising student teachers. These high school courses may be shifted to the extension division of the college.

Certainly the new curricular programs should be incorporated into the methods courses and the student teaching. It makes no sense for colleges to be turning out high school teachers who need remedial inservice training immediately upon graduation. Certainly many colleges will need to relieve their staffs from the load of teaching high school courses if they want to have them available to teach the courses recommended by the CUPM.

Another important feature of all the new curricular programs is that they have a much higher proportion of nonroutine problems, in comparison with the conventional programs. In a conventional textbook you will often find an illustrative example, and a recipe of about five or six steps for solving problems of the same type, followed by a series of 20 or 30 problems all exactly alike except for different numbers. This stereotypes a student's thinking in such a way that when you present him with a problem that is not quite like any in the book, he shrugs his shoulders and says, "I can't do this problem. I have never seen anything like it before."

In the new courses there is very little of this sort of thing. The student is continually confronted with new situations. We try to train him to be flexible in his thinking. We try to prepare him to attack with courage problems for which he has not been specifically instructed. There is very little rote memorization or technique for its own sake. There is a high density of ideas.

The students who emerge from these programs will not be any more gifted than our present ones, but will be much more mature mathematically. They will be bored stiff by the usual calculus course emphasizing manipulation of formulas. They will expect mathematics to deal with ideas. They will rebel if they are not exposed to an idea in mathematics until their junior year.

The new mathematics courses have some content which is not shared by the conventional courses. The students are given practice with the process of proof in both algebra and geometry. They will expect statements to be justified. While they may not know the term "ordered field," they will have had experience in proving such things as the rule of signs in multiplication from the field properties of the real number system. They will know such things as the associative, commutative, and distributive properties, and the existence of inverses with respect to addition and multiplication. Students will know the language of sets and

something about functions. They will have had considerable practice in working with inequalities and making estimates.

We do not yet know how much of this knowledge they will retain after entering college. It is certain, however, that this will depend largely on the extent to which this knowledge is used in their freshman courses. Students tend to learn what they are held responsible for in homework and examinations. No matter what you say in your lectures, if they can pass their examinations only by rote memorization and routine manipulation, then this is what they will practice on and learn.

Students, and the high schools they come from, will be strongly influenced by the placement examinations you use. If they are tested only on content of the traditional high school courses, they will feel that they are not profiting from the new ideas and skills which they have learned. They will report this back to their high schools, and this will discourage the schools from improving their curricula. I note that the College Entrance Examination Board and the Educational Testing Service are both changing their examinations to adapt to the new curricular materials. They plan to change a few questions each year, and expect by 1966 to be examining almost entirely on the new high school courses. Colleges which make up their own placement examinations should see to it that members of the staff who are responsible for these examinations are familiar with recent developments in school mathematics, and are revising the examinations accordingly.

Some schools will be experimenting with courses in probability and statistics, matrix algebra, numerical analysis, elementary number theory, and the like. Colleges may be faced with questions about recognition of such work in the high schools and arrangements for advanced standing. Probably the simplest and most practical policy is to allow credit or exemption to a student for any course which he can pass by examination on the college level. Colleges may make available to schools syllabi, lists of textbooks, and copies of past examinations, so as to inform schools of the level of achievement which would be necessary for college credit.

In general, no responsible group is advocating calculus for high schools except in advanced standing programs where there is a qualified staff. Experience seems to indicate that unless the high school staff is very well prepared, they would do better to offer courses which do not overlap very much with the first year or two of a normal college curriculum. Morale problems arise when students think they know something that they have studied in high school but find it necessary to repeat the course in college. Not so much harm is done if, for example, a student who has had a poorly taught course in probability and statistics in high school, takes it again in college at a higher level after he has already had calculus.

While it is true that we are now in a transitional period and that we can expect that some sort of equilibrium between the old and the new school mathematics will be reached within a few years, I don't believe that we can expect the

situation ever to be stabilized again. Now that we have established channels of communication between the research mathematician and the school teacher, we can expect to be in a continuous state of flux. As new mathematics is discovered, it is more likely that the implications for the school curriculum will be quickly recognized, and appropriate changes will continue to be made. It is important to realize that none of the new curricular programs uses any mathematics less than 55 years old, except for some material in probability and statistics which is about 40 years old. We are introducing geometric ideas which stem from Hilbert along with those from Euclid, and algebraic ideas from Dickson's era along with those of Euler's day. We are just beginning to get into the secondary school some of the ideas of Cantor which actually should be taught in kindergarten. We can expect, then, that the ferment which we have started will continue indefinitely.

There is an urgent need for experimentation on the college level of the kind proposed by CUPM. We do not know yet how to take advantage of the preparation which high school students are now getting. Certainly there will have to be some review for freshmen of the properties of an ordered field, of the concept of function, and of the technique of working with inequalities. We do not know how deeply we can treat these subjects. Can we go very far beyond a mere review?

The most serious question, I think, is how early can we give students a rigorous discussion of the real number system. This is crucial for any serious discussion of continuous functions or of integration. At present this is postponed until the advanced calculus course or the introduction to functions of a real variable. Because of the pressures from the physics and engineering departments to teach students as early as possible the techniques of calculus, we have no time to do these things adequately in our elementary course. When we have to begin from scratch, it becomes too long a digression for us to treat the algebraic properties of the real number system, and to give students adequate practice in working with inequalities and sequences as a preparation for calculus. We do not know yet what is the feasible balance between rigor and intuition. Is the learning of these concepts entirely a matter of maturity, or is it also a matter of adequate preparation? Probably the crucial concept is that of the completeness of the real number system, and the associated ideas of a Cauchy sequence, nested intervals, and least upper bound.

Obviously, the adjustment to high school students with the new type of course is going to make very serious demands on the college staffs. You will not be able to handle such students properly with the college teacher who has not even had a course in functions of a real variable. This only underlines the urgent importance of the work of CUPM. We will need to give colleges many kinds of help. New types of textbooks will be needed, we will need films and television courses for in-service training of college staffs, and we will need an expanded institute program on the college level. While in the long run the proposed program of a Doctor of Arts in mathematics may help us meet the man-

power shortage in college teaching, I doubt that we have time to wait for such an increased production of teachers on the college level. We need to take energetic action now.

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ANALOGUES OF POISSON'S SUMMATION FORMULA

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Some analogies between the theories of Hilbert space and of finite-dimensional vector spaces were mentioned by Weyl ([11], p. 34) and such analogies were probably the inspiration for two books [5] and [6]. Weyl made reference to the discrete Fourier transform, though not by name, and we should like here to take the matter up again and to discuss some analogies between Fourier transforms, Fourier series, and discrete Fourier transforms. Our discussion will include a proof of a discrete analogue of Poisson's summation formula. Our intention is by no means to think the matter through to the end, but rather *to keep the discussion on an elementary level.* An advanced treatment may be found in Loomis [14].

It seems to be easier to develop the analogies rather than to develop a single comprehensive theory that would include all cases at once. (A referee states that the true reason for the analogy is due to fairly deep group-theoretical facts, and that a single comprehensive theory is outlined by Mackey [15].)

The discrete Fourier transform is a stock-in-trade of the numerical analyst when he is engaged in practical Fourier analysis ([13], Ch. X). It is not always recognized that it deserves a respected place in pure mathematics also. There are also applications to mathematical statistics, including the design of statistical experiments [4], and to the generation of random numbers. (See [16], where there are further references.)

I am indebted to Dr. S. Wylie, Professor D. Rees, and Professor M. H. A. Newman for stimulating discussions sixteen years ago.

Notation and formulas for Fourier transforms and series. It would be pointless to mention the conditions of validity of the formulas concerning Fourier series and integrals since these formulas are not our main concern. The formulas concerning discrete Fourier transforms do not need encumbrance by such conditions. We need only say that we shall usually be talking about real or complex numbers and will not be concerned with abstract generalizations.

We shall use an asterisk to denote a transform of any of the three kinds. It should always be clear from the context which transform we are talking about.

Apart from minor changes in convention, the Fourier transform of a function $F(x)$, defined for all real x, is a function $F^*(y)$ defined by the integral

$$F^*(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(x) e^{ixy} dx.$$

There is the inversion formula

$$F(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F^*(y) e^{-ixy} dy,$$

and the Rayleigh-Parseval orthogonality or unitary relation

$$\int_{-\infty}^{\infty} |F^*(y)|^2 dy = \int_{-\infty}^{\infty} |F(x)|^2 dx.$$

The function $e^{-\frac{1}{2}x^2}$ is its own Fourier transform; in other words, it is an eigenfunction with eigenvalue 1. Finally, Poisson's summation formula (in one form) is

$$\sqrt{\alpha} \sum_{n=-\infty}^{\infty} F(n\alpha) = \sqrt{\beta} \sum_{n=-\infty}^{\infty} F^*(n\beta), \tag{1}$$

where $\alpha\beta = 2\pi$. (The above formulas may all be found in [10].)

We now quote some analogous formulas for Fourier series. Apart from minor variations in convention, the Fourier series of a sequence f_n, defined for all integers n, is (formally) a function $f^*(\theta)$, where

$$f^*(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}.$$

There is the inversion formula

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f^*(\theta) e^{in\theta} d\theta,$$

and the Rayleigh-Parseval formula

$$\frac{1}{2\pi} \int_0^{2\pi} |f^*(\theta)|^2 d\theta = \sum_{n=-\infty}^{\infty} |f_n|^2.$$

The eigenfunction problem seems to be artificial for Fourier series. An analogue of Poisson's summation formula is, for any positive integer t,

$$\frac{1}{t} \sum_{r=0}^{t-1} f^*(2\pi r/t) = \sum_{n=-\infty}^{\infty} f_{nt}. \tag{2}$$

(See [7] and [3], where applications are given.)

If we write z for $e^{i\theta}$ in the formulas for f^* and f_n, we get the Laurent series

and the Cauchy contour integral formula for the coefficients. These are more than an analogy with the above formulas.

The discrete Fourier transform. Apart from minor variations in convention [2], the discrete Fourier transform of a finite sequence a_r, defined for $r=0, 1, \cdots, t-1$ (in other words, for all the residues mod t), is a finite sequence a_s^* defined by

$$a_s^* = \frac{1}{\sqrt{t}} \sum_{r=0}^{t-1} a_r \omega^{rs} \qquad (\omega = e^{2\pi i/t}, \ s \text{ an integer}).$$

There is the inversion formula

$$a_r = \frac{1}{\sqrt{t}} \sum_{s=0}^{t-1} a_s^* \omega^{-rs},$$

and the Rayleigh-Parseval relation

$$\sum_{r=0}^{t-1} |a_r|^2 = \sum_{s=0}^{t-1} |a_s^*|^2.$$

(The reader should have no difficulty in proving these two formulas.)

Connections between the discrete Fourier transform and the theory of numbers are exemplified by the following comments. The eigenvalues of a circulix (circulant matrix) are given by the discrete Fourier transform of its top row multiplied by $\sqrt{t}$ (cf. [1], [6]). The matrix of the transformation is $\Omega = \{\omega^{rs}/\sqrt{t}\}$. The columns of Ω are the eigenvectors of every circulix. The trace of $\Omega\sqrt{t}$ is the Gaussian sum

$$\sum_{r=0}^{t-1} \omega^{r^2} = \tfrac{1}{2}(1+i)(1+i^{-t})\sqrt{t}.$$

More generally, by taking $f^*(\theta) = \exp[i\theta((t\theta)/(2\pi)+s)]$ in (2), we may deduce that the discrete Fourier transform of $\mu_r = \omega^{r^2}$ is

$$\mu_s^* = \tfrac{1}{2}(1+i)(1+i^{2s-t})\omega^{-\frac{1}{4}s^2}.$$

(The Gaussian sum is the special case $s=0$ ([8], p. 153).) The fourth power of Ω is the identity. The eigenvectors of Ω are the columns of the four matrices

$$V_v = I + i^v\Omega + i^{2v}\Omega^2 + i^{3v}\Omega^3 \qquad (v = 1, 2, 3, 4),$$

with eigenvalues i^{-v}. The multiplicities of these eigenvalues are the ranks of the matrices V_v. For $v=1, 2, 3, 4$, these ranks can readily be shown ([8], pp. 164–165) to be

$$[\tfrac{1}{4}(t-1)], \quad [\tfrac{1}{4}(t+2)], \quad [\tfrac{1}{4}(t+1)], \quad [\tfrac{1}{4}(t+4)],$$

and the trace of V_v is four times its rank. We may write V_v in the form

$$V_v = \delta_r^s + \frac{i^v\omega^{rs}}{\sqrt{t}} + i^{2v}\delta_r^{-s} + \frac{i^{3v}\omega^{-rs}}{\sqrt{t}},$$

where δ_a^b is Kronecker's delta ($\delta_a^b = 1$ if $a = b$, $\delta_a^b = 0$ if $a \neq b$).

Multidimensional transforms. Multidimensional Fourier transforms and series are analogous to the n-dimensional discrete Fourier transform

$$a_s^* = \frac{1}{\sqrt{\tau}} \sum_r a_r \omega_1^{r_1 s_1} \cdots \omega_n^{r_n s_n},$$

where $\tau = t_1 \cdots t_n$ and $t_1, \cdots, t_n$ are positive integers; $r = (r_1, \cdots, r_n)$; $s = (s_1, \cdots, s_n)$; $s_\nu = 0, 1, \cdots, t-1$, $\nu = 1, \cdots, n$; $\omega_\nu = \exp(2\pi i/t_\nu)$. The transform itself can be described as "to the moduli $t_1, \cdots, t_n$" or simply mod $t_1, \cdots, t_n$. The matrix of the transformation is the direct product of n matrices of the form of Ω [4]. The inversion and Rayleigh-Parseval formulas are

$$a_r = \frac{1}{\sqrt{\tau}} \sum_s a_s^* \omega_1^{-r_1 s_1} \cdots \omega_n^{-r_n s_n}, \qquad \sum_r |a_r|^2 = \sum_s |a_s^*|^2,$$

respectively, and can be readily verified. If $t_1, \cdots, t_n$ are mutually prime, then the n-dimensional Fourier transform can be expressed as an one-dimensional discrete Fourier transform [2]. This fact does not seem to have an analogue in Fourier transforms or series.

In the ordinary theory of multidimensional transforms and series, it is convenient to use the scalar-product notation in the index of the exponential. The corresponding notation for the present transform is the *scalar-indicial* $\omega^{rs} = \omega_1^{r_1 s_1} \cdots \omega_n^{r_n s_n}$. In this notation the multidimensional formulas take on almost the same appearance as the one-dimensional ones.

Some readers will have noticed that we are in effect talking about group characters, but our aim is to keep the arguments as elementary as possible. The knowledge of groups that will be needed, other than at the end of the paper, will be given now.

A finite Abelian group is a finite set of elements for which the sum of any two elements is defined and in the set, addition is associative and commutative, there is a zero element, and subtraction is always possible. An integral multiple of an element is defined in the obvious way. A subgroup is a subset that is a group, and is therefore also a finite Abelian group. The fundamental theorem for finite Abelian groups ([9], Sec. 45) can be expressed in the form

Every Abelian group R of order τ can, for some factorization of $\tau = t_1 \cdots t_n$, be regarded as a set of vectors $r = (r_1, \cdots, r_n)$, the law of combination being addition mod $t_1, \cdots, t_n$.

(The numbers $t_1, \cdots, t_n$ can in fact be selected to be powers of primes, though we shall not make use of this fact.) Another way of stating the theorem is that there is a *basis* in terms of which every element can be expressed linearly

and uniquely when the coefficients are residues modulo the orders of the basis elements. (The *order* of an element is the smallest positive integer which, when multiplied by the element, gives zero. On the other hand the order of a group is simply the number of elements in it.) The same group may have different bases having different values of n.

We shall now introduce a definition that depends on the particular basis selected, namely, that of orthogonality.

Orthogonality. If $\mathbf{r}$ and $\mathbf{s}$ are such that $\omega^{rs}=1$, then we shall say that $\mathbf{r}$ and $\mathbf{s}$ are *mutually orthogonal* (for a given basis). An equivalent definition is that

$$(\mathbf{r}, \mathbf{s}) = \sum_{\nu=1}^{n} (r_\nu s_\nu)/t_\nu \equiv 0 \pmod 1.$$

If G is a subgroup of R, then the set of all elements of R that are orthogonal to all elements of G can at once be seen to be another subgroup H. For short, we may write $H=\mathrm{Orth}_R(G|\text{ the basis})$, read "orthogonal in R to G given the basis." We shall prove that $G=\mathrm{Orth}_R(H|\text{ the basis})$, so that G and H may be called *mutually orthogonal*. It is possible for G to be orthogonal to itself, *isotropic* as it were. An example is the group consisting of all integer multiples of (2, 3), when $n=2$, $t_1=4$, $t_2=9$.

It is natural to define the scalar product of $\mathbf{r}$ and $\mathbf{s}$ as either $(\mathbf{r}, \mathbf{s})$ (mod 1) or, perhaps, as $(\mathbf{r}, \mathbf{s})\alpha$ (mod α) for any real number α, so that $\omega^{rs}=\exp(2\pi i(\mathbf{r}, \mathbf{s}))$. After these preliminaries we now come to the main result.

Discrete analogue of Poisson's summation formula. Let R be the finite Abelian group of order τ expressed as a set of vectors $\mathbf{r}$ as above. Let G be a subgroup and H the subgroup orthogonal to G, $H=\mathrm{Orth}_R(G|t_1, \cdots, t_n)$ in a self-explanatory notation. Let $a_\mathbf{r}$ be any real or complex function of the elements of R and let $a_\mathbf{s}^*$ be the discrete multidimensional Fourier transform of $a_\mathbf{r}$ mod $t_1, \cdots, t_n$. Then

$$\frac{1}{\sqrt{|G|}} \sum_{\mathbf{r}\in G} a_\mathbf{r} = \frac{1}{\sqrt{|H|}} \sum_{\mathbf{s}\in H} a_\mathbf{s}^*, \tag{3}$$

and, a little more generally,

$$\frac{1}{\sqrt{|G|}} \sum_{\mathbf{r}}^{\mathbf{r}-\mathbf{r}^{(0)}\in G} a_\mathbf{r}\omega^{\mathbf{r}\mathbf{s}^{(0)}} = \frac{1}{\sqrt{|H|}} \sum_{\mathbf{s}}^{\mathbf{s}-\mathbf{s}^{(0)}\in H} a_\mathbf{s}^*\omega^{\mathbf{r}^{(0)}(\mathbf{s}^{(0)}-\mathbf{s})}, \tag{4}$$

where $|G|$ and $|H|$ denote the orders of G and H, where $\mathbf{r}^{(0)}$ and $\mathbf{s}^{(0)}$ are any two elements of R. We shall find that $|G|\cdot|H|=\tau$.

The process of averaging over group elements is said by Weyl ([12], p. 185) to be a powerful instrument yielding unexpectedly far-reaching results. It is therefore not surprising that (4) has application outside pure mathematics [4]. The following result is of some interest in itself.

LEMMA. $\sum_{r\in G}\omega^{rs}=|G|$ *if* $s\in H$ *and is zero otherwise.*

Proof. We shall suppose that s does not belong to H since the lemma is trivial when it does. By the fundamental theorem for finite Abelian groups, G has a basis $g^{(1)}, \cdots, g^{(m)}$ so that every element of G is of the form $i_1g^{(1)}+\cdots+i_mg^{(m)}$, where the i_μ are integers, $0\leqq i_\mu<\gamma_\mu$ $(\mu=1, \cdots, m)$, $\gamma_1, \cdots, \gamma_m$ being the order of the basis elements. Clearly, $|G|=\gamma_1\cdots\gamma_m$.

Since s does not belong to H there must be a basis element $g^{(\mu_0)}$ of G such that $\omega^{g^{(\mu_0)}s}\neq 1$. We have

$$\sum_{r\in G}\omega^{rs}=\sum_{i_1,\cdots,i_m}\prod_{\mu=1}^{m}(\omega^{g^{(\mu)}s})^{i_\mu}=\prod_{\mu=1}^{m}\sum_{i=0}^{\gamma_\mu-1}(\omega^{g^{(\mu)}s})^{i_\mu}.$$

Now

$$\sum_{i_{\mu_0}=0}^{\gamma_{\mu_0}-1}(\omega^{g^{(\mu_0)}s})^{i_{\mu_0}}=(1-\omega^{g^{(\mu_0)}s\gamma_{\mu_0}})/(1-\omega^{g^{(\mu_0)}s})=0,$$

and the lemma is proved.

Next we shall prove that

$$\sum_{r}^{r-r^{(0)}\in G}a_r\omega^{rs^{(0)}}=\frac{|G|}{\sqrt{\tau}}\sum_{s}^{s-s^{(0)}\in H}a_s^*\omega^{r^{(0)}(s^{(0)}-s)}. \tag{5}$$

(For example, if $G=R$, we obtain the definition of a_s^* whereas if G consists only of the zero, $\mathbf{0}=(0, \cdots, 0)$, then we obtain the inversion formula for the discrete multidimensional Fourier transform.) We have

$$\begin{aligned}\sum_{r}^{r-r^{(0)}\in G}a_r\omega^{rs}&=\frac{1}{\sqrt{\tau}}\sum_{r}^{r-r^{(0)}\in G}\sum_{s^{(0)}\in R}a_s^*\omega^{r(s^{(0)}-s)}\\&=\frac{1}{\sqrt{\tau}}\sum_{s\in R}a_s^*\sum_{r}^{r-r^{(0)}\in G}\omega^{r(s^{(0)}-s)}\\&=\frac{1}{\sqrt{\tau}}\sum_{s\in R}a_s^*\sum_{r\in G}\omega^{(r^{(0)}+r)(s^{(0)}-s)}\\&=\frac{1}{\sqrt{\tau}}\sum_{s\in R}a_s^*\omega^{r^{(0)}(s^{(0)}-s)}\sum_{r}^{r\in G}\omega^{r(s^{(0)}-s)},\end{aligned}$$

and then (5) follows from the lemma. This result may be written as

$$\sum_{r\in G}a_{r+r^{(0)}}\omega^{rs^{(0)}}=\frac{|G|}{\sqrt{\tau}}\sum_{s\in H}a^*_{s+s^{(0)}}\omega^{-r^{(0)}(s+s^{(0)})}. \tag{6}$$

Apply the same result to H and a_s^* instead of to G and a_r. Let G' be the group of vectors orthogonal to H. (Clearly, G' contains G; we shall see later that

$G'=G$. This will justify our describing G and H as mutually orthogonal.) Then for any $u^{(0)}$ in R, we have

$$\sum_{r\in H}{}^{*} a_{r+r^{(0)}}\omega^{ru^{(0)}} = \frac{|H|}{\sqrt{\tau}} \sum_{u\in G'} a_{-u-u^{(0)}}\omega^{-r^{(0)}(u+u^{(0)})},$$

so that, after a few changes of variables, we get

$$\sum_{s\in H}{}^{*} a_{s+s^{(0)}}\omega^{-r^{(0)}(s+s^{(0)})} = \frac{|H|}{\sqrt{\tau}} \sum_{r\in G'} a_{-r+r^{(0)}}\omega^{-rs^{(0)}}.$$

Therefore, from (6), we have

$$\begin{aligned}\sum_{r\in G} a_{r+r^{(0)}}\omega^{rs^{(0)}} &= \frac{|G|\cdot|H|}{\tau} \sum_{r\in G'} a_{-r+r^{(0)}}\omega^{-rs^{(0)}} \\ &= \frac{|G|\cdot|H|}{\tau} \sum_{r\in G'} a_{r+r^{(0)}}\omega^{rs^{(0)}},\end{aligned} \tag{7}$$

because if r runs through G' so does $-r$, since G' is a group.

Let the subgroup orthogonal to G' be H', that to H' be G'', and so on. Clearly, $G\subseteqq G'\subseteqq G''\subseteqq\cdots$ and $H\subseteqq H'\subseteqq H''\cdots$. This process of successive orthogonal enlargement must terminate since R is finite. Let the terminating subgroups be G_0 and H_0. They are each orthogonal to the other. From (7) we get

$$\sum_{r\in G_0} a_{r+r^{(0)}}\omega^{rs^{(0)}} = \frac{|G_0|\cdot|H_0|}{\tau} \sum_{r\in G_0} a_{r+r^{(0)}}\omega^{rs^{(0)}}. \tag{8}$$

Therefore

$$|G_0|\cdot|H_0| = \tau \tag{9}$$

since the sum in (8) obviously does not vanish for all functions a_r.

Consider the special function $a_0=1$, $a_r=0$ if $r\neq 0$, and put $r^{(0)}=0$, $s^{(0)}=0$, $a_s^{*}=1$ in (7). We get $|G|\cdot|H|=\tau$. But $|G|\leqq|G_0|$, $|H|\leqq|H_0|$; so from (9) we get $|G|=|G_0|$, $|H|=|H_0|$, and therefore $G=G_0$, $H=H_0$.

We have therefore proved the general form (4) of the discrete analogue of Poisson's summation formula, and have also justified describing G and H as mutually orthogonal (given the basis).

Since the order of H is independent of the basis and is $|R|/|G|$, we are led to ask

(i) Is $\mathrm{Orth}_R(G|$ the basis) isomorphic with $\mathrm{Orth}_R(G|$ any other basis)?

(ii) Is $\mathrm{Orth}_R(G|$ the basis) isomorphic with the quotient group R/G?

Mr. Michael M. Crum has succeeded in answering these questions affirmatively in the following manner, by making a little more use of the theory of groups than does the preceding work.

For all $r\in R$,

$$\gamma_\mu(r, g^{(\mu)}) \equiv (r, \gamma_\mu g^{(\mu)}) \equiv (r, 0) \equiv 0 \pmod 1,$$

so that $\gamma_\mu(r, g^{(\mu)})$ is an integer and is defined mod γ_μ. Consequently, the equation

$$g(r) = \sum_\mu \gamma_\mu(r, g^{(\mu)}) g^{(\mu)}$$

meaningfully defines a function, linear in r, and whose values are in G. Since the $g^{(\mu)}$ are a basis in G, we have $g(r) = 0$ if and only if $\gamma_\mu(r, g^{(\mu)}) \equiv 0 \pmod{\gamma_\mu}$, i.e., if and only if $(r, g^{(\mu)}) \equiv 0 \pmod 1$, $(\mu = 1, \cdots, m)$, i.e., if and only if $r \in H$. If $g(r) = g(r^{(0)})$, then $g(r - r^{(0)}) = 0$ and r and $r^{(0)}$ must belong to the same coset $H^0 = H + r^{(0)}$ of H in R. So we can define a function of cosets by putting $g(H^0) = g(r^{(0)})$. Moreover, if H^0 and H^1 are two distinct cosets of H, we have $g(H^0) \neq g(H^1)$. Hence $H^0 \to g(H^0)$ is a homomorphism of the quotient group R/H into G. But R/H has the same number of elements as G has, so the homomorphism must be an isomorphism. Thus the quotient group of H is isomorphic with the group orthogonal to H, namely G. Similarly, the quotient group of G is isomorphic with H. This answers question (ii) affirmatively and *a fortiori* question (i). Also, since G and H are *mutually orthogonal*, we see that any subgroup G of an Abelian group R is isomorphic with R/H, where H is some subgroup of R isomorphic with R/G. This fact is presumably known, but I have not found a reference.

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ON THE CONSTRUCTION OF INVOLUTORY MATRICES

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1. Introduction. The actual construction of involutory matrices with elements in a field $\mathfrak{F}$, while of interest in itself, has a practical application in algebraic cryptography, Hill [4] and Levine [5]. For such applications the use of involutory matrices $X=X^{-1}$ has the cryptographic advantage that, for example, the same cipher machine can be used for enciphering and deciphering. Such a machine has, in fact, been patented by Weisner and Hill (U.S. Patent 1,845,947).

In this connection it is desired to construct involutory matrices of any order. Such matrices, with elements in a field, will apply to the cases of an "alphabet" with a prime number p of letters (for instance the 26 letters of the English alphabet plus 3 punctuation marks, giving a 29-letter alphabet), with the field then being the field of integers mod p. The use of such an alphabet has been illustrated by Stewart [7]. In the case of an alphabet with a composite number n, of letters (as 26) the involutory matrices will have elements in the ring of integers mod n, and the method of construction of the present paper will not yield all such matrices.

Hill [4] has given a formula producing symmetric involutory matrices, this being based on a result of Aitken [9]. The method proposed in this paper removes the restriction of symmetry and is a generalization of Aitken's result.

The more general problem of solving the matrix equation $X^m=A$ has been considered by many authors among whom may be mentioned Cayley [1], Sylvester [8], Frobenius [3], Dickson [2], Wedderburn [10], Weitzenbock [11], and Roth [6]. Generally speaking their solutions for X were obtained in canonical form X_0, it then being observed that more general solutions were matrices similar to X_0.

In this paper we give in some detail the structure of the matrices $X=PX_0P^{-1}$ in the particular case where $X^2=I$, these results being stated in Theorems 1 and 2.

2. Involutory matrices over a field not of characteristic 2. Let H be an $n\times n$ involutory matrix with elements in a field $\mathfrak{F}$, not of characteristic 2. Since $H^2=I$, the minimal equation of H is either $x-1=0$, $x+1=0$, or $x^2-1=0$. In the first two cases H is similar to, and indeed, equal to the identity matrix I_n or $-I_n$, respectively. In the third case H is similar to a diagonal matrix,

$$J=\begin{bmatrix} I_r & Z_{r,s} \\ Z_{s,r} & -I_s \end{bmatrix}, \tag{1}$$

where $r+s=n$, $0<s<n$. In this case we will say that H is a *nontrivial involutory matrix of signature s*. A construction for the nontrivial case is indicated in the following theorem.

THEOREM 1. *A necessary and sufficient condition that an $n\times n$ matrix H with elements in a field $\mathfrak{F}$, not of characteristic 2, be nontrivial involutory of signature s, $0<s<n$, is that $H=I_n-Q_2P_2$, where Q_2 is $n\times s$ and P_2 is $s\times n$ such that $P_2Q_2=2I_s$. (Alternatively $H=Q_1P_1-I_n$, where Q_1 is $n\times r$ and P_1 is $r\times n$ such that $P_1Q_1=2I_r$ and $r+s=n$.)*

The advantage of two forms is that the computation of H may be shortened by using that form in which r or $s\leqq\frac{1}{2}n$.

Proof. Necessity. We write $H=QJQ^{-1}$, where Q is any nonsingular matrix with elements in $\mathfrak{F}$. Let $Q=(Q_1'\,Q_2')$, where Q_2' is $n\times s$, and write

$$Q^{-1}=\begin{bmatrix}P_1\\P_2\end{bmatrix},$$

where P_2 is $s\times n$. Then

$$QQ^{-1}=Q_1'P_1+Q_2'P_2=I_n, \tag{2}$$

$$Q^{-1}Q=\begin{bmatrix}P_1Q_1' & P_1Q_2'\\P_2Q_1' & P_2Q_2'\end{bmatrix}=\begin{bmatrix}I_r & Z_{r,s}\\Z_{s,r} & I_s\end{bmatrix}, \tag{3}$$

$$H=QJQ^{-1}=Q_1'P_1-Q_2'P_2. \tag{4}$$

Replacing $Q_1'P_1$ in (4) by $I_n-Q_2'P_2$ and setting $Q_2=2Q_2'$ we have

$$H=I_n-Q_2P_2. \tag{5}$$

From (3) we observe that P_2 is $s\times n$ and Q_2 is $n\times s$ such that

$$P_2Q_2=2P_2Q_2'=2I_s. \tag{6}$$

In a similar manner we obtain the alternate form

$$H=Q_1P_1-I_n, \tag{7}$$

where P_1 is $r\times n$ and Q_1 is $n\times r$ such that

$$P_1Q_1=2P_1Q_1'=2I_r. \tag{8}$$

Sufficiency. Let $H=I_n-Q_2P_2$. Then

$$\begin{aligned}H^2&=(I_n-Q_2P_2)(I_n-Q_2P_2)=I_n^2-2Q_2P_2+Q_2P_2Q_2P_2\\&=I_n-2Q_2P_2+Q_2(2I_s)P_2=I_n.\end{aligned}$$

In the same way, $H^2=I_n$ if $H=Q_1P_1-I_n$. Hence H is involutory.

To show that H is nontrivial, we assume $H=I_n$. Then from (5) $Q_2P_2=Z_{n,n}$ and by (6) $P_2Q_2=2I_s$. Hence, $2Q_2=Q_2P_2Q_2=Z_{n,n}Q_2=Z_{n,s}$ and since $\mathfrak{F}$ is not of characteristic 2, it follows that $Q_2=Z_{n,s}$ in contradiction to $P_2Q_2=2I_s$. In the same way, if we assume that $H=-I_n$, then from (7) $Q_1P_1=Z_{n,n}$ which leads to a contradiction that $P_1Q_1=2I_r$.

3. Involutory matrices over a field of characteristic 2. If $\mathfrak{F}$ is of characteristic 2, then the minimal equation for an $n\times n$ involutory matrix H is either $x-1=0$ or $x^2-1=(x-1)^2=0$. In the first case, $H=I_n$, and in the second, H is similar to

$$J_1 = \begin{bmatrix} I_r & Z_{r,2s} \\ Z_{2s,r} & K_{2s} \end{bmatrix}, \tag{9}$$

where K_{2s} is the direct sum of s matrices of the form

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and $r+2s=n$, $0<s\leqq\frac{1}{2}n$.

In the latter case we will say that H is a nontrivial involutory matrix of order n and signature s. In this connection we prove the following:

THEOREM 2. *If $\mathfrak{F}$ is of characteristic 2, then a necessary and sufficient condition that H be a nontrivial involutory matrix of order n and signature s, $0<s\leqq\frac{1}{2}n$, is that $H=I_n+Q_2P_2$ where Q_2 is $n\times s$ of rank s and P_2 is $s\times n$ of rank s such that $P_2Q_2=Z_{s,s}$.*

Proof. Sufficiency. If $H=I_n+Q_2P_2$ then

$$H^2 = (I_n + Q_2P_2)(I_n + Q_2P_2) = I_n + Q_2P_2Q_2P_2 = I_n + Q_2Z_{s,s}P_2 = I_n.$$

Furthermore, H is nontrivial. For if we assume $H=I_n$, then $Q_2P_2=Z_{n,n}$. But by assumption Q_2 is $n\times s$ of rank s and hence has a left inverse, say R_2, of order $s\times n$. Whence $R_2Q_2P_2=P_2=Z_{s,n}$ which contradicts the assumption that P_2 is $s\times n$ of rank s.

Necessity. Since $H=QJ_1Q^{-1}$, J_1 as in (9), we write

$$Q = (Q_1'\,Q_2') \quad \text{and} \quad Q^{-1} = \begin{bmatrix} P_1' \\ P_2' \end{bmatrix}$$

where Q_2' is $n\times 2s$ and P_2' is $2s\times n$. Then

$$QQ^{-1} = Q_1'P_1' + Q_2'P_2' = I_n, \tag{10}$$

$$Q^{-1}Q = \begin{bmatrix} P_1'Q_1' & P_1'Q_2' \\ P_2'Q_1' & P_2'Q_2' \end{bmatrix} = \begin{bmatrix} I_r & Z_{r,2s} \\ Z_{2s,r} & I_{2s} \end{bmatrix}. \tag{11}$$

Hence,

$$H = QJQ^{-1} = (Q_1'\,Q_2') \begin{bmatrix} I_r & Z_{r,2s} \\ Z_{2s,r} & K_{2s} \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \end{bmatrix} = Q_1'P_1' + Q_2'K_{2s}P_2'. \tag{12}$$

From (10) we have $Q_1'P_1'=I_n-Q_2'P_2'$. Hence we write

$$H = I_n + Q_2'L_{2s}P_s', \tag{13}$$

where $L_{2s}=K_{2s}-I_{2s}=(0, u_1, 0, u_3, 0, \cdots, 0, u_{2s-1})$, u_k being the kth unit column vector of dimension $2s$.

Forming the product $Q_2' L_{2s}$, we have $Q_2' L_{2s}=(0, q_1, 0, q_3, \cdots, 0, q_{2s-1})$, where q_k is the kth column of Q_2'. It follows that

$$Q_2' L_{2s} P_2' = q_1 p_2 + q_3 p_4 + \cdots + q_{2s-1} p_{2s},$$

where p_k is the kth row of P_2'. Thus, if we form Q_2 by using only columns numbered $1, 3, \cdots, 2s-1$ from Q_2' and construct P_2 by using only the rows numbered $2, 4, \cdots, 2s$ from P_2', then $Q_2' L_{2s} P_2' = Q_2 P_2$. We write

$$H = I_n + Q_2 P_2. \tag{14}$$

Now the rank of P_2' and of Q_2' is $2s$ since by (11) $P_2' Q_2' = I_{2s}$. Hence the ranks of P_2 and Q_2 are both s, and if we use the notation of above we have $p_i q_j = \delta_{ij}$ where $i=2, 4, \cdots, 2s$, $j=1, 3, \cdots, 2s-1$. Therefore

$$P_2 Q_2 = Z_{s,s}. \tag{15}$$

In summary we state

THEOREM 3. *Counting both the trivial and the nontrivial cases, the number* $N(\mathfrak{F}, n)$ *of classes of dissimilar involutory matrices is given as follows*:

$$N(\mathfrak{F}, n) = 1 + n, \text{ when } \mathfrak{F} \text{ is not of characteristic } 2;$$

$$N(\mathfrak{F}, n) = 1 + [\tfrac{1}{2}n], \text{ when } \mathfrak{F} \text{ is of characteristic } 2.$$

Proof. In Theorem 1, in addition to the two trivial types, there are $n-1$ distinct types, each characterized by the value s with $0<s<n$.

In Theorem 2, in addition to the one trivial type, there are $[\frac{1}{2}n]$ classes, each characterized by the value of s, with $0<s\leqq\frac{1}{2}n$.

Illustration 1. ($\mathfrak{F}$ not of characteristic 2) We illustrate Theorem 1 by constructing a 5×5 involutory matrix of signature 2 over the field of integers mod 29. Since this requires that a 2×5 matrix P_2 and a 5×2 matrix Q_2 be found such that $P_2Q_2=2I_2$, we choose P_2 arbitrarily and determine Q_2.
Thus

$$P_2 = \begin{bmatrix} 1 & 1 & -1 & -3 & 2 \\ 1 & 2 & 4 & 2 & 0 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \end{bmatrix}, \quad \text{where} \quad P_2Q_2 = 2I_2,$$

lead to two linear systems in x_{ij} with three arbitrary parameters in each case. Among the solutions we choose

$$Q_2 = \begin{bmatrix} 2 & -4 \\ 0 & 4 \\ -1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$H = I - Q_2P_2 = \begin{bmatrix} 3 & 6 & -11 & 14 & -4 \\ -4 & -7 & 13 & -8 & 0 \\ 2 & 3 & 4 & -1 & 2 \\ -2 & -3 & -3 & 2 & -2 \\ -2 & -3 & -3 & 1 & -1 \end{bmatrix}.$$

Illustration 2. ($\mathfrak{F}$ of characteristic 2). As an illustration of Theorem 2 we construct a 5×5 involutory matrix of signature 2 over a field of characteristic 2. In this case we choose any 2×5 matrix P_2 of rank 2 with elements in the field, and find any 5×2 matrix Q_2 of rank 2 such that $P_2Q_2=Z_{2,2}$. Thus if

$$P_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \qquad Q_2 = [x_{ij}],$$

we are led to two linear systems in x_{ij} with three parameters. Among the solutions we choose

$$Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$H = I + Q_2P_2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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AN EXTENSION OF THE FARKAS THEOREM*

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1. Introduction. With the rise of interest in such areas as linear programming and the theory of games, new emphasis has been placed on the study of systems of linear inequalities. In a recent expository article based on lectures by A. W. Tucker, R. A. Good [4] gives the following statement of a theorem first proved in 1902 by J. Farkas [3].

FARKAS THEOREM. *Let* $x_0, x_1, \cdots, x_n$ *be* $n+1$ *vectors in (euclidean) m-space. If* $x \cdot x_0 \geqq 0$ *for every vector* x *satisfying all the inequalities* $x \cdot x_1 \geqq 0, \cdots, x \cdot x_n \geqq 0$, *then* x_0 *is a nonnegatively weighted sum of* $x_1, \cdots, x_n$, *that is,* $x_0 = \lambda_1 x_1 + \cdots \lambda_n x_n$ *with* $\lambda_1 \geqq 0, \cdots, \lambda_n \geqq 0$.

In addition to the original paper of Farkas and the above mentioned article of Good, proofs of the Farkas Theorem may be found, for example, in papers by H. Weyl [6] Theorem 3, p. 9, and K. Fan [2] Theorem 4, p. 108. In this paper, an application of some elementary techniques from functional analysis provides a simple proof of an extension of the Farkas Theorem to the case of

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infinite systems of homogeneous linear inequalities in a locally convex real linear space of arbitrary dimension.

Recall that euclidean m-space E^m is self-dual. That is, the space of all continuous linear functionals on E^m may be identified with E^m in the sense that if x^* is a continuous linear functional on E^m there exists a vector y in E^m such that for each x the value $x^*(x)$ is given by the dot product $y \cdot x$ and, conversely, each vector y in E^m defines a continuous linear functional whose value at x is $y \cdot x$. Because of this self-duality the Farkas Theorem can be rewritten in each of the following two equivalent forms.

THEOREM 1.1. *Let $x_0, x_1, \cdots, x_n$ be $n+1$ elements in E^m. If $x^*(x_0) \geqq 0$ for every continuous linear functional x^* on E^m such that $x^*(x_i) \geqq 0$ $(1 \leqq i \leqq n)$, then $x_0 = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \geqq 0$ $(1 \leqq i \leqq n)$.*

THEOREM 1.2. *Let $x_0^*, x_1^*, \cdots, x_n^*$ be $n+1$ continuous linear functionals on E^m. If $x_0^*(x) \geqq 0$ for every element x of E^m such that $x_i^*(x) \geqq 0$ $(1 \leqq i \leqq n)$ then $x_0^* = \sum_{i=1}^{n} \lambda_i x_i^*$ where $\lambda_i \geqq 0$ $(1 \leqq i \leqq n)$.*

The procedure employed in this paper is based upon the recognition that the solution set of an inequality $x^*(x) \geqq 0$ is, geometrically, a half-space and that the solution set of a system of such inequalities, being the intersection of half-spaces, is therefore a closed convex cone. This allows the application of a well-known separation theorem from the geometric theory of linear topological spaces to provide a short, direct proof of the Farkas Theorem.

Section 2, below, is devoted to a more detailed discussion of the above ideas. The proof of the Farkas Theorem in a form generalizing Theorem 1.1 is found in Section 3, Theorem 3.1. Also in Section 3 it is shown that if the space E satisfies a reflexivity condition then a similar extension of Theorem 1.2 is possible. An example is included to demonstrate that this additional assumption of reflexivity is necessary. Section 4 concludes this paper by posing some questions concerning still further generalizations of the Farkas Theorem.

2. Preliminaries. Included in this section are the definitions, notation and preliminary results which provide a minimum background for the discussion in Section 3. For concepts not specifically defined here, see for example, A. E. Taylor [5] or N. Bourbaki [1].

In what follows, unless otherwise mentioned, E denotes a locally convex real linear topological space, E^* the space of all linear functionals x^* which are continuous on E, and E^{**} the space of all linear functionals x^{**} on E^* which are continuous in whatever topology is provided for E^*. The notation $\{x: P\}$ means "the set of all x such that P." A subset C of a real linear space is a *convex cone with vertex θ if $C+C \subset C$ and $\alpha C \subset C$ for all $\alpha \geqq 0$*. If A is a nonempty subset of E, the *conical hull* of A, denoted by $A\angle$, is the set of all finite linear combinations, with nonnegative coefficients of elements of A. Clearly both $A\angle$ and $\overline{A\angle}$ (the closure of $A\angle$ in the topology of E) are convex cones with vertex θ. For x^* in E^* and α a real number, sets of the form $\{x: x \in E, x^*(x) \sim \alpha\}$ will be considered.

The symbol $\sim$ will be replaced by $=$, $\geqq$, $\leqq$, $>$ or $<$ as needed. In particular, $\{x: x\in E,\ x^*(x)=\alpha\}$ is a *closed hyperplane* in E and, if F is a subset of E such that $F \subset \{x: x \in E,\ x^*(x) \geqq \alpha\}$ (or $F \subset \{x: x \in E,\ x^*(x) \leqq \alpha\}$) and $F\cap\{x: x\in E,\ x^*(x)=\alpha\}$ is nonempty then $\{x: x\in E,\ x^*(x)=\alpha\}$ is called a *hyperplane of support* for F.

LEMMA 2.1. *Let C be a convex cone with vertex θ in E. If $C\subset\{x: x\in E, x^*(x)\geqq\alpha\}$ then $\alpha\leqq 0$ and $\{x: x\in E,\ x^*(x)=0\}$ is a hyperplane of support for C.*

Proof. Since $\theta\in C\subset\{x: x\in E,\ x^*(x)\geqq\alpha\}$ certainly $\alpha\leqq 0$ $(x^*(\theta)=0)$ and $C\cap\{x: x\in E, x^*(x)=0\}$ is nonempty. Suppose $y\in C$ and $y\notin\{x: x\in E, x^*(x)\geqq 0\}$. Then $x^*(y)<0$ and $\beta>0$ may be chosen large enough so that $x^*(\beta y)=\beta x^*(y)<\alpha$. Since $y \in C$, $\beta y \in C$ and $x^*(\beta y) \geqq \alpha$. This contradiction implies that $C\subset\{x: x\in E,\ x^*(x)\geqq 0\}$ and completes the proof of this lemma.

LEMMA 2.2. *If D is a closed convex nonempty set in E and K is a convex compact set in E, disjoint from D, then there exists an x_0^* in E^* and a real number α such that $D\subset\{x: x\in E,\ x_0^*(x)>\alpha\}$ and $K\subset\{x: x\in E,\ x_0^*(x)<\alpha\}$.*

Proof. See [1] Prop. 4, p. 73.

3. The extensions. If, in the statement of Theorem 1.1, E^m is replaced by a locally convex space E, the set $x_1, \cdots, x_n$ is replaced by any nonempty subset A of E, x^* is considered as any element from the set

$$A^* = \{x^*: x^* \in E^*,\ x^*(x) \geqq 0,\ x \in A\},$$

and x_0 is considered as an element of the set

$$B = \{x: x \in E,\ x^*(x) \geqq 0,\ x^* \in A^*\},$$

then it is easy to see that the Farkas Theorem as stated in Theorem 1.1 is a special case of the following theorem.

THEOREM 3.1. $B=\overline{A\angle}$.

Proof. The definition of B shows immediately that B is exactly the intersection of all the closed half-spaces $\{x: x\in E,\ x^*(x)\geqq 0\}$ which contain the set A. Thus, in particular, B is a closed convex cone with vertex θ containing A, and therefore $\overline{A\angle}$. Suppose $x_0\in B$ but $x_0\notin\overline{A\angle}$. Lemma 2.2 with $D=\overline{A\angle}$ and $K=\{x_0\}$ guarantees the existence of an x_0^* in E^* and a real number α such that $\overline{A\angle}\subset\{x: x\in E, x_0^*(x)>\alpha\}$ and $x_0^*(x_0)<\alpha$. Since $\overline{A\angle}$ is a convex cone with vertex θ, Lemma 2.1 shows that $\alpha\leqq 0$ and $\overline{A\angle}\subset\{x: x\in E,\ x_0^*(x)\geqq 0\}$. But then $\{x: x\in E,\ x_0^*(x)\geqq 0\}$ contains A and, therefore, B and, since $x_0\in B$, necessarily $x_0^*(x_0)\geqq 0$. This contradiction completes the proof.

If, in the statement of Theorem 1.2, E^m is again replaced by an arbitrary locally convex space E, the set $x_1^*, \cdots, x_n^*$ is replaced by any nonempty subset S^* of E^*, x is considered as any element from the set

$$P = \{x: x \in E,\ x^*(x) \geqq 0,\ x^* \in S^*\}$$

and x_0^* is considered as an element from the set

$$P^* = \{x^* : x^* \in E^*, x^*(x) \geqq 0, x \in P\},$$

then the generalization of Theorem 1.2 analogous to Theorem 3.1 would read

$$P^* = \overline{S^*Z}.$$

However, as will be shown later, this conclusion is not valid unless E satisfies a reflexivity condition.

Recall that a normed linear space E is *reflexive* if E can be identified with the space E^{**} of all linear functionals on E^* which are continuous in the norm topology of $E^*(\|x^*\| = \sup\{|x^*(x)| : \|x\| \leqq 1\})$ in the sense that the *canonical imbedding* J of E into E^{**} (defined by $J(x) = x^{**}$ where $x^{**}(x^*) = x^*(x)$ for all x^* in E^*) is a mapping onto all of E^{**}. Most of the standard works introducing functional analysis do not define the concept of reflexivity for spaces of a more general topological character. Bourbaki [1] defines reflexivity for a locally convex Hausdorff space E and requires that the space E^* be provided with the *strong* topology. In what follows there need be no concern with the specific topology assigned to E^* except that it be a locally convex topology. The following definition will suffice.

DEFINITION. *A locally convex space E satisfies a reflexivity condition if E^* is provided with a locally convex topology and, when E^{**} is the space of all linear functionals continuous on E^* in this topology, $E^{**} = J(E)$.*

THEOREM 3.2. *If E satisfies a reflexivity condition, then $P^* = \overline{S^*Z}$.*

Proof. Let $S^{**} = \{x^{**} : x^{**} \in E^{**}, x^{**}(x^*) \geqq 0, x^* \in S^*\}$ and $Q^* = \{x^* : x^* \in E^*, x^{**}(x^*) \geqq 0, x^{**} \in S^{**}\}$. If, in Theorem 3.1, E is replaced by E^*, A by S^*, and B by Q^*, the result is $Q^* = \overline{S^*Z}$. Since $J(E) = E^{**}$, $S^{**} = \{J(x) : J(x)(x^*) = x^*(x) \geqq 0, x^* \in S^*\}$. Thus

$$\begin{aligned} Q^* &= \{x^* : x^* \in E^*, x^{**}(x^*) = J(x)(x^*) = x^*(x) \geqq 0, J(x) \in S^{**}\} \\ &= \{x^* : x^* \in E^*, x^*(x) \geqq 0, x^* \in S^*\} = P^*. \end{aligned}$$

The following example shows that, in order to keep the complete generality of the set S^* in Theorem 3.2, the reflexivity assumption is necessary.

Let E be a nonreflexive real normed linear space. Then there exists an x_0^{**} in E^{**} such that $x_0^{**} \notin J(E)$. Choose the subset S^* of E^* specifically to be the set of all elements x^* in E^* such that $x_0^{**}(x^*) = 0$. S^* is a closed linear subspace of E^*. Since $x_0^{**} \notin J(E)$, x_0^{**} is not the zero functional and consequently, $S^* \neq E^*$. Let P and P^* be defined as for Theorem 3.2. The proof that $P^* \neq \overline{S^*Z}$ proceeds as follows:

Let ${}^0S^* = \{x : x \in E, x^*(x) = 0, x^* \in S^*\}$ and

$$({}^0S^*)^0 = \{x^* : x^* \in E^*, x^*(x) = 0, x \in {}^0S^*\}.$$

A. E. Taylor [5, pp. 225,232] states that $S^* \subset ({}^0S^*)^0$ but if E is nonreflexive, $S^* \neq ({}^0S^*)^0$. Suppose x_1 is any element of P. If $x^* \in S^*$ then, by definition of P, $x^*(x_1) \geqq 0$. Since S^* is a subspace, $-x^* \in S^*$ so also $(-x^*)(x_1) \geqq 0$. Thus, $x^*(x_1) = 0$ and $P \subset {}^0S^*$. Now select $x_0^* \in ({}^0S^*)^0$ with $x_0^* \notin S^*$. Then $x_0^{**}(x_0^*) \neq 0$ and $x_0^*(x) = 0$ for all $x \in {}^0S^*$. In particular, $x_0^*(x_1) = 0$. Since S^* is a closed linear subspace of E^* and $S^* \neq E^*$, each element x^* in E^* has a representation $x^* = [x_0^{**}(x^*)/x_0^{**}(x_0^*)]x_0^* + y_0^*$, $y_0^* \in S^*$. Since $x_1 \in P \subset {}^0S^*$, $y_0^*(x_1) = 0$ and, hence, for any element x^* in E^*, $x^*(x_1) = 0$. Because x_1 was selected arbitrarily from P, $P = \{\theta\}$ and $P^* = E^*$. S^* is a closed subspace of E^* so $\overline{S^* \angle} = S^* \neq E^* = P^*$.

4. Questions. It is clear that the closure requirements on the conical hulls in Theorems 3.1 and 3.2 are redundant if the hypotheses are restricted to those of Theorems 1.1 and 1.2 respectively. This observation suggests the question: Do the Theorems of Section 3 remain valid if these closure requirements are dropped? The answer to this question is: In general, no. To see this, consider the following elementary example.

Let $E = E^2$, the euclidean plane. Let the set A of Theorem 3.1 be the open first quadrant. Under the duality of E^2 the set A^* is identified with the closed first quadrant which is, therefore, also the set B. But $A \angle = A \cup \{\theta\} \neq B = \overline{A \angle}$.

The set A of the preceding example is not a closed set. If the set A of Theorem 3.1 is now assumed to be closed can the closure requirement on $A \angle$ be eliminated? That this question also has a negative answer can be seen as follows:

Let E be the Hilbert space, of all real sequences $x = \{\xi_1, \xi_2, \cdots\}$ such that $\|x\|^2 = \sum_{i=1}^{\infty} \xi_i^2 < \infty$. Let A consist of the elements $x_1, x_2, \cdots$ where $x_n = \{\delta_{n1}, \delta_{n2}, \cdots\}$ with $\delta_{ni} = 0$ for $i \neq n$ and $\delta_{nn} = 1$. Clearly A is a closed subset of E. $A \angle$ consists of all the elements $x = \{\xi_1, \xi_2, \cdots\}$ in E such that $\xi_i \geqq 0$ for all $i = 1, 2, \cdots$, and with at most a finite number of the coordinates ξ_i nonzero. Consider the sequence $y_1, y_2, \cdots$ of elements $y_k = \sum_{i=1}^{k} (1/i)x_i$ in A. This sequence converges in E to the element $w = \sum_{i=1}^{\infty} (1/i)x_i$ which, although an element of E, is not in $A \angle$. Thus, $A \angle$ is a proper subset of $\overline{A \angle}$. However, for any x^* in A^*, it is clear that $x^*(w) \geqq 0$ so $w \in B$.

The negative answers to the above questions lead to the next question which is still open. What restriction, weaker than finiteness and stronger than being closed, can be placed on A in order to replace $\overline{A \angle}$ by $A \angle$ in Theorem 3.1?

The counter-example following Theorem 3.2 showed that in order to keep the complete generality of the set S^*, a reflexivity condition on E was needed. Another question which is still unanswered is: What are minimum restrictions on S^* which will allow the removal of the reflexivity assumption in Theorem 3.2? By requiring that S^* be a finite set, K. Fan [3] Theorem 4, p. 108, proved Theorem 3.2 for an arbitrary real linear space E as a special case of a Farkas-type theorem for a finite system of nonhomogeneous linear inequalities.

The final question is: Can the above Theorem of Fan be generalized to infinite systems of nonhomogeneous inequalities?

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MATHEMATICAL NOTES

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THE STURM-PICONE THEOREM; PSEUDO-OSCILLATORY SOLUTIONS*

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This paper is divided into two parts. The first is concerned with the Sturm-Picone theorem, and the second presents a phenomenon occurring in the behavior of solutions of second-order linear differential equations which is believed to be noteworthy.

I. It will be recalled that the Sturm-Picone theorem compares the oscillation of solutions of differential equations of the form

$$\{r(x)y'\}' + p(x)y = 0, \tag{1.1}$$

and

$$\{r_1(x)z'\}' + p_1(x)z = 0. \tag{1.2}$$

To be specific, suppose that $r(x)$, $p(x)$, $r_1(x)$, and $p_1(x)$ are continuous on an open interval (α, β) and that $r(x)$ and $r_1(x)$ are positive there. Picone's extension of the Sturm comparison theorem may then be stated as follows:

THEOREM 1.1. *If $z(x)$ is a solution of* (1.2) *having consecutive zeros a and $b(\alpha<a<b<\beta)$ and if $p(x)\geqq p_1(x)$ and $r(x)\leqq r_1(x)$ (where not both $p(x)\equiv p_1(x)$ and $r(x)\equiv r_1(x)$ hold throughout $[a, b]$), then a solution $y(x)\not\equiv 0$ of* (1.1) *with the property $y(a)=0$ has a zero on the open interval (a, b) provided there is no subinterval of (a, b) on which $p(x)\equiv p_1(x)\equiv 0$.*

* The problems treated in this paper were suggested by Professor Walter Leighton.

† Work on this paper was begun in June 1960 while the author held a National Science Foundation Undergraduate Research Assistantship at Western Reserve University.

The usual proof* is moderately sophisticated.

In the present paper we consider the most important case: that in which $p(x)$ and $p_1(x)$ are positive on (a, b), and give an elementary proof of the theorem for this case. To that end we note that when both $r(x)$ and $p(x)$ are positive on $[a, b]$, a solution $y(x)$ of (1.1) has the following property:

LEMMA 1.1. *If $p(x)$ is positive on the closed interval $[a, b]$, then on the open interval (a, b) a solution $y(x)$ of the equation*

$$\{r(x)y'\}' + p(x)y = 0, \tag{1.3}$$

cannot have a maximum for negative values of $y(x)$, nor a minimum for positive values of $y(x)$.

We shall give a proof for the first part of the lemma. The proof of the second part is then immediate. Let us assume the lemma is false and suppose that $y(x)$ has a maximum at some point $x=x_0$ on (a, b) at which $y(x_0)<0$. Then $y'(x_0)=0$.

Note that because $[r(x)y'(x)]'=-p(x)y(x)>0$ near $x=x_0$, $r(x)y'(x)$ is strictly increasing with x near $x=x_0$. Since $r(x)$ is positive and $r(x_0)y'(x_0)=0$, it follows that $r(x)y'(x)<0$, and, hence, that $y'(x)<0$ for x less than x_0 and sufficiently near x_0. Accordingly, $y(x)$ cannot attain a maximum at $x=x_0$. From this contradiction we infer the truth of the lemma.

We are now prepared to prove the following variation of the Sturm-Picone theorem:

THEOREM 1.2. *Suppose that in equations* (1.1) *and* (1.2) *the coefficients $r(x)$, $p(x)$, $r_1(x)$, and $p_1(x)$ are continuous and positive on the interval (α, β). If $z(x)$ is a solution of* (1.2) *which has consecutive zeros at $x=a$ and $x=b$ $(\alpha<a<b<\beta)$ and if $p(x)\geqq p_1(x)$, $r(x)\leqq r_1(x)$ (where not both identities $p(x)\equiv p_1(x)$ and $r(x)\equiv r_1(x)$ hold throughout the interval $a\leqq x\leqq b$), then a solution $y(x)\not\equiv 0$ of* (1.1) *such that $y(a)=0$ has a zero on the open interval (a, b).*

To prove the theorem we shall employ the following lemma. We assume, without loss in generality, that $z(x)>0 (a<x<b)$ and that $y'(a)>0$.

LEMMA 1.2. *Under the conditions given in Theorem* 1.2 *the solution $y(x)$ will have a maximum on (a, b) no later than $z(x)$.*

The proof of the lemma involves the use of the following equation, which is readily verified:

$$[ry'z - r_1yz']_{x_1}^{x_2} = \int_{x_1}^{x_2} (r - r_1)y'z'dx + \int_{x_1}^{x_2} (p_1 - p)yzdx. \tag{1.4}$$

In (1.4) x_1 and x_2 are arbitrary numbers such that $a\leqq x_1<x_2\leqq b$. Suppose, first,

* See for example [1, p. 53] and [2, p. 210]. The reader's attention should also be directed to a paper by Leighton to appear in the Proceedings of the Amer. Math. Soc., in which some general theorems of this type are proved.

that $z(x)$ attains its maximum on (a, b) at $x=c$ and compute (1.4) with $x_1=a$ and $x_2=c$. We have

$$r(c)y'(c)z(c) = \int_a^c (r - r_1)y'z'dx + \int_a^c (p_1 - p)yzdx. \tag{1.5}$$

If $y(x)$ does not have a maximum on $a<x\leqq c$, the terms $y(x)$, $z(x)$, and $y'(x)$ are positive there, while $z'(x)$ is positive on $a<x<c$ and zero at $x=c$. Thus the right hand side of equation (1.5) is negative while the left hand side is positive. From this contradiction we infer the truth of the lemma.

The proof of Theorem 1.2 may now be achieved. Let $z(x)$ attain its maximum at $x=c$. If the conclusion of the theorem does not hold, we recall Lemma 1.1 and note that $y(x)$ and $z(x)$ are positive, and $y'(x)$ and $z'(x)$ are negative on the interval $c<x<b$. Next, evaluate (1.4) with $x_1=c$ and $x_2=b$. We have

$$-r_1(b)y(b)z'(b) = r(c)y'(c)z(c) + \int_c^b (r - r_1)y'z'dx + \int_c^b (p_1 - p)yzdx. \tag{1.6}$$

The right-hand member of (1.6) has a value $\leqq 0$. It follows that $y(b)\leqq 0$. Finally, it is seen at once that $y(b)=0$ can hold only if both $r(x)\equiv r_1(x)$, $p_1(x)\equiv p(x)(c\leqq x\leqq b)$. Unless these identities also hold on $a\leqq x\leqq c$, we have $y'(c)<0$ and, consequently, the strictly inequality $y(b)<0$ holds unless equations (1.1) and (1.2) are identical.

II. In this section we consider differential equations,

$$y'' + p(x)y = 0, \qquad [p(x) \textit{ continuous on } a \leqq x \leqq b] \tag{2.1}$$

that may have nonvanishing solutions which possess an infinity of maximum points on the interval $[a, b]$. The following example shows that the class of such equations is not empty.

Example. It is readily verified that the differential equation (2.1) with

$$p(x) = -\frac{5x(\sin \log x + \cos \log x)}{2 + x^3 \sin \log x} \quad (0 < x \leqq 1), \qquad p(0) = 0, \tag{2.2}$$

has the solution

$$y_1(x) = 2 + x^2 \sin \log x \quad (0 < x \leqq 1), \qquad y_1(0) = 2, \tag{2.3}$$

and that this solution has an infinity of maximum points (and, of course, of minimum points) neighboring $x=0+$.

Inasmuch as the values of x for which such solutions assume maximum values constitute an infinite set of points on a finite interval, there must be at least one accumulation point of critical values of x. We are concerned with those cases in which there are at most a finite number of such accumulation points on the interval and confine our attention to the behavior of solutions neighboring such a point. We may suppose without loss in generality that $x=a$ is the accumulation point. A solution which does not vanish in an interval $a<x<c$ but which

possesses an infinity of such maximum points in this interval will be termed *pseudo-oscillatory* neighboring $x=a$.

A pseudo-oscillatory solution neighboring $x=a$ of an equation (2.1) *is never a principal solution at $x=a$,* [4, p. 256] *and* [3, p. 261].

To prove this statement it will be sufficient to show that $y(a)=\lim_{x\to a} y(x)$ is not zero. Recall that a solution of (2.1) is, by definition, of class C' on $[a, b]$. Accordingly, $y'(a)$ exists and is equal to $\lim_{x\to a} y'(x)$. But inasmuch as $y'(x)$ vanishes infinitely often neighboring $x=a$, it follows that $y'(a)=0$. If $y(a)=0$ also, by the fundamental existence theorem, $y(x)\equiv 0$, contrary to hypothesis.

We conclude with the proof of the following result.

THEOREM 2.1. *If an equation* (2.1) *possesses a pseudo-oscillatory solution $y_1(x)$ near $x=a$, those solutions of the equation which are pseudo-oscillatory near $x=a$, and, except for the null solution, only those, are constant multiples of $y_1(x)$.*

That is to say that no linearly independent solution of the equation can have an infinity of maximum points near $x=a$.

To prove the theorem it is sufficient to note that a second solution $y(x)$ depends linearly on $y_1(x)$, if and only if, $y(x)=cy_1(x)$, where c is a constant. All such solutions, except the null solution, are evidently pseudo-oscillatory. For such a solution and no other is $y'(a)=0$. Hence the proof is complete.

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THE DIOPHANTINE EQUATION $x^2+b^2=y^5$

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1. In the solution of problem E1442 [1], it is conjectured that for fixed integral b, the equation

$$x^2 + b^2 = y^5, \quad \text{where} \quad (x, y) = 1, \tag{1}$$

has no integral solutions x and y when $|b| < 38$, except for the trivial solution corresponding to $1^2+0^2=1^5$. This bound is suggested by $41^2+38^2=5^5$. We shall prove this conjecture and shall also show that the next larger b for which equation (1) is solvable is 122, corresponding to $597^2+122^2=13^5$.

2. Let G be the domain of Gaussian integers. It is well known that unique factorization holds in G. We can show that if x and y satisfy (1), then $x+bi$ and $x-bi$ are relatively prime in G. Consequently, every rational prime factor of y (and hence, y itself) is expressible as $c\bar{c}$ for some c in G. It follows from the unique

factorization theorem that $x+bi=\epsilon(t+ui)^5$ for t and u rational integers and ϵ a unit in G. It is then not difficult to show that all solutions of (1) are given by

$$x = t^5 - 10t^3u^2 + 5tu^4, \qquad y = t^2 + u^2, \tag{2}$$

where t and u are any relatively prime rational integers of different parity satisfying

$$b = u^5 - 10u^3t^2 + 5ut^4. \tag{3}$$

3. We assume throughout 3 that $|b| \leqq 122$. We seek all such values of b for which equation (1) is solvable. We may write

$$b = qu, \qquad q = u^4 - 10u^2t^2 + 5t^4,$$

whence $q \equiv 0$ or 1 (mod 5), q is odd, and

$$t^2 = u^2 \pm \sqrt{\{\tfrac{1}{5}(4u^4 + q)\}}. \tag{4}$$

A. Suppose first that $q=1$. Then $b=u$, and (4) implies that for some integral w,

$$4b^4 + 1 = 5w^2. \tag{5}$$

Taking equation (5) modulo 8 and 5, in turn, we learn that b is odd and that if p is a prime dividing b, then 5 is a quadratic residue modulo p and therefore $p \equiv \pm 1$ (mod 10). Hence, the only possibilities for $|b|$ are

$$1, 11, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 121. \tag{6}$$

Now we consider equation (5) modulo 7, 11, 17, and 19, in turn; we find that $b \equiv \pm 1, \pm 3$ (mod 7); $0, \pm 1, \pm 5$ (mod 11); $\pm 1, \pm 2, \pm 4, \pm 6, \pm 7, \pm 8$ (mod 17); and $0, \pm 1, \pm 6, \pm 8, \pm 9$ (mod 19). These restrictions rule out everything listed in (6), except $|b| = 1$ or 11. Now equation (4) shows that $|b| = 1$ gives only the trivial solution and that $|b| = 11$ fails to give an integral t.

B. Suppose secondly that $|q| > 1$. Write

$$q = (u^2 - 5t^2)^2 - 5(2t^2)^2. \tag{7}$$

Let $p \neq 5$ be a prime divisor of q. If also $p \mid (u^2 - 5t^2)$, then $p \mid t$ and hence $p \mid u$, so that $(u, t) > 1$. Thus $p \nmid (u^2 - 5t^2)$, whence (from (7)) the Legendre symbol $(5/p) = 1$, and therefore $p \equiv \pm 1$ (mod 10). Hence, letting r be a solution of $r^2 \equiv 5$ (mod p), we see from (7) that $u^2 - 5t^2 \equiv \pm 2rt^2$ (mod p). Since $p \nmid tu$, we obtain

$$((5 \pm 2r)/p) = 1. \tag{8}$$

We list the primes $p < 122$ of the form $10k \pm 1$, for which (8) holds:

$$19, 41, 59, 61, 101. \tag{9}$$

The primes displayed in (9) and the prime 5, then, are the only possible prime divisors of q. We consider these primes in turn, noting from (7) that if $5 \mid q$, then also $5 \mid u$ and so $25 \mid b$.

(i) If $p=19$, q must be -19 because of the restriction on the size of b and because of the restriction on q modulo 5. Then $|u| \leqq 122/19$, and we consider $|u| = 1, 2, 3, 4, 5, 6$ successively. Only $|u| = 2$ gives an integral t in (4); this leads to the known equality $41^2+38^2=5^5$.

(ii) If $p=41$, then, as in (i), q is restricted to be 41 and $|u|$ restricted to be 1 or 2. Now $|u| = 1$ yields the solution in (i), but $|u| = 2$ fails to give an integral t in (4).

(iii) Similar considerations rule out the remaining primes listed in (9) except for $p=61$, when $|u| = 2$ gives an integral t in (4). We then obtain $597^2+122^2=13^5$.

(iv) For $p=5$, we have only to consider $|b| = 25, 50, 75, 100$. For each of these values of b, $|q|$ must be 5, with $|u|$ restricted accordingly. None of these yields an integral t in (4).

This proves the result stated in Section 1.

4. Remarks. It is clear from equation (3) that there exist arbitrarily large values of b for which (1) has solutions. Furthermore, for each given b, the above considerations indicate a finite procedure for determining whether equation (1) is solvable. The congruence conditions used greatly reduce the amount of computation required.

One may similarly consider $x^2+db^2=y^n$, for fixed d and fixed n. See, for example, Uspensky and Heaslet [2], or a thesis [3] of the author, wherein d is any integer such that the ring of algebraic integers in the field obtained by adjoining $\sqrt{(-d)}$ to the rational field is a unique factorization domain.

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n AND $n+1$ CONSECUTIVE INTEGERS WITH EQUAL SUMS OF SQUARES

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The equation $5^2=3^2+4^2$ can be considered as

(i) the simplest solution in positive integers of $z^2=x^2+y^2$,

(ii) the only solution in positive integers of $(y+2)^2=y^2+(y+1)^2$,

(iii) the simplest solution in positive integers of $x^2=y^2+(y+1)^2$,

(iv) the simplest solution in positive integers for the case $n=1$ of

$$x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n-1)^2 = y^2 + (y+1)^2 + (y+2)^2 + \cdots + (y+n)^2, \tag{1}$$

i.e. the case where the sum of n consecutive squares equals the sum of $n+1$ consecutive squares.

The complete set of solutions of $z^2=x^2+y^2$ involved in (i) is, of course, well known. In a recent paper, Brother Alfred [1] gave a set of solutions for the equations involved in (iii) and (iv) and raised the question whether the solutions he found were the only ones.

It is the purpose of this paper to show that the solutions found by Brother Alfred are indeed the only ones and to show two methods by which they can be obtained.

Since the equation involved in (iii) is a special case of equation (1), it is sufficient to restrict our attention from here on to equation (1), which, by use of the formula for the sum of the first n squares, can be rewritten as

$$nx^2 + n(n-1)x + \tfrac{1}{6}(n-1)n(2n-1) = (n+1)y^2 + (n+1)ny + \tfrac{1}{6}n(n+1)(2n+1) \tag{2}$$

or

$$nx^2 + n(n-1)x - n^2 = (n+1)y^2 + n(n+1)y. \tag{3}$$

Since the left-hand side is divisible by n, so must be the right-hand side so that we can let in the notation of Brother Alfred $y=ny'$, which, when substituted into (3), leads to

$$(x+n)(x-1) = n(n+1)y'(y'+1). \tag{4}$$

Letting $z=x-1$, (4) becomes

$$z(z+n+1) = n(n+1)y'(y'+1), \tag{5}$$

which shows that z must be divisible by $n+1$, i.e., $z=(n+1)z'$, so that (5) then takes the form

$$(n+1)z'(z'+1) = ny'(y'+1). \tag{6}$$

Finally letting $y'=z'+\alpha$, we obtain

$$z'^2 + (1-2n\alpha)z' - n\alpha(\alpha+1) = 0. \tag{7}$$

If z_1' and $-z_2'$ are the roots of this equation, then

$$z_1' - z_2' = 2n\alpha - 1 \quad \text{and} \quad z_1'z_2' = n\alpha(\alpha+1). \tag{8}$$

From (8) follows that

$$(z_1' - z_2')^2 + 4z_1'z_2' = (2n\alpha-1)^2 + 4n\alpha(\alpha+1) \tag{9}$$

or

$$(z_1' + z_2')^2 = 4n(n+1)\alpha^2 + 1. \tag{10}$$

Clearly (10) and consequently (1) will have a solution for each α for which the right-hand side is a perfect square, i.e., if we let $z_1'+z_2'=u$, we need to find all α such that u is an integer or

$$u^2 - 4n(n+1)\alpha^2 = 1, \tag{11}$$

which is Pell's equation, usually written

$$x^2 - Dy^2 = 1,$$

with $x=u$, $y=\alpha$, $D=4n(n+1)$.

Pell's equation is most conveniently solved by use of the infinite continued fraction expansion of $\sqrt{D}$, which in this case is given by

$$\sqrt{D} = \sqrt{\{4n(n+1)\}} = (2n, \overline{1, 4n}). \tag{12}$$

Since the length of the period is even (2 in this case), we know from the theory of continued fractions that all solutions of Pell's equation (11) are given by the numerators and denominators, respectively, of the convergents with even subscripts; i.e., if we denote the i'th convergent by

$$C_i = \frac{p_i}{q_i}, \tag{13}$$

then all solutions of equation (11) are given by

$$u = p_{2k}, \qquad \alpha = q_{2k}, \quad \text{where} \quad k = 1, 2, 3, \cdots.$$

The first solution, therefore, would be obtained for $k=1$ and can easily be found to be

$$u = p_2 = 2n + 1, \qquad \alpha = q_2 = 1.$$

Since the convergents in a continued fraction expansion are related recursively by

$$p_i = a_i p_{i-1} + p_{i-2},$$
$$q_i = a_i q_{i-1} + q_{i-2},$$

and since from (12) the partial quotients a_i with even subscript are all 1, and with odd subscript (>1) are all $4n$, we have for even i (or $i=2k$)

$$p_{2k} = p_{2k-1} + p_{2k-2},$$
$$p_{2k-1} = 4np_{2k-2} + p_{2k-3},$$
$$p_{2k-2} = p_{2k-3} + p_{2k-4},$$

and consequently

$$p_{2k} = (4np_{2k-2} + p_{2k-3}) + p_{2k-2} = (4n + 1)p_{2k-2} + p_{2k-3}$$
$$= (4n + 1)p_{2k-2} + (p_{2k-2} - p_{2k-4}) = (4n + 2)p_{2k-2} - p_{2k-4}.$$

Completely analogously we find

$$q_{2k} = (4n + 2)q_{2k-2} - q_{2k-4}.$$

This shows that the complete set of solutions of equation (11) is given by the following recursion formulae:

$$(14)\qquad \begin{aligned} u &= p_{2k} = (4n+2)p_{2k-2} - p_{2k-4}, \\ \alpha &= q_{2k} = (4n+2)q_{2k-2} - q_{2k-4}, \end{aligned}$$

with $p_2 = 2n+1$, $q_2 = 1$.

An explicit formula for the same solutions (see, for example, [2]) is given by

$$(15)\qquad \begin{aligned} u &= \tfrac{1}{2}[\{2n+1+\sqrt{(4n(n+1))}\}^m + \{2n+1-\sqrt{(4n(n+1))}\}^m], \\ \alpha &= \frac{1}{2\sqrt{(4n(n+1))}}[\{2n+1+\sqrt{(4n(n+1))}\}^m - \{2n+1-\sqrt{(4n(n+1))}\}^m], \end{aligned}$$

for $m = 1, 2, 3, \cdots$, from which we obtain the solutions to the original equation (1) as

$$(16)\qquad \begin{aligned} x &= \left(\frac{n+1}{2}\right)u + n(n+1)\alpha - \frac{n-1}{2}, \\ y &= n\left(\frac{u+2n\alpha-1}{2} + \alpha\right). \end{aligned}$$

The solutions for x and y can also be given recursively. The necessary formulae were derived by Brother Alfred [1], who in his paper listed also the first 5 solutions for each value of n up to 20.

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DEFINITIVE SOLUTIONS OF GENERAL QUARTIC AND CUBIC EQUATIONS

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Definitive solutions of the general quartic and cubic equations with real coefficients

$$(1)\qquad f_4(x) = x^4 + 4ax^3 + 6bx^2 + 4cx + d \equiv D_4(x)Q_4(x) + R_4(w) = 0,$$

$$(2)\qquad f_3(x) = x^3 + 3B_2x^2 + 3B_1x^1 + 2B_0 \equiv D_3(x)Q_3(x) + R_3(w) = 0,$$

result from the following division exercises.

Write the quotient $Q_i(x)$ and the remainder $R_i(w)$ when the divisor $D_i(x)$ of $f_i(x)$ is so selected that the reduced resolvent cubic equation in the form

$$(3)\qquad w^3 - pw - q = 0 \qquad (p, q \text{ real})$$

is obtained from $R_i(w) = 0$. For $i = 4$,

$$(4)\qquad D_4(x) = x^2 + 2[a - (w - b + a^2)^{1/2}]x + 2w + b + \frac{a(2w+b) - c}{-(w-b+a^2)^{1/2}},$$

$$(5)\quad Q_4(x) = x^2 + 2[a + (w - b + a^2)^{1/2}]x + 2w + b + \frac{a(2w + b) - c}{(w - b + a^2)^{1/2}},$$

$$(6)\quad R_4(w) = d - (2w + b)^2 + \frac{[a(2w + b) - c]^2}{w - b + a^2}.$$

Let the real number $2U$ be the largest positive root of $R_4(w) = 0$ when all its roots are real, the only real root when two of its roots are complex conjugates. Under these conditions, take $D_4(x) = 0$, and $Q_4(x) = 0$ and derive a quartic formula

$$(7)\quad x = \pm (2U - b + a^2)^{1/2} - a \pm \left[-2(U + b - a^2) - \frac{c - 3ab + 2a^3}{\pm(2U - b + a^2)^{1/2}}\right]^{1/2},$$

an expression free from all ambiguity, since it gives four and only four values of x, when the radicals properly receive their double signs.

When $2U = b - a^2$, the quartic formula (7) goes over into

$$(8)\quad x = -a \pm [-3(b - a^2) \pm \{(4U + b)^2 - d\}^{1/2}]^{1/2}.$$

Formulas (7) and (8) evaluate each root of the quartic equation as a function of real numbers. These quartic formulas correspond to a result given by Burnside and Panton [1] and represent a simplification of the formulas given in a note by Salzer [2] for his solution of the quartic equation. Thus, Problem 27 on page 144 of the 1881 edition of Burnside and Panton is The Definitive Solution of the General Quartic Equation for the applied mathematician and as such should be programmed for use on digital computers.

In order to set up a similar analysis for the general cubic equation ($i = 3$), repeat the division exercise.

$$(9)\quad D_3(x) = x - w + B_2,$$

$$(10)\quad Q_3(x) = x^2 + (w + 2B_2)x + w^2 + B_2 w + 3B_1 - 2B_2^2,$$

$$(11)\quad R_3(w) = w^3 - 3(B_2^2 - B_1)w - (3B_1 - 2B_2^2)B_2 + 2B_0.$$

Let $2U$ be numerically the largest real root of $R_3(w) = 0$. Under this condition, take $D_3(x) = 0$ and $Q_3(x) = 0$ and derive the following formulas for the roots of the general cubic equation given in (2):

$$(12)\quad x_{1,2} = -U - B_2 \pm \{-3(B_1 - B_2^2 + U^2)\}^{1/2},$$

$$(13)\quad x_3 = +2U - B_2.$$

Formulas (12) and (13) evaluate each root of the cubic equation as a function of real numbers.

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SKEW FIELDS OF PRIME CHARACTERISTIC

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1. When students are told that every finite division ring is a field, they often ask for examples of an infinite skew field of characteristic p. This note shows how to avoid general theorems on algebras and go directly to a matrix representation similar to that for the quaternions.

In the case of the quaternions, we use

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

and it soon is clear that we only need that $x \to \bar{x}$ is a nontrivial involution of C, the complex numbers, and that $a\bar{a}+b\bar{b}=0$ implies $a=b=0$, in order to show that these matrices form a noncommutative division algebra over the reals.

2. THEOREM. *If F is a field in which there exists a nonsquare, γ, if t is transcendental over F, and if σ denotes the involution $t \to 1/t$ of $F(t)$, then the matrices*

$$\begin{pmatrix} a & b \\ \gamma b^{\sigma} & a^{\sigma} \end{pmatrix},$$

where a and b are in $F(t)$, form a noncommutative division algebra over F (not over $F(t)$) with the usual operations on matrices.

Proof. It is easy to check that the system is an algebra over F. As with the quaternions, we have noncommutativity:

$$\begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & \dfrac{1}{t} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{t} \\ \gamma t & 0 \end{pmatrix} \neq \begin{pmatrix} t & 0 \\ 0 & \dfrac{1}{t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}.$$

It remains to show that $aa^{\sigma}-\gamma bb^{\sigma}=0$ implies $a=b=0$. Assume this is not so. Then, with $a(t)a(1/t)=\gamma b(t)b(1/t)$, we can change a, b into polynomials by multiplying both sides by an appropriate $d(t)d(1/t)$. Further cancellation allows us to assume that $a(t)$, $b(t)$ have no root in common in F. But then, as t is transcendental, we can substitute $t=1$ and find

$$a(1)^2 = \gamma b(1)^2.$$

Since γ is not a square in F, we must have $b(1)=0$, hence $a(1)=0$, so that $t-1$ is a common factor of $a(t)$ and $b(t)$, the desired contradiction.

3. We observe that if $F=GF(p^n)$, $p\neq 2$, the mapping $x\to x^2$ of F^* has kernel $\{+1, -1\}$, so that $(F^*)^2$ is a subgroup of index 2. Thus many γ are available in this case. If we want a case where char $F=2$, the easy way to get a nonsquare is to take $F=K(\gamma)$, where γ is transcendental over K, char $K=2$.

Finally, we remark on the proof that at first we took $\gamma=-1$, just as with the quaternions. This meant that we could only succeed when -1 was a nonsquare, so we had to go beyond this simple analogy with the quaternion case.

AN INVERSION INTEGRAL FOR A LEGENDRE TRANSFORMATION

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Recently Ta Li [2] has obtained an integral inversion formula for an integral transformation where the kernel involved a Chebyshev polynomial. An integral inversion formula also exists for a similar type of an integral transformation where a Legendre polynomial occurs as a kernel. We consider the integral equation

$$\int_x^1 P_n(t/x)g(t)dt = f(x)$$

and show that, under suitable hypotheses,

$$g(t) = \int_t^1 P_{n-2}(t/y)y^{2-n}(y^{-1}d/dy)^2[y^n f(y)]dy.$$

LEMMA. *If P_n is a Legendre polynomial and $0<x\leqq y$, $n\geqq 2$, then*

$$J_n(x, y) = \int_x^y P_n(t/x)P_{n-2}(t/y)dt = \tfrac{1}{2}y^{n-1}x^{-n}(y^2 - x^2). \tag{1}$$

Proof. From [1: 3.12(23)] with integers $k\geqq 0$, $n\geqq 0$, we have

$$\begin{aligned}\int_0^u t^k P_{n-2}(t/u)dt &= u^{k+1}\int_0^1 x^k P_{n-2}(x)dx \\ &= u^{k+1}\pi^{\frac{1}{2}}2^{-k-1}k!\left[\Gamma\left(\tfrac{1}{2}k - \tfrac{1}{2}n + 2\right)\Gamma\left(\tfrac{1}{2}k + \tfrac{1}{2}n + \tfrac{1}{2}\right)\right]^{-1}.\end{aligned} \tag{2}$$

It is noted that if k and n are both even or both odd this integral is zero for $0\leqq k<n-2$. Thus $J_n(x, y)$ can be simplified by writing

$$J_n(x, y) = \int_0^y P_n(t/x)P_{n-2}(t/y)dt - \int_0^x P_{n-2}(t/y)P_n(t/x)dt,$$

where the second integral consequently equals zero. Further, if P_n is expanded in terms of powers of (t/x) in the first integrand, all but two terms drop out leaving

$$J_n(x, y) = [(2n)!2^{-n}/(n!)^2] \int_0^y \{(t/x)^n - [n(n-1)/(4n-2)](t/x)^{n-2}\} P_{n-2}(t/y)dt.$$

Now (2) is used to evaluate these terms so that we have (1).

THEOREM. *If f and its first two derivatives are sectionally continuous for $0<x_0\leqq x\leqq 1$, $f(1-0)=f'(1-0)=0$, and P_n is a Legendre polynomial with $n\geqq 2$; then the solution to*

$$\int_x^1 P_n(t/x)g(t)dt = f(x), \tag{3}$$

for $0<t_0\leqq t<1$, is given by

$$g(t) = \int_t^1 P_{n-2}(t/y)y^{2-n}(y^{-1}d/dy)^2[y^n f(y)]dy. \tag{4}$$

Proof. Consider the integral $I(x)$ which is obtained by substituting (4) into the integrand of (3):

$$I(x) = \int_x^1 P_n(t/x)\left\{\int_t^1 P_{n-2}(t/y)y^{2-n}(y^{-1}d/dy)^2[y^n f(y)]dy\right\} dt.$$

Using Dirichlet's formula and the Lemma, this can be written

$$\begin{aligned} I(x) &= \int_x^1 J_n(x, y)y^{2-n}(y^{-1}d/dy)^2[y^n f(y)]dy \\ &= \tfrac{1}{2}x^{-n}\int_x^1 (y^2 - x^2)d\{(y^{-1}d/dy)[y^n f(y)]\}. \end{aligned}$$

Integration by parts and application of the conditions at $(1-0)$ gives

$$I(x) = -x^{-n}\int_x^1 d[y^n f(y)] = f(x).$$

Finally we note that the cases $n=0$, 1 which are not included in the theorem can be solved by differentiation with respect to x. This yields the pairs

$$\int_x^1 P_0(t/x)g(t)dt = f(x), \qquad g(t) = \int_t^1 (d/dy)^2[f(y)]dy;$$

$$\int_x^1 P_1(t/x)g(t)dt = f(x), \qquad tg(t) = \int_t^1 (d/dy)^2[yf(y)]dy.$$

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CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

A NOTE ON SUMS OF POWERS OF INTEGERS

L. CARLITZ, Duke University*

Allison [1] has proved that the identity (in n)

$$\left(\sum_{r=1}^{n} r^k\right)^p = \left(\sum_{r=1}^{n} r^m\right)^q \qquad (p > q)$$

holds only in the familiar case

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2.$$

Put

$$S_k(n) = \sum_{r=1}^{n} r^k.$$

It is known [2, p. 112] that $S_{2k-1}(n)$ is a polynomial in $S_1(n)$. Since

$$S_k(n) = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1},$$

where $B_{k+1}(x)$ is the Bernoulli polynomial of degree $k+1$, this is equivalent to the statement that $B_{2k}(x)$ is a polynomial in $B_2(x)$. We shall show that for $m \geqq 3$ and $k > 1$, $B_{km}(x)$ is not expressible as a polynomial in $B_m(x)$.

Assume that

$$B_{km}(x) = f(B_m(x)), \tag{1}$$

where $f(u)$ is a polynomial in u. Since $B_k(x+1) - B_k(x) = kx^{k-1}$, (1) yields

$$\begin{aligned} kmx^{km-1} &= f(B_m(x) + mx^{m-1}) - f(B_m(x)) \\ &= mx^{m-1}f'(B_m(x)) + \tfrac{1}{2}(mx^{m-1})^2 f''(B_m(x)) + \cdots. \end{aligned}$$

It follows that

$$x^{m-1} \mid f'(B_m(x)). \tag{2}$$

* Supported in part by National Science Foundation grant G 9425.

But by (1) $kB_{km-1}(x)=f'(B_m(x))B_{m-1}(x)$, so that (2) gives

$$x^{m-1} \mid B_{km-1}(x). \tag{3}$$

Since $B_{2n}(0)\neq 0$, while $B_{2n-1}(x)$ has a simple zero at $x=0$ it follows that (3) is impossible for $m\geqq 3$. We may state the following

THEOREM. *If $k>m>2$ then $B_k(x)$ cannot be exhibited as a polynomial in $B_m(x)$. Consequently if $k>m\geqq 2$ then $S_k(n)$ cannot be exhibited as a polynomial in $S_m(n)$.*

For the properties of Bernoulli polynomials assumed above see for example [3, Ch. 2].

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A NOTE ON THE METHOD OF VARIATION OF PARAMETERS

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Let $y_1(x), \cdots, y_n(x)$ be n solutions of the nth order linear homogeneous differential equation

$$y^{(n)}(x)+\sum_{r=1}^{n} a_r(x)y^{(n-r)}(x)=0 \qquad \left[y^{(r)}(x)=\frac{d^r y}{dx^r}\right] \tag{1}$$

which exist and are linearly independent on some open interval I of the real variable x containing the point $x=0$. If

$$W(x)=\det \| y_i^{(r)}(x) \| \quad (i=1,\cdots,n;\ r=0,\cdots,n-1)$$

is the Wronskian determinant of these solutions, then $W(x)\neq 0$ on I and it follows that, to any function which is n times differentiable on I and, in particular, to any solution $y(x)$ of the inhomogeneous linear equation

$$y^{(n)}(x)+\sum_{r=1}^{n} a_r(x)y^{(n-r)}(x)=f(x), \tag{2}$$

there corresponds a unique set of functions $\lambda_1(x), \cdots, \lambda_n(x)$ such that, for $x\in I$,

$$y^{(r)}(x)=\sum_{i=1}^{n} \lambda_i(x)y_i^{(r)}(x), \qquad (r=0,\cdots,n-1). \tag{3}$$

Since, on solving the linear algebraic equations (3), $\lambda_i(x)$ is expressed as a quotient of polynomials of differentiable functions, $\lambda_1(x), \cdots, \lambda_n(x)$ are themselves differentiable. Hence, differentiating each side of equation (3) with respect to x, we find

$$y^{(r+1)}(x)=\sum_{i=1}^{n} \lambda_i'(x)y_i^{(r)}(x)+\sum_{i=1}^{n} \lambda_i(x)y_i^{(r+1)}(x), \tag{4}$$

which, by virtue of equation (3), gives

$$\sum_{i=1}^{n} \lambda_i'(x) y_i^{(r)}(x) = 0, \qquad (r = 0, \cdots, n-2). \tag{5}$$

Putting $r=n-1$ in equation (4) and using the results (*vide* equations (1), (2))

$$y_i^{(n)}(x) = -\sum_{r=0}^{n-1} a_{n-r}(x) y_i^{(r)}(x),$$

$$y^{(n)}(x) = f(x) - \sum_{r=0}^{n-1} a_{n-r}(x) y^{(r)}(x),$$

together with equation (3), we obtain the further relation

$$\sum_{i=1}^{n} \lambda_i'(x) y_i^{(n-1)}(x) = f(x). \tag{6}$$

Equations (5), (6) can be rewritten in the form

$$\sum_{i=1}^{n} \lambda_i'(x) y_i^{(r)}(x) = f(x)\delta_{r,n-1},$$

where $\delta_{r,n-1}$ is the Kronecker delta. This is a set of n linear algebraic equations for the derivatives $\lambda_1'(x), \cdots, \lambda_n'(x)$. The solution is

$$\lambda_i'(x) = \frac{f(x)\phi_i(x)}{W(x)}, \tag{7}$$

where $\phi_i(x)$ is the cofactor of $y_i^{(n-1)}(x)$ in the determinant $W(x)$, i.e.

$$W(x) = \sum_{j=1}^{n} y_j^{(n-1)}(x)\phi_j(x).$$

Integrating equations (7) between the limits 0 and x, where $x \in I$, and making use of equation (3) with $r=0$, we obtain a particular integral of equation (2) in the form:

$$y(x) = \int_0^x u(x, \xi) f(\xi) d\xi, \tag{8}$$

where

$$u(x, \xi) = \sum_{i=1}^{n} \frac{y_i(x)\phi_i(\xi)}{W(\xi)},$$

provided that $\lambda_1(0) = \lambda_2(0) = \cdots = \lambda_n(0) = 0$. It follows from equation (3) that this proviso is satisfied if and only if

$$y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

and the solution (8) accordingly fulfills these conditions at $x=0$.

I am indebted to Dr. H. K. Farahat for most helpfully commenting on this note.

A GEOMETRIC CONSTRUCTION OF COMPOSITE FUNCTIONS

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Let f and g be real functions of a real variable and suppose that the domain of g contains the range of f. Then the composite function $F(x)=g(f(x))$ is defined for all x in the domain of f. The purpose of this note is to give a simple geometrical description of the graph of the function $F(x)$ which adds to the student's insight into the operation of composition.

In a 3-dimensional coordinate system we begin by graphing $y=f(x)$ and $z=g(y)$ in the planes $z=0$ and $x=0$, respectively. The problem is to describe the graph of $z=F(x)$ in the plane $y=0$.

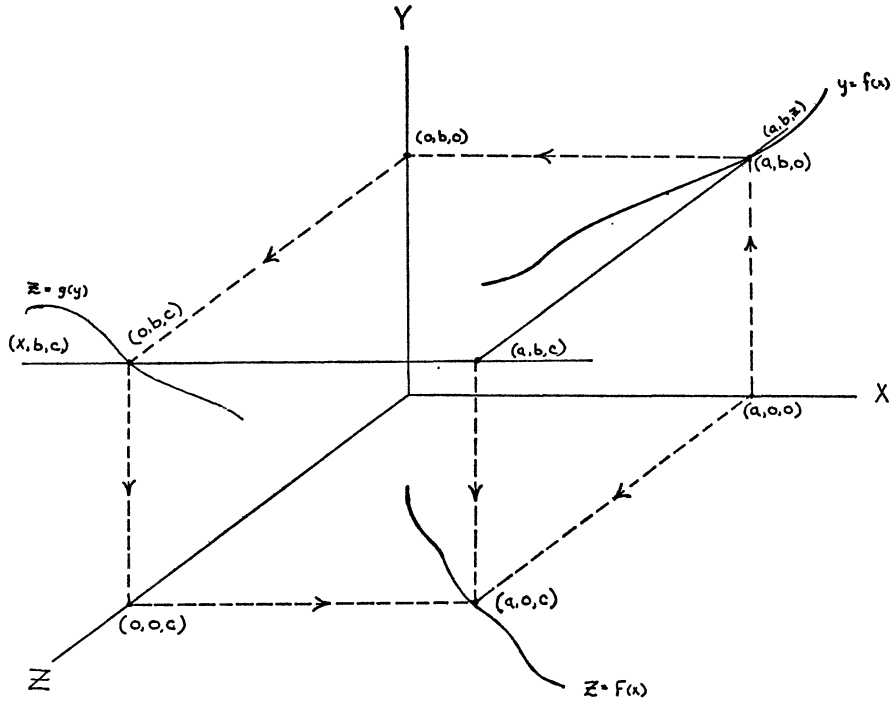


FIG. 1

Beginning at $x=a$ we follow the mapping $b=f(a)$, $c=g(b)$, and arrive at the conclusion that $(a, 0, c)$ belongs to the graph of $z=F(x)$.

The point $(a, 0, c)$ could also have been reached as follows: Construct the lines (a, b, z) and (x, b, c). These intersect at (a, b, c), and the projection of (a, b, c) onto the plane $y=0$ yields $(a, 0, c)$.

Now to find *all* the points in the graph $z=F(x)$ we consider the cylinders

$$S_1 = (x, f(x), z),$$
$$S_2 = (x, y, g(y)).$$

The intersection of these cylinders is a space curve C, and the projection of C onto the plane $y=0$ clearly yields the graph of $z=F(x)$.

This construction of the composite function in a 3-dimensional space also facilitates a graphical explanation of certain theorems on composite functions (e.g. the composition of continuous functions is continuous).

LINEAR FRACTIONAL TSCHIRNHAUS TRANSFORMATIONS IN ALGEBRAIC FIELDS

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If ρ is a root of an irreducible equation $p(x)=0$, of degree n over a field F, and η is a primitive number of $F(\rho)$, the classical method [1] of proving that ρ is a rational function of η leads to the result

$$\rho = \frac{\phi(\eta)}{f'(\eta)} \tag{1}$$

where $f(x)=0$ is the principal equation of η with coefficients in F and $\phi(x)$ is a polynomial whose coefficients are shown to be symmetric functions of ρ and its conjugates, and thus are numbers of F. It is the writer's conviction that every student in a course in algebraic numbers should be subjected, at least once, to the discipline of carrying through the above method for either a cubic or a quartic equation after which there should be available an easily applied check on the accuracy of the result. As will be shown, such a check is available for the cubic equation and, under certain conditions, for equations of higher degree.

THEOREM 1. *If* $p(x)=x^3+p_1x+p_2$ *is irreducible over* F, $p(\rho)=0$, *and* $\eta=a_0+a_1\rho+a_2\rho^2$ *is a primitive number of* $F(\rho)$, *then* ρ *is expressible in the form* $(a\eta+b)/(c\eta+d)$, *where* a, b, c, *and* d *are numbers of* F.

Proof. Since the theorem is obviously true when $a_2=0$, it will be assumed that $a_2\neq 0$. We may write

$$\begin{aligned}(x+\rho)\eta &= (x+\rho)(a_0+a_1\rho+a_2\rho^2)\\ &= a_0x - a_2p_2 + (a_1x+a_0-a_2p_1)\rho + (a_2x+a_1)\rho^2,\end{aligned}$$

the last expression being obtained by replacing ρ^3 by $-p_1\rho-p_2$. Since $a_2\neq 0$ we may put $x=-a_1/a_2$ and obtain an equation which may be solved for ρ as the quotient of two linear functions of η.

Normally, for $n=3$, the denominator of (1) is an irreducible quadratic function and the equality of the two fractional representations of ρ can be checked by cross-multiplication and simplification to $f(\eta)=0$. If, for example, $p(x)$

$=x^3-x+2$ and $\eta=1+\rho+\rho^2$, then $f(x)=x^3-5x^2+13x-13$, and the two expressions for ρ are respectively

$$\frac{-4\eta}{3\eta^2-10\eta+13} \quad \text{and} \quad \frac{\eta-3}{\eta-1}.$$

But if $p(x)=x^3-3x-1$ and $\eta=1+\rho^2$, $f(x)=x^3-9x^2+24x-17$ and the two expressions for ρ are

$$\frac{3\eta-6}{3\eta^2-18\eta+24} \quad \text{and} \quad \frac{1}{\eta-4},$$

the former reducing to the latter.

THEOREM 2. *If* $p(x)=x^n-p_1x^{n-2}-p_2x^{n-3}-\cdots-p_{n-2}x-p_{n-1}$ *is irreducible in* F, $p(\rho)=0$, $\eta=\sum_{i=0}^{n-1}a_i\rho^i$ *is a primitive number of* $f(\rho)$, *then* ρ *is expressible in the form* $(a\eta+b)/(c\eta+d)$ *if and only if*

$$p_i a_{n-2}^2 + a_{n-1}a_{n-i-2} - a_{n-2}a_{n-i-1} = 0, \qquad (i = 1, \cdots, n-3). \tag{2}$$

Proof. We write

$$(x+y\rho)\eta = (x+y\rho)\sum_{i=0}^{n-1} a_i\rho^i$$

and on the right replace ρ^n by $\sum_{i=1}^{n-1} p_i\rho^{n-i-1}$. The coefficients of $\rho^{n-1}, \cdots, \rho^2$ are respectively $a_{n-1}x+a_{n-2}y$, $a_{n-2}x+(a_{n-3}+a_{n-1}p_1)y$, $\cdots$, $a_2x+(a_1+a_{n-1}p_{n-3})y$. A necessary and sufficient condition that x and y exist such that these coefficients vanish and hence that ρ be expressible in the linear fractional form is that the matrix

$$\begin{bmatrix} a_{n-1} & a_{n-2} \\ a_{n-2} & a_{n-3}+a_{n-1}p_1 \\ a_{n-3} & a_{n-4}+a_{n-1}p_2 \\ \vdots & \vdots \\ a_{n-i-1} & a_{n-i-2}+a_{n-1}p_i \\ \vdots & \vdots \\ a_2 & a_1+a_{n-1}p_{n-3} \end{bmatrix}$$

be of rank less than 2 or, in other words, that

$$p_i a_{n-1}^2 + a_{n-1}a_{n-i-2} - a_{n-2}a_{n-i-1} = 0, \qquad (i = 1, \cdots, n-3).$$

For $n>4$ the practical value of the check becomes more and more questionable with larger values of n but, after all, who wants to work the problem for $n>4$? When $n=4$, the equations given by (2) reduce to the single equation

$$p_1 a_3^2 + a_3a_1 - a_2^2 = 0 \tag{3}$$

and checking becomes a relatively simple matter.

While ρ can easily be expressed as a polynomial in η [2] which can be equated to the fractional form in (1) thus providing a check for any value of n, this method requires the reduction of a polynomial of degree $2n-2$ in η to one of degree n. The method of this paper, when it can be used, involves no polynomial of degree greater than n.

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SEMIGROUP EXAMPLES IN INTRODUCTORY MODERN ALGEBRA

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Among the more important generalizations of the group concept is the theory of semigroups, that is, of sets which are closed under a single-valued binary operation and obey the associative law.

At the present time, one of growing prominence in the theory of semigroups, certain features of this theory are finding their way into the treatment of elementary modern algebra. A pioneer in this direction is Nathan Jacobson who made extensive use of semigroups in a particularly satisfying introduction to the theory of groups and rings [1]. More recently A. O. Lindstrum, Jr. has suggested a considerable use of semigroups in the teaching of modern algebra [2]. Noteworthy also is the practice of defining algebraic varieties in terms of semigroups. Several examples of such definitions have appeared in this MONTHLY, e.g. group, pseudo-ring and semi-lattice, vol. 62, 1955, pp. 698–702; ring, c-ring, w-ring, vol. 66, 1959, pp. 280–282; and near-ring (near-field), vol. 66, 1959, p. 23. Finally, there is the remarkable ease with which one can discover semigroups to illustrate many algebraic laws and such elements as units, inverses, zeros, idempotents, nilpotents and ideals.

In view of these considerations it might be profitable to preface the treatment of groups in introductory modern algebra with the definition of a semigroup and the use of examples of semigroups to illustrate the laws and elements considered in group theory.

The objective of this note is to make more readily available some representative examples of elementary semigroups. Many of the examples given will suggest to the reader other similar examples. We also give some of the uses of these examples as illustrations in elementary modern algebra. The examples of continuous semigroups on the unit interval, the unit circle, and the unit disk, are likewise important examples in the study of topological semigroups.

The terminology followed is that of Dr. A. D. Wallace [5]; to him and to the Tulane University Semigroup Seminars the author gratefully acknowledges his indebtedness.

A semigroup $\{S; \circ\}$ is a set of elements S which, under the single-valued

binary operation $\circ$, satisfies the closure law: $a \circ b \in S$ for all $a, b \in S$, and the associative law: $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in S$.

The semigroup examples enumerated are indexed consecutively $\{S_1; \circ\}$, $\{S_2; \circ\}, \cdots$. In referring to these semigroup examples as illustrations we shall write, e.g., "S_1" instead of the more precise "the semigroup $\{S_1; \circ\}$."

I. *Example.* Let $S_1 = J^+$, the set of positive integers, under the operation $\circ$ defined by $x \circ y = x + y$ (real addition), $x, y \in S_1$. $\{S_1; \circ\}$ is a semigroup.

II. *Example.* Let $S_2 = [-1, 1] = \{x \mid -1 \leqq x \leqq 1, x \text{ real}\}$ with $x \circ y = p$, where p is a fixed element of S_2. $\{S_2; \circ\}$ is a semigroup.

III. *Example.* Let $S_3 = [0, 1]$, the closed unit interval, with $x \circ y = \min\{x, y\}$. $\{S_3; \circ\}$ is a semigroup.

IV. *Example.* Let $S_4 = [0, 1]$ with $x \circ y = xy$ (real multiplication). $\{S_4; \circ\}$ is a semigroup.

In the foregoing examples it is trivial to show that the closure and associative laws hold.

V. *Example.* Let $S_5 = J^+$ with $x \circ y = x + y + xy$. $\{S_5; \circ\}$ is a semigroup. For this and the following examples closure and associativity can be readily verified.

VI. *Example.* Let $S_6 = [-1, 1]$ with

$$x \circ y = \left\{ \begin{array}{ll} xy & \text{if } x, y \in [\ 0, 1] \\ -xy & \text{if } x, y \in [-1, 0] \\ 0 & \text{otherwise} \end{array} \right\} = \tfrac{1}{2}(x \mid y \mid + y \mid x \mid).$$

$\{S_6; \circ\}$ is a semigroup.

Commutative (or Abelian) Law. A semigroup S which has the property that $a \circ b = b \circ a$ for all $a, b \in S$ is said to be a *commutative* (or *Abelian*) semigroup.

VII. *Example.* Let S_7 be any nonvoid set with $x \circ y = y$. $\{S_7; \circ\}$ is a noncommutative semigroup.

Units. A *right unit* of a semigroup S is an element $e' \in S$ such that $x \circ e' = x$ for all $x \in S$; a *left unit* is an element $e'' \in S$ such that $e'' \circ x = x$ for all $x \in S$; a *unit* is an element $e \in S$ such that $x \circ e = e \circ x = x$ for all $x \in S$. In S_7 every element x is a left unit; and if S_7 has more than one element then S_7 has no right unit.

VIII. *Example.* Let $S_8 = \{a, b\}$ with multiplication defined by the table

$\circ$	a	b
a	a	a
b	b	b

$\{S_8; \circ\}$ is a semigroup in which a and b are right units. The four semigroups of order 2 and the eighteen semigroups of order 3 are tabulated in [4], p. 151.

IX. *Example.* Let $S_9=\{z \mid |z|=1,\ z \text{ complex}\}$ with $z \circ w=(x,\ y) \circ (u,\ v)=(xu-yv,\ xv+yu)$ (complex multiplication). $\{S_9;\ \circ\}$ is a semigroup.

X. *Example.* Let $S_{10}=\{z \mid |z| \leqq 1,\ z \text{ complex}\}$ under complex multiplication. $\{S_{10};\ \circ\}$ is a semigroup.

XI. *Example.* Let $S_{11}=\{z \mid |z| \leqq 1,\ z \text{ complex}\}$ with $z \circ w=z|w|$. $\{S_{11};\ \circ\}$ is a semigroup. Each element of S_9 is a right unit of S_{11}.

If a semigroup S has both a right unit e' and a left unit e'' then $e'=e''$, for $e'' \circ e'=e'$ and $e'' \circ e'=e''$. Thus, if a unit of S exists it is unique. The unit of S_4 is 1; of S_9 and S_{10} is (1, 0). S_1 and S_5 have no unit.

XII. *Example.* Let S_{12} be the set of all the subsets of a given set S under the union composition $\cup$, where, for $A,\ B \subset S$, $A \cup B=\{x \mid x \in A \text{ and/or } x \in B\}$. $\{S_{12};\ \circ\}$ is a semigroup. The unit of S_{12} is the empty or null set. Similarly we have a semigroup under the intersection composition $\cap$, where $A \cap B=\{x \mid x \in A \text{ and } x \in B\}$.

Inverses. A *right inverse* of an element $x \in S$, a semigroup, with respect to e', a right unit of S, is an element $x' \in S$ such that $x \circ x'=e'$; a *left inverse* of an element x is an element $x'' \in S$ such that $x'' \circ x=e''$, a left unit of S; an *inverse* of an element x is an element $x^{-1} \in S$ such that $x \circ x^{-1}=x^{-1} \circ x=e$, the unit of S. In S_7 any element is a left inverse of a given element y; in S_8 the right inverses of a and b are a and b. If an element x of a semigroup has both a right inverse x' and a left inverse x'' then $x'=x''$, for $x'' \circ (x \circ x')=x'' \circ e'=x''$ and $(x'' \circ x) \circ x'=e'' \circ x'=x'$. Thus, if an inverse of an element of a semigroup exists this inverse is unique. In S_9 the inverse of $(x,\ y)$ is $(x,\ -y)$.

The set of all transformations (single-valued mappings) of a given set S under the composition $\circ$, where, for $\alpha,\ \beta \in S$, $\alpha \circ \beta=\alpha\beta$, the resultant of the transformations α and β, is a semigroup. A special case of this is the following

XIII. *Example.* Let S_{13} be the n^n transformations of a fixed set S of n elements into itself, with

$$\alpha \circ \beta=\begin{pmatrix}1 & 2 & \cdots & n\\ p_1 & p_2 & \cdots & p_n\end{pmatrix}\begin{pmatrix}1 & 2 & \cdots & n\\ q_1 & q_2 & \cdots & q_n\end{pmatrix}=\begin{pmatrix}1 & 2 & \cdots & n\\ r_1 & r_2 & \cdots & r_n\end{pmatrix},$$

where r_i is that one of the q's which corresponds to p_i in the first row of the second factor β. $\{S_{13};\ \circ\}$ is a semigroup. In this semigroup the inverse of

$$\begin{pmatrix}1 & 2 & \cdots & n\\ p_1 & p_2 & \cdots & p_n\end{pmatrix},$$

where the p's are (all) the n integers $1, 2, \cdots, n$ in any order, is

$$\begin{pmatrix}1 & 2 & \cdots & n\\ s_1 & s_2 & \cdots & s_n\end{pmatrix}$$

where, if $p_i=j$ then $s_j=i$. The other elements of S_{13} do not have inverses. Consider, for example, the case when $n=2$. We then have $n^n=2^2=4$ transformations, say

$$\epsilon=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix},\quad \alpha=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix},\quad \gamma=\begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix},$$

and let $S'_{13}=\{\epsilon, \alpha, \beta, \gamma\}$. $\{S'_{13}; \circ\}$ is a semigroup having the multiplication table

$\circ$	ϵ	α	β	γ
ϵ	ϵ	α	β	γ
α	α	ϵ	β	γ
β	β	γ	β	γ
γ	γ	β	β	γ

Note that $\epsilon^{-1}=\epsilon$, $\alpha^{-1}=\alpha$, and that β and γ do not have inverses ([1], pp. 16–17; [4], pp. 135–136).

A *group* is a semigroup with unit in which every element has an inverse. The semigroup S_9 is a group. The subset of S_{13} consisting of the $n!$ one-to-one transformations of the finite set S of n elements onto itself is a group (the symmetric group). Thus $\{(\epsilon, \alpha); \circ\}$ is a group, where $\epsilon, \alpha\in S'_{13}$.

Thus appears the appropriateness of the term *semigroup* (Fr. demi-groupe, Ger. Halbgruppe) meaning "half a group," part of a group, a notion which seems to have first appeared in the literature in a work of J. A. de Séguier in 1904 [3]. S_1 and S_5 are examples of semigroups satisfying only the two essential properties (closure and associativity) of the four group properties.

Zeros. A *right zero* of a semigroup S is an element $z'\in S$ such that $x\circ z'=z'$ for all $x\in S$; a *left zero* is an element $z''\in S$ such that $z''\circ x=z''$ for all $x\in S$; a *zero* is an element $z\in S$ such that $x\circ z=z\circ x=z$ for all $x\in S$. In S_7 every element is a right zero; in S_8, a and b are left zeros. If a semigroup has both a right zero and a left zero then these are equal. Thus, if a zero of a semigroup exists this zero is unique. (This is shown as in the case of the unit above.) The zero of S_2 is p; of S_6 is 0.

XIV. *Example.* Let $S_{14}=[\frac{1}{2}, 1]$ with $x\circ y=\max(\frac{1}{2}, xy)$. $\{S_{14}; \circ\}$ is a semigroup with zero $\frac{1}{2}$.

XV. *Example.* Let $S_{15}=[0, 1]\times[0, 1]$ with $(x, y)\circ(u, v)=(xu, yv)$ (coordinate-wise real multiplication). $\{S_{15}; \circ\}$ is a semigroup with zero (0, 0).

Idempotents. An element a of a semigroup S is said to be *idempotent* if $a^2=a\circ a=a$. Clearly, every unit is an idempotent, and every zero is an idempotent. The idempotents of S_8 are a and b; and in S_3, S_7 and S_{12} every element is idempotent.

Nilpotents. An element a of a semigroup S is said to be *nilpotent* if for some positive integer n, $a^n = z$, the zero of S. Clearly, every zero is a nilpotent element. Every element of S_{14} except the unit $(=1)$ is nilpotent.

Ideals. A *right ideal* is such a nonvoid set $A \subset S$, a semigroup, that $AS \subseteq A$, where $AS = A \circ S = \{a \circ s \mid a \in A,\ s \in S\}$; *a left ideal* is such a nonvoid set $A \subset S$ that $SA \subseteq A$; an *ideal* is such a nonvoid set $A \subset S$ that $(AS \cup SA) \subseteq A$. In S_4 and S_6, 0 is an ideal and any connected set containing 0 is an ideal. In S_{11}, $z_0 S_{11} = \{z_0 t \mid 0 \leqq t \leqq 1\}$, the closed segment $0P_0$ (Fig. 1), where $0 = (0, 0)$ and $z_0 = P_0$ is any element of S_{11}, is a right ideal of S_{11}; and $S_{11} z_0 = \{zt \mid z \in S_{11},\ 0 \leqq t \leqq |z_0|\}$, the closed disk with center 0 and radius $0P_0$, is a left ideal of S_{11}. Since $S_{11} S_{11} = S_{11}$

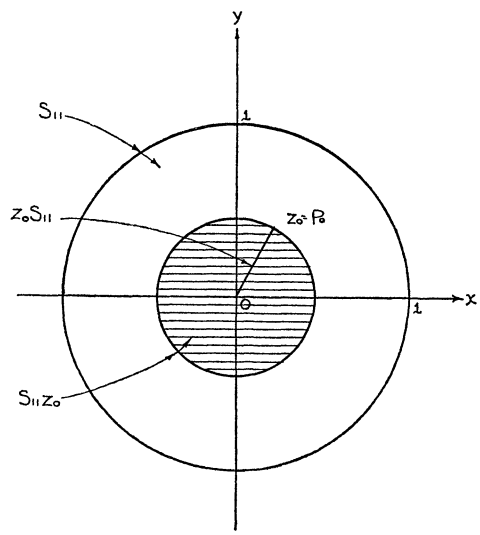


FIG. 1

and $z_0 S_{11} \subset S_{11} z_0$ for any $z_0 \in S_{11}$ we see that any left ideal $S_{11} z_0$ of S_{11} is likewise a right ideal of S_{11}, for $(S_{11} z_0) S_{11} = S_{11}(z_0 S_{11}) \subset S_{11}(S_{11} z_0) = S_{11} z_0$. $S_{11} z_0$ is also an ideal of S_{11} for

$$S_{11}(S_{11} z_0) \cup (S_{11} z_0) S_{11} = S_{11} z_0 \cup S_{11}(z_0 S_{11}) \subset S_{11} z_0 \cup S_{11}(S_{11} z_0) = S_{11} z_0.$$

XVI. *Example.* Let S_{16} be the set of all $n \times n$ matrices (n a fixed positive integer) with elements from the even integers and let the operation $\circ$ be matrix multiplication. $\{S_{16};\ \circ\}$ is a semigroup.

XVII. *Example.* Let $S_{17} = \{(x, y) \mid 0 \leqq x,\ y;\ x + y \leqq 1\}$ with $(x, y) \circ (u, v) = (xu,\ xv + y)$. $\{S_{17};\ \circ\}$ is a semigroup isomorphic with the semigroup of all 2×2 matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \qquad (0 \leqq x, y; x + y \leqq 1)$$

under matrix multiplication, the correspondence being

$$(x, y) \leftrightarrow \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

Let $z_0 = P_0$ be any element of S_{17}. Let $L = (0, 1)$ and $M = (1, 0)$ (Fig. 2). Draw QP_0 parallel to LM and draw RP_0 parallel to $0M$. Then

$$z_0 S_{17} = \{(x_0 x, x_0 y + y_0) \mid 0 \leqq x, y; x + y \leqq 1\},$$

the closed area RP_0Q, is a right ideal of S_{17}; and

$$S_{17} z_0 = \{(x x_0, x y_0 + y) \mid 0 \leqq x, y; x + y \leqq 1\},$$

the closed area $0P_0L$, is a left ideal, right ideal and ideal of S_{17}.

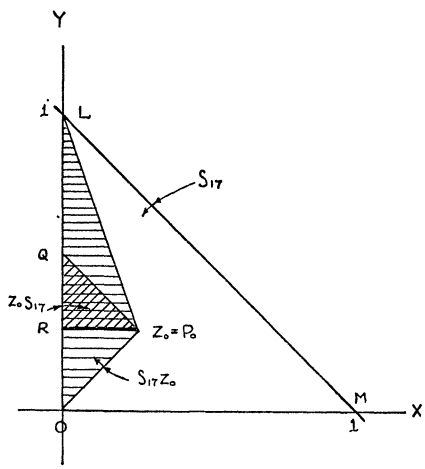


FIG. 2

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THE NEGATIVE BINOMIAL DISTRIBUTION AND THE INCOMPLETE BETA FUNCTION

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The negative binomial distribution is given by

$$f(p, q, t) = \frac{(p + q - 1)!\, t^q}{q!(p - 1)!(1 + t)^{p+q}}; \qquad q = 0, 1, 2, \cdots. \tag{1}$$

The expression in (1) is term number $q+1$ in the expansion of $[(1+t)-t]^{-p}$. The negative binomial distribution is frequently used in probability and statistics. Thus: "When the number of individuals per unit of space or time in repeated counts cannot be assumed to have the same expected value, they may represent a mixture of several homogeneous Poisson distributions. The number in each unit is restricted to the integers but this is not true of the expectations or means. In a mixture of Poissons, the means represent a positive continuous variate. The simplest frequency distribution which they might follow is the Eulerian or the Pearson type III curve, and if in fact they are so distributed the observations will conform to the negative binomial." [1]. See also [3].

As an actual example, consider the accompanying table, which gives the counts of red mites on apple leaves, data of Philip Garman [1]. The final column of this table gives the theoretical frequencies of a negative binomial distribution fitted to the observed data. The agreement is exceptionally good.

No. of mites per leaf	No. of leaves observed	Neg. binomial frequencies
0	70	69.49
1	38	37.60
2	17	20.10
3	10	10.70
4	9	5.69
5	3	3.02
6	2	1.60
7	1	.85
8+	0	.95
Total	150	150.00

Let us denote the sum of the first q terms of the distribution (1) by $F(p, q, t)$; thus

$$F(p, q, t) = \sum_{j=0}^{q-1} f(p, j, t). \tag{2}$$

It is the purpose of this note to establish a relation between the function F and the incomplete Beta function, a relation similar to that connecting the positive binomial distribution and the incomplete Beta function ([5] and [6]) and to that

connecting the Poisson distribution and the incomplete Gamma function ([4], p. 357).

Obviously $F(p, \infty, t) = 1$, and consequently we may write

$$1 - F(p, q, t) = \sum_{j=q}^{\infty} f(p, j, t). \tag{3}$$

Differentiating both sides of (3) with respect to t, we get

$$\begin{aligned} -\frac{\partial}{\partial t} F(p, q, t) &= \sum_{j=q}^{\infty} \frac{\partial}{\partial t} f(p, j, t) \\ &= \sum_{j=q}^{\infty} \frac{(p+j-1)!}{j!(p-1)!}\left[\frac{jt^{j-1}}{(1+t)^{p+j}} - \frac{(p+j)t^j}{(1+t)^{p+j+1}}\right] \\ &= \frac{(p+q-1)!t^{q-1}}{(q-1)!(p-1)!(1+t)^{p+q}} = \frac{t^{q-1}(1+t)^{-p-q}}{B(p, q)}. \end{aligned} \tag{4}$$

(All terms of the summation except the first cancel out.)

Integration yields

$$-F(p, q, t) = \frac{1}{B(p, q)} \int_0^t t^{q-1}(1+t)^{-p-q} dt + C. \tag{5}$$

If we set $t = (1-z)/z$, the integral in (5) becomes

$$\begin{aligned} \int_{1/(1+t)}^{1} z^{p-1}(1-z)^{q-1} dz &= \int_0^1 z^{p-1}(1-z)^{q-1} dz - \int_0^{1/(1+t)} z^{p-1}(1-z)^{q-1} dz \\ &= B(p, q) - B_{1/(1+t)}(p, q), \end{aligned} \tag{6}$$

where $B_{1/(1+t)}(p, q)$ is the incomplete Beta function. Equation (5) may now be written in the form

$$-F(p, q, t) = \frac{1}{B(p, q)} [B(p, q) - B_{1/(1+t)}(p, q)] + C, \tag{7}$$

and by setting $t = 0$ we find that $C = -1$. It follows quite readily that

$$F(p, q, t) = \frac{B_{1/(1+t)}(p, q)}{B(p, q)} = I_{1/(1+t)}(p, q), \tag{8}$$

where $I_{1/(1+t)}(p, q)$ is the incomplete Beta function ratio.

As an illustration let us consider the expansion of $(1.25-0.25)^{-3}$. Here $p = 3$, $t = 0.25$, $1/(1+t) = 0.8$. We shall take q successively equal to 1, 2, 3, 4; that is, we shall find the sum of 1, 2, 3, 4 terms in the expansion. Making use of [7], in the notation of which $x = 1/(1+t)$, we find

$$I_{0.8}(3, 1) = 0.5120000, \quad I_{0.8}(3, 2) = 0.8192000, \quad I_{0.8}(3, 3) = 0.9420800.$$

When $q > p$ we must use the formula $I_x(p, q) = 1 - I_{1-x}(q, p)$ ([7], p. vii). Thus,

$I_{0.8}(3, 4) = 1 - I_{0.2}(4, 3) = 1 - 0.0169600 = 0.9830400$. In this simple case it is easy to check results by comparing with the actual expansion:

$$(1.25 - 0.25)^{-3} = (1.25)^{-3}(1 - 0.2)^{-3}$$

$$= 0.512\left[1 + 3(.2) + \frac{3\cdot 4}{1\cdot 2}(.2)^2 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}(.2)^3 + \cdots\right]$$

$$= 0.512 + 0.3072 + 0.12288 + 0.04096 + \cdots .$$

The values obtained from the tables check exactly with the sums of terms from the expansion.

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MATHEMATICAL EDUCATION NOTES

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THE CUPM CATALOG SURVEY

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1. Background. About five years ago, the Division of Mathematics of the National Research Council, with support from the National Science Foundation, initiated a survey of Research Potential and Training in the mathematical sciences; the findings, commonly referred to as the Albert Report, were issued in 1957 and include a study of the research activities of about 1500 mathematicians, and a cross-sectional study of the organization and mathematical environment of about sixty universities offering training in mathematics at the doctorate level. Copies of the Albert Report may still be available from the Providence office of the American Mathematical Society.

As part of its general study of mathematics programs, the Committee on the Undergraduate Program is presently carrying out an extension of the 1955–57 survey with a different and somewhat larger sample of colleges, principally representing those that do not offer a doctorate program in mathematics. This survey is being carried out with the assistance of the National Opinion Research Center at the University of Chicago. As a guide in the design of this survey and to provide certain information which we felt was needed, CUPM asked one of its members, Professor Mosteller of Harvard, to supervise an examination of the mathematical listings in a stratified sample of 300 college catalogs. The complete report, prepared by Frederick Mosteller, Keewhan Choi, and Joseph Sedransk, may be obtained by members of the Association by writing to Professor R. J. Wisner, Michigan State University Oakland, Rochester, Michigan. The present note is intended to summarize the findings from this survey.

2. **Summary.** In attempting to draw conclusions from the data that result from an examination of catalogs, it is important to keep three points in mind: (i) a catalog description of a course may not be a faithful description of the course that was actually given, (ii) the available catalog may be out of date, and thus not represent the *present* program of a college, (iii) the courses listed in a catalog may be given infrequently, if at all.

It is clear that the data resulting from the catalog survey may not give a true picture of the courses in mathematics that are currently being given in the nation's colleges; it may provide a measure of the courses that they think they *should* give. It is worth observing that remarks (i) and (ii) are perhaps less likely to apply to a small college. Since the study was concerned with the over-all picture rather than with the situation at any individual college, we have felt that there may be some interest outside CUPM in the results of the study.

An analysis of the structure of the sample will be found in the report itself. It included 216 of the listed 1147 regular four-year colleges, and 54 of the listed 414 junior colleges. It was stratified in many ways, taking into account such aspects as geographic location, size, and support; in terms of student body, it took in about 70% of the total college student body of the nation.

It is easiest to summarize the information about junior colleges first. Most are quite small, with enrollments less than 500, and they offer no advanced mathematics. Among the courses offered for credit, one is apt to see courses in elementary algebra, plane geometry, intermediate algebra, trigonometry, college algebra, analytical geometry, mathematics of finance, and even such courses as shop mathematics, industrial mathematics and slide rule. About a third of the junior colleges offer no course in calculus. A surprising discovery was that about 12% of the students attending junior colleges apparently have access to a course in computer programming; this is slightly more than have access to a course in the theory of equations.

Turning to the four-year colleges, we find an expectedly different pattern. Again, most of the colleges are small; about 80% have fewer than 3000 students,

and only a handful have more than 7000 students. Although many of these also offer high school level mathematics courses, it is comforting to see that so many of these courses give no college credit.

Leaving the more detailed analysis to the report itself, the following brief comments may be of interest. About half of the colleges seem to offer something in the general area of modern algebra (e.g. algebraic structures, linear algebra, etc.). Until the results of the NORC survey are in, it is difficult to be more explicit about the nature and depth of these courses. The traditional theory of equations course (e.g. Dickson) is still available to about 50% of the total student body, often as the only course in algebra beyond college algebra. Those who are concerned about the *applied* vs *pure* debates will not be reassured by the discovery that about a third of the colleges offer no course in differential equations.

Some traditions die slowly; about a third of the colleges continue to offer separate courses in differential calculus and integral calculus, and about two-thirds offer a separate course in analytical geometry.

Textbook writers may give heed to the fact that about 75% of the colleges appear to give something called *advanced calculus*; judging however by course descriptions, most of these seem to be a continuation or completion of the elementary calculus course, and to lay no stress upon theory or upon modern aspects of analysis. Again, more detailed information on this point must await the completion of the NORC survey. In strong contrast to the junior college picture, only a tiny group of colleges (4%) offer no course in calculus.

A course in statistics is listed by about half of the colleges, but only about a quarter offer one with a calculus prerequisite and, as might be expected, these tend to concentrate among the larger colleges with more diverse programs. A course specifically in probability theory is much less likely to be listed, with or without a calculus prerequisite. On the other hand, courses in *finite mathematics* are burgeoning and are now apparently available to about 15% of the total student body.

To end this summary on a somewhat encouraging note, we note that 20% of the colleges list at least one *modern* course in at least three of the five basic areas—analysis, algebra, probability and statistics, geometry and topology, logic. The term *modern* is to be understood in a specific (charitable) interpretation explained more fully in the report itself. This group of colleges takes in about 50% of the total college population, showing the strong bias toward the larger schools. The representation of different areas of mathematics, however, is not uniform. A modern algebra course is available to 75% of the students, and one in probability or statistics to 60%, while each of the remaining fields reach about 40%.

For all additional details, the reader is referred to the detailed report; there he will find the list of courses used to code the catalogs, and the detailed tabulation of courses, either by colleges or by students, in a variety of different arrangements and subclasses.

FRESHMAN LOGIC: REVISITED

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In a recent issue of this Monthly Jon Wheatley [3] gave an excellent critique of so called symbolic logic as it is presented to mathematics students at a first year university level. In this note I should like to add my pen to Mr. Wheatley's in recognizing the inadequacies of many such introductions. He and I do disagree, however, as to what might be a better approach.

The introduction to symbolic logic fails in many modern freshman texts probably because of the overambitiousness of the author in wanting to provide an insight into mathematical thinking; an insight which requires a good deal more of logic than he has the space to include. Consequently he skips around hither and yon in the areas of traditional logic until he has some kind of incomplete picture for presentation to students who as of yet are untrained in critical thinking. Wheatley mentions such things as the implicit denial of the basic function of an axiom set, the subtle exclusion of the complete use of the logical conjunction, and the poor handling of the inclusive disjunction. To this list I would add the general misuse of truth tables.

Symbolic logic, if taught with the proper spirit, can be made even at the elementary level a beautiful and edifying study. Its areas of applications are so widespread that students in all disciplines should have an opportunity to savor its flavor. The use of truth tables introduces in nearly every case a sense of mechanization into this study; a mechanization which muddles the primary function of logical expression. Wheatley notes that we should never give to students muddled accounts of anything. Let me show how this is done with truth tables: Suppose that we ask a student to verify that a certain expression is a tautology. He will first draw a lot of columns on his paper and place in the first one the first letter of the expression. Then using the rules for disjunction, conjunction, implication, and negation he proceeds to examine piece by piece the entire expression. This process is rather like teaching a third grader how to differentiate polynomial functions; he can do it but does he know precisely what he has done?

I propose that we introduce the students to normal forms rather than to truth tables. Surely the conjunctive and disjunctive normal forms furnish the same information as the truth tables [1] and [2]; moreover, this information can be obtained more quickly and without this self-imposed mechanization. Also, the various theorems illustrating the relationships involving the four rules can be imprinted more firmly in the mind of the student, e.g. the conjunctive equivalent of the negated disjunction or the reduction of the implication to expressions involving the other three.

As for the rest of Wheatley's note, we agree in content but not in order of presentation. He suggests that after the introduction of truth tables one should discuss axiom sets and then proceed to various theorems, obtaining in the end "... a complete elementary system of symbolic logic, securely founded on

axioms and with the psychological origin of the axioms laid bare. Now, by interpreting this system in terms of propositions and connectivities, the relationship between mathematics and the world can be discussed."

I suggest that an elementary discussion be carried on more in the spirit of Hilbert and Ackermann [2] with the motivations placed first, followed by the axioms, and finally the normal forms.

The work can then be extended to set theory and to more sophisticated areas of logic, depending upon the object of the course, the time and experience of the instructor, and the interest of the students. The right kind of beginning is essential so that symbolic logic can truly be a tool rather than a somewhat uninteresting game.

References

1. L. E. De Noya, Normal Forms in the Sentential Calculus (to appear).
2. D. Hilbert and W. Ackermann, Principles of Mathematical Logic, Chelsea, New York, 1950.
3. J. Wheatley, this MONTHLY, **67** (1960) 916–917.

THE NEED FOR AN EMPHASIS UPON CREATIVE DEVELOPMENT IN THE TEACHING OF MATHEMATICS

WILLIAM EDWARD CHRISTILLES, St. Mary's University

At no time in America's history has the mathematical world been faced with a more critical need for competent mathematics teachers than at the present. Yet this need cannot most suitably be met by teachers who themselves lack the one great quality of a truly fine mathematical mind, the ability to create. Although there are other desirable qualities which are also characteristic of good mathematics teachers, the aspect of creativeness is too often overlooked and quite frequently given secondary consideration in the evaluation of prospective teachers, particularly on the secondary and elementary school level.

To overcome this difficulty, we must analyze the existing situation and attempt to correct it. Basically the difficulty seems to lie in the lack of emphasis upon creative thinking by many of our teachers who themselves do not possess this desirable trait. This fundamental weakness, which seems to be an inherent characteristic of the noncreative mind, is most likely due to the inability of such persons to appreciate fully both the beauty and far-reaching implications of an original creation. Yet this weakness must be observed and corrected if we are to continue at our present rate of scientific attainment. It is to be remembered that the ability to create is basic to all discovery.

Perhaps the noncreative teacher hesitates, subconsciously, to admit the existence of the student whose abilities exceed his own. A truly great teacher, however, upon recognizing a student with exceptional abilities will strive in every way to advance this student by making full use of his outstanding attributes. Unfortunately, a teacher who lacks both, great intellectual ability and creativeness himself, frequently is more of a hindrance to the brilliant student

than an aid, merely loading him with continually monotonous tasks. It is far better when the creative student, with guidance, is allowed to choose a certain topic, or topics, which he personally would like to investigate. It is even possible in many instances for young geniuses to perform some rather remarkable bits of research on their own. History relates one such instance in the case of Karl Friedrich Gauss [1], who was later to distinguish himself as one of history's greatest mathematicians. Another more recent example is George Bergman who, when only twelve years of age, wrote a fine mathematical paper entitled *A Number System with an Irrational Base* which was published in 1957 in the Mathematics Magazine [2].

It is of course clear that such excellence of performance by our youthful geniuses is most adequately obtained only when they are properly guided by instructors who themselves are men of the highest caliber. Since there is an acute shortage of such mathematics teachers, however, it is impractical to assume that we can fill every classroom situation with these teachers. The only realistic alternative is to use the limited number of such competent instructors in specially organized classes containing our most talented youth, thus enabling the full potential of our gifted youth to be utilized.

Certainly, the choice of an instructor to teach in these special classes should be based strictly upon professional competence. These men, or women, as the case may be, should be educated with regard to the highest of standards. They should be required to hold a bachelor's or preferably a master's degree in the field of mathematics with a well-rounded group of courses in calculus, advanced calculus and analysis, modern algebra, fundamental concepts of mathematics, foundations of geometry, number theory, and probability and statistics. Preferably, they should have at one time been engaged in mathematical research, possibly during the writing of a master's thesis. It would also be desirable that such prospective teachers minor in the sciences so that they could relate their knowledge of mathematics to some of its many applications.

In conclusion, the author feels that this goal can be reached only by determined effort and planning. To educate our gifted youth effectively is no easy task, but we can and must educate these youth to their maximum potential. It is only in this manner that we shall be able to continue at our present rate of scientific advancement.

References

1. James R. Newman, The World of Mathematics, Simon and Schuster, New York, 1956. Vol. 1, p. 298.

2. George Bergman, A number system with an irrational base, Math. Mag. **31** (1957) 98–110.

SAWYER ON EDUCATION

A reader of this department has written to recommend that other readers will enjoy the article by W. W. Sawyer of Wesleyan University, "The Reconstruction of Mathematical Education," which appeared in the December, 1960 issue of *The Journal of Engineering Education.*

ONTARIO COURSE CONTENT MATERIALS

The complete revised version of the 1960–1961 experimental teaching material "Mathematics 9" is now available from Service and Smiles Distributors, Lim., 9 Mayfair Mews, Toronto 5, Ontario.

The text has been published in two parts and the price is $2.00 per set, postpaid. Remittance should accompany the order. There is also available a limited number of copies of "Mathematics 10," Unit 1—Geometry and Unit 2—Algebra, which is being tried out by permission of the Ontario Department of Education in about twenty schools during the coming session. "Mathematics 10" is available from the above distributors at the price of $1.00 per set, postpaid.

—Announcement from the Ontario Mathematics Commission

RUTGERS PROGRAM FOR THE RE-TRAINING IN MATHEMATICS OF COLLEGE GRADUATE WOMEN

The program for re-training in mathematics of college graduate women is administered by Rutgers, The State University, under the sponsorship of the Ford Foundation. Initiated in January, 1961 and starting its first full year of courses with this academic year 1961–1962, the program was set up in recognition of the fact that college graduate women who wish to start or resume professional work after years of raising a family need guidance and re-training. Both aims are given attention so that these women will be able to contribute professionally to the full extent of their abilities. In the field of mathematics, which has developed so rapidly over the past two decades, training means not only relearning what has been forgotten, but learning new terminology and new concepts which constitute the basis of modern mathematics.

The re-training program is considered a pilot program and, in sponsoring it, the Ford Foundation hopes that other universities will be inspired and encouraged to offer special opportunities to college graduate women.

THE MIDWEST PROGRAM ON AIRBORNE TELEVISION AND INSTRUCTION

Midwest Airborne Television Headquarters, Purdue University, has announced that airborne television instruction was started on September 11, 1961. The program is under the sponsorship of the Ford Foundation. Classes in mathematics during the school year 1961–1962 will include elementary arithmetic for grade six and geometry. The total offering of classes for the school year 1961–1962 will include elementary school classes in science, French, Spanish, music, and social studies. High school classes in the program are: American history, geometry, American government and American literature, and biology.

The airborne television headquarters are also currently telecasting professional assistance material for teachers who will teach the courses, and also for the television instructors. Source units are available at the Midwest Airborne Television Headquarters. The source units provide objectives for each lesson and suggest preparatory and follow-up activities, description of content

and a vocabulary list. The headquarters will present a telecast, showing ways in which television can be used by classroom teachers. Viewers will see teachers and students from several elementary and secondary schools which are using a variety of television lessons.

A MONOGRAPH ON CUISENAIRE RODS

The Educational Practices Committee of the National Council of Independent Schools is planning a series of monographs based on the identification of outstanding pieces of work in member schools of the Council. One of the recent monograph publications is entitled, *An Introduction to Cuisenaire Rods*, with subtitle, *An Approach to the Teaching of Elementary Mathematics*; this monograph is written by William P. Hull, a member of the faculty of Shady Hills School in Cambridge, Massachusetts.

Individual copies of this monograph are available without charge to mathematics teachers as long as the supply holds out. Requests should be addressed to: National Council of Independent Schools, 84 State Street, Boston 9, Massachusetts.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1511. *Proposed by Underwood Dudley and Arnold Lebow, University of Michigan*

"This system of n linear equations in n unknowns," said the Great Mathematician, "has a curious property."

"Good heavens!" said the Poor Nut, "What is it?"

"Note," said the Great Mathematician, "that the constants are in arithmetic progression."

"It's all so clear when you explain it!" said the Poor Nut. "Do you mean like $6x+9y=12$ and $15x+18y=21$?"

"Quite so," said the Great Mathematician, pulling out his bassoon. "Indeed, the system has a unique solution. Can you find it?"

"Good heavens!" cried the Poor Nut, "I am baffled."

Are you?

E 1512. *Proposed by C. W. Trigg, Los Angeles City College*

(a) If the nine positive digits are arranged at random in a square array, what is the probability that the determinant of the array will be an even number?

(b) Will the probability be the same if the first nine nonnegative digits are similarly arranged and the determinant evaluated?

E 1513. *Proposed by Seymour Kass, Illinois Institute of Technology*

Let $P(x)$ and $Q(x)$ be differentiable on an open interval I. Show that between any two real zeros of $P(x)$ on I there exists a real zero of $P'(x)+P(x)Q'(x)$.

E 1514. *Proposed by R. P. Current and J. F. Leetch, Bowling Green State University*

Let n be a positive integer. For each real x, let $d(x)$ be the digit in the nth decimal place of the infinite decimal expansion of x (assume no expansion terminates with repeated 9's). Define $f(x)=1$ if $d(x)$ is odd, and $f(x)=0$ if $d(x)$ is even. Prove that f has intervals of continuity for every n.

E 1515. *Proposed by Joachim Lambek, McGill University, and Leo Moser, University of Alberta*

Show that for every positive integer n, the Euclidean plane, considered as a point set, can be decomposed into n congruent connected subsets.

SOLUTIONS

Bounds on the Perimeter of a Triangle

E 1481 [1961, 803]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

If r and R denote the inradius and circumradius of a triangle of perimeter $2s$, show that $2[r(r+4R)]^{1/2}<2s\leqq[4(r+2R)^2+2R^2]^{1/2}$.

Solution by Leonard Carlitz, Duke University. Since $r=\Delta/s$ and $4R=abc/\Delta$ we have

$$r(r+4R)=\Delta^2/s^2+abc/s=s^2-(\Sigma a^2)/2<s^2,$$

and the first part of the inequality is established.

Since also $9R^2\geqq\Sigma a^2$ (see Court, *College Geometry*, p. 71, Sec. 110) we have

$$(2R+r)^2+R^2/2=9R^2/2+r(r+4R)\geqq s^2.$$

and the second part of the inequality is established.

Also solved by Maurice Brisebois, D. I. A. Cohen, Erwin Just, D. C. B. Marsh, Michael Skalsky, Dale Woods, and the proposer. Late solutions by A. N. Aheart, Jane Evans, Robert Frucht, and Guy Torchinelli. Frucht sharpened the left inequality by showing that $2[3r(r+4R)]^{1/2}\leqq 2s$.

A Function Differentiable at Only One Point

E 1482 [1961, 803]. *Proposed by E. L. Spitznagel, Jr., Xavier University, Cincinnati, Ohio*

Many simple functions are differentiable everywhere except at a single point (e.g., $f(x)=|x|$). Find a function defined on the real line which is differentiable nowhere save at a single point.

I. *Solution by Underwood Dudley, University of Michigan.* If $g(x)$ is the characteristic function of the set of rationals, then $f(x)=x^2g(x)$ is discontinuous and hence not differentiable for $x\neq 0$. But $f'(0)=0$.

II. *Solution by D. A. Moran, University of Illinois.* Let g be any one of the well-known continuous nowhere differentiable functions defined on the real line. Then the function $f(x)=xg(x)$ is everywhere continuous but differentiable only at $x=0$. The proof of differentiability at $x=0$ is a simple matter involving only an examination of the difference quotient; non-differentiability at all $a\neq 0$ follows from the identity

$$[f(x)-f(a)]/(x-a)=g(x)+a[g(x)-g(a)]/(x-a).$$

More generally, if $\{a_1, a_2, \cdots, a_m\}$ is any set of distinct real numbers, the function

$$F(x)=g(x)\prod_{i=1}^{m}(x-a_i)$$

is differentiable only at $x=a_i$ $(i=1, 2, \cdots, m)$.

Also solved by Robert Andrews, L. K. Arnold, D. W. Bailey, J. D. Baum, J. L. Botsford, Maurice Brisebois, J. A. Brown, J. L. Brown, Jr., Frederick Chong, Martin Cohen, Romae J. Cormier, Gus DiAntonio, D. Drasin, W. A. Drumin, C. W. and L. K. Durst (jointly), Jules Fink, N. V. Glick, Michael Goldberg, J. C. Hickman, J. E. Homer, Jr. and J. E. Lange (jointly), R. T. Hood, Leo Jonker, Erwin Just and Norman Schaumberger (jointly), William Kantrowitz, J. F. Leetch, Gerald Leibowitz, Betty Levine, D. L. Linfield, Jiang Luh, R. A. McGuigan, Jr., D. C. B. Marsh, R. A. Melter, N. S. Mendelsohn, J. C. Mettauer, R. M. More, Leo Moser, William Moser, John Nichols, Albert Nijenhuis, Hugh Noland, Bruce Orcutt, Sidney Penner, A. L. Rabenstein, L. A. Ringenberg, H. D. Ruderman, James Sebastian, John Shaw, Oved Shisha, N. V. Shulman, D. L. Silverman, Michael Skalsky, R. P. Soni, H. E. Spencer, J. W. Toole, Julius Vogel, W. J. Wayne, Robert Weinstock, E. H. Weiss, Raymond Whitney, the proposer, and an anonymous solver. Late solutions by Béla Bollobás, Jack Chamberlain, Gus Di Antonio, H. J. Fletcher, John Herzog, A. P. Hillman and P. B. Manchester (jointly), G. C. Kenney II, Andrzej Makowski, Solomon Marcus, D. L. Muench, R. J. Oberg, C. F. Pinzka, C. P. Sequin, C. E. Skinner, E. P. Starke, and W. C. Waterhouse.

Starke pointed out that the problem appears, with solution, in Fulks, *Advanced Calculus*, p. 54.

The Euler ϕ-function

E 1483 [1961, 803]. *Proposed by M. V. Mielke, University of Wisconsin*

For any pair (a, b) of positive integers, show that there are infinitely many pairs (A, B) of positive integers such that $\phi(A)\equiv 0 \bmod a$, $\phi(B)\equiv 0 \bmod b$, $\phi(A+B+AB)\equiv 0 \bmod (a+b)$, where ϕ denotes the Euler ϕ-function.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Since $x\,|\,\phi(x^2)$ and $\phi(x)\,|\,\phi(xy)$, it suffices to exhibit $A=pa^2(a+b)^2$ and $B=qb^2(a+b)^2$, where p and q are arbitrary positive integers.

II. *Solution by D. L. Silverman, Beverly Hills, Calif.* Let A', B', and C be chosen so that $\phi(A') \equiv 0 \bmod a$, $\phi(B') \equiv 0 \bmod b$, $\phi(C) \equiv 0 \bmod (a+b)$. Then letting $A = pA'B'C$ and $B = qA'B'C$, where p and q are arbitrary integers, and observing that $\phi(M) \mid \phi(N)$ when $M \mid N$, we see that A and B satisfy the requirements of the problem.

Also solved by Leonard Carlitz, D. I. A. Cohen, Underwood Dudley, N. V. Glick, Leo Moser, Albert Nijenhuis, Barry Simon, Michael Skalsky, Dale Woods, David Zeitlin, and the proposer. Late solutions by Béla Bollobás, S. H. Greene, G. E. Keller, R. J. Oberg, C. F. Pinzka, and Guy Torchinelli.

Editorial Note. Re Solution II, that $\phi(x) \equiv 0 \bmod a$ has at least one solution follows from the fact that the sequence $\{na+1\}$ contains a prime.

A Maximum Triangle

E 1484 [1961, 803]. *Proposed by J. H. Heinbockel and R. R. Korfhage, University of North Carolina*

A given point is at distances $\sqrt{2}$, 2, and $\sqrt{3}-1$ from the vertices of a triangle. Find the maximum area of the triangle.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Label the given point P and the vertices A, B, C, with $PA = 2$, $PB = \sqrt{2}$, $PC = \sqrt{3}-1$. Let x denote the length of the perpendicular from P to AB. Then the optimum configuration gives an area of

$$(\sqrt{3} - 1 + x)\{(4 - x^2)^{1/2} + (2 - x^2)^{1/2}\}/2$$

for $0 \leqq x \leqq \sqrt{2}$. This area has a maximum of $(3+\sqrt{3})/2$ at $x = 1$.

II. *Solution by Michael Goldberg, Washington, D. C.* The calculus can be circumvented. We use the notation of the above solution. First note that if A, P, B are fixed, then for the maximum triangle we must have PC perpendicular to AB. Such considerations assure us that P is the orthocenter of triangle ABC. Denote PA, PB, PC by p, q, r. Now

$$\tan^2 B = (r + x)^2/(q^2 - x^2) \quad \text{and} \quad \cot^2 BAP = (p^2 - x^2)/x^2.$$

But $\tan B = \cot BAP$, whence

$$(r + x)^2/(q^2 - x^2) = (p^2 - x^2)/x^2$$

or

$$2rx^3 + (p^2 + q^2 + r^2)x^2 - p^2q^2 = 0.$$

Setting $p = 2$, $q = \sqrt{2}$, $r = \sqrt{3}-1$ this becomes

$$2(\sqrt{3} - 1)x^3 + (10 - 2\sqrt{3})x^2 - 8 = 0.$$

This has the root $x = 1$; the other roots, $x = 2.38$ and $x = 3.085$, are clearly extraneous. Hence the sought area is $\sqrt{3}(\sqrt{3}+1)/2 = (3+\sqrt{3})/2$.

Also solved by Leonard Carlitz, D. I. A. and S. P. Cohen (jointly), Bruce Feldmeyer, Erwin Just and Norman Schaumberger (jointly), H. P. Lam, J. F. Leetch, Esther A. Linfield, C. N. Mills, F. D. Parker, and L. A. Ringenberg. Late solutions by Ronald Alter, Jane Evans, L. D. Goldstone, S. H. Greene, Frank Herlihy, Roman Kaluzniaki, G. C. Kenny II, C. C. Oursler, C. F. Pinzka, Harold Simpson, and Guy Torchinelli.

Arranging Men, Women, and Dogs

E 1485 [1961, 804]. *Proposed by C. N. Bhaskaranandha, Chikmaglur, India*

In how many ways can a party of m men, w women, and d dogs be arranged in a row so that neither two women nor two dogs are together?

Solution by the proposer. Arrange the m men in any of the $m!$ possible ways. Divide the w women into r groups in any of the possible

$$w!\binom{w-1}{r-1}$$

ways. Insert these r groups among the men so that no two groups come side by side. It is evident that the r groups have to choose among the $m+1$ places available. It follows that, for a given r, there are

$$m!w!\binom{w-1}{r-1}\binom{m+1}{r}$$

ways of arranging the men and women. Introduce the dogs among the women so that no two women come together. Let us denote the r groups of women by $w_1, \cdots, w_r$, where w_k is a group containing w_k women. We then require $(w_1-1)+\cdots+(w_r-1)=w-r$ dogs to separate the women to satisfy the conditions of the problem. To do this take one of the P^d_{w-r} possible arrangements of the $w-r$ dogs and introduce a dog between every pair of adjacent women. We still have to allot places for the remaining $d-w+r$ dogs so that no two dogs are together. These $d-w+r$ dogs have to choose from the available $(m+w+1)-(w-r)=m+r+1$ places. It follows that the $d-w+r$ dogs can be arranged in P^{m+1+r}_{d-w+r} ways for each way of arranging the $w-r$ dogs. It follows that, for a given r, there are altogether

$$m!w!\binom{w-1}{r-1}\binom{m+1}{r}P^d_{w-r}P^{m+1+r}_{d-w+r} = m!w!d!\binom{w-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{d-w+r}$$

ways of arranging the party. The solution indicates that $r \geqq w-d$ and $d-w \leqq m+1$. Since r should not be greater than $m+1$, it follows that $w-d \leqq m+1$, and we see that arrangements can be made only if $|w-d| \leqq m+1$. Thus the required enumeration is

$$m!w!d!\sum_{r=w-d}^{w}\binom{w-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{d-w+r},$$

where $\binom{n}{t}=0$ whenever $t>n$ or $t<0$ for $n \geqq 0$.

Note 1. In the above solution we introduced the women first and then the dogs. Had we reversed the procedure, we would have obtained the sum as

$$m!w!d!\sum_{r=d-w}^{d}\binom{d-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{w-d+r}.$$

When $w-d\geqq 0$ this reduces to

$$m!w!d!\sum_{r=1}^{d}\binom{d-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{w-d+r}.$$

It follows that, for $0\leqq w-d\leqq m+1$,

$$\sum_{r=1}^{d}\binom{d-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{w-d+r}\equiv\sum_{r=w-d}^{w}\binom{w-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{d-w+r},$$

which might otherwise be a difficult identity to establish.

Note 2. The proposer has not been able to extend the solution of the problem to cover the case of more than three categories (men, women, dogs).

Also solved, similarly, by D. C. B. Marsh. Other approaches were formulated by Leo Jonker, Albert Nijenhuis, and Michael Skalsky. Late solution by S. H. Greene.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5019. *Proposed by W. M. McKeeman, Stanford University*

Consider the interval [0, 1] and a denumerable ordering of the rationals in it. Cover the nth rational with an interval of length $1/2^{n+1}$ with center at the rational. Then the total length covered is less than or equal to $\frac{1}{2}$ and there are numbers not covered. Find one.

5020. *Proposed by J. A. H. Hunter, Toronto, Canada*

(1) Show that the Diophantine equation $(x^2-e)(z^2-e)=(y^2-e)^2$ has infinitely many integral solutions for each value of e such that $p^2-2q^2=e$ has integral solutions.

(2) The quartic equation in (1) possesses also other solutions (e.g., $x=2$, $y=8$, $z=22$; $e=-6$). Determine the set $\{e\}$ for which solutions exist.

5021. *Proposed by W. A. J. Luxemburg and A. C. Zaanen, California Institute of Technology*

If B is a Banach space (*i.e.*, a complete normed linear space) and B^* its adjoint space, then it is a known theorem that a subset of B^* is weak*-compact (*i.e.*, $\sigma(B^*, B)$-compact) if and only if the subset is weak*-closed and bounded in norm.

(1) Show by means of an example that this theorem becomes false if the hypothesis that B is a Banach space is replaced by the hypothesis that B is a normed linear space.

(2) Show that if B is an arbitrary normed linear space, then a weak*-compact subset of B^* is bounded in norm if and only if the weak*-closure of its convex hull is weak*-compact.

5022. *Proposed by T. S. Nanjundiah, Central College, Bangalore, India*

Let

$$H_n = \sum_{\nu=1}^{n} \frac{1}{x+\nu-1}, \qquad G_n = \prod_{\nu=1}^{n}\left(1+\frac{x-1}{\nu}\right), \qquad n = 1, 2 \cdots.$$

Prove:

$$(1)\qquad \begin{aligned} -\frac{\Gamma'(x)}{\Gamma(x)} &< H_p + H_q - H_{pq} \leqq 1/x \qquad \text{for } x \geqq 1/2. \\ \frac{1}{\Gamma(x)} &> \frac{G_pG_q}{G_{pq}} \geqq x \qquad \text{for } \begin{cases} 0 < x < 1 \\ x > 1. \end{cases} \end{aligned}$$

(Problem E 819 [1948, 317] is the special case $x=1$.)

5023. *Proposed by D. J. Newman, Yeshiva University*

Let $P(z)$ be a nonconstant polynomial and for each positive number t define $L(t)$ to be the length of the curve $|P(z)|=t$. Prove that $L(t)$ is an increasing function.

5024. *Proposed by D. A. Moran, and J. J. Rotman, University of Illinois*

Prove that there exists a subset of the plane which meets every straight line in precisely two points.

Prove that such a set is 0-dimensional.

SOLUTIONS

Maximizing a Functional

4949 [1961, 181]. *Proposed by Richard Bellman, the RAND Corporation*

Consider the problem of maximizing the functional

$$J(\lambda_1, \cdots, \lambda_n) = \int_0^T \Big(\sum_{i=1}^n a_i\lambda_i(t) \Big) \exp \Big(\int_t^T \sum_{i=1}^n b_i\lambda_i(s)ds \Big) dt,$$

over all functions $\lambda_i(t)$ satisfying the conditions $\lambda_i(t) \geqq 0$, $\sum_{i=1}^n \lambda_i(t) = 1$, $0 \leqq t \leqq T$. Show that the solution can be obtained in terms of the solution of the differential equation $du/dt = \max_{1 \leqq i \leqq n} (a_i + b_i u)$, $u(0) = 0$, and thus determine the solution.

Solution by Fred Suvorov, Princeton University. We may write J in the form

$$\exp\Big(\int_0^T \sum_{i=1}^n b_i\lambda_i(s)ds \Big) \int_0^T \Big(\sum_{i=1}^n a_i\lambda_i(t) \Big) \exp\Big(- \int_0^t \sum_{i=1}^n b_i\lambda_i(s)ds \Big) dt.$$

As suggested, we consider J as a function of T and differentiate. (From now on, $J = u(T)$.)

$$\frac{du}{dT} = \sum_{i=1}^n b_i\lambda_i(T)u(T) + \sum_{i=1}^n a_i\lambda_i(T) \leqq \max_{1 \leqq i \leqq n} (b_i u(T) + a_i), \tag{1}$$

from our assumptions on the λ_i. Also $u(0) = 0$.

It is not yet clear that the function $u(T)$ for which (1) is an equality exists, but if it does, we claim that by defining λ_i to be the characteristic function of the set on which $du/dT = b_i u(T) + a_i$, we maximize the functional J for each T. (This is clear from (1).) We proceed to prove the existence of u.

Define the sequence of functions u_n as follows:

$$u_0(T) = 0, \qquad u_1(T) = a_{i_0} \cdot T,$$

where $a_{i_0} = \max a_i$ for $1 \leqq i \leqq n$. Having defined u_n, we define u_{n+1} as

$$u_{n+1}(T) = \int_0^T \max_{1 \leqq i \leqq n} (b_i u_n(t) + a_i) dt.$$

Now let $\delta_n(T) = |u_{n+1}(T) - u_n(T)|$, and

$$\begin{aligned} K_n(t) &= \max_{1 \leqq i \leqq n} (b_i u_n(t) + a_i) \\ &= \max_{1 \leqq i \leqq n} \{ b_i(u_n(t) - u_{n-1}(t)) + b_i u_{n-1}(t) + a_i \}. \end{aligned}$$

It is easily verified that $|K_n(t) - K_{n-1}(t)| \leqq 3b\delta_{n-1}(t)$, where $b = \max |b_i|$ for $1 \leqq i \leqq n$. Now

$$\begin{aligned} \delta_n(T) &= \Big| \int_0^T (K_n(t) - K_{n-1}(t)) dt \Big| \\ &\leqq \int_0^T | K_n(t) - K_{n-1}(t) | \, dt \leqq 3b \int_0^T \delta_{n-1}(t) dt. \end{aligned} \tag{2}$$

Combining (2) with $\delta_1(T) = \frac{1}{2} |a_{i_0}| t^2$, we get $\delta_n(T) \leqq |a_{i_0}| (3b)^{n-1} T^n / n!$.

$$\lim_{n\to\infty} u_n(T) = \sum_{n=1}^{\infty} (u_n(T) - u_{n-1}(T)), \tag{3}$$

and since $\sum_{n=1}^{\infty} \delta_n(T)$ is uniformly convergent on each finite T-interval, the limit (3) must exist. Then

$$u(T) = \lim_{n\to\infty} u_n(T)$$

is the required function.

Also solved by the proposer.

Series and Product Evaluation

4950 [1961, 181]. *Proposed by T. S. Nanjundiah, Central College, Bangalore, India*

Evaluate

$$\prod_{n=1}^{\infty}{}' \frac{n^3 + p^3}{n^3 - p^3}, \qquad \sum_{n=1}^{\infty}{}' \frac{n^2}{n^6 - p^6}, \qquad p = 1, 2, \cdots,$$

the accent indicating that $n = p$ is to be omitted.

The product is known for the case $p = 1$. See Bromwich, *Infinite Series* (1926), p. 313, Ex. 18; *Collected Papers of Ramanujan*, (1927), p. 322, Qu. 261.

Solution by the proposer. We note the identities

$$\frac{x^3 + a^3}{x^3 - a^3} = \frac{1 + a^2/x(x - a)}{1 + a^2/x(x + a)}. \tag{1}$$

$$\frac{x^2}{x^6 - a^6} = \frac{1}{6a}\left[\frac{1/x + 1/(x - a)}{x(x - a) + a^2} - \frac{1/x + 1/(x + a)}{x(x + a) + a^2}\right]. \tag{2}$$

Let us set:

$$u_{n,p} = (n^3 + p^3)/(n^3 - p^3) \quad (n \neq p), = 1 \ (n = p);$$
$$u_n = 1 + p^2/n(n - p) \quad (n \neq p), = u_{2p} \ (n = p);$$
$$v_{n,p} = n^2/(n^6 - p^6) \quad (n \neq p), = 0 \ (n = p);$$
$$v_n = \frac{1}{6p}\,\frac{1/n + 1/(n - p)}{n(n - p)\,p^2} \quad (n \neq p), = v_{2p} \ (n = p).$$

Then, using (1) and (2), we have

$$u_{n,p} = u_n/(u_{n+p}), \qquad v_{n,p} = v_n - v_{n+p}, \qquad n = 1, \cdots, p, \cdots. \tag{3}$$

Using the self-evident relations

$$\prod_{i=1}^{n} \frac{u_i}{u_{i+p}} = \prod_{i=1}^{p} \frac{u_i}{u_{i+n}}, \qquad \sum_{i=1}^{n} (v_i - v_{i+p}) = \sum_{i=1}^{p} (v_i - v_{i+n}),$$

and observing that $u_n \to 1$ and $v_n \to 0$ for $n \to \infty$, (3) yields

$$\prod_{n=1}^{\infty} u_{n,p} = \prod_{n=1}^{p} u_n, \qquad \sum_{n=1}^{\infty} v_{n,p} = \sum_{n=1}^{p} v_n. \tag{4}$$

On simplification of the right members, (4) reads:

$$\prod_{n=1}^{\infty}{}' \frac{n^3 + p^3}{n^3 - p^3} = (-1)^{p-1} \cdot \frac{3}{2} \prod_{n=1}^{p} \left(1 - \frac{p}{n} + \frac{p^2}{n^2}\right), \qquad \sum_{n=1}^{p}{}' \frac{n^2}{n^6 - p^6} = \frac{1}{12p^4}.$$

The latter result owes its simplicity to the relation $v_n + v_{p-n} = 0$, $(1 \leqq n \leqq p)$.

Also solved by J. J. C. Aldemira, and V. Vitek.

A Simplified Functional

4952 [1961, 182]. *Proposed by Lawrence Shepp, University of California at Berkeley*

Banach has shown (*Opérations Linéaires*, pp. 29–34) that there exists a method which satisfies certain natural axioms whose domain is the set of bounded sequences. His proof is unnecessarily long because he uses the functional:

$$P(S_n) = \operatorname*{glb.}_{\{i_1, i_2 \cdots, i_k\}} \limsup_n \sum_{j=1}^{k} \frac{S_{n+i_j}}{k} \text{ for bounded } \{S_n\},$$

where the same proof works for the simpler

$$q(S_n) = \operatorname*{glb}_k \limsup_n \sum_{j=1}^{n} \frac{S_{n+j}}{k}.$$

Prove:

$$P(S_n) = q(S_n) = \lim_k \limsup_n \sum_{j=1}^{k} \frac{S_{n+j}}{k}.$$

Editorial Note. The proposer, after providing a complete proof, writes that the problem has already been solved. See Jerison, *the set of all generalized limits of bounded sequences*, Canadian Journal of Mathematics, 9 (1957), 87–88.

Double Integral

4953 [1961, 299]. *Proposed by O. P. Aggarwal and Irwin Guttman, McGill University*

Show that

$$\iint_S e^{-(x^2+y^2)/2} dx dy = \iint_T e^{-(x^2+y^2)/2} dx dy,$$

where S is the square $0 \leqq (x, y) \leqq a$ and T is the triangle bounded by the coordinate axes and the line $x + y = a\sqrt{2}$.

Solution by John E. Vinson, Aerojet-General, Sacramento, Calif. Let $S_1(S_2)$ be the region of S above (below) the diagonal $y=x$. Rotation of S_2 through 45° and of S_1 through $-45°$ transforms S into T and leaves the distance r of any point from the original unchanged. Since the integrand is a function of r alone, equality is evident.

Note that $\exp[-(x^2+y^2)/2]$ can be replaced by any function of x^2+y^2 integrable over the given regions.

Also solved by A. N. Aheart, E. B. Andersen, C. B. Barfoot, T. J. Barth, P. T. Bateman G. G. Bilodeau, M. T. L. Bizley, Harry Björk, A. P. Boblétt, Edward Bregstone, Robert Breusch, D. R. Brillinger, J. A. Brown, J. L. Brown, Jr., W. G. Brown, Kenneth Brownstein, R. J. Buchler, F. P. Callahan, Jr., L. Carlitz, Ben Carter, A. G. Clark, Robert Comiskey, K. M. Das, G. Di Antonio, Irma Esrig, Harley Flanders, B. A. Fusaro, R. W. Gammon, G. Gilbert and P. G. Heyda, Todd Gitlin, Cornelius Groenewoud, W. C. Guenther, H. A. Heckart, J. C. Hickman, Richard Holt, D. G. Kabe, M. S. Klamkin, A. G. Konheim, R. E. Lynch, D. C. B. Marsh, Immanuel Marx, M. D. Mavinkurve, Herman Meyer, J. B. Muskat, M. F. Neuts, C. F. Pinzka, B. F. Plybon, K. A. Post, Gordon Raisbeck, J. Raleigh, M. M. Rao, P. R. Rider, L. A. Ringenberg, G. S. Rogers, David Sachs, Paul Schaefer, E. M. Scheuer, Marlow Sholander, F. C. Smith, Bob Snell, W. G. Spohn, O. E. Stanaitis, L. Steinberg, Elaine Tatham and James Siu, V. Vitek, W. C. Waterhouse, Charles Wexler, David Zeitlin, and the proposer.

An Integral Inequality

4954 [1961, 299]. *Proposed by D. J. Newman, Yeshiva University*

Show that, if $f(x) \geqq 0$ is convex, then

$$\int_0^\infty f^2 \leqq \frac{2}{3} \max f \int_0^\infty f,$$

and that $\frac{2}{3}$ is best possible.

Solution by Lawrence Shepp, University of California, Berkeley. The more general result:

$$\int_0^\infty f^{\alpha+\beta}(x)dx \leqq \frac{\alpha+1}{\alpha+\beta+1} \max f^\alpha \int_0^\infty f^\beta(x)dx$$

is still a very special case of the following theorem.

Let f be as above (convex, positive) and let F, L be increasing on $[0, \infty)$ and absolutely continuous, let $F(0)=0$ and let $G=F\cdot L$, then

$$\int_0^\infty G'(f(x))dx \leqq L(\max f) \int_0^\infty F'(f(x))dx.$$

If $F'(0)=0$, $F'(x)>0$ for $x>0$, finite equality holds if and only if, for some a, $b>0$

$$f(x) = \begin{cases} b(1-x/a) & 0 \leqq x \leqq a \\ 0 & x > a. \end{cases}$$

To prove the theorem, we have

$$F(f(t)) = -\int_t^\infty F'(f(u))f'(u)du \leqq -f'(t)\int_t^\infty F'(f(u))du.$$

Multiplying by $L'(f(t))$ and adding $L(f(t))F'(f(t))$, we obtain

$$\begin{aligned} G'(f(t)) &= L(f(t))F'(f(t)) + L'(f(t))F(f(t)) \\ &\leqq L(f(t))F'(f(t)) - L'(f(t))f'(t)\int_t^\infty F'(f(u))du. \end{aligned}$$

Integrating from 0 to ∞ gives

$$\begin{aligned} \int_0^\infty G'(f(t))dt &\leqq -\int_0^\infty (L(f(t))\int_t^\infty F'(f(t_1))dt_1)'dt \\ &\leqq L(\max f)\int_0^\infty F'(f(t))dt, \end{aligned}$$

as required. The other assertions are easy to verify.

Also solved by Robert Breusch, K. Brownstein, C. C. Farrington and E. W. Cheney, Harley Flanders, Wendell L. Jones, M. D. Mavinkurve, Walter Noll, H. Onishi and J. E. Potter, Marlow Sholander, W. G. Spohn, L. Steinberg, J. H. van Lint and K. A. Post, and the proposer.

A Summation Problem

4955 [1961, 299]. *Proposed by T. L. Saaty, Office of Naval Research, Washington, D. C.*

Show that

$$\sum_{k=2}^{n}\binom{n}{k}(1-\beta)^k\beta^{n-k}\sum_{j=1}^{k-1}\frac{1}{j} = \sum_{j=1}^{n-1}\frac{(1-\beta^j)(1-\beta^{n-j})}{j}.$$

Solution by C. Ligtmans, Technical University, Eindhoven, Netherlands. Using Newton's binomial formula we have

$$(ax+b)^n - x(a+b)^n + b^n(x-1) = \sum_{k=2}^{n}\binom{n}{k}a^kb^{n-k}x(x^{k-1}-1).$$

Dividing by $x(x-1)$ we get

$$\begin{aligned} &\frac{(ax+b)^n-(a+b)^n}{x-1} - \frac{(ax+b)^n-b^n}{x} \\ &= a\cdot\frac{(ax+b)^n-(a+b)^n}{(ax+b)-(a+b)} - a\cdot\frac{(ax+b)^n-b^n}{(ax+b)-b} = \sum_{k=2}^{n}\binom{n}{k}a^kb^{n-k}\frac{x^{k-1}-1}{x-1}. \end{aligned}$$

We can write this in the form

$$a\sum_{j=1}^{n}(ax+b)^{j-1}(a+b)^{n-j} - a\sum_{j=1}^{n}(ax+b)^{j-1}b^{n-j} = \sum_{k=2}^{n}\binom{n}{2}a^kb^{n-k}\sum_{j=1}^{k-1}x^{j-1}.$$

Integrating both sides of this identity with respect to x from 0 to 1 we derive

$$\sum_{j=1}^{n} \frac{\{(a+b)^{n-j} - b^{n-j}\}\{(a+b)^j - b^j\}}{j} = \sum_{k=2}^{n} \binom{n}{k} a^k b^{n-k} \sum_{j=1}^{k-1} \frac{1}{j}.$$

The required result is obtained by taking $a = 1-\beta$ and $b = \beta$.

Also solved by M. T. L. Bizley, L. Carlitz, S. H. Greene, A. G. Konheim, R. G. Nath, Franklin C. Smith, David Zeitlin, and the proposer. One solution was unsigned.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Convex Figures. By I. M. Yaglom and V. G. Boltyanskii (Library of the Mathematical Circle—vol. 4). Translated by Paul J. Kelly and Lewis F. Walton. Holt, Rinehart and Winston, New York, 1961. xv+301 pp. $4.75.

The authors of *Convex Figures* (largely plane figures) had two aims in view. The first is to provide an introduction to a subject that has been a field of active research since about the turn of the present century. The edition of 1919 of Cajori's History of Mathematics does not mention convex figures; this subject, however, is currently listed as a separate branch of mathematics in the authoritative *Mathematical Reviews*.

The second aim of the authors is to maintain the tradition that the geometry of Euclid is the first deductive science the young becomes acquainted with, where he can first exercise his own imagination, and test his ability to make use of such a discipline. For it so happens that convex geometry obtains its unexpected and quite useful results by having recourse, for the most part, to nothing more elaborate than the Elements of Euclid. This point of view of the authors is eloquently and convincingly supported by Herbert Busemann in his *Foreword.*

Readers familiar with modern geometry of the triangle and the circle, may be surprised to see what curious company those old friends find among convex plane figures, such as the Reuleaux triangle.

The American edition incorporates the additions to—and the revisions of—the original edition (1951) made by the editors of German translation (1956), in consultation with the Russian authors. Those authors provided additional material for the present edition.

NATHAN ALTSHILLER COURT
University of Oklahoma

Unified Calculus and Analytic Geometry. By Earl D. Rainville. Macmillan, New York, 1961. 724 pp. $8.50.

After an introductory chapter on analytic geometry which includes circles and polar coordinates, a well-written account of the notions of function, limit, and continuity is offered before the derivative is approached on page 69. The differential calculus of algebraic functions is followed by integration of such functions. No attempt is made to prove the fundamental theorem but its meaning is clarified by examples. The calculus of the remaining nonalgebraic elementary functions is treated next. The title appears to be justified because sprinklings of analytic geometry occur where such material is most useful. There is no paucity of coverage. The usual topics are there, and a number of topics are treated more thoroughly than in some texts. Instances of this are the discussions of odd and even functions, singular points under curve tracing, the error function, and the extended treatment of three-dimensional geometry. There is an abundance of well-selected exercises throughout. It was probably a wise decision of the author to give no exposition of vector analysis. The last seventy-five pages cover a large segment of the usual course on elementary differential equations.

The order of topics and the careful exposition indicate the success of the author "to effect a balance between the desire for rigor and the cold fact of the probable state of mathematical maturity of the student." This book should be considered carefully for possible use in first year college mathematics courses.

C. E. SPRINGER
The University of Oklahoma

Linear Algebra and Group Theory. By V. I. Smirnov. Revised, adapted and edited by Richard A. Silverman. McGraw-Hill Book Company, New York, 1961. x+464 pp. $12.50.

This book is a translation, with a certain amount of editing, of the sections on linear algebra and group theory from Smirnov's six-volume "Course of Higher Mathematics." The earlier chapters cover the usual material on determinants, matrices, linear transformations of a vector space, orthogonal and unitary transformations, quadratic forms and derivation of the Jordan normal form of a matrix. In addition there is a chapter on infinite dimensional spaces, an introduction to group theory, a long chapter on group representations and characters and a final chapter on continuous groups. Considerable attention is paid to representations of the orthogonal, unitary and Lorentz groups and to the role these play in various problems of mathematical physics.

The book is written more for the applied mathematician and physicist than for the algebraist. The language is that of classical rather than that of abstract algebra. For example, everything is done over the complex numbers and the word field is not used. Determinants are treated in the opening chapter and play a more central role than is usual in recent American texts on linear algebra. However, the exposition is excellent and the book gives support to the thesis that fashions in the order and language of presentation are less important

than over-all clarity of writing. Anyone who believes that the more abstract approach to algebra should be postponed until the student has acquired some maturity through dealing with more concrete situations, will find this an admirable text. It is less likely to appeal to those who favor an early introduction to abstract algebra. Its value has been enhanced by the inclusion of exercises, a total of more than four hundred problems, which did not occur in the Russian original. Answers are given for about half the computational problems and hints for the more difficult problems requiring proofs. These problems, together with the book's clarity of exposition, make it highly suitable for a text book at the junior or senior level.

D. C. MURDOCH
University of British Columbia

Special Functions of Mathematical Physics. By Harry Hochstadt. Holt, Rinehart and Winston, Inc., New York, 1961. 78 pp. $2.50.

The problem of introducing students to the mass of formulae and information collected under the heading *Special Functions* seems at first sight almost impossible. But, as the author of this monograph points out, the task is made much easier by the existence of at least two excellent reference works: Magnus and Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics and the Bateman Manuscript Project (Erdelyi and others, Higher Transcendental Functions).

The teacher of special functions need not give an encyclopaedic list of results; he can refer to the standard references and must only give a feeling for the motivation of the study of special functions and for the types of results that the student can reasonably expect. These two aims are very well achieved in the present monograph.

The first chapter treats topics in the theory of orthogonal polynomials. It introduces the notion of completeness, recursion formulae, quadrature formulae, the Weierstrass Theorem and also nice side-lights, like the interpretation of the zeros of the Jacobi polynomials as equilibrium positions of charges interacting according to an inverse first power law. Chapters 2, 3, and 4 deal with the Legendre, Bessel and Mathieu functions, introduced as solutions of equations arising by separation of variables in the Laplace equation. Noteworthy is the consistent treatment of the Bessel functions as limits of Legendre functions (plane = limit of sphere).

The mathematical prerequisites for the book are a knowledge of calculus (including functions of several variables) and an elementary knowledge of determinants. It should be within reach of most physics and engineering students in the junior year.

The typography is very pleasing. The only misprint of consequence that the reviewer could discover was in the third equation on page 69 ($\frac{1}{3}$ in place of $\frac{1}{4}-$, for $c_1=0$ in the second equation).

W. H. J. FUCHS
Cornell University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

De Paul University: Dr. Stephen Wainger and Messrs. A. Z. Czarnecki, John Synowic, Dean Heller and Jacob Towber have been appointed Instructors.

University of Notre Dame: Dr. Hugo Steinhaus, Wroclaw University, Poland, has been appointed Professor; Drs. R. E. Clay, University of Notre Dame, Hermann Heineken, Frankfurt University, Germany, T. E. Stewart, Institute for Advanced Study, and Mr. Klaus Hoechsmann, University of Notre Dame, have been appointed Assistant Professors; Professor Wladimir Seidel is on leave for the academic year 1961–62 at Wayne State University; Associate Professor Irving Glicksberg is on leave for the academic year 1961–62 at the Institute for Advanced Study.

Professor A. A. Albert, University of Chicago, has been appointed Dean of the Division of the Physical Sciences.

Mr. Joseph Beer, Syracuse University, has accepted a position as Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Dr. R. E. Carr, California Institute of Technology, has been appointed Research Geophysicist and Acting Head of the Research Division of the Geophysical Institute at the University of Alaska.

Mr. Conrad Dietz, University of North Dakota, has accepted a position as Programmer with Rocketdyne, Canoga Park, California.

Mr. R. A. Estes, University of Kansas, has been appointed Instructor at Otterbein College.

Mr. M. V. Fiondella, University of New Mexico, has been appointed Instructor at the University of Connecticut.

Dr. Stanley Frank, St. John's River Junior College, has accepted a position in the Intelligence Systems Department of the Mitre Corporation, Bedford, Massachusetts.

Dr. Irving Gerst, Radio Corporation of America, New York, New York, has been appointed Professor at the State University of New York, Long Island Center.

Mr. L. K. Gilchrist, Ohio State University, has been appointed Instructor at Phillips University.

Assistant Professor J. R. Gorman, United States Naval Academy, has been promoted to Associate Professor.

Mr. A. B. Harper, Jr., Mutual Benefit Life Insurance Company, Newark, New Jersey, has accepted a position as Actuarial Assistant with the Union Mutual Life Insurance Company, Portland, Maine.

Mr. P. M. Kahn, University of Michigan, has accepted a position with the Equitable Life Assurance Society of the United States, New York, New York.

Assistant Professor Lola F. Kiser, Birmingham-Southern College, has been promoted to Associate Professor.

Mr. M. R. Luttrell, Western Union Telegraph Company, Georgia, has been appointed Electronic Technician Instructor at the Tennessee Valley State Technical School, Huntsville, Alabama.

Professor Ingram Olkin, University of Minnesota, has been appointed Professor at Stanford University.

Mr. J. McD. Ortega, Stanford University, has accepted a position with the Sandia Corporation, Albuquerque, New Mexico.

Mr. Charles Parish has been appointed Instructor at Ball State Teachers College.

Associate Professor B. E. Rhoades, Lafayette College, has been promoted to Professor.

Visiting Professor F. S. Nowlan, College of William and Mary, died September 8, 1961. He was a member of the Association for 29 years.

Associate Professor L. M. Reagan, University of Wichita, died December 29, 1961. He was a member of the Association for 20 years.

SUMMER CONFERENCE ON HISTORY OF MATHEMATICS

The National Science Foundation is supporting a Summer Conference for College Teachers of the History of Mathematics at the University of Michigan, June 25 through July 20. Professor Phillip S. Jones, Director, and Professor Carl B. Boyer of Brooklyn College will lecture and advise for the period of the conference. Participants will have access to the University's outstanding collection of source materials in this area of study, as well as hear lectures by visiting scholars. For further information and application forms, write Professor P. S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-FIFTH ANNUAL MEETING OF THE ASSOCIATION

The Forty-fifth Annual Meeting of the Mathematical Association of America was held at the Sheraton-Gibson Hotel, Cincinnati, Ohio, from Wednesday to Friday, January 24 to 26, 1962, in conjunction with the annual meeting of the American Mathematical Society. There were registered 1492 persons, including 1033 members of the Association. This was the first meeting at which attendance of members exceeded one thousand.

Sessions of the Association were held on Wednesday afternoon, Thursday morning, and on Friday morning and afternoon in the Roof Garden of the Sheraton-Gibson Hotel. Presiding officers were Second Vice-President R. A. Rosenbaum for the session on "Logic" on Wednesday afternoon, Professor Ernst Snapper for the session on "Real Analysis—A First Course?" on Thursday morning, President A. W. Tucker for the business meeting, First Vice-President A. S. Householder for the session on "Geometry: Two Approaches" on Friday morning, President A. W. Tucker for the retiring presidential address, and Dean W. L. Duren, Jr. for the session on "Geometry and the College and Secondary Curricula" on Friday afternoon. The Program Committee for the meeting consisted of G. M. Merriman, Chairman; T. M. Apostol, H. W. Kuhn, R. A. Rosenbaum, and Ernst Snapper.

FIRST SESSION OF THE ASSOCIATION
LOGIC

Some Recent Developments in Logic, by Professor L. A. Henkin, University of California, Berkeley, and the Institute for Advanced Study, Princeton.

Early efforts of logicians were directed toward fundamental questions of consistency, completeness, and decidability. In the 1930's a considerable change in the direction of research was brought about by basic discoveries of Gödel and Church. We here attempt a rough classification and brief description of recent areas of logical investigation, under the following headings: Decision problems, foundations of set theory, recursive functions and constructive mathematics, proof theory, model theory, foundations of geometry, logico-algebraic problems.

Logic in College Curricula, by Professor Hartley Rogers, Jr., Massachusetts Institute of Technology.

The author presents facts and ideas to be considered by a mathematics department when it decides whether or not, and in what form, logic should be made available to undergraduates. The nature and goals of logic are discussed. Various current attitudes toward logic are analyzed. Content of a possible upperclass course in logic is outlined.

Logic in School Curricula, by Professor F. B. Fitch, Yale University.

Symbolic logic is viewed as a subject that can easily be taught at the elementary and high school levels, and as a powerful tool for increasing ability in abstract thinking. Attention is given to the relevance of symbolic logic to mathematics, science, ethics, and philosophy. A detailed description is presented of a kind of symbolic logic that has been successfully taught at the sixth-grade level in a public school. This is a "natural-deduction" or "subordinate-proof" type of logic, using the so-called Polish notation, and taught by way of a teaching-machine program.

SECOND SESSION OF THE ASSOCIATION
REAL ANALYSIS—A FIRST COURSE?

A First Course in Analysis (*CUPM*), by Professor L. W. Cohen, University of Maryland.

The first course in analysis contemplated by CUPM is designed as the beginning of an Honors Program. It takes into account the greater mathematical maturity induced by the new high school curriculum, the spread of Honors Courses for freshmen and sophomores, and the penetration of the undergraduate program by pieces of graduate mathematics. The additive properties of the classes of derivatives and integrals prepare for a section on linear algebra, which in turn prepares for the calculus of functions of several variables. The unifying concept is that of mappings, linear and non-linear, of $R^{(m)}$ into $R^{(n)}$. The material is developed mathematically with applications used for motivation or interpretation. The course requires at least two years.

Reformation in Advanced Calculus, by Professor N. E. Steenrod, Princeton University.

Mathematics students at the junior level are better prepared now than ten years ago. The improving freshman-sophomore courses have absorbed much material once taught in advanced calculus. The following reorganized program for advanced calculus is a natural continuation of unified calculus; and it covers much basic mathematics all students need. Briefly, the study of functions is extended from 1 and 2 to many dimensions. Linear transformations come first. Then the Frechet derivative (Jacobian) of a mapping is shown to give the best linear approximation. Vector fields, tensor fields, and exterior forms transform covariantly using Jacobians. Finally the exterior derivative leads to the generalized Stokes'-Green's theorems on integration.

Critique, Rebuttal, Discussion from the Floor; moderated by Professor B. J. Pettis, University of North Carolina.

Annual Business Meeting of the Association and the Association's First Award for Distinguished Service to Mathematics (see report below).

THIRD SESSION OF THE ASSOCIATION
GEOMETRY: TWO APPROACHES

Geometric Algebra, by Professor Ernst Snapper, Indiana University.

The term "Geometric Algebra" was coined by Artin. It designates the use of geometry to give geometric meaning to algebra. "Linear Geometry," that is, the small part of Euclidean Geometry which deals with hyperplanes, their angles and their distances, has been particularly successful in this respect. Linear Geometry has been good for algebra, but harbors a danger to geometry, if instructors sell it as "The New, Modern Geometry."

Linear Geometry is exciting when used to geometrize algebra. Linear Geometry, however, constitutes only a small part of geometry and a pretty dull part to boot.

Axiomatics, by Professor Marshall Hall, Jr., California Institute of Technology.

In 1893 J. J. Sylvester posed the problem: "Given n points (in the real plane) with the property that a line through any two points contains a third point of the set, must the points all lie on one line?" The answer to this question is no in the complex plane, and in planes whose coordinates have finite characteristic. After forty years the problem was solved by T. Grünwald but not until 1948 was the essential part of the problem made clear by H. S. M. Coxeter who gave an elegant proof based solely on the Pasch axiom of order. It has been conjectured by Warren White that the Pasch axiom and the Sylvester property are equivalent.

Recent work related to the Pasch axiom and the Sylvester problem are considered. Also the problem of replacements for Euclid's superposition axioms are discussed.

Retiring Presidential Address: The Narrow Mathematician, by Professor C. B. Allendoerfer, University of Washington.

This address will be published in this MONTHLY soon.

FOURTH SESSION OF THE ASSOCIATION
GEOMETRY AND THE COLLEGE AND SECONDARY CURRICULA

Linear Algebra for Underclassmen, by Professor Arno Jaeger, University of Cincinnati.

A fervent plea for an undergraduate course aiming at the following goals:

(a) to show the way in which contemporary mathematicians think and act,
(b) to eliminate prospective majors who merely enjoy solving simple numerical routine problems without being really gifted in mathematics,
(c) to prepare talented majors for graduate work.

These considerations point to the need for an introductory course in abstract algebra in which a few carefully chosen topics are taught thoroughly instead of a great variety of them superficially. The conditions that the notions selected should be representative of algebra and abstractions of well-understood, more concrete notions lead to groups, vector spaces and Euclidean vector spaces, i.e. to vectorial linear geometry.

Axiomatic Geometry for Undergraduates, by Professor H. S. M. Coxeter, University of Toronto.

Undergraduates who have some familiarity with Euclid's Elements readily recognize two interwoven threads in the development: affine geometry and absolute geometry. The axioms of order, proposed by Pasch and improved by later authors, provide a common basis for these. Two further axioms then yield affine geometry. Alternatively, the axioms of congruence (Hilbert's, as improved by R. L. Moore) yield absolute geometry, which can be extensively developed before introducing either the Euclidean or the hyperbolic axioms of parallelism. For projective geometry the axioms proposed by Veblen and Young are still as good as any. The finite geometry $PG(2, 5)$ provides a suitable model for verifying consistency.

Affine Geometry for the Secondary School Curriculum, by Professor Howard Levi, Columbia University.

It is proposed that the study of affine geometry be the high school student's introduction to formal geometry. In the first place it is easier to treat this geometry soundly than the more elaborate Euclidean geometry. In the second place affine geometry is the geometry which is naturally linked to vector spaces, and it is in the setting of vector spaces that a substantial portion of the student's later encounters with geometry will take place.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday morning in Parlor I of the Sheraton-Gibson Hotel in Cincinnati with 33 members present. Among the items of business transacted were the following:

The Board approved the appointment by President Tucker of the following Nominating Committee for 1962: J. C. Oxtoby, Chairman; W. L. Duren, Jr., and R. S. Pierce.

The Board re-elected Professor E. A. Cameron as a member of the Finance Committee for a four-year term ending in 1965.

The Board also elected as members of the Editorial Board of the MATHEMATICS MAGAZINE Professor Rothwell Stephens as Editor of the Teaching of Mathematics department and Professor H. M. MacNeille as Associate Editor in charge of Current Papers and Books to replace Professor Homer V. Craig who had requested to be relieved of this position.

Upon the recommendation of the Committee on Publications, the Board authorized all expository writing, including Carus Monographs, Slaught Papers, and MAA Studies to have a uniform remuneration for the authors at the rate of $6 per printed page for new articles and $3 per printed page for revised articles and reprints; this new rate of remuneration replaces all previous actions of the Board regarding payment of honoraria for such writing.

The Board authorized the Committee on the Undergraduate Program in Mathematics (CUPM) to establish a consultative service to colleges, a so-called Consultants Bureau, to serve in an advisory capacity to assist departments of mathematics in their discussions with their own administrations to solve certain of their problems, and to upgrade their course offerings. It also authorized CUPM to continue the studies of the Library Committee to prepare annotated and classified book lists for colleges which would be of use to the Consultants and to others such as the Visiting Lecturers. The Board authorized the appointment of a new Panel on Mathematical Environments and Surveys. The Board also approved a proposal to the National Science Foundation for continuation of support of CUPM during 1962 to 1964.

The Board approved the following schedule of future meetings: University of British Columbia, August 27–29, 1962; University of California, Berkeley, January 26–28, 1963; University of Colorado, August 26–28, 1963; Miami, Florida, January 1964; University of Michigan, August 1964; Denver, Colorado, January 1965; Cornell University, August 1965; Rutgers-The State University, New Brunswick, New Jersey, August 1966.

The Board elected to membership in the Association the first applicants for academic and corporate memberships and authorized establishment of the first reciprocity agreement, the latter with the Société Mathématique de Belgique (for further details see page 248 of the March issue of this MONTHLY).

The Board authorized the establishment of a standing committee on the Chauvenet Prize and that the award (for expository writing) be offered annually instead of every three years, as is presently done, and that a certificate be awarded to the recipient along with the cash award, provided, however, that an award shall not be made if there is no suitable recipient.

The Board approved the establishment of a standing Committee on Educational Media with subcommittees on Television, Films, and Programmed Learning, respectively.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, January 25, 1962 in the Roof Garden of the Sheraton-Gibson Hotel, Cincinnati, Ohio, with President Tucker presiding. The Association's first Award for Distinguished Service to Mathematics was made to Dr. Mina S. Rees, Dean of Graduate Studies of the City University of New York. The citation was read by Professor Wallace Givens, Chairman of the Committee on the Award for Distinguished Service to Mathematics; the award was presented by President Tucker. Dean Rees expressed her deep appreciation to the Committee and the Board of Governors for having awarded her this high honor and recalled some of the significant events which established and later increased federal support for mathematics. (For further details on this award see the February issue of this MONTHLY, pages 185–187.)

The Secretary reported the membership of the Association was 11,455, an increase of 1258 since the corresponding date last year.

The balloting for officers in which 2615 votes were cast resulted in the election of Professor H. S. M. Coxeter of the University of Toronto as First Vice-President for the two-year term 1962–1963, and of Professors T. M. Apostol of the California Institute of Technology and Hans Rademacher of the University of Pennsylvania as Governors for the three-year term 1962–1964.

The Secretary then reported on some of the actions taken by the Board of Governors the previous day. He expressed, on behalf of the Association, the deep appreciation to all those who had contributed so greatly to the success of the meeting. Special thanks were expressed to Professor I. A. Barnett, the Chairman of the Committee on Arrangements, for providing so much for the pleasure and comfort of the attending members, and to Dr. Gordon L. Walker, Executive Director of the Society, for his most painstaking attention to the many details requiring coordination of the activities of the Society and Association.

The amendments to the By-laws which were printed on pages 1031–1032 of the December 1961 issue of this MONTHLY were unanimously adopted.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Monday, January 22, through Thursday, January 25. The thirty-fifth Josiah Willard Gibbs Lecture was delivered by Professor C. N. Yang of the Institute for Advanced Study on Monday evening at 8:00 P.M. on "Symmetry Principles in Modern Physics." Professor Bertram Kostant of the Massachusetts Institute of Technology delivered an invited address entitled "A Survey of Lie Group Representation Theory" on Monday at 2:00 P.M. Professors C. B. Morrey, Jr., Jürgen K. Moser, and Edward Nelson spoke on various topics in Partial Differential Equations at a session held Tuesday at 10:00 A.M.; and Professors W. L. Chow and Bernard M. Dwork spoke on various topics in Algebraic Geometry at a session held Wednesday at 10:00 A.M.

The Cole Prize was awarded by the American Mathematical Society to Professor Kenkichi Iwasawa for his paper "Gamma Extensions of Number Fields," and to Professor B. W. Dwork for his paper "On the Rationality of the Zeta Function of an Algebraic Variety."

The Conference Board of the Mathematical Sciences held a meeting on Friday at 8:30 A.M. in the Roof Garden, at which reports from the officers of the Conference Board were presented and plans and activities discussed.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of I. A. Barnett, Chairman; H. L. Alder, A. J. MacIntyre, W. E. Restemeyer and J. W. T. Youngs.

Registration headquarters were located in the Ballroom Foyer of the Sheraton-Gibson Hotel. Book exhibits were located in the ballroom on the Ballroom Floor. The Mathematical Sciences Employment Register was maintained in Parlors O, P, and Q. The Sheraton-Gibson Hotel was the official hotel for the meeting with the overflow crowd being accommodated at the Netherlands Hilton and the Sinton just around the corner.

At 6:00 P.M. on Wednesday, a bar was opened in the Roof Garden of the Sheraton-Gibson Hotel to give members and their guests a chance to be together. At 7:00 P.M. following the cocktail hour, a complimentary dinner for 600 was served in the Roof Garden for members and adult guests of the Association and Society. Tickets were available at the registration desk on a first come, first served basis for a service charge of $1.50 each. The dinner was made possible through the generosity of numerous local educational and industrial institutions, and life insurance companies. Professor Marshall Hall, Jr., acted as Toastmaster. A welcome on behalf of the University of Cincinnati was extended by President Walter C. Langsam. There were also brief addresses by Professor Deane Montgomery, as President of the American Mathematical Society, Professor A. W. Tucker, as President of the Mathematical Association of America, and Professor C. B. Allendoerfer.

HENRY L. ALDER, *Secretary*

REPORT OF THE TREASURER FOR THE YEAR 1961

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1961. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

The Treasurer is pleased to report a substantial increase in the General Fund due to a surplus in the Current Fund, contributions from members of the Association and an increase in the market value of the securities held by the Association. This fund serves as the endowment fund of the Association and an increase in the fund serves to insure the financial stability of the Association.

There has been a considerable decrease in the temporary funds held by the Treasurer. All funds held on behalf of the Conference Board of the Mathematical Sciences have been turned over to that organization. The balances in several other funds have decreased as the funds granted have been expended to support some of the activities conducted by the Association.

ASSETS OF THE ASSOCIATION	JANUARY 1, 1961	DECEMBER 31, 1961
M & T Trust Company, Buffalo	$119,699	$ 42,152
Savings Accounts	89,550	6,017
Securities	388,041	415,603
	$597,290	463,772
FUNDS OF THE ASSOCIATION		
Current Fund	$ 4,316	$ 1,086
MATHEMATICS MAGAZINE	—	861
Carus Fund	35,066	42,664
Chace Fund	9,062	7,157
Houck Fund	11,012	12,236
Chauvenet Fund	1,671	1,850
Dunkel Fund	21,410	26,711
General Fund	77,316	148,285
	$159,853	$240,850

Visiting Lecturers	$ 79,580	$ 39,499
Secondary School Lecturers	52,091	72,433
CUPM (Ford Grant)	41,340	20,759
CUPM (NSF Grant)	164,809	71,567
High School Contests (Deficit)	5,828	1,581
Survey Non-Teaching Mathematical Employment	3,768	2,928
Production of Films	8,711	13,505
Washington Office	48,732	—
Survey European Mathematical Education	3,980	650
Institutes	10,254	—
Conference Board Survey	30,000	—
	$597,290	$463,772

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1962

OFFICERS

President, A. W. TUCKER, Princeton University (1961–1962)
First Vice-President, H. S. M. COXETER, University of Toronto (1962–1963)
Second Vice-President, R. A. ROSENBAUM, Wesleyan University (1961–1962)
Editor, F. A. FICKEN, New York University (1962–1966)
Secretary, H. L. ALDER, University of California, Davis (1960–1964)
Treasurer, H. M. GEHMAN, University of Buffalo (1958–1962)
Associate Secretary, L. J. MONTZINGO, JR., University of Buffalo (1958–1962)

General Offices: University of Buffalo, Buffalo 14, New York.
Executive Director, H. M. GEHMAN.

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

W. L. DUREN, Jr., University of Virginia (1957–1962)
G. B. PRICE, University of Kansas and Conference Board of the Mathematical Sciences, Washington, D. C. (1959–1964)
C. B. ALLENDOERFER, University of Washington (1961–1966)

Elected Members of Finance Committee (not already otherwise members of the Board):

E. A. CAMERON, University of North Carolina (1962–1965)

Governors at Large

T. M. APOSTOL, California Institute of Technology (1962–1964)
R. C. BUCK, University of Wisconsin (1960–1962)
J. G. KEMENY, Dartmouth College (1960–1962)
P. R. HALMOS, University of Michigan (1961–1963)
J. L. KELLEY, University of California, Berkeley (1961–1963)
HANS RADEMACHER, University of Pennsylvania (1962–1964)

Sectional Governors (July 1, 1959–June 30, 1962)

Illinois, ROTHWELL STEPHENS, Knox College
Iowa, H. T. MUHLY, State University of Iowa
Louisiana-Mississippi, ARTHUR OLLIVIER, Mississippi State College
Maryland-D. C.-Virginia, R. P. BAILEY, Naval Academy, Annapolis
Michigan, R. M. THRALL, University of Michigan
Minnesota, W. S. LOUD, University of Minnesota
Philadelphia, ALBERT WILANSKY, Lehigh University
Southern California, P. B. JOHNSON, University of California, Los Angeles
Texas, W. T. GUY, JR., University of Texas

Sectional Governors (July 1, 1960–June 30, 1963)

Allegheny Mountain, A. B. CUNNINGHAM, West Virginia University
Indiana, M. E. SHANKS, Purdue University
Kentucky, W. C. ROYSTER, University of Kentucky
Metropolitan New York, C. T. SALKIND, Polytechnic Institute of Brooklyn
Nebraska, EDWIN HALFAR, University of Nebraska
Northern California, C. D. OLDS, San Jose State College
Oklahoma, O. H. HAMILTON, Oklahoma State University
Rocky Mountain, M. L. MADISON, Colorado State University
Wisconsin, R. D. WAGNER, University of Wisconsin

Sectional Governors (July 1, 1961–June 30, 1964)

Kansas, PAUL EBERHART, Washburn University
Missouri, R. J. MICHEL, Southeast Missouri State College
New Jersey, H. O. POLLAK, Bell Telephone Laboratories, Inc.
Northeastern, D. E. RICHMOND, Williams College
Ohio, R. R. STOLL, Oberlin College
Pacific Northwest, D. C. MURDOCH, University of British Columbia
Southeastern, H. S. THURSTON, University of Alabama
Southwestern, J. B. GIEVER, New Mexico State University
Upper New York State, HARRIET F. MONTAGUE, University of Buffalo.

COMMITTEES OF THE ASSOCIATION*

EXECUTIVE COMMITTEE

A. W. TUCKER, *Chairman* (1961–1962); H. L. ALDER (1960–1964), H. S. M. COXETER (1962–1963), F. A. FICKEN (1962–1966), H. M. GEHMAN (1958–1962), R. A. ROSENBAUM (1961–1962), all *ex officio*.

* Terms of office of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees no terms of office are listed, since they are automatically discharged at the expiration of the President's term of office which is at the Annual Meeting in January 1963.

FINANCE COMMITTEE

A. W. TUCKER, *Chairman* (1961–1962), *ex officio*; C. B. ALLENDOERFER (1961–1963), E. A. CAMERON (1962–1965), H. L. ALDER (1960–1964), *ex officio*, H. M. GEHMAN (1958–1962), *ex officio*.

ADVISORY COMMITTEE FOR A SURVEY OF NONTEACHING MATHEMATICAL EMPLOYMENT

MORRIS OSTROFSKY, *Chairman*; PAUL ARMER, T. E. CAYWOOD, CHURCHILL EISENHART, WALLACE GIVENS, A. S. HOUSEHOLDER, Z. I. MOSESSON, G. B. PRICE, G. B. THOMAS, JR.

COMMITTEE ON ADVISEMENT AND PERSONNEL

J. S. FRAME, *Chairman* (1961–1963); A. H. BOWKER (1961–1962), C. R. PHELPS (1961–1963), MINA S. REES (1961–1963), S. A. ROBERTSON (1961–1962), C. E. SEALANDER (1961–1963).

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

WILLIAM FELLER, *Chairman* (1960–1962); R. H. BING (1962–1964), IVAN NIVEN (1961–1963).

COMMITTEE ON HIGH SCHOOL CONTESTS*

C. T. SALKIND, *Chairman* (1961–1963); W. H. FAGERSTROM, Director, H. L. ALDER (1961–1964), A. J. COLEMAN (1961–1962), W. H. SCHMIDT (1961–1963), SISTER MARY FELICE (1961–1962), E. E. STROCK (1961–1964).

COMMITTEE ON HIGH SCHOOL TEACHER'S CONTEST

HARLEY FLANDERS, *Chairman*; L. A. HENKIN, P. S. JONES, E. E. MOISE, R. E. K. ROURKE, H. W. SYER.

COMMITTEE ON INSTITUTES

E. A. CAMERON, *Chairman* (1961–1962); E. G. BEGLE (1961–1963), W. T. GUY, JR. (1961–1962), WILLIAM H. L. MEYER (1961–1962), D. A. PAGE (1961–1963).

COMMITTEE ON PRODUCTION OF FILMS

L. A. HENKIN, *Chairman*; C. B. ALLENDOERFER, E. G. BEGLE, L. W. COHEN, GEORGE SPRINGER, R. L. WILDER.

COMMITTEE ON PUBLICATIONS

R. P. DILWORTH, *Chairman* (1962–1964); R. P. BOAS (1962–1964), C. W. CURTIS (1962–1964), W. E. FERGUSON (1962–1964), W. T. GUY, JR. (1960–1962), I. I. HIRSCHMAN, JR. (1960–1962), R. D. JAMES (1962–1964), IVAN NIVEN (1960–1962), F. A. FICKEN (1962–1966), *ex officio*, R. E. HORTON (1961–1963), *ex officio*, H. M. GEHMAN (1961–1962), *ex officio*.

Subcommittee on Carus Monographs: IVAN NIVEN, *Chairman* (1960–1962); R. P. BOAS (1962–1964).

Subcommittee on MAA Studies in Mathematics: C. W. CURTIS, *Chairman* (1962–1964); R. D. JAMES (1962–1964).

Subcommittee on Slaught Papers and Miscellaneous Publications: I. I. HIRSCHMAN, JR., *Chairman* (1960–1962); W. E. FERGUSON (1962–1964), W. T. GUY, JR. (1960–1962).

Subcommittee on Expository Writing: R. P. DILWORTH, *Chairman* (1962–1964); C. W. CURTIS (1962–1964), F. A. FICKEN (1962–1966), all *ex officio*.

* Terms of office of members of this committee expire on August 31 of last year of service listed.

COMMITTEE ON RETENTION OF HIGH SCHOOL MATHEMATICS TEACHERS

ROY DUBISCH, *Chairman*; R. E. GASKELL, EDWIN HALFAR, B. E. MESERVE, E. D. NICHOLS, HENRY SWAIN.

COMMITTEE ON SECONDARY SCHOOL LECTURERS

C. O. OAKLEY, *Chairman* (1960–1962); R. N. BRADT (1962–1964), W. E. BRIGGS (1961–1963), MRS. JEWELL H. BUSHEY (1961–1963), H. M. GELDER (1962–1964), J. G. HERRIOT (1961–1963), MISS J. M. HILL (1962–1964).

COMMITTEE ON SECTIONS

L. J. MONTZINGO, JR., *Chairman* (1958–1962), *ex officio*; I. L. BATTIN (1959–1962), P. B. JOHNSON (1961–1964), A. W. MCGAUGHEY (1959–1963), MRS. A. B. WALTCHER (1962–1965).

COMMITTEE ON THE AWARD FOR DISTINGUISHED SERVICE TO MATHEMATICS

R. J. WALKER, *Chairman* (1960–1962); MINA S. REES (1962–1964), G. B. THOMAS, JR. (1961–1963).

COMMITTEE ON THE CHAUVENET PRIZE FOR THE PERIOD 1959–1961

WALTER RUDIN, *Chairman*; ERNST SNAPPER, R. L. WILDER.

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

D. E. RICHMOND, *Chairman* (1960–1962); L. E. BUSH, Director (1958–1962), J. M. H. OLMSTED (1961–1963), GIAN-CARLO ROTA (1962–1964).

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

R. C. BUCK, *Chairman* (1959–1962); E. G. BEGLE (1959–1963), ROY DUBISCH (1962–1964), DAVID GALE (1962–1964), W. T. GUY, JR. (1959–1962), L. A. HENKIN (1962–1964), J. G. KEMENY (1959–1963), E. E. MOISE (1962–1964), J. C. MOORE (1962–1964), FREDERICK MOSTELLER (1959–1962), H. O. POLLAK (1959–1962), G. B. PRICE (1959–1963), I. M. SINGER (1962–1964), PATRICK SUPPES (1959–1963), R. J. WALKER (1959–1963), A. D. WALLACE (1959–1962), R. J. WISNER, *ex officio*.

COMMITTEE ON VISITING LECTURERS

R. A. ROSENBAUM, *Chairman* (1959–1962); R. E. GASKELL (1962–1964), R. E. JOHNSON (1959 1962), SEYMOUR SCHUSTER (1962–1964), C. L. SEEBECK, JR. (1960–1962), ROTHWELL STEPHENS (1962–1964), D. W. WESTERN (1962–1964).

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES*

R. M. THRALL, *Chairman* (1959–1962, AMS); E. K. RITTER (1960–1963, SIAM), J. W. WEIHE (1961–1964, MAA).

JOINT COMMITTEE ON PLACES OF MEETINGS

G. L. WALKER, *Chairman*; H. L. ALDER, H. M. GEHMAN, J. W. GREEN, all *ex officio*.

* Terms of office of members of this committee expire on February 28 of last year of service listed.

JOINT COMMITTEE ON THE DOCTOR OF ARTS DEGREE

E. E. MOISE, *Chairman*; H. M. DAY, P. R. HALMOS, A. D. WALLACE.

NOMINATING COMMITTEE FOR 1962

J. C. OXTOBY, *Chairman*; W. L. DUREN, JR., R. S. PIERCE.

PLANNING COMMITTEE FOR THE FIFTIETH ANNIVERSARY OF THE ASSOCIATION

C. B. ALLENDOERFER, *Chairman*; H. L. ALDER, H. M. GEHMAN, F. L. GRIFFIN, C. A. HUTCHINSON, E. J. MCSHANE, R. J. WALKER.

REPRESENTATIVES OF THE ASSOCIATION

On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
P. S. JONES (1960–1962).
On the American Council on Education:
H. L. ALDER, *ex officio*, A. W. TUCKER, *ex officio*.
On the Conference Board of the Mathematical Sciences:
H. L. ALDER, *ex officio*, A. W. TUCKER, *ex officio*.
On the Council of the American Association for the Advancement of Science:
S. S. CAIRNS (1960–1962), B. W. JONES (1962–1964).
On the Governing Council of Mu Alpha Theta:
R. A. GOOD (1961–1963).
On the National Research Council:
W. L. DUREN, JR. (July 1, 1959–June 30, 1962).
On the National Society of Professional Engineers' Committee on the Professional Engineer's Merging Role with the Scientist in Technology:
MORRIS OSTROFSKY
On the U. S. Commission on Mathematical Instruction:
HENRY VAN ENGEN (July 1, 1959–June 30, 1962), R. J. WALKER (July 1, 1961–June 30, 1965).

ACADEMIC AND CORPORATE MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendments adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in Cincinnati on January 24, 1962, elected to membership in the Association the first applicants for academic and corporate memberships. Approval for election to membership was given to the following 47 applicants for academic and 2 applicants for corporate membership:

Academic Membership

Adelphi College
University of Alaska
Albertus Magnus College
Auburn University
Brigham Young University
University of British Columbia
University of Buffalo
Carnegie Institute of Technology
Case Institute of Technology
Connecticut College for Women
Dartmouth College
University of Denver
Florida Presbyterian College
Fordham University
Georgetown University
Harvey Mudd College
College of the Holy Cross
University of Houston
Idaho State College
University of Illinois, Urbana
Illinois Institute of Technology

University of Kansas
Kansas State Teachers College, Emporia
Knox College
Marquette University
Massachusetts Institute of Technology
University of Miami, Florida
University of Michigan
Michigan State University
University of Mississippi
Monmouth College, New Jersey
University of New Hampshire
New Mexico State University
Occidental College
Ohio Wesleyan University
Princeton University
University of Rochester
St. Bonaventure University
Seattle University
University of South Carolina
Vanderbilt University
Vassar College
Western Michigan University
Western Reserve University
College of William and Mary in Virginia
Williams College
Yeshiva University

Corporate Membership

Boeing Scientific Research Laboratories
International Business Machines Corporation

HENRY L. ALDER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

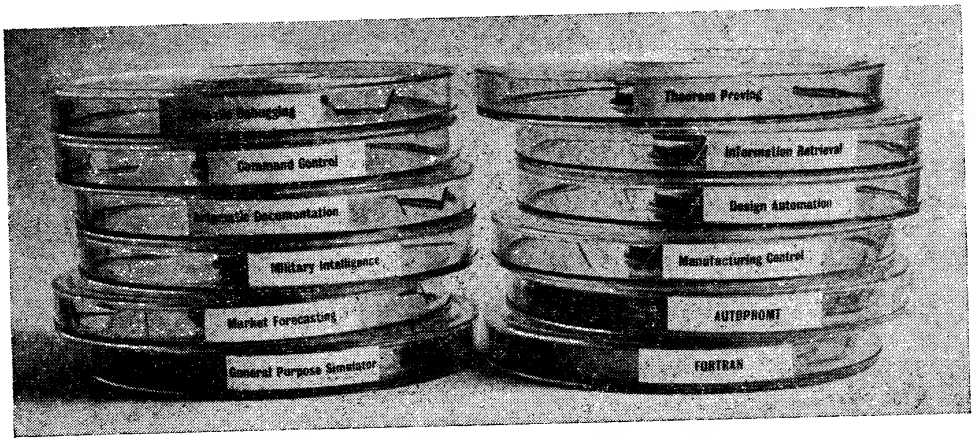
Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, May 5, 1962.
ILLINOIS, North Central College, Naperville, May 11–12, 1962.
INDIANA, Butler University, Indianapolis, May 5, 1962.
IOWA
KANSAS, Bethel College, North Newton, April 28, 1962.
KENTUCKY, University of Kentucky, Lexington, Spring, 1962.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, U. S. Naval Weapons Laboratory, Dahlgren, Virginia, April 28, 1962.
METROPOLITAN NEW YORK, University Heights Campus, New York University, May 12, 1962.
MICHIGAN
MINNESOTA, College of St. Teresa, Winona, May 12, 1962.
MISSOURI, Missouri School of Mines, Rolla, April 28, 1962.
NEBRASKA
NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.
NORTHEASTERN, November 24, 1962.
NORTHERN CALIFORNIA
OHIO, Muskingum College, New Concord, May 5, 1962.
OKLAHOMA
PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.
PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.
ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, May 4–5, 1962.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN, University of New Mexico, Albuquerque, April 27–28, 1962.
TEXAS
UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, May 5, 1962.
WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

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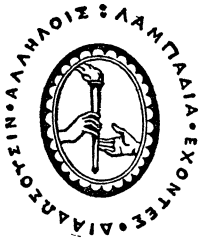
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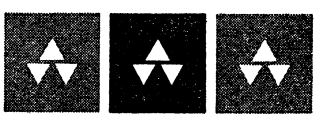
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RECOMMENDATIONS ON THE UNDERGRADUATE MATHEMATICS PROGRAM FOR ENGINEERS AND PHYSICISTS

The Committee on the Undergraduate Program in Mathematics (CUPM)* is a committee of the Mathematical Association of America and is supported in part by the National Science Foundation. The general purpose of this committee is to develop a broad program of improvement in the undergraduate mathematics curriculum of the nation's colleges and universities.

The CUPM Panel on Physical Sciences and Engineering† has been primarily concerned with the curriculum as it relates to the training of engineers and physicists. In this report we present our recommendations on required and optional mathematics courses for students majoring in these fields. Later reports will deal with other aspects of scientific training.

To help bring about the curriculum changes implied by these recommendations the Panel encourages appropriate conferences, experimental programs and textbook writing, and will support all worthy efforts in these directions.

Background. One reason for the current effort on the undergraduate program is the rapid change in the mathematical world and in its immediate surroundings. Three aspects of this change have a particular effect on the undergraduate training in physical sciences and engineering. The first is the work being done in improving mathematics education in the secondary school. Several programs for this have already had considerable effect and can be expected to have a great deal more. Not only can we hope that soon most freshmen expecting to take a scientific program will have covered precalculus mathematics, but, perhaps more important, they will be accustomed to care and precision of mathematical thought and statement. Of course, not all students will have this level of preparation in the foreseeable future, but the proportion will be large enough to enable us to plan on this basis. Students with poorer preparation may be expected to take remedial courses without credit before they start the regular program.

This improved preparation obviously means that we will be able to improve the content of the beginning calculus course, since topics which take time in the first two years will have been covered earlier. More than that, however, it means that the elementary calculus course will have to take a more sophisticated attitude in order to keep the student from laughing at a course in college which is less careful mathematically than its secondary-school predecessors.

The second aspect of change in mathematics which confronts us is the expansion in the applications of mathematics. There is a real "revolution" in engineering—perhaps "explosion" is an even better description than "revolu-

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tion," because, as it turns out, several trends heading in different directions are simultaneously visible. One is a trend toward basic science. The mathematical aspect of this trend is a strengthening of interest in more algebraic and abstract concepts. An orthogonal trend is one towards the engineering of large systems. These systems, both military and nonmilitary, are of ever-increasing complexity and must be optimized with regard to such factors as cost, reliability, maintenance, etc. Resulting mathematical interests are probability and statistics, and linear algebra. A further trend, in part a consequence of the preceding two, is a real increase in the variety and depth of the mathematical tools which interest the engineer. In general, engineers are finding that they need to use new and unfamiliar mathematics of a wide variety of types.

A third factor is the arrival of the electronic computer. It is having its effect on every phase of science and technology, all the way from basic research to the production line. In mathematics it has, for one thing moved, some techniques from the abstract to the practical field; for example, some series expansions, iterative techniques, and so forth. Then too, computers have led people to tackle problems they would never have considered before, such as large systems of linear equations, linear and nonlinear programing, and Monte Carlo methods. Many of these new techniques require increased sophistication in mathematics.

An additional factor entering from another direction must also be mentioned. Mathematicians in the United States have in recent years become much more closely involved with areas adjacent to their own research. Of the many factors which enter here, we may mention the greatly increased interest of mathematicians at all levels in education, the rapid growth of mathematical employment in industry, the spread of research and consulting contracts into the universities, and the development of a number of mathematical disciplines, such as information theory, that have many applications but are not classical applied mathematics. There is thus a real desire among mathematicians and physical scientists to cooperate in matters of education.

The above conclusions, and the recommendations that constitute the body of this report, were formulated by the Panel after extensive consultation with mathematicians, physicists, and engineers. In engineering, in particular, representatives of many fields and many types of institutions were consulted,* as well as officials of the American Society for Engineering Education. The Recommendations for Physicists were drawn up in close collaboration with the Commission on College Physics.

In considering the recommendations which follow, it is crucial to examine what has been our attitude toward certain ideas which inevitably occupy a

* Some of the results of a conference with engineers are embodied in four addresses delivered at a Conference on the Engineering Curriculum which was held under the auspices of this Panel in March, 1961. These addresses were published in the December, 1961, (vol. 52, no. 3) issue of the *Journal of Engineering Education*, and copies may be obtained from Professor R. J. Wisner, CUPM, Michigan State University Oakland, Rochester, Michigan.

central position in any discussion of mathematical education. Among these are mathematical sophistication and mathematical rigor, motivation, and intuition. Now it is a fact that mathematical rigor—by which we mean an attempt to prove essentially everything that is used—is not the way of life of the physicist and the engineer. On the other hand, mathematical sophistication—which means to us careful and clear mathematical statements, proofs of many things, and generally speaking a broad appreciation of the mathematical blocks from which models are built—is desired by, and desirable for, all students preparing for a scientific career. How does one choose what is actually to be proved? It seems to us that this is related to the plausibility of the desired result. It is unwise to give rigor to either the utterly plausible or the utterly implausible, the former because the student cannot see what the fuss is all about, and the latter because the most likely effect is rejection of mathematics. The moderately plausible and the moderately implausible are the middle ground where we may insist on rigor with the greatest profit; the great danger in the overzealous use of rigor is to employ it to verify only that which is utterly apparent.

Let us turn next to the subject of motivation. Motivation means different things to different people, and thus requires clarification. One aspect of motivation is concerned with the difference between mathematics and the applications of mathematics, between a mathematical model and the real world. For many engineers and physicists, motivation of mathematical concepts can be supplied by formulating real situations which lead to the construction of reasonable models that exhibit both the desirability and the usefulness of the mathematical concept. Thus, motion of a particle, or growth of a bacterial culture, may be used as physical motivation for the notion of a derivative. It is also possible, of course, to give a *mathematical* motivation for a new mathematical concept; the geometric notion of a tangent to a curve also leads to the notion of derivative, and is quite enough motivation to a mathematician. Since both kinds of motivation are meaningful to a large group of students, we feel that both should appear whenever relevant. It is certainly a matter of individual taste whether one or both motivations should precede, or perhaps follow, the presentation of a mathematical topic. In both cases, however, it is necessary to be very clear in distinguishing the motivating mathematical or physical situation from the resulting abstraction.

Physical and mathematical examples which are used as motivation, as well as previous mathematical experience, help to provide *intuition* on the mathematical concept under consideration. By "intuition" we mean a process of guessing the mathematical properties and limitations of a mathematical abstraction by analogy with known properties of the mathematical or physical objects which motivated this abstraction. Intuition should lead the way to rigor whenever possible; neither can be exchanged or substituted for the other in the development of mathematics.

A mathematics course for engineers and physicists must involve the full spectrum from motivation and intuition to sophistication and rigor. While the

relative emphasis on these various aspects will forever be a subject for debate, no mathematics course is a complete experience if any of these is omitted.

Introduction to Recommendations. This report presents a program for the undergraduate mathematical training of engineers and physicists.

Obviously, no single program of study can be the best one for all types of students, all institutions, and all times. It is therefore important that anyone expecting to make use of the present recommendations understand the assumptions underlying them. The following comments should make these assumptions clear, and also explain some other features of the recommendations.

1. This is a program for *today*, not for several years in the future. Programs somewhat like this are already being given at various places, and the sample courses we outline are patterned after existing ones. We assume a good but not unusual background for the entering freshman.

Five or ten years from now the situation will undoubtedly be different—in the high schools, in research, in engineering practice, and in such adjacent areas as automatic computation. Such differences will necessitate changes in the mathematics curriculum, but a good curriculum can never be static, and it is our belief that the present proposal can be continually modified to keep up with developments. The material encompassed here will certainly continue, however, to be an important part of the mathematical training needed by engineers and physicists.

2. The programs we recommend may seem excessive in the light of what is now being done at many places, but it is our conviction that this is the minimal amount of mathematics appropriate for students who will be starting their careers four or five years from now. We recognize that some institutions may simply be unable to introduce such a program very soon. We would hope that such places regard the program as something to work toward. The members of the Panel are willing to be consulted on such matters and to give what advice they can as to a suitable modification of the basic program to fit special needs.

3. Beyond the courses *required* of all students there must be available considerable flexibility to allow for variations in fields and in the quality of students. The advanced material whose availability we have recommended can be regarded as a main stem that may have branches at any point. Also, students may truncate the program at points appropriate to their interests and abilities.

4. The order of presentation of topics in mathematics and some related courses is strongly influenced by two factors:

a. The best possible treatment of certain subjects in engineering and physics requires that they be preceded by certain mathematical topics.
b. Topics introduced in mathematics courses should be used in applications as soon afterwards as possible.

To attain these ends, coordination among the mathematics, engineering, and physics faculties is necessary, and this may lead to course changes in all fields.

5. The Recommendations are of course the responsibility of CUPM. In cases where it seems of interest and is available, we have indicated the reaction of the groups of engineers and physicists who were consulted. For convenience we will refer to them as "the consultants."

List of Recommended Courses. It is desirable that all calculus prerequisites, including analytic geometry, be taught in high school. At present it may be necessary to include some analytic geometry in the beginning analysis course, but all other deficiencies should be corrected on a noncredit basis.

The following courses should be available for undergraduate majors in engineering and physics:

1. *Beginning analysis* (12 semester hours). As far as general content is concerned this is a relatively standard course in calculus and differential equations. There can be many variations of such a course in matters of rigor, motivation, arrangements of topics, etc., and textbooks have been and are being written from several different points of view. Beyond the specific items mentioned below we make no recommendations on these matters.

The course should contain the following topics:

a. An intuitive introduction of four to six weeks to the basic notions of differentiation and integration. This serves the dual purpose of filling in the student's intuition for the more sophisticated treatment to come, and preparing for immediate applications to physics.
b. Theory and technique of differentiation and integration of functions of one real variable, with applications.
c. Infinite series, including Taylor series expansions.
d. A brief introduction to differentiation and integration of functions of two or more real variables.
e. Topics in differential equations, including the following: linear differential equations with constant coefficients and first order systems—linear algebra (including eigenvalue theory, see 2 below) should be used to treat both homogeneous and nonhomogeneous problems; first order linear and nonlinear equations, with Picard's method and an introduction to numerical techniques.
f. Some attempt should be made to fill the gap between the high school algebra of complex numbers and the use of complex exponentials in the solution of differential equations. In particular some work on the calculus of complex valued functions of a real variable should be included in items b and c.
g. Students should become familiar with vectors in two and three dimensions and with the differentiation of vector valued functions of one variable. This material can obviously be correlated with the course in linear algebra (see below).
h. Theory and simple techniques of numerical computation should be introduced where relevant. Further comments on this point, applying to the whole program, are made at the end of this list of courses.

We feel that the above comments on beginning analysis describe sufficiently a familiar course. The remaining courses in our list are less generally familiar. Hence the brief descriptions of courses 2 through 9 are supplemented in an Appendix* by detailed outlines of sample courses of the kind we have in mind.

2. *Linear algebra* (3 semester hours). A knowledge of the basic properties of n-dimensional vector spaces has become imperative for many applied fields as well as for progress in mathematics. Since this subject is so fundamental and since its development makes no use of the concepts of calculus, it should appear very early in the student's mathematical training. We recommend a course with strong emphasis on the geometrical interpretation of vectors and matrices and with applications to mathematics (see items 1-d and 1-g above), physics and engineering. Topics should include the algebra and geometry of vector spaces, linear transformations and matrices, linear equations (including computational methods), quadratic forms and symmetric matrices, and elementary eigenvalue theory.

It may be desirable, for mathematical or scheduling purposes, to combine beginning analysis and linear algebra into a single coordinated course to be completed in the sophomore year.

3. *Probability and statistics* (3 semester hours). Basic topics in probability theory, both discrete and continuous, as well as an introduction to statistics, are essential to the training of the modern engineer in every branch of engineering. We recommend a course based on the notions of sample spaces and random variables, and including, *inter alia*, an introduction to limit theorems and stochastic processes, and to estimation and hypothesis testing. Certain fields of engineering (e.g., electrical, industrial) may require an additional three semester hours.

4. *Functions of several variables* (3 semester hours). This course, continuing the material of item 1-d, covers the calculus of vector functions of one variable, of functions of several variables, and of vector fields, and concludes with a brief introduction to boundary value problems. Examples from the physical world should be used whenever possible as motivation and applications.

5. *Ordinary differential equations* (3 semester hours). This intermediate course continues the work of item 1-e into further topics important to applications, including linear equations with variable coefficients, boundary value problems, rudimentary existence theorems, asymptotics, and an introduction to nonlinear problems. We recommend especially an interest in the qualitative behavior of solutions and numerical processes as well as in analytic techniques.

6. *Functions of a complex variable* (3 semester hours). This course presup-

* Available on request from Professor R. J. Wisner, CUPM, Michigan State University Oakland, Rochester, Michigan.

poses somewhat more mathematical maturity than courses 4 and 5 and so would ordinarily be taken after them, even though they are not prerequisites as far as subject matter is concerned. In addition to the usual development of integrals and series, there should be material on multivalued functions, contour integration, conformal mapping, and integral transforms.

7. *Partial differential equations* (3 semester hours). Derivation, classification, and solution techniques of boundary value problems.

8. *Elements of real variable theory* (3 semester hours). A rigorous treatment of basic topics in the theory of functions of a real variable.

9. *Algebraic structures* (3 semester hours). The theory of groups, rings, and fields.

The development of high-speed computers has made it necessary for the appliers of mathematics to understand the path from mathematical theory through programing logic to numerical results. Mathematics courses for engineers and physicists should be aware of this need. In particular, courses 1 through 7 should contain, where it is suitable and applicable, mathematical topics motivated by the desire to relate mathematical understanding to computation. It is especially desirable that the student see the possibility of significant advantage in combining analytical insight with numerical work. Indications of such opportunities are scattered throughout the course outlines in the Appendix.

The above list of courses is the result of careful consideration by the Panel and the consultants. The brief description given here and the detailed sample courses in the Appendix are based rather strictly on the interests of engineers and physicists. We realize that the nature of the institution and the requirements of other users of mathematics as well as of the mathematics majors may influence the specific offerings.

Recommended Program for Engineers.

A. *Courses to be required of all students.*

1. Beginning analysis. This recommendation needs no comment.
2. Linear algebra. The great majority of the consultants felt that this is important material that all engineers should have during the first two years.
3. Probability and statistics. All students should have at least a three-semester hour course in probability and statistics. *A six-semester hour course should be available for students with special interests.* The consultants agreed on the value of probability and statistics to an engineer, but there was considerable disagreement among the consultants as to the advisability of requiring them of all students. However, the members of our Panel are unanimously and strongly of the opinion that these subjects will soon pervade all branches of engineering and that now is the time to begin preparing students for this development.

B. *Courses recommended for students intending to go into research and development.*
 4. Functions of several variables.
 5. Ordinary differential equations.
 6. Functions of a complex variable.

 The consultants agreed to the value of this material, but some preferred that it be completed within the junior year. The Panel is convinced that an adequate presentation requires a minimum of nine semester hours, which could, of course, be taken in one year if desired. The order in which courses 4 and 5 are taken is immaterial, except as they may be coordinated with other courses. If they are to be presented to the students in a fixed order, one may wish to take advantage of this by some adjustment of the time schedules and choice of topics.

C. *Courses which should be available for theoretically minded students capable of extended graduate study.*
 7. Partial differential equations.
 8. Elements of real variable theory.

D. There was considerable discussion of the complex of problems involving the design of systems. Such problems concern the achievement of desired performance at minimum cost, maximum reliability, etc. The special mathematical topics appropriate to this field are linear and dynamic programing, game theory, and calculus of variations. There is at present much doubt as to the desirability of these topics at the undergraduate level, and we make no recommendation concerning them.

E. The computing area also received much discussion. The Committee and the consultants agreed that the computational implications associated with the various mathematical topics are best integrated in courses in these topics. We therefore make no recommendation for a separate course in the computational area.

Recommended Program for Physicists.

A. *Courses to be required of all students.*
 1. Beginning analysis.
 2. Linear algebra. Like the engineers, the physicists felt that this material is essential.
 4. Functions of several variables. This should be taken in the sophomore year if possible, and in any event no later than the first part of the junior year.
 5. Ordinary differential equations.

B. *Additional courses, in order of preference.* Students contemplating graduate work should be required to take a minimum of three to nine semester hours of these courses.
 3. Probability and statistics. The value of requiring this course in the undergraduate training of all physicists is not as well established as it is for engineers.

6. Functions of a complex variable.
8. Elements of real variable theory.
9. Algebraic structures. A student would ordinarily take only one of these last two courses, but the consultants were strong in the belief that a good physicist should have at least one course in abstract mathematics to see how a mathematician thinks and to experience the limitations of non-rigorous reasoning.
7. Partial differential equations.

C. The undergraduate physicist has less concern with computing techniques than does the engineer. The same general principle holds, however, that the computational viewpoint should appear in mathematics courses wherever appropriate.

EDUCATION FOR APPLIED MATHEMATICS*

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At present there are three ways in which a mathematically trained person can earn a living. These are either teaching, or the application of statistics, or employment in the government or industry as a mathematician. It will be convenient to confine the term "applied mathematics" to this last possibility, although the words themselves could have wider significance. The appearance of applied mathematics as a possible profession is a postwar development and is associated with the availability of large scale automatic computing machines. These machines have permitted the application of a wide variety of mathematical concepts by eliminating the burden of computation and by providing the necessary accuracy.

It seems to me to be quite essential that this new vocation be established on a proper professional level. At present trained personnel is in short supply and, naturally, efforts are being made to utilize people of limited educational background. The various attempts to justify this obscure the fact that the results obtained are quite poor.

One main hope of a counter-trend would be in proper education. Here again, however, there are weaknesses and the basic problem is one that applies to a much broader area than education for applied mathematics. There are serious faults in the present position of science in education and most of the difficulties

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associated with education for applied mathematics are aspects of certain more universal problems. It is worthwhile, then, to examine briefly the broader situation.

Consider, then, the relation of education to our culture as a whole. One aspect of a culture consists of the skills and technology which it utilizes for the production of the necessities and niceties of life, and it is clear that education must always have the responsibility of maintaining the technology of the culture of which it is a part. In many cultures, this aspect of education is taken care of by an apprentice-type procedure in which the apprentice follows the action of the master or journeyman, and a purely imitative action may be adequate.

On the other hand, our present western civilization involves a tremendous technology which is based on immense scientific knowledge. This technology changes rapidly and a purely imitative procedure would soon become obsolete. Education for such a scientific technology must include, therefore, a fundamental understanding of the scientific knowledge on which the technology is based. The student is given an understanding of the situation and decides on the appropriate action. The distinction between professional action and subprofessional action is precisely in this, that professional action is based on an understanding of the situation to be dealt with, and subprofessional action is based on following rules laid down by others and for which understanding is not necessary. In many technological fields this distinction is clear—for instance, the difference between an electronics engineer and a repairman.

In our rapidly changing technology, the difficulty with subprofessional activity is that specific techniques become obsolescent. There is a considerable effort to try to lower activities in applied scientific fields from the professional to the subprofessional level for reasons of available manpower and economics. Efforts are made to induce people to enter these fields at the subprofessional level with immediate rewards. This can be, however, a critical disservice to these people because the techniques they learn may well become obsolete and, since their training does not permit immediate readjustment, they may be in difficulty. Because these people do not receive all the education they could effectively use, this situation also affects adversely the total capability of the country. This is a very serious matter in applied mathematics and I am strongly opposed to the efforts to introduce people into this field at the subprofessional level.

Now, education always has the responsibility of maintaining the technology of its culture, but this is by no means the only aim of education. For education should permit one to apprehend all the intellectual achievements of the human race, both past and present and should give to the individual the possibilities of an intellectual life rooted where he wishes in this complex. It is important, therefore, to appreciate that scientific education should not only be subject to technological needs, but should also be concerned with the fact that scientific knowledge represents perhaps the most extraordinary intellectual achievement of the human race. The understanding of nature that is represented by science is something extremely valuable in itself and would be of great educational

importance even if rockets and satellites and airplanes and automobiles, and all these other gadgets were not available.

Science, indeed, should be viewed as a broad panorama and the physical sciences represent one particular range of peaks of achievement. Mathematics is associated with these physical sciences in a very special way, and other branches of knowledge have viewed this association with a certain envy. Of course, the technological consequences of science are based on all the sciences and, therefore, knowledge of these is necessary for technological maintenance and advancement. But, since we are concerned with applied mathematics, we will emphasize the relationship of mathematics and the physical sciences.

There are three ways in which mathematics and the physical sciences are related. The first relation is utilitarian. The results and information of these sciences are formulated in mathematical terms and consequently, to apply this scientific knowledge, one must use mathematics. The second way is conceptual. The ideas and procedures of these sciences have on their theoretical side a mathematical origin and the understanding of mathematical concepts is essential for an understanding of these sciences. Finally, mathematics itself is creative. It develops concepts and procedures for their own sake. It explores in this way the possibilities of the human mind. All three of the aspects are each as closely interrelated as the various tissues of a living organism. One does brutal violence to all when one tries to separate any one out. Nor can science itself be neatly butchered into mathematics, physics, chemistry, etc. The living product of the work of Archimedes, Euclid, Galileo, Newton, and Einstein is a whole which should be recognized.

The proper approach to science as a part of education, therefore, is to emphasize the investigation of nature for its own sake and the tremendous range of accomplishment and understanding which has been achieved. It should be appreciated as a whole and relative to mathematics all three aspects should be clear. This entity has a very great appeal to many young people provided it is presented in the proper light as an intellectual accomplishment and with due regard to its historical origin.

One's understanding of the individual subject is, of course, tremendously enhanced by an appreciation of its relation with various other subjects and the way in which it originated. Surely the student of mathematics should understand the way in which the concepts of mathematical analysis first appeared in our efforts to understand the concept of motion; how the development of modern ideas of linear operators was inter-related with the development of the theory of quantum mechanics and the historical relationship between topology and celestial mechanics.

In this broad picture there is a proper place for the utilitarian aspect of mathematics. It widens the contact of the student with many fields and it increases his understanding of both mathematical ideas and the sciences themselves by increasing his capability of accomplishment. It is also of great educational value. But such educational values are always associated with an in-

creased understanding. They are not associated with techniques for their own sake; i.e., techniques taught without an appreciation of their scientific basis. For example, one can consider a course devoted to the use of a specific compiler for a computer which gives no understanding of basic programming, but simply lists the rules which must be followed in order to use this compiler. Such a course may be appropriate as on-the-job training in a business establishment. It is of no educational value, however, and its practical value will disappear with the compiler on which it is based.

It seems to me that the basic discussion I have just given has a much wider professional significance than reference to any one profession would indicate. But I will now consider examples only from the use of computers which is associated with applied mathematics. It is fair to say that the use of computers is typical of many technological fields, and is greatly expanding. By reason of its contact with other subjects—in particular, mathematics—the computer is affected by and influences many developments. It is important, of course, in mathematical education because of its relation to applied mathematics. There is every reason to believe that the number of students in applied mathematics will be as large as either of the other two fields—teaching or statistics.

An electronic computer can, in general, be applied whenever we have a logical pattern of action which can be precisely specified. The use of a computer will permit us to explore all the possibilities of this logical pattern. For example, engineering designs can often be formulated in terms of differential equations and these differential equations can be regarded as describing a pattern of infinitesimal motion. In a computer a mathematical model can be set up which has this infinitesimal pattern of motion. For design purposes this mathematical model is often far more effective than an actual working model.

In the design of a structure, one frequently needs large systems of simultaneous algebraic linear equations and the solution of such a system is essential for design procedures. Such systems occur in many other situations. The process for solving such a system can be described as a logical pattern of action and can be readily realized on a computer.

There are many other algorithms which can be realized on modern electronic computers. For example, it is possible to program the process of eliminating a variable between polynomial equations. This possibility has been known theoretically for a long time, but the amount of computation and the difficulty of avoiding errors made it only of theoretical interest until computers became available.

To utilize these procedures properly does require a good basic mathematical understanding. For instance, if one wishes to solve differential equations on a digital electronic computer, one must understand such mathematical phenomena as truncation error, round-off error and stability. If one does not have a measure of the validity of one's answer the answer is meaningless.

This applies also to the solution of simultaneous equations. One must know how ill-conditioned the system of equations is. In practice most systems are

poorly conditioned and the routine use of a formal procedure will frequently yield a meaningless result.

Thus, even from the purely utilitarian point of view the use of simply prescribed procedures without understanding is not of practical value. The field is not suitable for such subprofessional activities.

It is true that properly understood and properly controlled, a computer yields a tremendous amount of understanding of a situation which has been properly analyzed. This basic analysis, however, must be performed on a completely professional level. Where a computer is involved professional understanding must involve not only the situation which is studied but also the tools that are used. One very important set of tools are those of mathematical analysis. The other set is that of numerical analysis and computer programming. It is not practical to delegate professional responsibility. If the situation is so complex that one person cannot accept responsibility for all three aspects of the subject, it may be necessary to use a team of professional people, but the team must have joint and individual responsibility for the whole enterprise. On the proper professional level the new electronic computers represent a tremendous increase in professional capability, but this is true only if the people involved are on the proper professional level.

Of course, people with such a proper education are relatively rare and since there are such immediate tremendous demands, efforts are made to involve people at lower educational levels, thus cutting down the future supply. Then, of course, shortcuts and substitutes are repeatedly offered. Engineers are led to believe that a course in the use of a compiler is a substitute for mathematical understanding; that mere mass computation can be a substitute for mathematical understanding. None of these substitutes, however, can be effective. Ultimately the field will have to be on a proper professional level.

It is clear, therefore, that an applied mathematician needs a good basic understanding of the mathematics which he uses, but actually he needs an even broader understanding than this. He must have a good appreciation of the achievements represented by the physical sciences. He must have, therefore, a good education.

We have pointed out that there are pressures tending to encourage a subprofessional approach to applied mathematics, even though this approach is of very limited practical value. It is a grave disservice to these young people to encourage them to truncate their basic education for immediate rewards and with the opportunity for purely subprofessional work. Education in applied mathematics should proceed to the point where the student is capable of acting in a professional manner; i.e., come to his own conclusions on the basis of his understanding of the situation. Furthermore, scientific education of an appropriate intellectual character should be the basis for this education.

Many people have urged on different occasions the type of training and education which I have referred to above, but in our American culture there are tendencies which are highly unfavorable to developing this appropriate type of

education. The most obvious of these has often been discussed. There is certainly a strong trend in our society toward a very limited local culture. Many of our people find themselves bound by an horizon outlined by commercialized sports, commercialized entertainment, and commercialized social procedures. The sole effect of scientific knowledge is represented by gadgetry and even here meaningless styles are emphasized at the expense of utility. It is, of course, a form of life that is easy to accept during periods of economic prosperity, and probably one must always expect that the majority of people will form their life in a pattern of this type. The basic difficulty is, of course, not that a majority will follow such a line, but there is a strong tendency for all to follow it.

Such a limited local culture makes no intellectual demands and isolates intellectual activity from normal activities of the civilization. It is notorious, of course, that this type of local pattern has an adverse effect on education as a whole. Intellectual demands on the student are either lowered or eliminated and the school system is used for many noneducational purposes. Efforts to increase the intellectual content of education are resisted on the ground of cost—"What good is it?"—and even those who wish to improve the situation must use narrow utilitarian arguments.

This is by no means the only adverse influence relative to a proper basis for scientific education. There is a certain development in higher education which also has had unfortunate effects. This development can be most readily understood if one traces out its history. There are certain independent traditions in American higher education and there always has been a sense of responsibility for basic professional training. Many of our universities were started as seminaries and we have always had a need for professional, military and engineering training. The need to support such training represents a demand on various sciences which permits one to evaluate the capabilities of the universities in various fields. By the year 1900 a reasonably widespread network of higher education had appeared, and these new universities invited comparison with European equivalents. The comparison showed that the American counterparts were definitely inferior in many departments to the European. The comparison could be clearly made in such fields as mathematics and the physical sciences, but the actual discrepancy appeared in a much wider form. When this situation was appreciated, the universities adopted policies intended to develop competence in specialized fields. Corresponding to specific disciplines, departments were given considerable autonomy in order to permit them to achieve this objective; i.e., competence in their fields. As a result of this program the technical capabilities of American universities rose to a point which was second to none. This was, of course, a very desirable development and has resulted in a tremendous improvement in, for example, creative mathematics.

On the other hand such specialization has had regrettable effects as well. The local autonomy tends to emphasize ability in one field and to cut off the contacts of the specialists with other fields. There are other losses as well. One effect of specialization is to cut off a subject from its own historical origin. The

effort of the specialist is inevitably directed toward presenting a perfected picture of the present state of his subject. He will also try to present a single systematic treatment which, while undoubtedly of great interest from certain points of view, nevertheless tends to be disconnected from all other intellectual endeavors. The specialist wants to present a perfected, crystallized structure replacing the more characteristic living structure of the historical subject bound by many ligaments to human culture as a whole. This tendency is easy to trace in such subjects as mathematics and even in specialties within mathematics. It is interesting to note that it also appears in new subjects; in computer programming, for instance, every effort is made to give a definitive form to programming procedures even on very limited experience.

The intellectual vigor which results from specialization frequently obscures certain associated losses in educational values. Many students are dissatisfied with the isolated presentation of pristine purity, divorced from all other matters and even from its origin. It is natural to ask why this subject was considered and only a few are satisfied with the game for its own sake. In any case, isolation will result in educational loss and perhaps the worst loss is that of those who accept it.

On the other hand the tremendous conflict in which we are engaged has tended to counteract both the tendency to a local culture and the tendency toward detailed specialization. Enormous resources have been applied and, consequently, forms have become more fluid and a tremendous expansion in technical applications has occurred. These have produced many repercussions throughout both the social and the educational structure. The number of scientifically educated persons with professional status has increased tremendously in many communities. Consequently, there have been increased demands for the proper education of children by parents who are capable of appreciating the character of the education their children receive. Another interesting development is the increase in adult education, by far one of the most desirable developments both from the technical and cultural point of view. The complex demands of the government defense effort permits people who wish to despecialize to do so in increasing numbers. The appearance of new professions requires the development of crossfield education which counteracts the effect of specialization.

Of course, there is a need for intelligent direction of these counter influences. The easiest argument to use to increase the intensity of educational effort is the purely utilitarian reason. This can lead to an emphasis on techniques of no educational value rather than to an increase in basic understanding. It seems to me as if the essential element would be a complete awareness by educators of this situation.

This complex interaction of forces is characteristically American. A tendency appears which requires a certain specific action. The appropriate action takes place and then goes beyond what one may consider the proper limit. New forces appear and interact with the situation. In the specific matter which we are discussing; that is, a proper basis for scientific education, we may see a result in

which commercial interests guide educational procedures into lines which will make our new professions nothing but trivial extensions of the local barbarous culture which is often thought to be American. On the other hand the demands and interests of the new professions may give us a large proportion of highly educated and properly educated people with scientific training which would immeasurably increase the strength of our country.

We have been discussing the appearance of the profession of applied mathematics in the period following the war. It is, of course, very desirable that a proper professional level be set up for this new vocation and one readily sees that one is dealing here with a more general problem which is critical for education as a whole.

One should have as a clear-cut objective in education, the proper appreciation of science as an intellectual achievement and as an opportunity for future achievement. Successful scientific education must not be limited to merely the maintenance of a technology, but should provide for the student adequate understanding so that his actions can have a proper professional character. We have noted that relative to such an objective there are unfortunate trends toward a purely local culture with relatively narrow objectives which exercise an extremely adverse influence on education. Specialization in specific scientific fields which was a necessary development to produce competence, can also have adverse effects, isolating the subject from others, from its own historical background and cutting it off from general cultural developments. On the other hand the present struggle has released resources producing new communities which are critical of purely local educational standards and also has created exigencies which can counteract over-specialization.

Thus we see a struggle which is characterized on our side by a type of development which is peculiar to our heritage. The result is a complicated compromise between many tendencies and it is typical that one would find it very difficult to predict what the outcome would be. Our social and governmental structure has been developed on the basis of many such compromises. It is interesting to notice that the totalitarian states have been set up on the basis of one or, at the most, two violent adjustments and have resulted in a relatively simplified social structure consisting in general of two strata. The upper strata inevitably contains the technically and professionally competent personnel of the country. The remaining strata consists of a large unorganized but carefully controlled majority. The objectives of national effort, then, are formulated in the upper strata and it is now well understood that all forms of education in these states are directed toward national objectives, and their system is not subject to local diversion for other purposes. To the extent that this educational system has been maintained on egalitarian standards it is capable of producing tremendous technological results. We have also learned to appreciate the extent to which counter-specialization has appeared in advanced education in Russia in response to national objectives. Many distinguished mathematicians are associated with different applied objectives.

Unfortunately there can be no question as to the nature of their objectives. At the close of World War II they used their military ability to enslave many nations economically and they have become dependent on this setup. Now a slave-master must eliminate hope and to the people the Soviets have conquered we represent hope. Thus they must effectively eliminate us or withdraw from their slave empire. There has never been any doubt as to the choice they have made.

The only response to such a situation of which our government is capable is to spend money. It can apply really large resources by this method. The difficulty is that such spending should be intelligently directed, taking into account the rapid change in scientific and technical knowledge. Unfortunately, to a considerable extent, this direction is lacking. The men who rise to high administrative positions in our society in either business or government are in most cases specialists in marketing, corporate, legal or financial relations, or in politics. Naturally, they are not educated to appreciate the precise manner in which research and development policies should be constructed in order to take into account the complex developments in science which now occur. These men surround themselves with buffer administrative personnel and the latter, of course, endeavor to prove that precise formulation of policies in technical areas is not essential. Even when money is spent on research it is believed that day-to-day understanding is totally unnecessary, and that when a "break-through" occurs the chief will be "briefed" on it.

It is hardly necessary to point out that at present the specialist in technical matters is in general even more at sea relative to the specialities of the administrators whether they are marketing considerations or purely political ones. It certainly would be very desirable if policy experience in defense matters could be formulated by cooperating teams of individuals, each with high competence in a specific field. But it must be appreciated that our specialists on both sides of the fence do not have the type of education that would make this cooperation effective. It is also true that specialists in general would have little sympathy with such a cooperative idea. This is the point where a serious handicap is imposed by the concentration of our present education on individual subjects and its failure to provide a broad educational background.

The failure to supervise the defense effort properly as a whole, from the technical point of view, has resulted in a rather erratic course with dangerous tendencies. The result frequently is not one for good professional development. Many companies specialize in large one-shot contracts. Lucrative opportunities are offered to people whose education is incomplete and the work tends to be done on an *ad hoc* basis rather than in a proper professional way. This certainly disturbs the possibilities for normal professional development and many people who could have a good professional career and whose talents, if properly developed, would be very valuable for the defense effort are pulled from one position to another mainly for financial reasons and never get a chance at a proper professional development. I think the effects of this on education are very ob-

vious. It is certainly difficult to expect a young man to ignore immediate lucrative offers and develop himself professionally when the remuneration from the professional activity may definitely be lower than one can get immediately. It is hardly necessary to add that such a situation places a considerable premium on educational shortcuts. Surely a knowledge of a technique which will permit a person to accept a high paying job without the inconvenience of a basic education is apparently advantageous in these circumstances.

To me it seems clear that a measure of the erratic nature of high administrative policies relative to scientific research and development is given by the chaotic nature of possible professional development in this field of applied mathematics and in related fields. If these policies had proper balance and a proper evolution under wise guidance one would have an appropriate professional development and our culture might even be strengthened rather than disturbed by the defense effort.

Of course, there are many who have struggled against this chaos. There are cases where policy has been carefully formulated, taking into account the present situation of scientific knowledge and the arts. It is very interesting to observe, when one examines such a situation, the effect it has on professional development. I know of cases where the need to respond to certain criteria has resulted in a very effective self-education. It seems to me that the main hope of improvement must be in regard to education and in the outlook of educators. The most obvious aspect of the situation is that the educators must educate themselves to an appreciation of the unity of all intellectual activity. There really isn't any basic difficulty here. I certainly know of a number of mathematicians who have had a great deal of interest in linguistics. In the specific region of the relation of mathematics and sciences it is interesting to notice how much greater the interest of physicists in mathematics has become recently, and there has certainly been a certain amount of interest in physics by mathematicians. This is definitely a region where joint research would be very valuable. Incidentally, the chemists have recently been very interested in both physics and mathematics.

In mathematics itself it is probably desirable to emphasize both the utilitarian and conceptual ideas which tie in with science and technology. Such connections make mathematics an integral part of present scientific development and contribute to the general intellectual character of scientific education. In all cases one would like to recommend an emphasis on the historical development of the subject as distinct from a crystallized current picture.

Again, I would like to repeat that the most valuable thing is an appreciation by educators of the situation. In addition, however, there is clearly a need for general orientation courses relative to the sciences. This is specifically true in regard to the physical sciences as a whole. Introductory courses nowadays deal with only one specific subject, such as physics or chemistry, and a beginning physics student can take a complete course in physics without hearing the words "differential equation." One should distinguish carefully between the technical

details of a particular subject and the general objectives. The student should know the purpose of a course even before he takes it. The basic problems that one must consider can be stated before the solution is given.

It would be very desirable to see efforts at cooperation across the standard field lines. Such a joint effort could take the form of courses or seminars or research efforts. Another very desirable course change would be an emphasis on the historical origin of the subject. This does require a certain amount of study and appreciation on the part of the teacher himself, but it is rewarding. In connection with cross-field efforts involving technology one should also have courses devoted to the utilitarian aspects of mathematics discussed in the proper intellectual manner. Such courses would throw considerable light on many of our theoretical courses and greatly increase their intellectual value. For example, studies of circuits and linear transforms would be very valuable adjuncts to courses in the theory of functions of a complex variable. Courses on numerical analysis, programming, and automatic computation procedures should be an effective complement to advanced calculus and real variable theory.

Finally, there is one general educational policy that should be emphasized relative to courses with specific subject matter. One should always be capable of distinguishing between courses which present basic scientific fundamentals and increase understanding, and those courses which are substitutes for on-the-job training. Educators should resist any tendency to substitute job training for basic education.

The present struggle adds a note of immediate urgency to the problem of scientific education. It represents lines of stress imposed on our complicated American pattern and, of course, it is impossible to know what the final result will be. We are organized in many groups and each group has its own ideology and certainly one cannot tell whether a good alignment will result or a poor one, in view of these stresses and difficulties. But this has always been our case. Can anyone tell whether our national motto—*E Pluribus Unum*—really proclaims a solution, or simply states a problem?

SOME RESULTS ON ORDER-CONVEXITY

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1. Introduction. G. Birkhoff defines an order-theoretical concept analogous to convexity ([1] p. 22). This paper presents several results concerning order convex sets and their extreme points. Theorem II is analogous to a theorem of V. L. Klee [3], while Theorem III is an order-theoretic analogue of the Krein-Mil'man theorem [4] ([2] p. 84).

2. Let $\langle X, \leqq \rangle$ be a partially ordered set. For any $a, b \in X$ we call the set $\{x \in X \mid a \leqq x \leqq b\}$ the *segment* determined by a and b and denote it by $\sigma(a, b)$. A subset A of X is *order-convex* iff for all $a, b \in A$, $\sigma(a, b) \subseteq A$.

Let Γ be the family of all order-convex subsets of X. Since $\langle 2^X, \subseteq \rangle$ is a complete lattice and Γ is closed under intersections, it follows ([1] p. 50) that Γ is also a complete lattice. For any $A \subseteq X$ let $\mathfrak{C}(A) = \bigcap \{C \in \Gamma \mid A \subseteq C\}$ be the *order-convex hull* of A. Since Γ is a complete lattice, the operator $\mathfrak{C}$ has the following properties ([1] pp. 49–50):

(i) for all $A \subseteq X$, $A \subseteq \mathfrak{C}(A)$,

(ii) for all $A, B \subseteq X$, if $A \subseteq B$, then $\mathfrak{C}(A) \subseteq \mathfrak{C}(B)$,

(iii) for all $A \subseteq X$, $\mathfrak{C}\mathfrak{C}(A) = \mathfrak{C}(A)$,

(iv) for all $A \subseteq X$, $A \in \Gamma$ iff $A = \mathfrak{C}(A)$.

THEOREM I. *For all* $A \subseteq X$, $\mathfrak{C}(A) = \bigcup \{\sigma(a, b) \mid a, b \in A\}$.

Proof. Let $U = \bigcup \{\sigma(a, b) \mid a, b \in A\}$. $\mathfrak{C}(A)$ is order convex. Hence $\sigma(a, b) \subseteq \mathfrak{C}(A)$ for all $a, b \in \mathfrak{C}(A)$. By (i) $A \subseteq \mathfrak{C}(A)$. $\therefore U \subseteq \mathfrak{C}(A)$.

Since $a, b \in \sigma(a, b)$, $A \subseteq U$. Suppose $c, d \in U$ and $x \in \sigma(c, d)$. Then for some $a_1, b_1, a_2, b_2 \in A$, $c \in \sigma(a_1, b_1)$ and $d \in \sigma(a_2, b_2)$. Hence, $a_1 \leqq c \leqq x \leqq d \leqq b_2$ and $x \in \sigma(a_1, b_2)$. Thus $x \in U$ and U is therefore order-convex. But this, along with $A \subseteq U$, implies that $\mathfrak{C}(A) \subseteq U$.

3. A point $p \in A \subseteq X$ is an *extreme point* of A iff for all $a, b \in A$, $p \in \sigma(a, b)$ implies $p = a$ or $p = b$. Let $\mathfrak{E}(A)$ denote the set of extreme points of A. The following theorem characterizes extreme points of order-convex sets.

THEOREM II. *If* $p \in \mathfrak{C}(A)$, *the following assertions are equivalent*:

(1) $p \in \mathfrak{E}\mathfrak{C}(A)$.

(2) $B \subseteq \mathfrak{C}(A)$ *and* $p \in \mathfrak{C}(B)$ *implies* $p \in B$.

(3) $B \subseteq A$ *and* $p \in \mathfrak{C}(B)$ *implies* $p \in B$.

Proof. (1)$\Rightarrow$(2) Suppose $p \notin B$. Then there exist $a, b \in B$ such that $p \in \sigma(a, b)$, $p \neq a$, $p \neq b$. But $B \subseteq \mathfrak{C}(A)$. This contradicts (1).

(2)$\Rightarrow$(3) This follows trivially, since by (i) $A \subseteq \mathfrak{C}(A)$.

(3)$\Rightarrow$(1) Suppose for some $a, b \in \mathfrak{C}(A)$, $p \in \sigma(a, b)$. By Theorem I there exist $a_1, b_1, a_2, b_2 \in A$ such that $a \in \sigma(a_1, b_1)$ and $b \in \sigma(a_2, b_2)$. Then $p \in \sigma(a_1, b_2)$. Let $B = \{a_1, b_2\}$. Then $B \subseteq A$ and $p \in \mathfrak{C}(B) = \sigma(a_1, b_2)$. Hence by (3), $p \in B$. Therefore $a_1 = p$ or $p = b_2$. But $a_1 \leqq a \leqq p \leqq b \leqq b_2$. Hence $p = a$ or $p = b$.

4. If $a \in A \subseteq X$, let $L(A, a) = \{x \in A \mid x \leqq a\}$ and $U(A, a) = \{x \in A \mid a \leqq x\}$. We will call A *locally complete* iff for any $a \in A$, $L(A, a)$ contains a minimal element and $U(A, a)$ contains a maximal element.

THEOREM III. *A subset* A *of* X *is locally complete iff* $\mathfrak{C}\mathfrak{E}(A) = \mathfrak{C}(A)$.

Proof. If $\mathfrak{C}\mathfrak{E}(A) = \mathfrak{C}(A)$, choose any $x \in A$. Since by (i) $A \subseteq \mathfrak{C}(A) = \mathfrak{C}\mathfrak{E}(A)$ and by Theorem I $\mathfrak{C}\mathfrak{E}(A) = \bigcup \{\sigma(e, f) \mid e, f \in \mathfrak{E}(A)\}$, there exist $e, f \in \mathfrak{E}(A)$ such that $x \in \sigma(e, f)$. If $x \notin \mathfrak{E}(A)$, e is a minimal element of $L(A, x)$ and f is a maximal

element of $U(A, x)$. If $x \in \mathcal{E}(A)$ we may choose e and f such that these statements are true. Thus A is locally complete.

If A is locally complete and $x \in \mathcal{C}A$, then $x \in \sigma(a, b)$ for some $a, b \in A$. Let e be a minimal element of $L(A, a)$ and f be a maximal element of $U(A, b)$. Then $e, f \in \mathcal{E}(A)$, and $x \in \sigma(e, f)$ since $e \leqq a \leqq x \leqq b \leqq f$. Hence $x \in \mathcal{C}\mathcal{E}(A)$ and $\mathcal{C}(A) \subseteq \mathcal{C}\mathcal{E}(A)$.

But by definition $\mathcal{E}(A) \subseteq A$. Hence by (ii) $\mathcal{C}\mathcal{E}(A) \subseteq \mathcal{C}(A)$. This completes the proof.

COROLLARY. *If a subset A of X is a complete lattice, then* $\mathcal{C}\mathcal{E}(A) = \mathcal{C}(A)$.

Proof. Every complete lattice is locally complete.

5. For $A \subseteq B \subseteq X$ define $\mathfrak{a}(A, B) = \{C \mid A \subseteq C \subseteq B$ and C is order convex$\}$.

THEOREM IV. *If $\mathcal{C}A \subseteq B$, $\mathfrak{a}(A, B)$, ordered by set inclusion, has a maximum element. Such a maximal element, M, is unique iff $M = \cup\mathfrak{a}(A, B)$.*

Proof. Clearly $\mathcal{C}A \in \mathfrak{a}(A, B) \neq \phi$. Let $\{C_\alpha\}_{\alpha \in I}$ be an ascending chain of elements of $\mathfrak{a}(A, B)$. Clearly $A \subseteq \cup_{\alpha \in I} C_\alpha \subseteq B$. Suppose $a, b \in \cup_{\alpha \in I} C_\alpha$. Then for some $\alpha, \beta \in I$, $a \in C_\alpha$ and $b \in C_\beta$. Without loss of generality, take $C_\alpha \subseteq C_\beta$. Then $a \in C_\beta$. Since C_β is order convex, $\sigma(a, b) \subseteq C_\beta \subseteq \cup_{\alpha \in I} C_\alpha$. $\therefore \cup_{\alpha \in I} C_\alpha$ is order-convex and is thus an upper bound on the chain which belongs to $\mathfrak{a}(A, B)$. Then by Zorn's Lemma, $\mathfrak{a}(A, B)$ has a maximum element M.

If $M = \cup\mathfrak{a}(A, B)$, M is clearly unique. If not, there exists a point $a \in \cup\mathfrak{a}(A, B) \setminus M$. Then $a \in C_a$ for some $C_a \in \mathfrak{a}(A, B)$. Then $A \cup \{a\} \subseteq \mathcal{C}(A \cup \{a\}) \subseteq C_a \subseteq B$. Then by the first paragraph of the proof, $\mathfrak{a}(A \cup \{a\}, B)$ has a maximal element N. Clearly $N \in \mathfrak{a}(A, B)$. Suppose $C \in \mathfrak{a}(A, B)$ and $N \subseteq C$. Then $a \in C$ and therefore $C \in \mathfrak{a}(A \cup \{a\}, B)$. Thus $N = C$. N is then a maximal element of $\mathfrak{a}(A, B)$ different from M.

T. G. McLaughlin has shown that for any preassigned cardinal $\aleph$ it is possible to construct a partially ordered set X and subsets A and B of X such that the set of maximal elements of $\mathfrak{a}(A, B)$ has cardinality $\aleph$.

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THE DIOPHANTINE EQUATION $x^2+y^2+z^2=m^2$

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The close connection between the equations $x^2+y^2=z^2$ and $x^4+y^4=z^4$ leads one to suspect a close connection between the title equation and the unsolved equation $x^4+y^4+z^4=m^4$. In this paper, it will be shown that these equations are indeed intimately related. The main portion of the paper, however, is concerned with an exposition and survey of what is known to date about the equation $x^2+y^2+z^2=m^2$. Early references may be found in Dickson [1].

1. Solution. We always take x, y, z and $m>0$, and we are concerned only with rational integral solutions where $(x, y, z)=1$, for the imprimitive solutions where $(x, y, z)>1$ can be obtained from the primitive ones by multiplication. By congruences (mod 8), one sees that x and y can be taken even, with z and m odd. The following theorem has long been known.

THEOREM 1. *Given $x^2+y^2+z^2=m^2$ with z odd and $(x, y, z)=1$, one can find integers u, v, w and t such that*

$$\begin{aligned} x &= 2(uw - vt), \\ y &= 2(ut + vw), \\ z &= u^2 + v^2 - w^2 - t^2, \\ m &= u^2 + v^2 + w^2 + t^2. \end{aligned} \tag{1}$$

Carmichael [2], in 1915, attempted a proof which was incomplete. Dickson [3], in 1920, was the first to give a satisfactory proof. Skolem [4], in 1941, gave a proof which was essentially correct. The following proof is a completion of that given by Skolem. It yields an algorithm to find u, v, w and t from x, y, z and m which is much more direct than the algorithm obtained from Dickson's [3] proof.

Proof. Set $x_1=\frac{1}{2}x$ and $y_1=\frac{1}{2}y$. Then

$$x_1^2 + y_1^2 = \left(\frac{m+z}{2}\right)\left(\frac{m-z}{2}\right).$$

Set $f=(x_1, y_1)$, $f_1=(f, \frac{1}{2}(m+z))$ and $f_2=(f, \frac{1}{2}(m-z))$. By an easy argument one sees that $(f_1, f_2)=1$, and $f=f_1\cdot f_2$. Set $x_2=x_1/f$, $y_2=y_1/f$, $z_1=(m+z)/2f_1^2$ and $z_2=(m-z)/2f_2^2$. Then

$$x_2^2 + y_2^2 = z_1\cdot z_2,$$

where $(x_2, y_2)=1$. Note that z_1 and z_2 are not necessarily relatively prime.

Let

$$x_2 + iy_2 = \prod_{j=1}^{n} \pi_j$$

be a factorization into gaussian primes. Note that x_2+iy_2 cannot be divisible by a rational prime p. For if so, then $(x_2+iy_2)/p=a+ib$, $x_2=pa$, $y_2=pb$, a contradiction, as $(x_2, y_2)=1$. Hence, none of the π_j's is a rational prime $\equiv 3 \pmod 4$.

Now

$$x_2 - iy_2 = \prod_{j=1}^{n} \bar{\pi}_j$$

and

$$z_1 \cdot z_2 = \prod_{j=1}^{n} (\pi_j \bar{\pi}_j), \qquad z_1 = \prod_{j=1}^{m} (\pi_j \bar{\pi}_j), \qquad z_2 = \prod_{j=m+1}^{n} (\pi_j \bar{\pi}_j)$$

if we write the π_j's in a suitable order. Set

$$u_1 + iv_1 = \prod_{j=1}^{m} \pi_j, \qquad w_1 + it_1 = \prod_{j=m+1}^{n} \pi_j.$$

Then

$$z_1 = (u_1 + iv_1)(u_1 - iv_1), \qquad z_2 = (w_1 + it_1)(w_1 - it_1),$$
$$(u_1 + iv_1)(w_1 + it_1) = x_2 + iy_2.$$

Since, given any $\bar{\pi}_j$ dividing u_1-iv_1, we cannot have $\bar{\pi}_j(u_1+iv_1) \mid (x_2+iy_2)$, (for then a rational prime would divide $x_2 + iy_2$), we have $u_1 + iv_1 = \text{G.C.D.}(z_1, x_2+iy_2)$. Similarly, $w_1+it_1=\text{G.C.D.}(z_2, x_2+y_2 i)$.

Setting $u=f_1u_1$, $v=f_1v_1$, $w=f_2w_1$ and $t=f_2t_1$, we now easily obtain equations (1). An example in which all the letters introduced occur nontrivially is $12^2+84^2+5^2=85^2$. The algorithm for obtaining u, v, w and t from x, y, z and m is simply Euclid's algorithm applied to $\frac{1}{2}(m+z)$ and x_2+y_2i, and also to $\frac{1}{2}(m-z)$ and x_2+y_2i, along with a few other operations obvious from the proof.

Tables of solutions for $m \leqq 100$ are given in Steiger [5]; and for $m \leqq 207$, in Miksa [6].

In addition to the evenness of x and y, one can also show that if $m \equiv 0 \pmod 3$, then none of x, y and z are $\equiv 0 \pmod 3$; and if $m \not\equiv 0 \pmod 3$, then exactly two of x, y and z are $\equiv 0 \pmod 3$.

Rational parametrizations were given in references [7] through [11]. Sierpinski [12] gives another solution. Dickson's [13] solution is equivalent to (1). Skolem [4] showed that primitive solutions occur as orthogonal triples (see Dickson [14], Buquet [11]). These triples can occur with distinct rows and columns, as shown by

17	16	52	2	34	53
32	−47	4	43	38	−26
46	28	−23	46	−37	22

Parity arguments show that such triples cannot be magic. Reitan [15] discussed chains of solutions.

2. Uniqueness. The following theorem was discovered empirically by Steiger [5] in 1956.

THEOREM 2. *If the parameters u, v, w and t of equations* (1) *are subjected to the conditions*

$$\text{(a)}\ uw > vt, \qquad \text{(b)}\ u^2 + v^2 > w^2 + t^2,$$

$$(c_1)\ u \geqq 1,\ v \geqq 0, \qquad (c_2)\ w \geqq 1,\ t \geqq 0, \qquad (c_3)\ t + v \geqq 1,$$

$$\text{(d)}\ u + v + w + t \equiv 1 \pmod 2,$$

$$\text{(e)}\ (u^2 + v^2,\ w^2 + t^2,\ ut + vw) = 1,$$

$$\text{(f)}\ t = 0 \rightarrow u \leqq v, \qquad \text{(g)}\ v = 0 \rightarrow w \leqq t,$$

then each primitive solution of $x^2+y^2+z^2=m^2$ is obtained once and only once.

Proof. First we show that each solution is obtained at least once under conditions (a) through (g). Since x and z are >0, we have: (a) $uw>vt$, and (b) $u^2+v^2>w^2+t^2$. From the proof of Theorem 1, we can take $u+iv$ in the first quadrant excluding the imaginary axis. Hence, (c_1) $u\geqq 1$ and $v\geqq 0$. It may happen that $\arg(x_2+iy_2)<\arg(u_1+iv_1)$. By our proof, note that we obtain v_1+iu_1 from y_2+ix_2. But then $\arg(y_2+ix_2)>\arg(v_1+iu_1)$, so that by possibly interchanging x and y we can assure that $\arg(x_2+iy_2)\geqq\arg(u_1+iv_1)$. Now from $x=2(uw-vt)$, $w+it$ cannot lie in the second quadrant, and from $y=2(ut+vw)$, $w+it$ cannot lie in the third quadrant. Finally, as $\arg(x_2+iy_2)\geqq\arg(u_1+iv_1)$, we must have $\arg(w+it)\geqq 0$, so that $w+it$ lies in the first quadrant, excluding the imaginary axis. Thus, (c_2) $w\geqq 1$, $t\geqq 0$. If v and t were zero simultaneously, then $y=0$. Hence, (c_3) $t+v\geqq 1$. Modulo 2, $u^2\equiv u$, $v^2\equiv v$, $w^2\equiv w$ and $t^2\equiv t$. Hence, (d) $1\equiv m=u^2+v^2+w^2+t^2\equiv u+v+w+t \pmod 2$. If we let $d\,|\,(u^2+v^2,\ w^2+t^2,\ ut+vw)$, then $d\,|\,m$, $d\,|\,z$, $d\,|\,y$ and hence $d\,|\,x$, so $d=1$. Thus we have (e) $(u^2+v^2,\ w^2+t^2,\ ut+vw)=1$. Finally, if $t=0$, by interchanging x and y if necessary, we can have $u\leqq v$. Similarly, if $v=0$, we can have $w\leqq t$. These last are conditions (f) and (g). These seven conditions are similar to those stated by Steiger [5], who used these conditions to construct his table of solutions.

The proof below that these conditions insure that each solution is obtained only once is due to W. H. Mills.

We start from the equations

$$(x + iy)/2 = (u + iv)(w + it),$$

$$(z + m)/2 = u^2 + v^2 = (u + iv)(u - iv).$$

Note that $w+it$ and $u-iv$ are relatively prime; for if a gaussian prime divided both, it would divide w^2+t^2 and u^2+v^2, and would also divide $v(w+it)+t(u-iv)=ut+vw$, which is impossible by condition (e). Hence $u+iv$ is the G.C.D. of

$\frac{1}{2}(x+iy)$ and $\frac{1}{2}(z+m)$, a definite complex number determined up to a unit. But as it lies in the first quadrant, it is determined uniquely. A similar proof holds for $w+it$. Now we use conditions (c), (f) and (g), in an obvious manner, to assure us that we obtain only one of the two possible sets of parameters obtained from the above equations by interchanging x and y. The conditions (a) through (g) are not independent, condition (c_3) being a consequence of (c_1), (c_2) and (f).

3. Application. Since one can go from the variables x, y, z and m to the parameters u, v, w and t, it is easy to see that the equation $x^4+y^4+z^4=m^4$ is equivalent to the four simultaneous quadratic equations:

$$\begin{aligned} x^2 &= 2(uw - vt), \\ y^2 &= 2(ut + vw), \\ z^2 &= u^2 + v^2 - w^2 - t^2, \\ m^2 &= u^2 + v^2 + w^2 + t^2. \end{aligned} \tag{2}$$

One can satisfy any three of these equations, giving x, y, z and m in terms of other parameters. This was done by Escott [16] and Fauquembergue [17]. Incidentally, Werebrusow's [18] identity, as corrected by Padhy [19], which reads

$$(3 + 4t^2)^4 = (-1 + 4t^2)^4 + (2 + 4t)^4 + [8(2t - 1)(2t^2 + t + 1)]^2,$$

is a special case of Escott's identity

$$(m^2 + mn + n^2)^4 = (mn)^4 + (mn + n^2)^4 + [m(m + n)(m^2 + mn + 2n^2)]^2,$$

obtained by taking $n=-1-2t$, $m=-1+2t$. Aside from this, no solution is contained in any other. For instance, Escott's [16] formula does not represent $33^4=31^4+4^4+[8\cdot 64]^2$; and Fauquembergue's formulas,

$$\begin{aligned} (a^4 + 2b^4)^4 &= (a^4 - 2b^4)^4 + (2a^3b)^4 + (8a^2b^6)^2 \\ &= (2a^2b^2)^4 + (2a^3b)^4 + (a^8 - 4a^4b^4 - 4b^8)^2, \end{aligned}$$

cannot represent $7^4=3^4+2^4+48^2$. The derivation of these identities is explained in Xeroudakes [20]. It seems unlikely that one would be able to find a complete solution of three of the above equations (2).

It is an interesting fact, however, that one can indeed find a complete solution of the last two equations. To do this, one subtracts the third from the fourth, and obtains

$$m^2 - z^2 = 2(w^2 + t^2) = (w + t)^2 + (w - t)^2.$$

By a short congruence argument (mod 8) on the third and fourth equations one shows that u and v have opposite parity, while w and t must be even. Thus, setting $r=w+t$ and $s=w-t$ gives a transformation reversible in integers, $w=\frac{1}{2}(r+s)$ and $t=\frac{1}{2}(r-s)$.

Then $r^2+s^2+z^2=m^2$, with r and s even. Moreover, $(r, s, z)=1$, for if p is a prime and $p\,|\,(r, s, z)$, then $p\,|\,r+s$ and $p\,|\,r-s$. So $p\,|\,2w$ and $p\,|\,2t$. But $p\neq 2$, as

z is odd, so $p \mid w$ and $p \mid t$. Thus $p \mid x^2$, $p \mid y^2$ and $p \mid z^2$, which is impossible. Thus we can apply our parametric solution:

$$\begin{aligned} r &= 2(UW - VT), \\ s &= 2(UT + VW), \\ z &= U^2 + V^2 - W^2 - T^2, \\ m &= U^2 + V^2 + W^2 + T^2. \end{aligned}$$

Solving now for u^2+v^2, we find $u^2+v^2=(U^2+V^2)^2+(W^2+T^2)^2$. Thus, u^2+v^2 is some representation of the sum of two squares $(U^2+V^2)^2+(W^2+T^2)^2$, and if we let u^2+v^2 run over all representations, we clearly obtain all the solutions of the last two of equations (2), namely

$$\begin{aligned} z &= U^2 + V^2 - W^2 - T^2, \\ m &= U^2 + V^2 + W^2 + T^2, \\ w &= UW - VT + UT + VW, \\ t &= UW - VT - UT - VW, \end{aligned} \qquad u^2 + v^2 = (U^2 + V^2)^2 + (W^2 + T^2)^2.$$

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ON QUASI-ORTHOGONAL NUMBERS

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1. Introduction. Let m, n, p, and s be positive integers or zero, and let x be a complex variable. We consider polynomials of degree n,

$$P(n, x) = \sum_{m=0}^{n} A_n^m x^m \tag{1}$$

and triangular matrices of their coefficients,

$$A(n) = \begin{bmatrix} A_0^0 & & & & \\ A_1^0 & A_1^1 & & & \\ A_2^0 & A_2^1 & A_2^2 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n^0 & A_n^1 & A_n^2 & \cdots & A_n^n \end{bmatrix}. \tag{2}$$

Two sets of numbers A_n^m, and B_n^m, are said to be quasi-orthogonal, according to [1] p. 32, if $A(n)B(n) = I$, i.e. if

$$\sum_{s=m}^{n} A_n^s B_s^m = \delta_n^m, \tag{3}$$

where δ_n^m is the Kronecker-δ. In order to complete the definition of the numbers as given by (1) we assume that $A_n^m = 0$ for $n < 0$, $m < 0$, and $n < m$. Examples of quasi-orthogonal numbers will be given later. The aim of our study is to find recurrence relations for quasi-orthogonal numbers.

2. Q-polynomials and generalized Stirling numbers. Let M and N be two functions of the variable n, such that $M(0) \neq 0$ and for n a positive integer or

zero $N(n) \neq 0$. We define

$$Q(x; M, N, n) = Q(x, n) = \prod_{m=1}^{n} [M(m) + N(m)x], \qquad n \geqq 1 \tag{4}$$

$$Q(x, 0) = M(0). \tag{5}$$

Since Q is a polynomial of degree n, we can write

$$Q(x, n) = \sum_{s=0}^{n} A_n^s x^s,$$

where the numbers A_n^s are called the *generalized Stirling numbers of the first kind.* Conversely, since $Q(x, n)$ is a polynomial of degree n, and $Q(x, 0)$ is a constant, we can write

$$x^n = \sum_{m=0}^{n} B_n^m Q(x, m), \tag{6}$$

$$1 = B_0^0 Q(x, 0) = B_0^0 M(0). \tag{7}$$

The numbers B_n^m are called the *generalized Stirling numbers of the second kind.* We shall prove that the numbers A_n^m and B_n^m are quasi-orthogonal. According to (4), (5), (6), and (7) we have,

$$x^n = \sum_{s=0}^{n} B_n^s Q(x, s) = \sum_{s=0}^{n} B_n^s \sum_{m=0}^{s} A_s^m x^m.$$

But since A_s^m and B_n^s are zero for $n < s$, and $s < m$, we can write

$$x^n = \sum_{s=0}^{n} \sum_{m=0}^{n} x^m B_n^s A_s^m = \sum_{m=0}^{n} x^m \sum_{s=m}^{n} B_n^s A_s^m,$$

thus

$$\sum_{m=0}^{n} B_n^s A_s^m = \delta_n^m.$$

We now look for a recurrence relation satisfied by the numbers A and B. Since

$$Q(x, n + 1) = Q(x, n)[M(n + 1) + N(n + 1)x],$$

we have

$$Q(x, n + 1) = \sum_{m=0}^{n+1} A_{n+1}^m x^m = [M(n + 1) + N(n + 1)x] \sum_{m=0}^{n} A_n^m x^m$$

$$= M(n + 1) \sum_{m=0}^{n} A_n^m x^m + N(n + 1) \sum_{m=0}^{n} A_n^m x^{m+1}.$$

Equating the coefficients of equal powers of x, we have

$$A_{n+1}^{m} = M(n+1)A_n^m + N(n+1)A_n^{m-1},$$

or changing n into $n-1$,

$$A_n^m = M(n)A_{n-1}^m + N(n)A_{n-1}^{m-1}. \tag{8}$$

On the other hand

$$x^{n+1} = x\,x^n = \sum_{m=0}^{n} B_n^m x Q(x, n). \tag{9}$$

But

$$Q(x, m+1) = [M(m+1) + N(m+1)x]Q(x, m),$$

so that, since $N(m+1) \neq 0$,

$$xQ(x, m) = \frac{1}{N(m+1)}[Q(x, m+1) - M(m+1)Q(x, m)].$$

Substituting into (9), we obtain,

$$x^{n+1} = \sum_{m=0}^{n+1} B_{n+1}^m \cdot Q(x, m) = \sum_{m=0}^{n} B_n^m [Q(x, m+1) - M(m+1)Q(x, m)]/N(m+1).$$

Equating the coefficients of equal polynomials, we have

$$B_{n+1}^m = B_n^{m-1}/N(m+1) - M(m+1)B_{n-1}^{m-1}/N(m+1),$$

or, changing n into $n-1$,

$$B_n^m = -\frac{M(m+1)}{N(m+1)} B_{n-1}^m + \frac{1}{N(m+1)} B_{n-1}^{m-1}. \tag{10}$$

We can state this result in the following form:

THEOREM I. *If A_n^m and B_n^m are quasi-orthogonal numbers, where both, A_n^m and B_n^m, are zero for $n<0$, $m<0$, and, $n<m$, and if the numbers A_n^m satisfy the recurrence relation*

$$A_n^m = M(n)A_{n-1}^m + N(n)A_{n-1}^{m-1}, \tag{11}$$

where $N(n) \neq 0$ for all positive integer values of n and $n=0$, then the numbers B_n^m satisfy the recurrence relation

$$B_n^m = -\frac{M(m+1)}{N(m+1)} B_{n-1}^m + \frac{1}{N(m+1)} B_{n-1}^{m-1}. \tag{12}$$

Conversely, under the same conditions, if B_n^m satisfy (12), *then A_n^m satisfy* (11).

3. Examples. (i) For $N=M=1$, we have

$$Q(x, n) = (1 + x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m.$$

Thus $A_n^m = \binom{n}{m}$, $A_0^0 = \binom{0}{0} = 1$, the A-numbers are the binomial coefficients satisfying the recurrence relation $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$. Taking $y = x+1$, we have

$$x^n = (y - 1)^n = \sum_{m=0}^{n} B_n^m (1 + x)^m = \sum_{m=0}^{n} B_n^m y^{n-m} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} y^{n-m},$$

so that $B_n^m = (-1)^m \binom{n}{m}$ and $B_n^m = -B_{n-1}^m + B_{n-1}^{m-1}$.

(ii) For $N=1$, and, $M(s) = -s+1$, we have,

$$Q(x, n) = x(x - 1)(x - 2) \cdots (x - n + 1) = (x)_n = \sum_{m=1}^{n} St_n^m x^m.$$

Thus $A_n^m = St_n^m$, the ordinary Stirling numbers of the first kind. On the other hand

$$x^n = \sum_{m=1}^{n} st_n^m (x)_m.$$

Thus $B_n^m = st_n^m$, the ordinary Stirling numbers of the second kind. For an exhaustive study of ordinary Stirling numbers we refer to [2], Chapter IV. The recurrence relations are, $St_n^m = -(n-1)St_n^m + St_{n-1}^{m-1}$, and, $st_n^m = mst_{n-1}^m + st_{n-1}^{m-1}$.

4. R-polynomials and the C-D numbers. We now look for quasi-orthogonal numbers satisfying more general relations than (11) and (12). Using the same method as before, we consider the four functions K, L, and M, N, of the variable n, such that $N(n) \neq -1$, and $L(n) \neq 0$, for all n positive integers and $n=0$, and define

$$R(x, n) = \sum_{m=0}^{n} C_n^m x^m, \tag{13}$$

$$R(x, n + 1) = [K(n + 1) + L(n + 1)x]R(x, n) + \sum_{m=0}^{n} [M(m + 1) + N(m + 2)x]C_n^m x^m, \tag{14}$$

$$R(x, 0) = K(0). \tag{15}$$

By substituting (13) into (14) and equating equal powers of x, we obtain

$$C_{n+1}^m = [K(n + 1) + M(m + 1)]C_n^m + [L(n + 1) + N(m + 1)]C_n^{m-1},$$

or, changing n into $n-1$, we have

$$C_n^m = [K(n) + M(m + 1)]C_{n-1}^m + [L(n) + N(m + 1)]C_{n-1}^{m-1}. \tag{16}$$

On the other hand, since $R(x, n)$ is a polynomial of degree n, we can write,

$$x^n = \sum_{m=0}^{n} D_n^m R(x, m). \tag{17}$$

It follows that

$$x^n = \sum_{s=0}^{n} D_n^s R(x, s) = \sum_{s=0}^{n} D_n^s \sum_{m=0}^{s} C_s^m x^m.$$

Since C_n^m and D_n^m are zero for $n < m$, we have

$$x^n = \sum_{s=0}^{n} D_n^s \sum_{m=0}^{n} C_s^m x^m = \sum_{s=0}^{n} \sum_{m=0}^{n} D_n^s C_s^m x^m = \sum_{m=0}^{n} x^m \sum_{s=m}^{n} D_m^s C_s^m,$$

so that

$$\sum_{s=m}^{n} D_n^s C_s^m = \delta_n^m, \tag{18}$$

which proves that the numbers C and D are quasi-orthogonal to each other. To find a recurrence relation between the D-numbers, we operate as before,

$$x^{n+1} = \sum_{m=0}^{n} D_n^m x R(x, m). \tag{19}$$

But

$$R(x, m+1) = [K(m+1) + L(m+1)x]R(x, m) + \sum_{p=0}^{m} [M(p+1) + N(p+2)x] C_m^p x^p;$$

hence, since $L(m+1) \neq 0$,

$$xR(x, m) = \left\{ \frac{R(x, m+1)}{L(m+1)} - \frac{K(m+1)}{L(m+1)} - \frac{1}{L(m+1)} \sum_{p=0}^{m} [M(p+1) + N(p+2)x] C_p^m x^p \right\}.$$

Substituting into (19), we have,

$$x^{n+1} = \sum_{m=0}^{n+1} D_n^m \left\{ \frac{R(x, m+1)}{L(m+1)} - \frac{K(m+1)}{L(m+1)} R(x, m) - \frac{1}{L(m+1)} \sum_{p=0}^{m} [M(p+1) + N(p+2)x] C_p^m x^p. \right\}$$

Let us assume first that $L = 1$. In this case we obtain, as before

$$x^{n+1} = \sum_{m=0}^{n+1} D_{n+1}^{m} R(x, m) = \sum_{m=0}^{n} D_{n}^{m} R(x, m+1) - \sum_{m=0}^{n} D_{n}^{m} K(m+1) R(x, m)$$

$$(20) \qquad - \sum_{p=0}^{m} x^{p} [M(p+1) + N(p+2)x] \sum_{m=p}^{n} D_{n}^{m} C_{m}^{p}.$$

According to (18) the last sum reduces to δ_n^p, so that

$$\sum_{m=0}^{n+1} D_{n+1}^{m} R(x, m) = \sum_{m=0}^{n} D_{n}^{m} R(x, m+1) - \sum_{m=0}^{n} D_{n}^{m} K(m+1) R(x, m) - M(n+1)x^{n}$$

$$- N(n+2)x^{n+1},$$

$$= \sum_{m=0}^{n} D_{n}^{m} R(x, m+1) - \sum_{m=0}^{n} D_{n}^{m} K(m+1) R(x, m)$$

$$- M(n+1) \sum_{m=0}^{n} D_{n}^{m} R(x, m) - N(n+2) \sum_{m=0}^{n+1} D_{n+1}^{m} R(x, m).$$

Equating equal R-polynomials, we have

$$D_{n+1}^{m} = D_{n}^{m-1} - K(m+1) D_{n}^{m} - M(n+1) D_{n}^{m} - N(n+2) D_{n+1}^{m}.$$

Regrouping terms, changing n into $n-1$ and dividing through by $1+N(n+2) \neq 0$, we obtain

$$(21) \qquad D_{n}^{m} = -\frac{K(m+1) + M(n)}{1 + N(n+1)} D_{n-1}^{m} + \frac{1}{1 + N(n+1)} D_{n-1}^{m-1}.$$

In the case where $L = L(m+1) \neq 1$, the last sum in (20) requires the evaluation of

$$\sum_{m=p}^{n} D_{n}^{m} C_{m}^{p} / L(m+1).$$

This sum cannot be equal to δ_n^p, for if it were, the matrix $D(n)$ would have two inverse matrices, i.e. $C(n)$, according to (18), and the matrix of general term $C_n^m/L(m+1)$, which is impossible. It follows that the sum would be equal to either δ_{n-a}^p, where a is a positive integer, since $p \leqq n$, or to some function $G(n, p)$. Assuming that the sum can be evaluated, by substitution into (20) we would find in both cases recurrence relations that are not of the form of (11), (12) and (21), but much more complicated. We shall not study these cases. Let then $N(n+1)+1 = P(n)$. The result for the case $L = 1$ can be stated as follows:

THEOREM II. *If C_n^m and D_n^m are quasi-orthogonal numbers, where both C_n^m and D_n^m are zero for $n<0$, $m<0$, $n<m$, and if the numbers C_n^m satisfy the recurrence relation*

$$(22) \qquad C_{n}^{m} = [K(n) + M(m+1)] C_{n-1}^{m} + P(m) C_{n-1}^{m-1},$$

with $P(m) \neq 0$, for all positive integer values of m and $m = 0$, then the numbers D_n^m satisfy the recurrence relation

$$D_n^m = -[K(m+1) + M(n)]D_{n-1}^m/P(n) + D_{n-1}^{m-1}/P(n). \tag{23}$$

Conversely, under the same conditions, if the numbers D_n^m satisfy (23), *the numbers C_n^m satisfy* (22).

Remark I. If instead of taking $L = 1$, we had taken $L = A = \text{Const.}$, we would have obtained a similar result.

Remark II. If $N = 0$, and $P = 1$, we have, $C_n^m = [K(n) + M(m+1)]C_{n-1}^m + C_{n-1}^{m-1}$ and $D_n^m = -[K(m+1) + M(n)]D_{n-1}^m + D_{n-1}^{m-1}$. The correspondence between the coefficients of C_{n-1}^m and D_{n-1}^m can be considered as a linear transformation T. Let the coefficient of the number D_{n-1}^m be the transform of the coefficient of the number C_{n-1}^m. We can write symbolically, (a and b being constants)

$$\text{(i)} \qquad TK(n) = -K(m+1),$$

$$T[aK_1(n) + bK_2(n)] = -aK_1(m+1) - bK_2(m+1),$$

$$\text{(ii)} \qquad TK(m+1) = -K(n).$$

Comparing (i) and (ii) we see that $T^2 = I$, where I is the identity operator. More generally we have

$$T[aK(n) + bM(m+1)] = -aK(m+1) - bM(n).$$

Remark III. By comparing the recurrence relations for A_n^m and C_n^m, given by relations (11) and (22) we see that it is not possible to obtain (11) from (22) or (22) from (11). In the general case there is no relationship between the numbers A_n^m and C_n^m. However, if in (11) we take $N(n) = 1$, and in (22) we take $M(m+1) = 0$, and, $P(m) = 1$, we obtain from both relations the simplified recurrence formula

$$A_n^m = M(n)A_{n-1}^m + A_{n-1}^{m-1}.$$

5. Examples. (i) If $K(n) = n+1$, and, $M(m+1) = m^2$, $C_n^m = (m^2+n+1)C_{n-1}^m + C_{n-1}^{m-1}$, and, $D_n^m = -[(n-1)^2+(m+1)+1]D_{n-1}^m + D_{n-1}^{m-1}$. $K(0) = 1$, thus

Values of C_n^m

$n\backslash m$	0	1	2	3	4
0	1				
1	2	1			
2	6	6	1		
3	24	36	14	1	
4	120	240	162	28	1

Values of D_n^m

$n\backslash m$	0	1	2	3	4
0	1				
1	−2	1			
2	6	−6	1		
3	−36	48	−14	1	
4	396	−612	230	−28	1

(ii) If $K(n) = n$, $N(m+1) = m+1$, $P(m) = m+2$, then $C_n^m = (n+m+1)C_{n-1}^m + (m+2)C_{n-1}^{m-1}$, and $D_n^m = -(n+m+1)D_{n-1}^m/(n+2) + D_{n-1}^{m-1}/(n+2)$.

Values of C_n^m

$n\backslash m$	0	1	2	3	4
0	1				
1	2	3			
2	6	18	12		
3	24	108	144	60	
4	120	720	1440	1200	360

Values of D_n^m

$n\backslash m$	0	1	2	3	4
0	1				
1	−2/3	1/3			
2	1/2	−1/2	1/12		
3	−2/5	3/5	−1/5	1/60	
4	1/3	−2/3	1/3	−1/18	1/360.

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GRAPHICAL SOLUTIONS OF PROBLEMS INVOLVING THE INCIDENCE OF POINTS, LINES AND PLANES

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Introduction. In this paper we shall introduce a transformation T which will transform the Euclidean n-space E_n into another space P_n which lies in an Euclidean plane E. Next we shall show that the incidence properties of P_n are identical with those of E_n and thus it is possible to study the space E_n by studying the space P_n. In particular some problems involving incidence of points, lines and planes will be solved by graphical methods.

In order to be able to solve these problems graphically, we also have to study certain properties of the space P_n which arise from the nature of transformation T.

In proving these properties we have used both analytical and synthetic methods. However, everything proved in this paper could be proved using only synthetic methods which, in some cases, are longer and more tedious.

Introduction of the Transformation. Consider n parallel lines $x_1, \cdots, x_n$ in the Euclidean plane such that lines x_i and x_{i+1} are one unit apart, for all i from 1 to $n-1$, and consider another line o perpendicular to these n lines. This line intersects the n parallel lines in n points $O_1, \cdots, O_n$.

Next consider a point $A(a_1, \cdots, a_n)$ in E_n. Let O_iA_i be line segments on the lines x_i, whose lengths are equal to the numerical values of a_i, from O_i to the right being considered the positive direction on the line x_i. Thus to every point $A(a_1, \cdots, a_n)$ will correspond a unique set of n points $A_1, \cdots, A_n$ of the Euclidean plane E, lying on the n lines $x_i, \cdots, x_n$, respectively.

Two other points $B(b_i)$ and $C(c_i)$ of E_n will go into two other sets B_i and C_i, i ranging from 1 to n. If these 3 points are collinear in E_n, it follows easily, from the analytical definition of collinear points in E_n, that the lines A_jA_{j+1}, B_jB_{j+1}

and C_jC_{j+1} are concurrent at the points $x_{j,j+1}$ for all j from 1 to $n-1$. These $n-1$ points are taken as the image of the line AB. Every other point D of the line AB has the property that the lines D_jD_{j+1} pass through the points $x_{j,j+1}$ for all j from 1 to $n-1$, and conversely, any point G having this property will belong to the line AB. The $n-1$ points $x_{j,j+1}$ are called the $n-1$ independent components of the line AB.* It may happen that the lines A_jA_{j+1} and B_jB_{j+1} are parallel, and consequently, $X_{j,j+1}$ will be a point at infinity. Then, if A, B and C are collinear, C_jC_{j+1} has to be parallel to A_jA_{j+1} also.

The images of a point, a line, point on a line and two parallel lines, under this transformation, are given in Figure 1.†

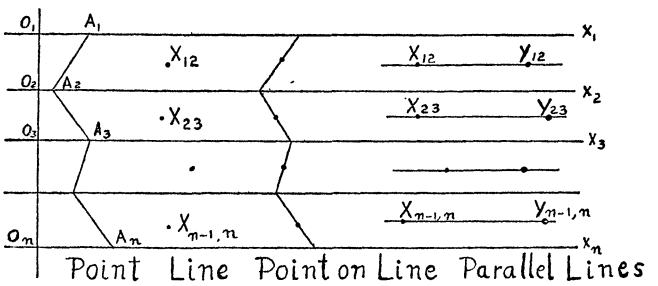


FIG. 1

The images of points and lines in E_n, determined by the above transformation, constitute the space P_n. The lines $x_i, \cdots, x_n$ are called the axes of the space P_n.

k-planes in P_n. A k-plane in $E_n(k \leqq n)$ is defined to be the set of all points on all the lines joining the points of a given $(k-1)$-plane to a point Q not on the given $(k-1)$-plane. If $k=n$, the k-plane is called a k-space.

A line is defined to be a 1-plane. The ordinary plane is the same as a 2-plane.

There are other ways to define a k-plane in E_n. For example, a k-plane is determined by $k+1$ points which do not belong to any $(k-1)$-plane.

* The n points $A_1, \cdots, A_n$, lying on the parallel lines $x_1, \cdots, x_n$ respectively, determine $\frac{1}{2}n(n-1)$ lines A_jA_k where j and k range over all integers from 1 to n, and $j \neq k$. The n points $B_1, \cdots, B_n$ determine another set of lines B_jB_k. The intersection of the corresponding lines A_jA_k and B_jB_k determine $\frac{1}{2}n(n-1)$ points x_jx_k. These $\frac{1}{2}n(n-1)$ points are not independent. The points x_jx_{j+1} (the independent components of the line AB) are an independent subset of the above set. Given these n points we can always construct the other points of the set.

These $\frac{1}{2}n(n-1)$ points form, in plane E, a configuration which contains $\frac{1}{2}n(n-1)$ vertices, $\frac{1}{8}n(n-1)(n-2)$ sides, and 3 vertices lying on every side. These configurations are a subclass of the class of polyhedral configurations.

† It is clear that one or all of the independent components of a line may lie on some of the lines $x_i, \cdots, x_n$. It is also clear that infinite points of E_n could either have all their n coordinates infinite or some of them finite. This leads to special cases that could be studied easily.

However, in this paper we shall not study any special cases unless needed in solving a particular problem.

Also an $(n-1)$-plane in n-space could be represented analytically as a linear equation in n unknowns $x_1, x_2, \cdots, x_n$.

Since all these definitions are equivalent, we shall use the definition which is most convenient in any particular problem.

The image of a k-plane in E_n, under transformation T, is defined to be a k-plane in P_n.

Next we shall study some of the properties of a k-plane in P_n.

a) Consider an $(n-1)$-plane in the space P_n. There exists a line in this $(n-1)$-plane such that all the independent components of this line coincide with each other in the Euclidean plane E.

The easiest way to prove this result is to consider the equation $\sum_{i=1}^{n} a_i x_i = a_0$ as the equation of an $(n-1)$-plane in E_n, whose image is the given $(n-1)$-plane in P_n.

Consider the point $(x_1, x_1+h, \cdots, x_1+(n-1)h)$. This point belongs to the $(n-1)$-plane if

$$x_1 = \frac{a_0 - h(a_2 + 2a_3 + \cdots + (n-1)a_n)}{\sum a_i}, \qquad \sum a_i \neq 0.$$

For any given value of h, the above equation gives a unique point of the $(n-1)$-plane. In particular let $h=0$ and $h=1$. Find the image of these two points and the line determined by these two. It is clear that all the independent components of this line coincide. Since for any $(n-1)$-plane it is possible to find two such points, the image of the $(n-1)$-plane always include a line with the above property. It is also clear that all the other points determined by other values of h belong to this line.

If $\sum a_i = 0$, such a line still exists, but it is at infinity and, as far as graphical problems are concerned, does not play any part in this paper. Therefore, to save space, we have excluded that case.

b) Consider an $(n-2)$-plane G in the n-space P_n. G includes a line such that the first t independent components of this line coincide in a single point in the Euclidean plane E, and the remaining $n-1-t$ components coincide in another point of E. This is true for any given $1 \leqq t \leqq n-2$.

The proof is similar to the proof of a) and, to save space, will not be given here.

A very simple proof exists, for the case $n=4$. Since we are going to use this property in a later problem we will discuss the proof, when $n=4$.

Since a 2-plane is determined by two intersecting lines, the lines (x_{12}, x_{23}, x_{34}) and (y_{12}, y_{23}, y_{34}), which intersect at the point (a_1, a_2, a_3, a_4), Figure 2, determine the 2-plane in 4-space. By examining the figure, it is obvious that the line (z_{12}, z_{23}, z_{34}) belongs to the plane (intersects two lines of the plane), and has the property that z_{23} and z_{34} coincide. In a similar way we could find a line (u_{12}, u_{23}, u_{34}) where u_{12} and u_{23} coincide.

Such a line is called an axis of the plane and is represented by p_1, p_2.

c) The $(n-1)$ components of the image of a line in n-space, are the images of the projections of this line over the $(n-1)$ coordinate planes x_1x_2, x_2x_3, $\cdots$, $x_{n-1}x_n$.

This follows immediately from the definition of the transformation T.

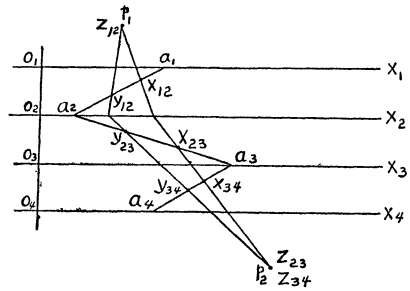


FIG. 2

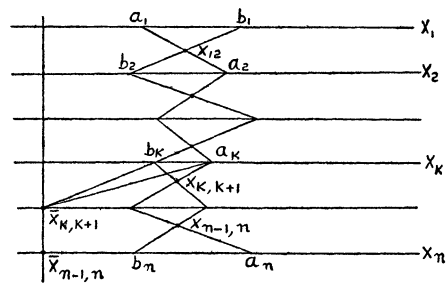


FIG. 3

Graphical solutions of certain problems.

Problem 1. Given a line t in an n-space, find its projection over one of the coordinate k-planes. $(k<n-1)$.

Solution. Without loss of generality we can assume that the given coordinate k-plane is determined by the $x_1, \cdots, x_k$ axes. Projection of a point $A(a_1, \cdots, a_k, \cdots, a_n)$ of the line t, over this k-plane, will be $A'(a_1, \cdots, a_k, 0, \cdots, 0)$.

Projection of the point $B(b_1, \cdots, b_n)$ will be $B'(b_1, \cdots, b_k, 0, \cdots, 0)$. $A'B'$ determines the projection of AB.

Now if the image of AB has the independent coordinates $x_{12}, x_{23}, \cdots, x_{n-1,n}$ the image of $A'B'$ will be determined by the coordinates $x_{12}, x_{23}, \cdots, x_{k-1,k}, \bar{x}_{k,k+1}, \cdots, \bar{x}_{n-1,n}$. See Figure 3.

Problem 2. In a 3-space, draw a line t through a given point $A(a_1, a_2, a_3)$ and intersecting two given skew lines t_1 and t_2.

Solution. Suppose the coordinates of the images of t_1 and t_2 are x_{12}, x_{23} and y_{12}, y_{23} respectively (Figure 4).

The problem amounts to the following. In the Euclidean plane, take a point P on the line x_2. Connect P to x_{12} to find P', connect P' to y_{12} to find P'', connect P'' to y_{23} to find P''' and connect P''' to x_{23} to find $\overline{P}$ on x_2. If P and $\overline{P}$ coincide P' and P''' could be taken as the coordinates of the line t. If P and $\overline{P}$ do not coincide we have to find another point R on x_2 having this property.

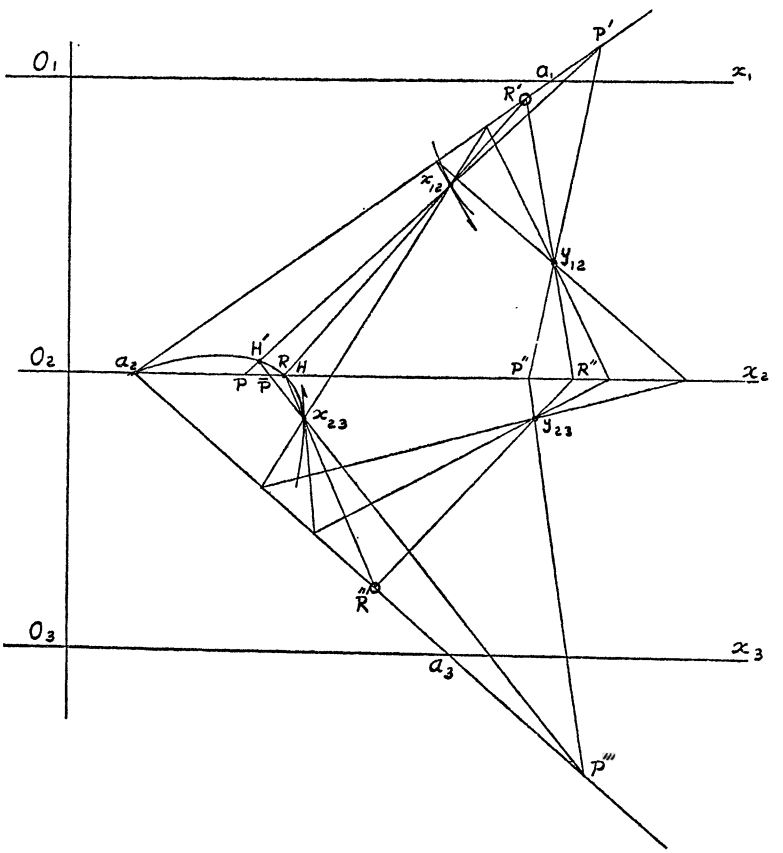


FIG. 4

Now as P moves over x_2, P and $\overline{P}$ generate two projective pencil of lines through x_{12} and x_{23}, respectively. The intersection of the corresponding lines of these two pencils is a point conic. The intersection of this point conic with the line x_2 is the point R which we are looking for.

It is clear that this conic will, in general, always intersect the line x_2, since the point a_2 obviously belongs to this conic. The other point of intersection will be the answer to the problem.

The geometric construction for finding this point of intersection is well known, but it can also be worked out graphically.

We notice that x_{12} and x_{23} belong to the conic and we can construct the tangents to the conic at those points. So if three points of a conic and tangents at two of these points are given we can approximate a conic to these conditions. The intersection of this conic and x_2 is the point H. We find $\overline{H}$. If H and $\overline{H}$ coincide the problem is solved. If they do not coincide, we have found another point of the conic and consequently we can draw a more accurate conic. We can continue this until we find a conic accurate enough to give us the point R, as accurately as needed. For most practical purposes we do not have to repeat this process for more than two or three times.

It should be noticed that a graphical solution of Problem 2 is only an approximation.

Problem 3. Given three skew lines in a 3-space, draw a line parallel to one and intersecting the other two.

Solution. The problem is quite similar to the previous problem and, to save space, the complete solution will not be given here. It should be only noticed that this problem is a special case of Problem 2, where the ordinary point A is replaced by a point at infinity.

Looking back at the solution of Problems 2 and 3, it can be immediately proved that in general the problem does not have a solution in any n-space where $n>3$. (Unless the two lines are in a 3-space which is a subset of the given n-space). However a more general problem suggests itself:

Problem 4. Given two skew lines in any n-space, it is possible to draw a line intersecting these lines and such that its projection over two of the coordinate planes (2-planes) be parallel to two given lines, one lying in each of the planes.

Solution. If we find the image of the given lines under the transformation T, we notice that the problem is almost the same as Problem 3. The proof follows from the fact that, in general, the construction is always possible. There are, of course, some special cases, but we omit them here.

Problem 5. Solve 4 linear equations in 4 unknowns, graphically.

Solution. Consider the following set of four linear equations in four unknowns:

$$ax_1 + bx_2 + cx_3 + dx_4 = e, \tag{1}$$

$$a'x_1 + b'x_2 + c'x_3 + d'x_4 = e', \tag{2}$$

$$a''x_1 + b''x_2 + c''x_3 + d''x_4 = e'', \tag{3}$$

$$a'''x_1 + b'''x_2 + c'''x_3 + d'''x_4 = e'''. \tag{4}$$

Any two of these equations determine a plane in 4-space. Consider the two planes determined by the first and the second pairs, respectively. If these two

planes do not lie in the same 3-space, they have one and only one point in common, and the coordinates of this point constitute the solution of the system. If they lie in the same 3-space, then either they intersect or they are parallel, and the system has infinitely many solutions or none.

Therefore the problem of solving this system of four equations graphically reduces to the following problem: Given two planes in 4-space, construct their point or line of intersection, if such a point or line exists.

We consider two such planes and find their intersections.

Consider two planes A and A' determined by their axes p_1p_2 and $p_1'p_2'$ and the points (a, b, c, d) and (a', b', c', d'), respectively, (Figure 5).

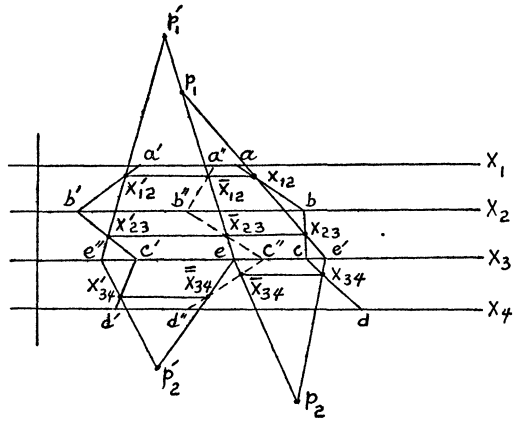


FIG. 5

In the plane E_2 (plane of the figure), draw two lines passing through p_1 and p_1' and intersecting ab and bc at the points x_{12} and x_{23} and $a'b'$ and $b'c'$ at the points x_{12}' and x_{23}' respectively, and with the additional property that the lines connecting x_{12} to x_{12}' and x_{23} to x_{23}' are parallel to the x_2 axis.

This construction is always possible and unique.* Connect the point p_1 to p_1' to determine the points $\bar{x}_{12}$, $\bar{x}_{23}$ and e. Connect the points e and e' to p_2 and e and e'' to p_2' to determine the points $\bar{x}_{34}$, x_{34}, $\bar{\bar{x}}_{34}$ and x_{34}'.

It will be noticed that the three points x_{12}, x_{23}, and x_{34} constitute the image of a line in the plane A, and hence $\bar{x}_{12}$, $\bar{x}_{23}$ and $\bar{x}_{34}$ also constitute the image of a line belonging to the plane A. ($\bar{x}_{12}$, $\bar{x}_{23}$, $\bar{x}_{34}$ is parallel to one of the lines of the plane and has a point in common with the plane.)

Similarly $\bar{x}_{12}$, $\bar{x}_{23}$ and $\bar{\bar{x}}_{34}$ is the image of a line in plane A'. Now consider the point (a'', b'', c'', d''). It belongs to both these lines and hence to both planes, therefore, it is the point of intersection of the two planes.

In case the Euclidean line connecting $\bar{\bar{x}}_{34}$ to $\bar{x}_{34}$ is parallel to x_3 axis, the point (a'', b'', c'', d'') is a point at infinity.

In case the points e, p_2 and p_2' are collinear and $\bar{x}_{34}$ coincides with $\bar{\bar{x}}_{34}$, the two

* See Problem E1066, this MONTHLY, 61 (1954) 47.

planes have a line in common, namely the line whose image is $\bar{x}_{12}$, $\bar{x}_{23}$ and $\bar{x}_{34}$. This happens in case the two planes lie in the same 3-space.

The second step in solving the problem. Given two linear equations (1) and (2) in four unknowns, find the image of the plane they determine.

To do this we find three points of the plane and map them on E_2 (plane of the figure). We then find the axis of the plane and consequently its graph.

Since there are two equations and four unknowns, it is possible to select two of the coordinates of the point arbitrarily and find the other two. In particular, these two coordinates are selected in such a manner that the computations will be simplified. For example, consider $x_1=0$ and $x_2=0$. Then

$$cx_3 + dx_4 = e,$$
$$c'x_3 + d'x_4 = e'.$$

Hence,

$$x_3 = \frac{\begin{vmatrix} e & d \\ e' & d' \end{vmatrix}}{\begin{vmatrix} c & d \\ c' & d' \end{vmatrix}}, \qquad x_4 = \frac{\begin{vmatrix} c & e \\ c' & e' \end{vmatrix}}{\begin{vmatrix} c & d \\ c' & d' \end{vmatrix}}, \qquad x_1 = 0,\ x_2 = 0,$$

so one point of the plane is

$$\left(0, 0, \frac{\begin{vmatrix} e & d \\ e' & d' \end{vmatrix}}{\begin{vmatrix} c & d \\ c' & d' \end{vmatrix}}, \frac{\begin{vmatrix} c & e \\ c' & e' \end{vmatrix}}{\begin{vmatrix} c & d \\ c' & d' \end{vmatrix}}\right).$$

Another convenient point is determined as follows:

$$ax_1 + bx_2 + cx_3 + dx_4 = e,$$
$$a'x_1 + b'x_2 + c'x_3 + d'x_4 = e',$$
$$x_1 = x_2, \qquad x_2 = x_3.$$

The solution of this system is

$$x_1 = x_2 = x_3 = \frac{\begin{vmatrix} e & d \\ e' & d' \end{vmatrix}}{\begin{vmatrix} a+b+c & d \\ a'+b'+c' & d' \end{vmatrix}} \quad \text{and} \quad x_4 = \frac{\begin{vmatrix} a+b+c & e \\ a'+b'+c' & e' \end{vmatrix}}{\begin{vmatrix} a+b+c & d \\ a'+b'+c' & d' \end{vmatrix}}.$$

The image of this point is shown in Figure 6.

Finally consider the point determined by the following equations:

$$ax_1 + bx_2 + cx_3 + dx_4 = e,$$
$$a'x_1 + b'x_2 + c'x_3 + d'x_4 = e',$$
$$x_2 = x_1 + k, \qquad x_3 = x_1 + 2k, \qquad x_4 = x_1 + 3k.$$

These five equations give us:

$$k = \frac{\begin{vmatrix} e & a+b+c+d \\ e' & a'+b'+c'+d' \end{vmatrix}}{\begin{vmatrix} b+2c+3d & a+b+c+d \\ b'+2c'+3d' & a'+b'+c'+d' \end{vmatrix}},$$

$$\bar{x}_1 = \frac{\begin{vmatrix} e & b+2c+3d \\ e' & b'+2c'+3d' \end{vmatrix}}{\begin{vmatrix} a+b+c+d & b+2c+3d \\ a'+b'+c'+d' & b'+2c'+3d' \end{vmatrix}},$$

$$\bar{x}_2 = \bar{x}_1 + k, \qquad \bar{x}_3 = \bar{x}_1 + 2k, \qquad \bar{x}_4 = \bar{x}_1 + 3k.$$

The image of this point is given in Figure 6.

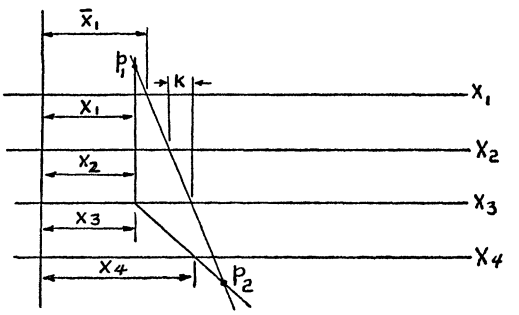


FIG. 6

It will be noticed from the figure that these two last points determine the axis of the plane. This axis and the first point determine the plane uniquely, and the rest of the solution follows.

To summarize the above results: Consider four equations. Take two of them at a time and plot them by finding three points of each plane. Construct the point of intersection of these two planes. The coordinates of this point, which can be measured, constitute the solution of the system. If the two planes are parallel or intersect, the problem has none or infinitely many solutions.

The problem of solving three linear equations in three unknowns is similar to the previous problem, but somewhat simpler. The reader could discover the construction for himself.

These five problems are, of course, examples. It is clear that one could find many more problems involving incidence of points, lines and n-planes which could be solved by the above methods.

ON A DEFINITION OF ORDINAL NUMBERS

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1. Introduction. In this paper we present a definition of ordinals which has the advantage of simplicity and of being an obvious extension of the definition of natural numbers as elements of the smallest set A such that

(i) The empty set $0 \in A$ and

(ii) if $x \in A$, then $x+1 \in A$ (where $x+1 = x \cup \{x\}$).

By adding a third condition, the set of all ordinals, Q, is defined first and then an ordinal is defined as an element of Q or Q itself.

Just as the above definition of natural numbers incorporates the principle of ordinary induction, our definition of ordinals incorporates the principle of transfinite induction. This being the single most powerful and intuitive tool for proving properties of ordinals, its availability from the definition greatly simplifies the development.

This definition, however, seems possible only in certain types of set theory, i.e. those permitting the existence of very large sets. For this reason we first give a brief description of the axiom system in which we work. Later in Section 6 we discuss more fully this system and its relation to others.

2. The axiom system. The development in this paper is based on an axiom system like the one in the appendix of Kelley [3], which to some extent follows Gödel [1] and Morse [4]. We assume the existence of sets of the form $\{x: x \text{ has property } P\}$ where P is any property, in particular the existence of a universe $\mathfrak{U} = \{x: x = x\}$. Our axiom of abstraction, which is due to A. P. Morse, reads:

$$u \in \{x: x \text{ has property } P\} \text{ iff } u \text{ has property } P \text{ and } u \in \mathfrak{U}.$$

Every set is a subset of $\mathfrak{U}$, but not necessarily an element of $\mathfrak{U}$.

We introduce now some definitions and notation which will be used in the subsequent development.

2.1. DEFINITION. 0 *denotes the empty set.*

2.2. DEFINITION. $\sigma A = \bigcup_{x \in A} x$.

2.3. DEFINITION. $\pi A = \bigcap_{x \in A} x$. *Note that* $\sigma 0 = 0$, $\pi 0 = \mathfrak{U}$.

2.4. DEFINITION. $\{x\} = \pi\{A: x \in A\}$.

With this definition x is the only element of $\{x\}$ only if $x \in \mathfrak{U}$. Otherwise, $\{x\} = \pi 0 = \mathfrak{U}$ and $x \notin \{x\}$.

2.5. DEFINITION. *A family of sets A is called a nest iff $x \in A$, $y \in A$, implies $x \subset y$ or $y \subset x$.*

2.6. DEFINITION. $A \sim B = \{x: x \in A, x \notin B\}$.

3. Definition of the ordinals. The set of natural numbers, ω, is defined as the smallest set A having the following properties:

(i) $0 \in A$, (ii) $x \cup \{x\} \in A$ whenever $x \in A$.

A natural extension of ω is the set of all ordinals Q defined as the smallest set having, in addition to (i) and (ii) above, the property:

(iii) If B is a nest, $B \subset A$, and $B \in \mathfrak{U}$, then $\sigma B \in A$. The smallest set having property P is naturally defined as the intersection of all sets having property P. However, since none of the sets having property P may be a point in $\mathfrak{U}$, some care must be taken in formulating this definition. As a matter of fact in such a case $\pi\{A : A \text{ has property } P\} = \pi 0 = \mathfrak{U}$. The definition in 2.4 of $\{x\}$ when $x \notin \mathfrak{U}$ is a good example of this.

Now it can be shown that if A satisfies (ii) and (iii) above, then $A \notin \mathfrak{U}$. For this reason we define:

3.1. DEFINITION. $Q = \{x : x \in A$ *for every* A *satisfying properties* (ii) *and* (iii) *above*$\}$.

Note that (i) follows from (iii) since $\sigma 0 = 0$.

3.2. DEFINITION. x *is an ordinal iff* $x \in Q$ *or* $x = Q$.

It follows immediately from 3.1 that Q itself satisfies (ii) and (iii).

One of the main advantages of this definition is that it incorporates the principle of transfinite induction, i.e. in order to show that all elements of Q have some property P, it is enough to prove that:

(a) If $x \in Q$ and x has property P, then $x \cup \{x\}$ has property P.

(b) If B is a nest of elements of Q having property P and $B \in \mathfrak{U}$, then σB has property P.

To see this, let $A = \{x : x \in Q$ and x has property $P\}$. Then because of (a) and (b) A satisfies (ii) and (iii) above and hence $Q \subset A$.

4. Elementary properties of ordinals. Throughout this section we suppose $a \in Q$, $b \in Q$.

4.1. DEFINITION. t *is transitive iff* $s \in t$ *implies* $s \subset t$.

4.2. THEOREM. a *is transitive.*

Proof. By induction. (a) Let the theorem hold for $x \in Q$ and suppose $t \in x \cup \{x\}$. Then $t \in x$ or $t = x$ and in either case $t \subset x \cup \{x\}$. (b) Let A be a nest of elements for which the theorem holds and suppose $t \in \sigma A$. Then $t \in x$ for some $x \in A$ and hence $t \subset x \subset \sigma A$.

4.3. THEOREM. $a \subset Q$.

Proof. By induction.

4.4. THEOREM. *If* $t \subset a$, *then* $t \notin t$.

Proof. By induction. (a) Let the theorem hold for $x\in Q$ and suppose $t\subset x\cup\{x\}$. If $t\in t$, then $t\in x$ or $t=x$ and by 4.2 $t\subset x$, contradicting the induction hypothesis. (b) Let A be a nest of elements for which the theorem holds and suppose $t\subset\sigma A$. If $t\in t$, then $t\in x$ for some $x\in A$ and by 4.2 $t\subset x$, again a contradiction.

4.5. COROLLARY. $Q\notin Q$.

Proof. Suppose $Q\in Q$. Since $Q\subset Q$, 4.4 implies $Q\notin Q$.

4.6. THEOREM. *Suppose* $t\in a$, $s\in a$, $t\neq s$. *Then* $t\subset s$ *or* $s\subset t$, *and* $t\in s$ *iff* $t\subset s$.

Proof. By induction. (a) Let the theorem hold for $x\in Q$ and suppose $t\in x\cup\{x\}$, $s\in x\cup\{x\}$, $t\neq s$. Then $t\in x$ or $t=x$, $s\in x$ or $s=x$. Three cases are possible:

(i) $t\in x$, $s\in x$. In this case the theorem holds by the induction hypothesis.

(ii) $t=x$, $s\in x$. By 4.2 $s\subset x$ whence $s\subset t$. If $t\in s$, then $t\in t$, a contradiction by 4.4.

(iii) $t\in x$, $s=x$. By an argument analogous to that in (ii) we have $t\subset s$, $t\in s$, $s\notin t$.

(b) Let A be a nest of elements for which the theorem holds and suppose $t\in\sigma A$, $s\in\sigma A$, $t\neq s$. Then $t\in x$ for some $x\in A$ and $s\in y$ for some $y\in A$. Since A is a nest, either $x\subset y$ or $y\subset x$. If $x\subset y$, then $t\in y$ and $s\in y$, and if $y\subset x$, then $t\in x$ and $s\in x$. In both cases the theorem holds by hypothesis.

4.7. THEOREM. $\sigma a\in Q$.

Proof. By 4.6 and 4.3 a is a nest and $a\subset Q$. Since also $a\in\mathfrak{U}$, $\sigma a\in Q$ by Definition 3.1.

4.8. THEOREM. *If* t *is transitive and* $t\subset a$, *then* $t=a$ *or* $t\in a$.

Proof. By induction. (a) Let the theorem hold for $x\in Q$ and suppose t is transitive, $t\subset x\cup\{x\}$. If $t\subset x$, then by assumption $t=x$ or $t\in x$, and hence $t\in x\cup\{x\}$. If $t\not\subset x$, then $x\in t$ and since t is transitive, $x\subset t$. But $x\in t$ and $x\subset t$ imply $x\cup\{x\}\subset t$. Therefore $x\cup\{x\}=t$. (b) Let A be a nest of elements for which the theorem holds and suppose t is transitive, $t\subset\sigma A$. There are two possibilities:

(i) $t\subset x$ for some $x\in A$. Then by assumption $t=x$ or $t\in x$. If $t\in x$, then $t\in\sigma A$. If $t=x$ and x is the largest element in A, then since A is a nest, $t=x=\sigma A$. If $t=x$ and x is not the largest element in A, then for some $y\in A$, $x\subset y$ and $x\neq y$. Then $t\subset y$, $t\neq y$, and by hypothesis $t\in y$. Hence $t\in\sigma A$.

(ii) For every $x\in A$, $t\not\subset x$. Then there exists $\alpha\in t\sim x$, and since $t\subset\sigma A$, for some $y\in A$ we have $\alpha\in y$. We show also $x\in y$: Since A is a nest, $x\subset y$ or $y\subset x$, but if $y\subset x$, then $\alpha\in x$, a contradiction. Thus $x\subset y$, $x\neq y$. By 4.2 x is transitive and hence by the hypothesis of induction $x\in y$. Now by 4.6, since $\alpha\notin x$, we have $x\subset\alpha$. Since $\alpha\in t$ and t is transitive, $\alpha\subset t$ and hence $x\subset t$. Since this is true for every $x\in A$, we have $\sigma A\subset t$ and therefore $t=\sigma A$.

The following theorem shows that the ordinals are well-ordered.

4.9. THEOREM. *If* $0 \neq A \subset Q$, *then* $\pi A \in A$.

Proof. Let $x = \pi A$. x is transitive since every element of A is an ordinal and so transitive by 4.2. For every $y \in A$, $x \subset y$ and by 4.7 either $x = y$ or $x \in y$. If $x \neq y$ for every $y \subset A$, then $x \in y$ for every $y \in A$ and $x \in \pi A = x$, contradicting 4.4. Therefore $x = y$ for some $y \in A$.

4.10. THEOREM. $a \subset b$ *or* $b \subset a$.

Proof. By 4.9 $a \cap b = a$ or $a \cap b = b$, so $a \subset b$ or $b \subset a$.

4.11. THEOREM. $a \in b$ *iff* $a \subset b$ *and* $a \neq b$.

Proof. Suppose $a \in b$. Then by 4.2 $a \subset b$ and by 4.4 $a \neq b$. Now suppose $a \subset b$ and $a \neq b$. Then by 4.7, a being transitive, $a \in b$.

4.12. THEOREM. *If* $t = \sigma a$, *then either* $a = t$ *or* $a = t \cup \{t\}$.

Proof. We first check that (i) $t \subset a \subset t \cup \{t\}$. For, if $s \in t$, then $s \in x$ for some $x \in a$. By 4.2 $x \subset a$ so that $s \in a$. Next, let $s \in a$. Then $s \subset \sigma a = t$. By 4.3 $s \in Q$ and by 4.2 s is transitive. Hence by 4.7 and 4.8 $s = t$ or $s \in t$. (i) being checked, there are now two possibilities:

(1) $a \subset t$. Then $a = t$.

(2) $a \not\subset t$. Then $t \in a$. Hence $t \cup \{t\} \subset a$ and $a = t \cup \{t\}$.

4.13. THEOREM. $\sigma Q = Q$.

Proof. (i) $Q \subset \sigma Q$. For let $x \in Q$. Then by 3.1 $x \cup \{x\} \in Q$. Since $x \in x \cup \{x\}$, we have $x \in \sigma Q$. (ii) $\sigma Q \subset Q$. Let $x \in \sigma Q$. Then $x \in y$ for some $y \in Q$ and by 4.3 $x \in Q$.

The next theorem helps explain the Burali-Forti paradox in this system. The paradox states that the largest ordinal is less than its successor ($Q \in Q \cup \{Q\}$).

4.14. THEOREM. $Q \notin \mathfrak{U}$.

Proof. Suppose $Q \in \mathfrak{U}$. Since $Q \subset Q$ and by 4.3 Q is a nest, we have $\sigma Q \in Q$ by 3.1. But $\sigma Q = Q$ by 4.13, so $Q \in Q$, contradicting 4.5.

5. Classical notation. The following definitions introduce the classical notation.

5.1. DEFINITION. $x + 1 = x \cup \{x\}$.

5.2. DEFINITION. $x - 1 =$ *the* $y \in Q$ *such that* $y + 1 = x$.

5.3. DEFINITION. x *is a limit ordinal iff* x *is an ordinal,* $x \neq 0$, *and* $x = \sigma x$.

5.4. DEFINITION. $x < y$ *iff* $x \in y$.

5.5. DEFINITION. $\sup A = \sigma A$ *if* $A \subset Q$.

$\inf A = \pi A$ *if* $0 \neq A \subset Q$, $\quad Q$ *if* $A = 0$.

The following theorems for $x \in Q$, $y \in Q$, are reformulations of results in Section 4.

5.6. THEOREM. $x < x+1$.

5.7. THEOREM. $x+1 \neq 0$.

5.8. THEOREM. *If x is not a limit ordinal, $x-1=\sigma x$.*

5.9. THEOREM. *If x is a limit ordinal and $y<x$, then $y+1<x$.*

5.10. THEOREM. $x \leqq y$ *iff* $x \subset y$.

5.11. THEOREM. *If $x<y$, then $x+1 \leqq y$.*

5.12. THEOREM. *If $x+1=y+1$, then $x=y$.*

5.13. THEOREM. *If $0 \neq A \subset Q$, then* inf $A \in A$.

6. Remarks on the axiom system. Generally speaking, there are two main approaches to axiomatizing set theory. The essential distinction between the two lies in the treatment of the so-called axiom of abstraction, which is really an infinite axiom scheme, and which states that for any given property there exists a set whose members are just those entities which possess that property. Russell's paradox, based on the set $\{x: x \notin x\}$, shows that the axiom as stated leads to a contradiction.

In the Zermelo-Fraenkel type of set theory [2, 5], which we shall call type I, this difficulty is avoided by placing restrictions on the defining property. The axiom, loosely formulated, becomes:

$u \in \{x: x \in A$ and x has property $P\}$ iff $u \in A$ and u has property P, where A is some already existing set.

In the Morse type of set theory [4], which we shall call type II, no restriction is placed on the property, but the axiom is amended to read:

$u \in \{x: x$ has property $P\}$ iff u has property P and $u \in A$ for some A.

Thus those sets which can belong to another set assume a particular importance. They will be called points. (In standard texts these points are called sets and general sets are called classes.)

The Bernays-Gödel type of set theory [1, 3] is essentially of type II. There no axiom of abstraction is used (so that the axiom scheme is finite), but the following metatheorem is proved:

If all quantifiers appearing in the definition of the property P apply only to points and u is a point then

$$u \in \{x: x \text{ has property } P\} \text{ iff } u \text{ has property } P.$$

For the main developments of set theory the differences between the two types are largely a matter of language; existence of a set in a system of type I translates into being a point in a system of type II. For example, the Russell paradox is avoided because in type I the set $\{x: x \notin x\}$ does not exist, while in

type II, although it does exist, it is not a point. Systems in type II, however, have a technical advantage because they permit discussion of certain sets, such as the universe and the set of all ordinals (the largest ordinal), which don't exist in systems of type I. Thus the definition of ordinals given in this paper could not have been possible in a system of type I.

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EXPANSIVE SELF-HOMEOMORPHISMS OF A COMPACT METRIC SPACE

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1. Introduction. Symbolic flows ([6], [7], [2]) have the remarkable property of being *expansive*† (see Example 1 below). Since 1950 several papers ([8], [9], [4], [5], [1]) have been written which are concerned, at least partially, with this property as possessed by a self-homeomorphism of a metric space. Most of the papers have been primarily investigations of the asymptotic properties of the orbits of pairs of points, but our main concern in this paper is of a somewhat different nature. In Section 3 we show that an arc cannot carry an expansive self-homeomorphism, and we use this to show that the possession of an expansive self-homeomorphism is not a topological property. We include a theorem which gives a sufficient condition for the property to be preserved under a homeomorphism. Section 3 also contains a discussion of some of the special properties of the set of expansive self-homeomorphisms considered as a subset of the group of all self-homeomorphisms of a compact metric space. In Section 4 we show (see Theorem 5) that there is a certain uniformity associated with an expansive homeomorphism, and Theorem 6 is concerned with the set of expansive constants (see the definitions below).

* This paper was prepared while the author was a Summer Research Participant at the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

† Utz first used the term *unstable homeomorphism*, but the term *expansive homeomorphism* (suggested by Gottschalk and Hedlund) seems rather natural and descriptive of the notion and is being used currently.

2. Definitions and examples. Unless otherwise stated, we shall suppose that X is a metric space with metric ρ and that ϕ is a homeomorphism of X onto X. ϕ is said to be *expansive* provided there exists $d_\phi > 0$ such that $x, y \in X$ with $x \neq y$ implies $\rho(\phi^n(x), \phi^n(y)) > d_\phi$ for some integer n. The constant d_ϕ is called an *expansive constant* for ϕ. Unless confusion is likely to result, we shall use d rather than d_ϕ. Henceforth we shall assume that ϕ is expansive with expansive constant d.

The set $\sum_{-\infty}^{+\infty} \{\phi^i(x)\}$ is called the *orbit* of x under ϕ and denoted by $0(x)$. Two points, x and y, are said to be *positively (negatively) asymptotic* if for each $\epsilon > 0$, there exists an integer N such that $n > N$ ($n < N$) implies $\rho(\phi^n(x), \phi^n(y)) < \epsilon$.

Example 1. Let X be the set of all functions which map the integers into the set $\{a, b\}$. If $f, g \in X$, define

$$\rho(f, g) = \frac{1}{1 + \max [m \mid f(i) = g(i) \text{ for } |i| < m]}.$$

For $h \in X$ define $\phi(h) = k$ where $k(i) = h(i+1)$ for each integer i. Under these definitions X is a compact metric space and ϕ is a homeomorphism of X onto X. This is an example of a symbolic flow. Clearly ϕ is expansive; for if $f \neq g$, then $\rho(\phi^m(f), \phi^m(g)) = 1$ for some integer m. Hence each positive number less than 1 is an expansive constant for ϕ. The fact that symbolic flows are expansive has been recognized for several years and has provided the impetus for the study of expansive homeomorphisms in a more general setting beginning with Utz's paper [8].

Example 2. If X is the reals with metric $\rho(x, y) = |x - y|$ and $\phi(x) = 2x$, then $\rho(\phi^n(x), \phi^n(y)) = 2^n \rho(x, y)$. Thus ϕ is expansive and each positive real number is an expansive constant for ϕ. We observe that every pair of points is negatively asymptotic. However X is not compact, and most of the theorems below require the space to be compact; therefore, in the remaining examples the space is compact.

Example 3. Let X consist of the real numbers 0, $1/n$ ($n = 1, 2, 3, \cdots$), and $1 - 1/n$ ($n = 3, 4, 5, \cdots$) with metric $\rho(x, y) = |x - y|$. Let $\phi(x) = x$ if $x = 0, 1$; if $x \neq 0, 1$, then let $\phi(x)$ be the number in X which is immediately to the left of x. Each positive number less than $\frac{1}{6}$ is an expansive constant for ϕ. Each point except 1 is positively asymptotic to 0, and each point except 0 is negatively asymptotic to 1.

If we identify 0 and 1, then each pair of points is doubly asymptotic; i.e., both positively and negatively asymptotic. As indicated by Theorem 3 of [1], this situation cannot occur if the space is compact and dense-in-itself. If the space is compact and dense-in-itself, the theorem asserts that for each point x and each $\epsilon > 0$, there exists $y \in U(\epsilon, x)$ such that x and y are not doubly asymptotic.

We note that the essential features of this example are also presented by

$\phi(x)=x^2$ defined on the unit interval and then restricted to the orbit closure of $\frac{1}{2}$.

Example 4. Let X be the space of Example 3, and let $Y=X\times X$. All of the points of Y are either in the interior of or on the square S with vertices (0, 0), (1, 0), (1, 1), and (0, 1). Define certain subsets of Y as follows:

$$A=\{(0,0),(1,0),(1,1),(0,1),(\tfrac{1}{2},\tfrac{1}{2})\};\qquad B=S\cap Y-A;$$

for $n>2$. Let $S_n=\{y\mid y\in Y$ and y is on the square with vertices $(1/n, 1/n)$, $(1-1/n, 1/n)$, $(1-1/n, 1-1/n)$, and $(1/n, 1-1/n)\}$. Now define ϕ: $Y\to Y$ by

$$\phi(x)=\begin{cases}x \text{ if } x\in A,\\ y \text{ if } x\in B\{x\in S_n\} \text{ and } y \text{ is the point in } B\{S_n\} \text{ which is}\\ \text{immediately ahead of } x \text{ in a counterclockwise direction.}\end{cases}$$

Again each positive number less than $\frac{1}{6}$ is an expansive constant. Each point of Y interior to S is a periodic point, so Y has countably many periodic points. The fact (shown by Utz [8]) that if a space is compact, then there are at most countably many periodic points may be seen as follows: the number of fixed points under ϕ is finite, for otherwise ϕ is not expansive at the limit points of the fixed points; the number of points of period n is thus finite because they are fixed under ϕ^n and ϕ^n is also expansive [8].

The space in Example 1 is dense-in-itself, but the spaces in Examples 3 and 4 above are not. None of the three spaces, however, is connected. Thus the question arises as to whether a compact connected space can carry an expansive self-homeomorphism. This question was answered in the affirmative by Williams [9].

3. Let X and Y be metric spaces with metrics ρ and ρ', respectively.

THEOREM 1. *If g is a homeomorphism of X onto Y such that g^{-1} is uniformly continuous, then $g\phi g^{-1}$ is an expansive self-homeomorphism of Y.*

Proof. By the uniform continuity of g^{-1} there exists $\alpha>0$ such that $\rho'(g(x_1), g(x_2))\leqq\alpha$ implies $\rho(x_1, x_2)<d$; thus $\rho(x_1, x_2)>d$ implies $\rho'(g(x_1), g(x_2))>\alpha$. If y_1 and y_2 is an arbitrary pair of distinct points in Y, then $g^{-1}(y_1)$ and $g^{-1}(y_2)$ are in X, and hence there exists an integer n such that $\rho(\phi^n g^{-1}(y_1), \phi^n g^{-1}(y_2))>d$. Therefore $\rho'(g\phi^n g^{-1}(y_1), g\phi^n g^{-1}(y_2))>\alpha$, which establishes the theorem. See the discussion following Theorem 4 for an example which shows that the conclusion is not necessarily true if the uniform continuity of g^{-1} is dropped from the hypothesis.

If X is compact, then Y is also compact and g^{-1} is therefore uniformly continuous. Hence $g\phi g^{-1}$ is an expansive self-homeomorphism of Y. In particular this shows that if X is compact, then the conjugate of an expansive self-homeomorphism by any self-homeomorphism of X is an expansive self-homeomorphism of X. Therefore the set A of expansive self-homeomorphisms of a compact metric space X is a self-conjugate subset of the group G of all self-homeomorphisms of X. However, A is not necessarily a subgroup because if X is nondiscrete, then

the identity transformation is not expansive. By using the metric ρ_1 on G defined by $\rho_1(f, g) = \sup\{\rho(f(x), g(x)) \text{ for } x \in X\}$ we prove the following theorem.

THEOREM 2. *If X is compact and infinite and if $m \neq n$, then $\rho_1(\phi^m, \phi^n) > d$. Also if there exist self-homeomorphisms arbitrarily near ϕ, then there exist expansive self-homeomorphisms arbitrarily near ϕ.*

Proof. X contains a nonperiodic point p (see the proof of Theorem 2 in [1]). Thus $p \neq \phi^{n-m}(p)$, and therefore there exists an integer k such that $\rho(\phi^k(p), \phi^{k+n-m}(p)) > d$; i.e., $\rho(\phi^m(\phi^{k-m}(p)), \phi^n(\phi^{k-m}(p))) > d$. Hence $\rho_1(\phi^m, \phi^n) > d$.

The second part of the theorem follows from the observations that $g\phi g^{-1}$ is expansive and $g\phi g^{-1}$ can be made arbitrarily near ϕ by selecting g sufficiently close to ϕ.

We have need of the following theorem later, and perhaps it is of some interest in itself.

THEOREM 3. *If $A \subset X$, $X - A$ is finite, and ϕ is a self-homeomorphism of X which is expansive on A, then ϕ is expansive on X.*

Proof. We shall show that ϕ is expansive on $A \cup \{x\}$ where $x \in X - A$. The proof may then be completed by using induction for finitely many steps. There is at most one point $p \in A$ such that $\rho(\phi^m(x), \phi^m(p)) \leqq \frac{1}{2}d$ for all integers m; if p and q are two such points, then $\rho(\phi^m(p), \phi^m(q)) \leqq \rho(\phi^m(p), \phi^m(x)) + \rho(\phi^m(x), \phi^m(q)) \leqq \frac{1}{2}d + \frac{1}{2}d = d$ for all m, which contradicts the expansiveness of ϕ on A. If a point p as described above exists, let $0 < c < \rho(x, p)$; otherwise let $c = \frac{1}{2}d$. It is clear that c is an expansive constant for ϕ on $A \cup \{x\}$.

THEOREM 4. *There are no expansive self-homeomorphisms of an arc.*

Proof. If there were an expansive self-homeomorphism of an arc, then, by Theorem 1, there would be an expansive self-homeomorphism ϕ of the unit interval of the reals. Then $\phi(0) = 0$ and $\phi(1) = 1$ or $\phi(0) = 1$ and $\phi(1) = 0$. We shall suppose that $\phi(0) = 0$ and $\phi(1) = 1$, for otherwise ϕ^2 (which is expansive) has this property. As remarked earlier, ϕ can have only a finite number of fixed points, and we suppose that p is the smallest positive fixed point. Let $0 < a < p$ and suppose $\phi(a) > a$ (the proof is similar if $\phi(a) < a$). The sequences $\{\phi^n(a)\}$ and $\{\phi^{-n}(a)\}$ converge to p and 0, respectively. Pairs of points in the open interval $(0, p)$ which are close together stay close together since they are asymptotic to the end points. Thus ϕ is not expansive.

Jakobsen and Utz [5] have shown that a closed 2-cell cannot carry an expansive self-homeomorphism, but it is an open question as to whether a closed n-cell $(n > 2)$ can carry one.

We use the above theorem to show that the uniform continuity of g^{-1} cannot be dropped from the hypothesis of Theorem 1. Define X and Y by

$$X = \{x \mid x \geqq 0\}, \qquad Y = \{y \mid 0 \leqq y < 1\}$$

with metric $\rho(z, w) = |z - w|$ for both X and Y. Let $f(x) = 2x$ and $g(x) = x/(x+1)$.

Then f is an expansive self-homeomorphism of X, and g is a homeomorphism of X onto Y, but $g^{-1}(y)=y/(1-y)$ is not uniformly continuous. Furthermore $gfg^{-1}(y)=2y/(y+1)$ is not an expansive self-homeomorphism of Y. In fact, there are no expansive self-homeomorphisms of Y; for each self-homeomorphism h of Y can be extended to a self-homeomorphism ϕ of the unit interval I. If h is expansive on Y, then, by Theorem 3, ϕ is expansive on I; this is contrary to the above theorem. Thus X and Y are homeomorphic, X can carry an expansive self-homeomorphism but Y cannot; hence the ability to carry an expansive homeomorphism is not a topological invariant.

4. There is a type of uniformity associated with ϕ as indicated by the following theorem.

THEOREM 5. *If X is compact, then for each θ such that $0<\theta\leqq d$ there exists a positive integer $k(\theta)$ such that $\rho(x, y)>\theta$ implies $\rho(\phi^n(x), \phi^n(y))>d$ for some n such that $|n|\leqq k$.*

Proof. If the theorem is not true, then there exists θ such that $0<\theta\leqq d$ and, for each positive integer i, there exists x_i and y_i with $\rho(x_i, y_i)>\theta$ and $\rho(\phi^n(x_i), \phi^n(y_i))\leqq d$ for each n such that $|n|\leqq i$. We can assume that the sequences $\{x_i\}$ and $\{y_i\}$ converge to x and y, respectively. We observe that $x\neq y$. Let n be an arbitrary integer and let $N=|m|$. Hence $\rho(\phi^m(x_i), \phi^m(y_i))\leqq d$ for $i\geqq N$, so that $\rho(\phi^m(x), \phi^m(y))\leqq d$. Since m was arbitrary, this contradicts the fact that d is an expansive constant for ϕ, and thus completes the proof.

If d is an expansive constant for ϕ and $0<\theta<d$, then of course θ is an expansive constant for ϕ. Since a compact metric space is bounded, the set of expansive constants for ϕ is a bounded set of positive real numbers, and hence has a least upper bound.

THEOREM 6. *If X is compact and if θ is the least upper bound of the expansive constants for ϕ, then θ is not an expansive constant for ϕ.*

Proof. Let $\epsilon_i=1/i$ for $i=1, 2, 3, \cdots$. Since $\theta+\epsilon_i$ is not an expansive constant for ϕ, then for each i there exist $x_i'\neq y_i'$ such that $\rho(\phi^n(x_i'), \phi^n(y_i'))\leqq\theta+\epsilon_i$ for each integer n. Also for each i there exists k_i such that $\rho(\phi^{k_i}(x_i'), \phi^{k_i}(y_i'))>d$. Let $x_i=\phi^{k_i}(x_i')$ and $y_i=\phi^{k_i}(y_i')$. We can assume that the sequences $\{x_i\}$ and $\{y_i\}$ converge to x and y, respectively, and we note that $x\neq y$. Let m be an arbitrary integer and α an arbitrary positive number, and choose p, q, and η with the following properties: (1) $\epsilon_p<\frac{1}{3}\alpha$; (2) $\rho(u, v)<\eta$ implies $\rho(\phi^m(u), \phi^m(v))<\frac{1}{3}\alpha$; (3) $n>q$ implies $\rho(x, x_n)<\eta$ and $\rho(y, y_n)<\eta$. Let $t>\max(p, q)$; then

$$\rho(\phi^m(x), \phi^m(y))\leqq\rho(\phi^m(x), \phi^m(x_t))+\rho(\phi^m(x_t), \phi^m(y_t))+\rho(\phi^m(y_t), \phi^m(y))$$
$$\leqq\tfrac{1}{3}\alpha+\theta+\tfrac{1}{3}\alpha+\tfrac{1}{3}\alpha=\theta+\alpha.$$

Thus $\rho(\phi^m(x), \phi^m(y))\leqq\theta$, and therefore θ is not an expansive constant for ϕ.

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AN ELEMENTARY DEVELOPMENT OF THE JORDAN CANONICAL FORM

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It is the purpose of this paper to present a comparatively simple development of the Jordan canonical form for a linear operator on a finite dimensional vector space over an algebraically closed field. The arguments employed will not depend on determinants or on characteristic polynomials or on principal ideals in the ring of polynomials over the scalar field. The only sophisticated concept applied is the uniqueness of the dimension of the vector space; eigenvectors and eigenvalues enter only incidentally.

Let A be a linear operator on a finite dimensional vector space V over an algebraically closed field F, and for each scalar λ in F, let V_λ be the set of vectors in V which are annihilated by some power of the operator $A-\lambda$. (For brevity λ is used here to denote the operator λI where I is the identity on V.) Clearly the V_λ are vector subspaces of V.

LEMMA 1. *There are at most finitely many scalars λ for which V_λ is nonzero and V is the direct sum of all the nonzero V_λ.*

Proof. First we show that the subspaces V_λ span V. Let V' be the subspace spanned by the V_λ and let N be the dimension of V. For an arbitrary vector z there exist scalars $\alpha_0, \alpha_1, \cdots, \alpha_N$, not all zero, such that $\sum_0^N \alpha_i(A^i z) = 0 = (\sum_0^N \alpha_i A^i)z$. ($A^0$ is understood to be the identity operator on V.) Thus there exists a nonzero polynomial p with scalar coefficients such that $p(A)z = 0$; let $n(z)$ be the smallest positive integer n for which there exists a polynomial q of degree n such that $q(A)z = 0$.

We will show that z is in V' by induction on $n(z)$. If $n(z) = 1$, there is a scalar λ such that $(A - \lambda)z = 0$ and z is in $V_\lambda \subset V'$. Assume that v is in V' if $n(v) \leqq m - 1$ and set $n(z) = m$. There is a polynomial p of degree m such that $p(A)z = 0$. Because F is algebraically closed there are scalars $c, \lambda_1, \lambda_2, \cdots, \lambda_m$ such that $p(A) = c(A - \lambda_1) \cdots (A - \lambda_m)$ and $[(A - \lambda_1) \cdots (A - \lambda_m)]z = 0$. Either all the λ_i are equal, or two or more of them are different; in the former case z is in $V_\lambda \subset V'$ where $\lambda_i = \lambda$. In the latter case suppose that $\lambda_i \neq \lambda_j$. By the inductive hypothesis $(A - \lambda_i)z$ and $(A - \lambda_j)z$ are in V', and V' contains the vector $(\lambda_i - \lambda_j)^{-1}[(A - \lambda_j)z - (A - \lambda_i)z] = z$. This completes the induction; since z was an arbitrary vector in V it follows that $V = V'$.

It suffices only to show by contradiction that the V_λ are linearly independent; the conclusion that only finitely many V_λ are nonzero then follows from the hypothesis that V is finite dimensional.

Suppose z is a nonzero vector in V_λ and $z = \sum_1^n z_i$ where z_i is in V_{λ_i} and all the λ_i are different from λ. Observe that each V_{λ_i} reduces A and reduces any polynomial in A. Then we can assume that $(A - \lambda)z = 0$ and $z \neq 0$ by replacing z with $(A - \lambda)^e z$ and replacing z_i with $(A - \lambda)^e z_i$ for an appropriate positive integer e if necessary. For certain positive integers m_i, $[(A - \lambda_1)^{m_1} \cdots (A - \lambda_n)^{m_n}]z = 0$. Now $(A - \lambda_n)z = (A - \lambda)z + (\lambda - \lambda_n)z = (\lambda - \lambda_n)z$; repeated application of this principle shows that $[(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_n)^{m_n}]z = 0$. But $(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_n)^{m_n}$ is a nonzero scalar and z is a nonzero vector, a contradiction. This completes the proof.

LEMMA 2. *If A is a linear operator on V such that $V_0 = V$ (i.e., each vector in V is annihilated by some power of A) then there exist nonzero vectors $x_1, \cdots, x_r$ such that the nonzero vectors in the collection $\{x_i, Ax_i, A^2x_i, \cdots\}$ constitute a linear basis of V.*

Proof. For any vector z in V, $A^{\dim V}z = 0$; otherwise the vectors $z, Az, \cdots, A^{\dim V}z$ would be linearly independent by a trivial argument ([2], pp. 109–110). Thus $A^{\dim V} = 0$ on V; let n be the nonnegative integer for which $A^n \neq 0$ and $A^{n+1} = 0$.

We shall prove the lemma by induction on n. If $n = 0$, then $A = 0$ and any linear basis of V would suffice for the x_i. Suppose that the conclusion is valid if $n \leqq m - 1$ and set $n = m$. Let N be the null space of A; then $AN \subset N$ and A implements in a natural manner a linear operator A' on the quotient space V/N. Clearly $A'^m = 0$ on V/N and by the inductive hypothesis there exist vectors $x_i + N$ in the quotient space V/N such that the nonzero vectors in the collection

$\{x_i+N,\ A'(x_i+N),\ A'^2(x_i+N),\ \cdots\}$ constitute a linear basis of V/N (it does not matter which representative x_i is selected in the coset x_i+N).

We claim that the nonzero vectors in the collection $\{x_i,\ Ax_i,\ A^2x_i,\ \cdots\}\cap N$ are linearly independent. Suppose $y_1, \cdots, y_q$ are such vectors and $\alpha_1, \cdots, \alpha_q$ are scalars such that $\sum_1^q \alpha_i y_i = 0$. Then there exist nonzero vectors $Y_1, \cdots, Y_q$ in the collection $\{x_i+N,\ Ax_i+N,\ A^2x_i+N,\ \cdots\}$ such that $\sum_1^q \alpha_i Y_i = N$ in V/N and hence $\alpha_1 = \cdots = \alpha_q = 0$. Extend the nonzero vectors in the collection $\{x_i,\ Ax_i,\ A^2x_i,\ \cdots\}\cap N$ to a linear basis of N by adjoining vectors $z_1, \cdots, z_s$ if necessary. It suffices to show that the nonzero vectors in the collection $\{z_1, \cdots, z_s,\ x_i,\ Ax_i,\ \cdots\}$ form a basis of V.

To prove independence let $u_1, \cdots, u_t$ be nonzero vectors in $\{z_1, \cdots, z_s,\ x_i,\ Ax_i,\ \cdots\}$ such that $\sum_1^t \alpha_i u_i = 0$ for some nonzero scalars $\alpha_1, \cdots, \alpha_t$; by construction all the u_i must be in N, which is impossible. To prove completeness let u be an arbitrary vector in V. By construction, in V/N we have $(u+N) - \sum_1^p \beta_i U_i = N$ for appropriate scalars $\beta_1, \cdots, \beta_p$ and appropriate nonzero vectors $U_1, \cdots, U_p$ in $\{x_i+N,\ Ax_i+N,\ A^2x_i+N,\ \cdots\}$. Since $\{z_1, \cdots, z_s,\ x_i,\ Ax_i,\ \cdots\}$ contains a basis of N it is clear that its span contains u. This completes the induction and Lemma 2 is proved.

Let us return to the general case in which V is not necessarily V_0. By the method presented in [2] p. 112 a basis can be selected for each V_λ such that the reduction of A to the subspace V_λ gives rise to a matrix composed of Jordan matrices down its diagonal and zeros elsewhere. By Lemma 1 these bases together constitute a basis for V such that the operator A gives rise to the matrix which is the Jordan canonical form of A. (For uniqueness of the Jordan canonical form see [2] pp. 111, 114).

Observe that if A were a nilpotent operator on an infinite dimensional vector space V the construction given in the proof of Lemma 2 would still be valid provided Zorn's axiom is applied where it is obviously needed. Likewise if in Lemma 1 A were an operator on an infinite dimensional vector space V over an algebraically closed field such that for each vector z in V there would exist a nonzero polynomial p with scalar coefficients for which $p(A)z = 0$, then V would still be the direct sum of all the V_λ (however in this event infinitely many V_λ might be nonzero). The details are left to the reader.

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MATHEMATICAL NOTES

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AN INTEGRAL INVOLVING TWO LEGENDRE FUNCTIONS

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In this note we evaluate an integral which involves both the Legendre function $\mathcal{P}_\nu^\mu$ and the Legendre function on the cut P_ν^μ. If $0 \leqq x \leqq y$ then

$$\int_x^y (t^2 - x^2)^{(\mu-1)/2} \mathcal{P}_{\nu+\mu}^{1-\mu}(t/x)(y^2 - t^2)^{(\mu-1)/2} P_{\nu-\mu}^{1-\mu}(t/y)dt$$

$$= 2^{1-2\mu}[\Gamma(2\mu)]^{-1}x^{-\nu-\mu}y^{\nu-\mu+1}(y^2 - x^2)^{2\mu-1},$$

and the integral exists for $\operatorname{Re}\mu > 0$.

Let $I(x, y)$ denote the integral and substitute $v = t/y$, $u = y/x$. The resulting integral is a function of u,

$$\int_{1/u}^1 (u^2v^2 - 1)^{(\mu-1)/2} \mathcal{P}_{\nu+\mu}^{1-\mu}(uv)(1 - v^2)^{(\mu-1)/2} P_{\nu-\mu}^{1-\mu}(v)dv = J(u) = x^{1-2\mu}u^{-\mu}I(x, ux),$$

which can be evaluated by considering it as of convolution type with respect to the Mellin transformation [2], or [1] 6.1(13). Thus for $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > -2$, $\operatorname{Re}(\nu+2\mu) < 1$, $0 < \operatorname{Re} s < \min[1, \operatorname{Re}(1-\nu-2\mu), \operatorname{Re}(2+\nu)]$,

$$\mathfrak{M}\{J(u); s\} = \int_1^\infty v^{s-1}(v^2 - 1)^{(\mu-1)/2} \mathcal{P}_{\nu+\mu}^{1-\mu}(v)dv \int_0^1 v^{-s}(1 - v^2)^{(\mu-1)/2} P_{\nu+\mu}^{1-\mu}(v)dv,$$

and these two resulting integrals can be evaluated from tables [1] 18.2(3), 18.1(5). After simplification

$$\mathfrak{M}\{J(u); s\} = 2^{-2\mu}[\Gamma(2\mu)]^{-1}B[\tfrac{1}{2}(1 - s - \nu - 2\mu), 2\mu],$$

which is the same as the Mellin transform [1] 6.2(32), 6.1(3) for $\operatorname{Re}\mu > 0$, $\operatorname{Re} s < \operatorname{Re}(1-\nu-2\mu)$ of

$$g(u) = u^{\nu-2\mu+1}(u^2 - 1)^{2\mu-1}, \quad u > 1; \quad = 0, \quad u < 1.$$

Substituting this function for $J(u)$ and solving for $I(x, y)$ leads to the desired result. The restrictions on μ and ν can be finally relaxed by consideration of the analyticity of $g(u)$ and of $J(u)$.

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SOME REMARKS ON ANTI-COMMUTATIVE ALGEBRAS

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1. Introduction. The main purpose of this note is to show that certain theorems which hold for Lie algebras do not extend to arbitrary anticommutative algebras.

2. Definitions. We shall be concerned throughout with linear algebras of finite dimension over a field of characteristic zero. The algebras are not assumed initially to be associative, but their vectors x, y do satisfy the anticommutative condition $yx = -xy$. Thus, $x^2 = 0$ and all ideals are two-sided. The term algebra will always refer to such algebras.

If $e_1, \cdots, e_n$ is a basis for an arbitrary algebra then the constants of structure C_{ij}^k $(i, j, k = 1, \cdots, n)$ are defined, as usual, by the equation

$$e_i e_j = C_{ij}^k e_k.$$

For the algebras considered here, $C_{ii}^k = 0$ and $C_{ji}^k = -C_{ij}^k$.

If the vectors of an algebra R also satisfy the Jacobi condition

$$x(yz) + y(zx) + z(xy) = 0,$$

then R is a *Lie algebra*. The main properties of these algebras are easily accessible and may be found, for example, in [1].

Following Schafer [3] (where original references can be found) an algebra R is *simple* if it is not commutative and if R, $\{0\}$ are its only ideals; it is *semi-simple* if it is the direct sum of simple ideals; and its *radical* Γ is the smallest ideal such that R/Γ is trivial or semi-simple. In particular, the trivial algebra $\{0\}$ is neither simple nor semi-simple, and every simple algebra is semi-simple.

Note that with the definitions given in [1], $\{0\}$ is semi-simple but not simple (Section 6, nos. 1, 2) and so should be excluded from Proposition 2 (Section 6, no. 2).

Semi-simple Lie algebras are often defined by a different but equivalent condition, and we shall say, therefore, that an algebra is *insolvable* if $\{0\}$ is its only solvable ideal. An equivalent condition is that $\{0\}$ be the only commutative ideal.

3. Some results. Those results in the following list which do not refer to Lie algebras can be proved very simply.

T.1. *All solvable ideals of an algebra R are contained in a unique maximum solvable ideal (M.S.I.)L and R/L is insolvable.*

T.2. *Every semi-simple algebra is insolvable.*

T.3. *Every insolvable Lie algebra is semi-simple.*

T.4. *The radical of a Lie algebra is its M.S.I.*

T.5. *Every ideal and every quotient algebra of a semi-simple algebra is semi-simple.*

T.6. *Every ideal of an insolvable Lie algebra is insolvable.*

T.7. *Every quotient algebra of an insolvable Lie algebra is insolvable.*

T.8. *Every algebra of dimension* 1, 2 *is solvable.*

T.9. *Every nontrivial insolvable algebra is not solvable and therefore has dimension not less than three.*

4. First example. In this section the algebra R, defined by

$$e_2e_3 = e_1, \qquad e_3e_1 = e_2, \qquad e_2e_1 = e_3, \qquad e_1e_4 = e_2,$$

where $e_1, \cdots, e_4$ is a basis,* is studied and its ideals determined. We shall then be able to show that some of the results, T. 1–T. 9, do not extend to arbitrary algebras.

The algebra R has no one-dimensional ideals. For if $x = x^1e_1 + x^2e_2 + x^3e_3 + x^4e_4$ spans such an ideal then, by considering the products xe_4, xe_3, xe_2, xe_1 in turn, it is found that $x^1 = x^2 = x^3 = x^4 = 0$.

To show that there are no two-dimensional ideals the following method is used. Suppose, if possible, that M is a two-dimensional ideal, and choose a basis $f_1, \cdots, f_4$ for R such that $M = ((f_1, f_2))$, the expression $((x, y))$ denoting the subspace spanned by the vectors x, y. If C_{ij}^k and $D_{\alpha\beta}^{\gamma}(i, j, k, \alpha, \beta, \gamma = 1, \cdots, 4)$ are the constants of structure of R, relative to the bases $e_1, \cdots, e_4$ and $f_1, \cdots, f_4$ respectively, then

$$D_{\alpha\beta}^{\gamma} = C_{ij}^{k}\, q_k^{\gamma}\, p_{\alpha}^{i}\, p_{\beta}^{j},$$

where $f_\alpha = p_\alpha^i e_i$, $(q_i^\alpha) = (p_\alpha^i)^{-1}$, and (p_α^i) is nonsingular. Since M is an ideal, $D_{\alpha'\beta}^{\gamma''} = 0$ $(\gamma'' = 3, 4; \alpha' = 1, 2)$ and therefore $C_{ij}^{k} q_k^{\gamma''} p_{\alpha'}^{i} p_{\beta}^{j} = 0$. Multiplying by q_m^β, we have

$$C_{ij}^{k} q_k^{\gamma''} p_{\alpha'}^{i} = 0 \qquad (\gamma'' = 3, 4; \alpha' = 1, 2). \tag{1}$$

Conversely, if a nonsingular matrix (p_α^i) can be found so that (1) is satisfied, then a two-dimensional ideal exists. (This method for determining ideals clearly applies to ideals and algebras of any dimension, the only practical limitation coming from the amount of calculation involved.) It is a straightforward matter to verify that no such matrix (p_α^i) exists for R and so there are no two-dimensional ideals.

Finally, the subspace $N = ((e_1, e_2, e_3))$ is the only three-dimensional ideal. For if M is another one, then $M \cap N$ is an ideal and so has dimension 3. Hence $M = N$.

The only ideals of R, therefore, are R, N, $\{0\}$, and neither R nor N is solvable. It follows that R is insolvable, but cannot decompose into a direct sum of simple ideals. Thus, R is not semi-simple. Moreover, R is its own radical since

* When the structure of an algebra is defined by means of the products of its basis vectors, zero products will be omitted, as will a product e_je_i if e_ie_j is given. Similar remarks apply to the constants of structure C_{ij}^k.

neither R/N nor $R/\{0\}$ is semi-simple or $\{0\}$, but the M.S.I. of R is $\{0\}$. Finally, R/N is commutative and hence not insolvable.

These results show that T. 3, T. 4 and T. 7 do not extend to arbitrary algebras.

5. Second example. By methods similar to those of Section 4 it can be shown that the algebra B, defined by

$$e_1e_2 = e_1, \quad e_1e_5 = e_2, \quad e_2e_3 = e_2, \quad e_2e_4 = e_3, \quad e_3e_4 = e_4,$$

for some basis $e_1, \cdots, e_5$, has no commutative ideals other than $\{0\}$, and is therefore insolvable. The derived algebra $B^2 = ((e_1, \cdots, e_4))$, however, is an ideal of B and is not insolvable. For $((e_1))$ is a solvable ideal of B^2. This shows that T. 6 does not extend to arbitrary algebras.

6. Perfect algebras. If an algebra P is identical with its derived algebra P^2 then P is *perfect*. Every such algebra always possesses two ideals X such that $PX = X$, namely P and $\{0\}$. It may be of interest to show that not every ideal necessarily satisfies this condition. For the algebra P, defined by

$$e_1e_2 = e_1, \quad e_1e_3 = e_2, \quad e_1e_4 = e_3, \quad e_2e_3 = e_4, \quad e_2e_4 = e_5,$$

for some basis $e_1, \cdots, e_5$, is clearly perfect. But $X = ((e_5))$ is the centre and $PX = \{0\}$ so that $PX \neq X$. The following elementary theorems show why it was necessary to construct an algebra of dimension not less than five.

THEOREM 1. *Every three-dimensional perfect algebra is simple.*

Proof. Let X be a one-dimensional ideal of a perfect algebra P, and e_1, e_2, e_3 a basis for P such that $X = ((e_1))$. Then e_1e_2 and e_1e_3 belong to X and are therefore linearly dependent. This implies that dim $P^2 \leqq 2$, a contradiction. Similarly if $Y = ((e_1, e_2))$ is an ideal then $P^2 \subset Y$ which is impossible.

Denoting the centre of P by $C(P)$ we have

THEOREM 2. *If P is an algebra of dimension n, such that* dim $P^2 = m$ *and* dim $C(P) = r$, *then* $n - r \geqq \frac{1}{2}(k+1)$, *where* $k = (8m+1)^{\frac{1}{2}}$.

Proof. The derived algebra P^2 can be created by $n-r$ vectors at most, so that dim $P^2 \leqq \frac{1}{2}(n-r)(n-r-1)$. Hence $2m \leqq (n-r)(n-r-1)$ which leads to

$$[n - r - \tfrac{1}{2}(1 + k)][n - r - \tfrac{1}{2}(1 - k)] \geqq 0.$$

The proof is completed by observing that $k \geqq 1$ and $r \leqq n$.

In particular, if B is perfect and $m = n = 4$ then $k^2 = 33$ and so $r = 0$.

There are no perfect Lie algebras of dimensions 4 or 5 so that an example of a Lie algebra, similar to that given earlier in this section, must have at least dimension 6. The Lie algebra defined by

$$e_1e_2 = 2e_3, \quad e_1e_3 = 2e_2, \quad e_2e_3 = 2e_1, \quad e_1e_4 = e_5, \quad e_1e_5 = e_4,$$

$$e_2e_4 = e_5, \quad e_5e_2 = e_4, \quad e_3e_4 = e_4, \quad e_5e_3 = e_5, \quad e_4e_5 = e_6,$$

with centre $((e_6))$, suffices.

It is clear that not every subalgebra of a perfect algebra need be perfect since every one-dimensional subspace is a commutative subalgebra. However,

THEOREM 3. *Every quotient of a perfect algebra is perfect.*

Proof. If X is an ideal of a perfect algebra P it is sufficient to prove that $P/X \subset (P/X)^2$. This follows from the fact that the mapping f of P onto P/X, defined by $f(p) = X+p$, is a homomorphism, and $f(P^m) \subset (f(P))^m$ for all $m \geqq 1$. Hence $f(P^2) \subset (P/X)^2$ and so $P/X \subset (P/X)^2$ since P is perfect.

7. Genus of a perfect algebra. The genus and nullity of a complex Lie algebra have been defined by Knebelman [2], but the definitions apply immediately to any linear algebra. An algebra B has *nullity* r if there are r vectors $x_1, \cdots, x_r$ in B such that the subspace generated by $x_1, \cdots, x_r$ and all possible products of $x_1, \cdots, x_r$ is the whole algebra, and if there is no set of $r-1$ vectors which will generate the algebra in this way. The *genus* of B is the difference

$$\dim B - \text{nullity } B.$$

Knebelman showed that the genus of a perfect (complex) Lie algebra of dimension n is $n-2$. We give now an example of a four-dimensional algebra which is perfect but whose genus is 1. This is the algebra P defined by

$$e_1e_2 = e_1, \qquad e_3e_4 = e_2, \qquad e_3e_2 = e_3, \qquad e_4e_2 = e_4,$$

relative to a basis $e_1, \cdots, e_4$. It is easily verified that the determinant $|x\ y\ xy\ x(xy)|$ is zero for all vectors x, y so that no two vectors x, y generate P. The three vectors e_1, e_3, e_2+e_4 do generate P, however, and so P has genus 1.

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FINITELY GENERATED MODULES OVER PRINCIPAL IDEAL DOMAINS

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It is well known that every finitely generated module over a principal ideal domain is a direct sum of cyclic submodules. The object of this note is to provide a short simple proof of this result. The ideas are similar to those of Rado's proof of the corresponding theorem for finitely generated Abelian groups [1].

Let M denote a finitely generated module over a principal ideal domain R. Let the letters x, y denote elements of M, and the letters a, b, c elements of R. If $a \neq 0$, let $f(a)$ denote the number of prime factors occurring in a factorization of a (counting multiplicities); put $f(0) = \infty$. With the understanding that every integer is less than ∞, we see that, if a divides b, then $f(a) \leqq f(b)$, equality holding

if and only if a and b are associates. In particular, $f(a)=0$ if and only if a is a unit. For an element x of M, the set of all a for which $ax=0$ is an ideal of R; we call a generator of this ideal the *order* $o(x)$ of x. $o(x)$ is determined to within associates.

The symbol $\{y_1, \cdots, y_m\}$ denotes the submodule generated by $y_1, \cdots, y_m$. If $\{y_1, \cdots, y_m\}=M$, and the relation $a_1y_1+\cdots+a_my_m=0$ implies that $a_iy_i=0$ for every i, the y_i are said to constitute a *basis* of M.

LEMMA. *Let* $M=\{x_1, \cdots, x_m\}$, *and* $a_1x_1+\cdots+a_mx_m=0$, *where not all the* a_i *are* 0. *Then there are elements* $y_1, \cdots, y_m$ *of* M *such that* $M=\{y_1, \cdots, y_m\}$, *and the order of* y_1 *divides every* a_i.

Proof. We notice first the truth of the result if one of the a_i is a unit. For example, if a_1 is a unit, x_1 is a linear combination of the other x_i, so that we may take $y_1=0$, $y_i=x_i$, $i=2, \cdots, m$.

Let $s=\sum f(a_i)$, the sum being taken over those i for which a_i is nonzero; then s is a nonnegative integer. If $s=0$, at least one a_i is a unit; we therefore proceed by induction on s. If only one a_i is nonzero, the result is trivial, so we may assume that a_1 and a_2 are nonzero, and are not units. Let b be their greatest common divisor. There exist elements b_1, b_2, c_1, c_2 of R such that $a_1=bb_1$, $a_2=bb_2$, $b=c_1a_1+c_2a_2$. Then, $b_1c_1+b_2c_2=1$, so that

$$x_1 = b_2(c_2x_1 - c_1x_2) + c_1(b_1x_1 + b_2x_2),$$
$$x_2 = -b_1(c_2x_1 - c_1x_2) + c_2(b_1x_1 + b_2x_2).$$

Hence,

$$M = \{c_2x_1 - c_1x_2, b_1x_1 + b_2x_2, x_3, \cdots, x_m\},$$
$$b(b_1x_1 + b_2x_2) + a_3x_3 + \cdots + a_mx_m = 0.$$

Since $f(b)\leqq f(a_1)<f(a_1)+f(a_2)$, the induction hypothesis applies, and provides us with elements $y_1, \cdots, y_m$ which generate M, such that $o(y_1)$ divides $b, a_3, \cdots, a_m$. Since b divides a_1 and a_2, $o(y_1)$ divides all a_i.

THEOREM. *M has a basis.*

Proof. The result being trivial if M can be generated by one element, we may suppose that M can be generated by a set of m elements, where $m>1$, and that the result is true for all modules which can be generated by $m-1$ elements. There exists a number n (possibly ∞) which is the least possible value of $f(o(x))$ for elements x which can occur in generating sets of M containing m elements. Let $M=\{x_1, \cdots, x_m\}$, where $f(o(x_1))=n$.

If $M=\{x_1\}+\{x_2, \cdots, x_m\}$ (direct sum), the induction hypothesis gives a basis for $\{x_2, \cdots, x_m\}$ which together with x_1 forms a basis for M. In the contrary case, we have $a_1x_1+\cdots+a_mx_m=0$, where $a_1x_1\neq 0$. Let b be the greatest common divisor of a_1 and $o(x_1)$. Then, $f(b)<f(o(x_1))$, for else b and $o(x_1)$ would be associates, so that $o(x_1)$ would divide a_1, contradicting the fact that $a_1x_1\neq 0$.

There exist c_1, c_2 such that $b=c_1a_1+c_2o(x_1)$. Then,

$$bx_1 + c_1a_2x_2 + \cdots + c_1a_mx_m = 0.$$

By the lemma, $M=\{y_1, \cdots, y_m\}$, where $o(y_1)$ divides b, so that $f(o(y_1)) \leqq f(b) < n$, an impossibility.

Remark. The usual definition of a basis requires that it contain only nonzero elements. Such a basis may be obtained from a basis as defined above by dropping any zero elements; hence the existence theorem is still valid with the more restricted definition.

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CONGRUENCES INVOLVING SUMS OF BINOMIAL COEFFICIENTS AND A FORMULA OF JENSEN*

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1. Carlitz [2] has used infinite series to establish the congruence

$$(1) \qquad S(a, c, b, n) = \sum_{k=0}^{n} \binom{a+bk}{k} \binom{c+b(n-k)}{n-k} \equiv \binom{a+c+bn}{n} \pmod{b},$$

for arbitrary integers b and a, c rational numbers whose denominators are prime to b. Adem [1] proved the congruence in the case where b is prime.

It may be of interest to note that (1) is an immediate consequence of Jensen's formula [3], [5]

$$(2) \qquad \sum_{k=0}^{n} \binom{a+bk}{k} \binom{c-bk}{n-k} = \sum_{k=0}^{n} \binom{a+c-k}{n-k} b^k,$$

since this yields at once

$$(3) \qquad S(a, c, b, n) = \binom{a+c+bn}{n} + \sum_{k=1}^{n} \binom{a+c+bn-k}{n-k} b^k.$$

We also remark that Jensen proved (2) using finite methods based on the recurrence relation satisfied by the series S.

More generally from Jensen's formula we have the congruence

$$(4) \qquad S(a, c, b, n) \equiv \sum_{k=0}^{p-1} \binom{a+c+bn-k}{n-k} b^k \pmod{b^p}$$

where p is any integer $1 \leqq p \leqq n$.

2. In view of the usefulness of Jensen's formula it is of interest to show how a series transformation developed by the writer [4] yields a variation of (3).

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we also conclude from (10) that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+c+bk}{n}\binom{a+c+bk}{k}^{-1}S(a,c,b,k)$$

$$(12)\qquad =\sum_{k=0}^{n}(-1)^k\binom{a+c+bk-k}{n-k}S(a,c,b,k)=(-1)^n(n+1)b^n.$$

The interesting transformation (9) should yield many other valuable such relations, the only problem being the summing of $F(a, b, n)$ as given by (5) in closed form. For instance, is it true that by proper choice of $f(k)$ we could obtain Jensen's formula from (9)?

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A MODIFIED WALLIS PRODUCT AND SOME APPLICATIONS

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1. Introduction. For $n=1, 2, \cdots$, let

$$b_n(c)=\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot 2n}\sqrt{(n+c)}$$

where $c(\geqq -1)$ is independent of n. If $c=0$, we obtain $b_n(0)$, the well-known Wallis product, which tends to $\pi^{-\frac{1}{2}}$ as n tends to infinity (e.g. [4], p. 351.) In [2] the author showed that under certain conditions, $\pi^{-\frac{1}{2}}$ is either an upper or lower bound for $b_n(c)$. The following theorem is an improvement of the previous result. Some applications will be given in the next section.

THEOREM 1. *For all positive integers n*

$$(1)\qquad \begin{aligned} b_n(c) &< 1/\sqrt{\pi}, && \text{if } c\leqq \tfrac{1}{4},\\ &> 1/\sqrt{\pi}, && \text{if } c\geqq (n+1)/(4n+3).\end{aligned}$$

Proof. For every fixed c, $b_n(c)$ tends to the same limit as n tends to infinity. Now $b_{n+1}(c)\gtreqless b_n(c)$ if and only if $(1-4c)n+1-3c\gtreqless 0$. It follows that $b_n(c)$ is a strictly increasing function of n if $c\leqq\frac{1}{4}$. Hence $b_n(c)<\pi^{-\frac{1}{2}}$ if $c\leqq\frac{1}{4}$. On the other hand, if m is any positive integer and $c\geqq(m+1)/(4m+3)$, then $b_m(c)\geqq b_{m+1}(c)$.

Since $(m+1)/(4m+3)$ is a strictly decreasing function of m, we see that $b_n(c) > b_{n+1}(c)$ for all $n \geqq m+1$. Therefore $b_m(c) > \pi^{-\frac{1}{2}}$.

2. Applications. *A. Gamma functions.* Various approximations have been derived for the ratio $a_n' = \Gamma(\frac{1}{2}n)/\Gamma(\frac{1}{2}(n-1))$. (See, e.g. [1] and [5].) We shall see that fairly close upper and lower bounds can be obtained for a_n' by means of Theorem 1. We will state the corresponding result explicitly, however, for the quantity $a_n = a_n' \sqrt{[2/(n-1)]}$, because the bounds for a_n look more "symmetrical." Incidentally a_n is a well-known constant in statistics ([2] p. 484).

THEOREM 2. *For all integers* $n \geqq 2$,

$$\sqrt{[(2n-3)/(2n-2)]} < a_n < \sqrt{[(2n-2)/(2n-1)]}. \tag{2}$$

Further, for all odd integers $n \geqq 3$, *an improved lower bound for* a_n *is* $(\frac{1}{2}n)\sqrt{[(2n-2)(2n+1)]}$.

Proof. Since $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we see that for $n = 2m+2$, where $m = 1, 2, \cdots$, $a_n = (1/b_m(c))\sqrt{[2(m+c)/\pi(2m+1)]}$, and for $n = 2m+1$, $a_n = b_m(c)\sqrt{[\pi m/(m+c)]}$. Using (1), we show that (2) holds for $n = 4, 6, 8, \cdots$, and for $n = 3, 5, 7, \cdots$, a_n has the same upper bound but the better lower bound stated above. Direct comparison shows that (3) holds also for $n = 2$.

Remark. Binet ([1], pp. 207–208) showed, in our notation, that

$$\left\{\frac{n-1}{n}\left(1 + \frac{1}{2(n+2)} + \frac{1\cdot 3^2}{2\cdot 4(n+2)(n+4)} + \cdots\right)\right\}^{1/2} = a_n \leqq 1. \tag{3}$$

His upper bound is obviously not so good as ours. It can be shown that for $n \geqq 3$, the lower bound given in (2) is greater than the approximation for a_n, obtained from (3) by taking only the first two terms in the series expansion.

Wise obtained a series expansion for $N!/(N-n)!$ ([5], p. 514 (1.6)). As a consequence, we obtain a very good approximation for a_n, i.e.,

$$\sqrt{[(2n-3)/(2n-2)]}[1 + \tfrac{1}{4}(2n-3)^2].$$

Computation showed that for most positive integers n, e.g., $n \geqq 7$, this approximation is smaller than the upper bound and greater than both lower bounds given in Theorem 2.

B. Beta functions. Let $B_n' = B(n, n) = \Gamma(n)\Gamma(n)/\Gamma(2n)$. Upper and lower bounds for B_n' can be obtained by using Theorem 2. We state

COROLLARY. *Let* $B_n = 2^{2n-1}\sqrt{[(2n-1)/2\pi]}B_n'$; *then for* $n \geqq 1$,

$$\sqrt{[(4n-3)/(4n-2)]} < B_n < \sqrt{[(4n-2)/(4n-1)]}.$$

Proof. Use the fact

$$B_n = \sqrt{[\tfrac{1}{2}(2n-1)]}\Gamma(n)/\Gamma(n+\tfrac{1}{2}) = \sqrt{[2/(2n-1)]}\Gamma(n)/\Gamma(n-\tfrac{1}{2}).$$

It follows that $B_n = a_{2n}$.

Remark. Using a result of Wise ([5], p. 516 (3.3)) we obtain

$$B_n = \sqrt{[(4n-3)/(4n-2)]}\,w(n-\tfrac{3}{4}), \quad \text{where}$$
$$w(t) = 1 + 1/(2^6t^2) - 19/(2^{13}t^4) + \cdots.$$

The first factor is the same as our lower bound.

Acknowledgment. The author wishes to thank the referee for suggesting a better proof for the Corollary.

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A NOTE ON VANDERMONDE'S CONVOLUTION

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Using the method of finite differences, H. W. Gould [1], [2] proved the following identity:

$$\sum_{k=0}^{n}\binom{\alpha+\beta k}{k}\frac{\alpha}{\alpha+\beta k}\cdot\binom{\gamma+\beta(n-k)}{n-k}\frac{\gamma}{\gamma+\beta(n-k)} = \binom{\alpha+\gamma+\beta n}{n}\frac{\alpha+\gamma}{\alpha+\gamma+\beta n}, \tag{1}$$

valid for any numbers α, β, γ, which for $\beta = 0$ reduces to the well-known Vandermonde's convolution

$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\gamma}{n-k} = \binom{\alpha+\gamma}{n}.$$

The purpose of this note is to give a short proof of (1), based on Lagrange's formula for the inversion of power series. We state without proof Lagrange's formula [3]. If $w = f(z) = z/\phi(z) = \sum_{n=1}^{\infty} a_n z^n$ is an analytic function, regular in a neighborhood of the point $z=0$, and satisfying the condition $a_1 = f'(0) \neq 0$, then the equation $f(z) = w$ has a unique solution, regular in a neighborhood of the point $w=0$, of the form

$$z = g(w) = \sum_{n=1}^{\infty}\frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\{\phi(z)\}^n\right]_{z=0}.$$

More generally, if $F(z)$ is an analytic function, regular in the neighborhood of the point $z=0$, then there is a neighborhood of the point $w=0$ in which

$$F(z) = F(0) = \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(F'(z)\{\phi(z)\}\right)^n\right]_{z=0}. \tag{2}$$

With the Lagrange's formula (2) given, the proof of (1) becomes embarrassingly simple. Choosing $\phi(z)=(1+z)^{\beta}$ and $F(z)=(1+z)^{\alpha}$ where α and β are any complex numbers, and applying (2), we obtain:

$$\begin{aligned}(1+z)^{\alpha} &= 1+\sum_{n=1}^{\infty}\frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\{\alpha(1+z)^{\alpha-1}(1+z)^{\beta n}\}\right]_{z=0}\\ &= 1+\sum_{n=1}^{\infty}\frac{w^n}{n!}\alpha(\alpha+\beta n-1)_{n-1} = \sum_{n=0}^{\infty} w^n\binom{\alpha+\beta n}{n}\frac{\alpha}{\alpha+\beta n}\\ &= \sum_{n=0}^{\infty} w^n A_n(\alpha,\beta)\end{aligned} \tag{3}$$

where $A_n(\alpha,\beta)=\binom{\alpha+\beta n}{n}\alpha/(\alpha+\beta n)$. Applying (2) with $\phi(z)=(1+z)^{\beta}$, $F(z)=(1+z)^{\gamma}$, we obtain in a similar way

$$(1+z)^{\gamma} = \sum_{n=0}^{\infty} w^n A_n(\gamma,\beta). \tag{4}$$

Using (2) once more, with $\phi(z)=(1+z)^{\beta}$ and $F(z)=(1+z)^{\alpha+\gamma}$ we get

$$(1+z)^{\alpha+\gamma} = \sum_{n=0}^{\infty} w^n A_n(\alpha+\gamma,\beta). \tag{5}$$

But since $(1+z)^{\alpha+\gamma}=(1+z)^{\alpha}(1+z)^{\gamma}$, it follows that

$$\left(\sum_{n=0}^{\infty} A_n(\alpha,\beta)w^n\right)\left(\sum_{n=0}^{\infty} A_n(\gamma,\beta)w^n\right) = \sum_{n=0}^{\infty} A_n(\alpha+\gamma,\beta)w^n. \tag{6}$$

Equating the coefficients of w^n in (6), we obtain

$$\sum_{k=0}^{n} A_k(\alpha,\beta)A_{n-k}(\gamma,\beta) = A_n(\alpha+\gamma,\beta)$$

which is the identity (1).

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CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

ON SERIES WHICH ARISE FROM A CONTINUATION OF THE ZETA FUNCTION

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The simplest method of continuing $\zeta(s) = \sum_1^\infty n^{-s}$ to the left of Re $s = 1$ consists in writing

$$2^{-s}\zeta(s) = \sum_1^\infty (2n)^{-s}$$

so that

$$\zeta(s) - 2\cdot 2^{-s}\zeta(s) = (1 - 2^{1-s})\zeta(s) = \sum_1^\infty (-1)^{n-1}n^{-s}. \tag{1}$$

The last series converges for Re $s>0$, so that (1) defines $\zeta(s)$ in this region. This procedure can be carried out for an arbitrary integer a greater than 1 by writing

$$a^{-s}\zeta(s) = \sum_1^\infty (an)^{-s}$$

so that

$$(1 - a^{1-s})\zeta(s) = \sum_1^\infty \beta_n n^{-s}, \qquad \beta_n = \begin{cases} 1 & \text{if } a\nmid n \\ 1 - a & \text{if } a \mid n \end{cases}. \tag{2}$$

For any such a the series converges for Re $s>0$ so that (2) will likewise define $\zeta(s)$ in this region.

Since $\zeta(s)$ has a simple pole at $s=1$ and is regular elsewhere, the Laurent expansion about this point is of some interest. This can be obtained by using (1) as was shown by the author and S. Chowla [1]. The expansion can be obtained similarly from (2) wherein it is necessary to evaluate the derivatives of $\sum_1^\infty \beta_n n^{-s}$ at $s=1$. The purpose of this note is to indicate how the series $\sum_1^\infty \beta_n n^{-1}\log^k n$ can be evaluated. For integral $k\geqq 0$ write

$$\sum_{n\leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + \gamma_k + o(1)$$

where γ_k is a constant and γ_0 equals the Euler constant γ. Hence

$$\sum_{n\le ax}\frac{\log^k n}{n}=\frac{(\log ax)^{k+1}}{k+1}+\gamma_k+o(1). \tag{3}$$

To obtain the desired series it is necessary to subtract from this a times each term which arises from an n divisible by a. This is

$$\begin{aligned} a\sum_{n\le x}\frac{(\log an)^k}{an} &= \sum_{t=0}^{k}\binom{k}{t}\log^{k-t}a\sum_{n\le x}\frac{\log^t n}{n} \\ &= \sum_{t=0}^{k}\binom{k}{t}\log^{k-t}a\left[\frac{\log^{t+1}x}{t+1}+\gamma_t+o(1)\right] \\ &= \frac{(\log ax)^{k+1}}{k+1}-\frac{\log^{k+1}a}{k+1}+\sum_{t=0}^{k}\binom{k}{t}\gamma_t\log^{k-t}a+o(1). \end{aligned} \tag{4}$$

Subtracting (4) from (3) and letting $x\to\infty$ gives the

THEOREM. *For integral a and k, $a\ge 2$, $k\ge 0$,*

$$\sum_{n=1}^{\infty}\frac{\beta_n\log^k n}{n}=\frac{\log^{k+1}a}{k+1}-\sum_{t=0}^{k-1}\binom{k}{t}\gamma_t\log^{k-t}a$$

where the summation on the right is zero for $k=0$.

This includes the interesting special case when $k=0$, namely

$$\log a=1+\frac{1}{2}+\cdots+\frac{1-a}{a}+\frac{1}{a+1}+\cdots+\frac{1-a}{2a}+\cdots.$$

These results were obtained in quite a different way by Kluyver [2] and the principal result of his paper follows by solving the equations in the above theorem successively for γ_t, $t=0, 1, 2, \cdots$. In particular for $k=1$ one has

$$\sum_{n=1}^{\infty}\frac{\beta_n\log n}{n}=\frac{\log^2 a}{2}-\gamma\log a$$

or

$$\gamma=\frac{1}{2}\log a-\sum_{n=1}^{\infty}\frac{\beta_n}{n}\log_a n.$$

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A NUMERICAL METHOD FOR SOLVING SCALAR EQUATIONS

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In this paper we present a simple numerical technique for solving a scalar equation $f(x)=0$. Besides simplicity, it has the advantage that it is straightforward, converges reasonably fast, and never fails; i.e., it never fails in the sense that given any $f(x)$ (subject to certain continuity conditions) on an interval $[a, b]$ the method will produce the first zero of $f(x)$ to the right of $x=a$. In fact, if x_0 is any point of $[a, b]$, the method produces the first zero of $f(x)$ to the right of x_0 if $f(x_0)>0$ and the first to the left if $f(x_0)<0$. If there is no zero on $[a, b]$, the method gives this information. Now *a priori* knowledge of the number or nature of the roots is needed.

We present the result formally below as a theorem, but for teaching purposes the method can be presented graphically and intuitively as follows: First assume that $f(x)$ is defined on $[a, b]$ and that $f(a)>0$. Clearly, $f(x)=0$ is equivalent to $x=\phi(x)$ for every $\lambda\neq 0$ where $\phi(x)=x+f(x)/\lambda$, and if λ is sufficiently large and positive, $\phi(a)>a$ and $\phi'(x)>0$ on $[a, b]$, so $\phi(x)$ is monotone increasing. Draw the graphs of $y=x$ and $y=\phi(x)$ (with $\phi(a)>a$ and ϕ monotone increasing) on the same coordinate system, choose any x_0 in $[a, b]$ and observe that the numerical process defined by $x_{m+1}=\phi(x_m)$ is equivalent to the geometric process of drawing a line vertically from x_0 meeting $y=\phi(x)$ at $(x_0, \phi(x_0))$, then horizontally from $(x_0, \phi(x_0))$ to meet $y=x$ at $(x_1, \phi(x_0))$, vertically from $(x_1, \phi(x_0))$ to meet $\phi(x)$ again, etc. It is then obvious that this process leads to the first zero to the left of x_0 if $\phi(x_0)<x_0$ and the first zero to the right of x_0 (provided there is one) if $\phi(x_0)>x_0$. When $\phi(x_0)>x_0$, the sequence fails to converge if and only if there is no zero on $[x_0, b]$, in which case we arrive at $x_N>b$ for some N. This also shows why the sequence is monotone. The formal proof is merely a rigorous statement of the geometrical argument.

THEOREM. *Assume that*

(i) $f(x)$ *is defined and continous on the finite interval* $[a, b]$,

(ii) $f(a)>0$ *and* $f(x')<f(a)$ *for some* x' *in* $[a, b]$,*

(iii) $$\operatorname*{Inf}_{a\leqq x_1<x_2\leqq b}\frac{f(x_2)-f(x_1)}{x_2-x_1}=-\lambda_0$$ *is finite.*†

Let λ *be arbitrary except that* $\lambda\geqq\lambda_0$ *and consider the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}=x_n+f(x_n)/\lambda \tag{1}$$

where x_0 *is an arbitrary number in* $[a, b]$. *(If* $x_N>b$ *for some* N, *we consider* $\{x_n\}$ *as defined, but terminating with* x_N.) *The sequence* $\{x_n\}$ *has the following properties:*

* This is not really a restriction, for if $f(a)<0$ we simply apply the method to $[-f(x)]$. Also, if $f(x)\geqq f(a)>0$ for all x there are certainly no zeros on $[a, b]$.

† Note that (iii) automatically holds if $f'(x)$ exists and is bounded below on $[a, b]$, and that in this case $-\lambda_0=\operatorname{Inf}_{[a, b]}f'(x)$.

1. *If $f(x_0)>0$, $\{x_n\}$ is nondecreasing. It converges if and only if $f(x)$ has a zero on $[x_0, b]$. If it does not converge, $x_N>b$ for some N. If it does converge, it converges to the first zero of $f(x)$ to the right of x_0.*

2. *If $f(x_0)<0$, $f(x)$ necessarily has a zero on $[a, x_0]$, $\{x_n\}$ is nonincreasing, and it converges to the first zero of $f(x)$ to the left of x_0.*

3. *If $\lambda_2>\lambda_1\geqq\lambda_0$, the sequence using λ_1 converges more rapidly than that using λ_2. (Thus one should choose λ as close to λ_0 as is practically feasible.)*

COROLLARY 1. *Suppose $[a_1, b_1]$ is a subset of $[a, b]$ such that $f(a_1)>0, f(b_1)<0$, and r is the only zero of $f(x)$ on $[a_1, b_1]$. Assume also that $f'(x)$ exists and is negative in some neighborhood of r. Let*

$$-\bar{\lambda}_0 = \operatorname*{Inf}_{a_1\leqq x_1<x_2\leqq b_1} \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

and

$$-K = \operatorname*{Sup}_{a_1<x<b_1} \frac{f(x)}{x-r}.$$

Then if x_0 is any point of $[a_1, b_1]$ and $x_{n+1}=x_n+f(x_n)/\lambda$ where $\lambda\geqq\bar{\lambda}_0$, we have $0<K\leqq\bar{\lambda}_0\leqq\lambda$ and

$$|x_n-r| \leqq \left(1-\frac{K}{\lambda}\right)^n |x_0-r| \qquad \text{for all } n.$$

COROLLARY 2. *If there are only a finite number of zeros on $[a, b]$, and*

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

is bounded on $a\leqq x_1<x_2\leqq b$, it is possible to obtain all of them as follows: By using $f(x)$ we can get every zero which lies to the right of an x_0 where $f(x_0)>0$ or to the left of an x_0 where $f(x_0)<0$. Let these zeros be denoted by $r_1, r_2, \cdots, r_m$. Then unless $f(x)\geqq 0$ for all x in $[a, b]$, in which case $r_1, \cdots, r_m$ is the complete set of zeros, there is an r_j and an $a_1>r_j$ such that $f(a_1)<0$ and $f(x)$ has no zeros other than $r_1, \cdots, r_j$ on $[a, a_1]$. Then using $g(x)=-f(x)$ on $[a_1, b]$ we get every zero of $g(x)$ which lies to the right of an x_0 where $g(x_0)>0$ or to the left of an x_0 where $g(x_0)<0$. But this is the set of those zeros of $f(x)$ on $[a_1, b]$ which lie to the left of an x_0 where $f(x_0)>0$ or to the right of an x_0 where $f(x_0)<0$. This set combined with $r_1, \cdots, r_m$ is the totality of zeros of $f(x)$ on $[a, b]$. Finally, one can check whether or not there are any additional zeros between any given two, say r_1 and r_2, by picking any x_0 in $r_1<x_0<r_2$ and applying the above to both $f(x)$ and $g(x)=-f(x)$; since no values of $f(x)$ outside $[r_1, r_2]$ are of any concern, it does not matter that the hypothesis $g(a)>0$ does not hold.*

* If (ii) fails for $g(x)$, it has no zeros on $[a_1, b]$.

Proof. Define $q(x)=x+f(x)/\lambda$. From (ii), $\lambda_0>0$. Consider the case $f(x_0)>0$. It is clear that $\{x_n\}$ is increasing and $x_N>b$ for some N if there is no zero to the right of x_0. Also, if there is a zero and r is the first one to the right of x_0 it is clear that $x_{n+1}\geqq x_n$ as long as $x_n\leqq r$. By definition $x_0<r$. Suppose $x_k\leqq r$ for some k. $q(x)$ is nondecreasing, since if $x_2>x_1$,

$$q(x_2)-q(x_1)=(x_2-x_1)\left[1+\frac{f(x_2)-f(x_1)}{\lambda(x_2-x_1)}\right]\geqq(x_2-x_1)\left[1-\frac{\lambda_0}{\lambda}\right]\geqq 0.$$

Therefore

$$x_k\leqq r\Rightarrow x_{k+1}=q(x_k)\leqq q(r)=r.$$

Thus by induction, $x_n\leqq r$ for all n. We now know that $\{x_n\}$ is nondecreasing in any case, and $x_n<r$ for all n. Thus $\{x_n\}$ converges if and only if there is a zero above x_0, and if it converges, $\lim_{n\to\infty}x_n=r$ where r is the first zero above x_0.

Part 2, the case $f(x_0)<0$, is entirely similar.

We will prove part 3 only for the case $f(x_0)>0$. The other case is entirely similar. It suffices to show that if $\lambda_0\leqq\lambda_1<\lambda_2$ and $x_k\geqq y_k$, then $x_{k+1}\geqq y_{k+1}$ where

$$x_{k+1}=x_k+f(x_k)/\lambda_1,\qquad y_{k+1}=y_k+f(y_k)/\lambda_2.$$

Since $\lambda_1<\lambda_2$ and $f(y_k)\geqq 0$, $y_{k+1}\leqq y_k+f(y_k)/\lambda_1$. But the function $x+f(x)/\lambda_2$ is nondecreasing so

$$x_k\geqq y_k\Rightarrow x_{k+1}=x_k+f(x_k)/\lambda_1\geqq y_k+f(y_k)/\lambda_1\geqq y_{k+1}.$$

Proof of Corollary 1. As long as $|x_n-r|\neq 0$,

$$x_{n+1}-r=(x_n-r)\left[1+\frac{f(x_n)}{\lambda(x_n-r)}\right].$$

Now

$$\frac{f(x_n)}{x_n-r}\leqq\operatorname*{Sup}_{a_1\leqq x\leqq b_1}\frac{f(x)}{x-r}=-K,$$

and $K>0$ since $f(x)/(x-r)<0$ for all $x\neq r$ and $f'(x)<0$ in some neighborhood of $x=r$. Also

$$-K=\operatorname*{Sup}_{a_1\leqq x\leqq b_1}\frac{f(x)}{x-r}\geqq\operatorname*{Inf}_{a_1\leqq x\leqq b_1}\frac{f(x)-f(r)}{x-r}\geqq-\lambda_0,$$

so we have $0<K\leqq\lambda_0\leqq\lambda$. Therefore, for all n

$$0\leqq 1+\frac{f(x_n)}{\lambda(x_n-r)}\leqq\left(1-\frac{K}{\lambda}\right)\leqq 1,$$

$$|x_{n+1}-r|\leqq(1-K/\lambda)|x_n-r|,$$

$$|x_n-r|\leqq(1-K/\lambda)^n|x_0-r|.$$

The proof of Corollary 2 is essentially contained in its statement.

We note, finally, that it may sometimes be worthwhile to compute a new λ_0 after the region in which a root lies has been narrowed down. By 3, this increases the rate of convergence, especially if the curve crosses the root with zero slope. In the case of Corollary 1, when $[a_1, b_1]$ is small, K and λ_0 are almost equal so the convergence is rapid. If one starts with an $x_0^{(1)}$ below r and an $x_0^{(2)}$ above r and recomputes λ_0 for each successive interval $[x_n^{(1)}, x_n^{(2)}]$ inside which r is known to be, we have essentially Newton's method, with its corresponding rapid convergence. An alternative is to recompute λ_0 until K/λ_0 is within some specified distance of 1, and just iterate from there on; e.g., once $1-K/\lambda \leqq 0.1$, each iteration gives one more decimal place accuracy.

TWO-SIDED TESTS VERSUS ONE-SIDED TESTS

IRWIN GUTTMAN AND W. A. O'N. WAUGH, McGill University

1. Purpose of this note. Introductory courses in statistics must often be given to students whose main aim is to acquire some techniques for data-analyzing. These students may not have the time nor the mathematical background to cover topics like sufficient statistics, "best" estimates, and the Neyman-Pearson fundamental lemma. Nevertheless it seems desirable that they should at least be shown a situation in which the power of one test can be compared with that of another. Most such courses have a prerequisite of some calculus, and it is our purpose in this note to show by very elementary means how the power of a well-known two-sided test can be shown to be always less than or equal to that of the related one-sided test.

2. The example. Suppose that we draw a sample $x_1, \cdots, x_n$ from a normal population with mean μ and variance 1, in order to test, at a significance level α, the null hypothesis $H_0: \mu = 0$ against the alternative $H_1: \mu = \mu_1 > 0$. Then (as is well-known) the Neyman-Pearson fundamental lemma provides the uniformly most powerful test

$$\text{(2.1)} \qquad \begin{cases} \text{reject } H_0 \text{ if } \bar{x} \geqq n^{-1/2} z_\alpha, \\ \text{accept } H_0 \text{ otherwise.} \end{cases}$$

where z_α is the point exceeded with probability α, for a standard normal distribution, and $\bar{x}$ is the sample mean.

Suppose on the other hand that we wish to test the same null hypothesis H_0 against the alternative $H_1': \mu \neq 0$. Then the compromise usually adopted is that the test should be

$$\text{(2.2)} \qquad \begin{cases} \text{reject } H_0 \text{ if } |\bar{x}| \geqq n^{-1/2} z_{\alpha/2}, \\ \text{accept } H_0 \text{ otherwise.} \end{cases}$$

We shall prove by elementary means and without using the Neyman-Pearson lemma that the power of the test (2.2) is not greater than the power

of (2.1) for any value of $\mu_1 > 0$. Let $\Phi(t)$ be the cumulative distribution function of the standard normal variable. Then it is easy to show that the power of the test (2.1) is given by

$$P_1 = 1 - \Phi(z_\alpha - n^{1/2}\mu_1), \tag{2.3}$$

and that of the test (2.2) by

$$P_2 = 1 - \Phi(z_{\alpha/2} - n^{1/2}\mu_1) + \Phi(-z_{\alpha/2} - n^{1/2}\mu_1). \tag{2.4}$$

If we write $n^{1/2}\mu_1 = x$ then what we wish to demonstrate is that $P_1 > P_2$ for any $x > 0$. If we define

$$\Psi(x) = \Phi(z_{\alpha/2} - x) - \Phi(-z_{\alpha/2} - x) - \Phi(z_\alpha - x) \tag{2.5}$$

then this is the same as $\Psi(x) > 0$ for all $x > 0$. Differentiating (2.5) we obtain

$$\begin{aligned} \Psi'(x) &\equiv (2\pi)^{-1/2}\{\exp[-\tfrac{1}{2}(z_{\alpha/2} + x)^2] + \exp[-\tfrac{1}{2}(z_\alpha - x)^2] \\ &\qquad - \exp[-\tfrac{1}{2}(z_{\alpha/2} - x^2)]\} \\ &\equiv (2\pi)^{-1/2}\exp[-\tfrac{1}{2}(z_{\alpha/2} - x)^2]\{\exp[-2xz_{\alpha/2}] \\ &\qquad + \exp[\tfrac{1}{2}(z_{\alpha/2} - z_\alpha)(z_{\alpha/2} + z_\alpha - 2x)] - 1\}. \end{aligned} \tag{2.6}$$

Now the factor in the braces is clearly positive for $x = 0$ and decreases monotonically to -1 as $x \to \infty$. Hence it vanishes just once, say for $x = x_0$, which is the only zero of $\Psi'(x)$ in $0 \leqq x < \infty$. Further, $\Psi(0) = 0$, while $\Psi'(x)$ is nonnegative throughout $(0, x_0)$ so that the stationary value is a maximum and must be positive. Since $\Psi(x) \to 0$, as $x \to \infty$, and $\Psi'(x) < 0$ for all $x > x_0$ we see that $\Psi(x)$ tends to the limit zero from above. Hence $\Psi(x) > 0$ for all positive x.

ON THE GENERALIZED INVERSE OF AN ARBITRARY LINEAR TRANSFORMATION

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In 1920 E. H. Moore [9] (see also [10]) introduced the concept of a generalized inverse of an arbitrary (rectangular and/or singular) matrix. The notion was later studied independently from different points of view by J. von Neumann [11], K. O. Friedrichs [5], A. Bjerhammer [2, 3], and R. Baer [1]. In 1955 a thorough investigation of the idea was made by R. Penrose [12], and in the following year R. Rado [15] showed that the basic definitions of Moore and Penrose were equivalent. Also, several applications of the concept have been made (see, for example, [6, 13, 14]), and related results have been considered, (see, for example, [4, 8, 16]).

The purpose of this present note is to provide a treatment of the generalized inverse of an arbitrary linear transformation suitable for use in the classroom.

Let $\mathfrak{V}$ and $\mathfrak{W}$ be finite dimensional (left) vector spaces over a division ring. Let T be a linear transformation of $\mathfrak{V}$ into $\mathfrak{W}$. Finally, suppose that S is a linear transformation of $\mathfrak{W}$ into $\mathfrak{V}$ such that

$$TST = T. \tag{1}$$

We begin by considering some immediate consequences and applications of (1).

First, if T is a one-to-one mapping of $\mathfrak{V}$ onto $\mathfrak{W}$, then it is evident that $S = T^{-1}$, the unique inverse of T.

Second, if β is a vector of $\mathfrak{W}$, then the equation

$$xT = \beta$$

has a solution if and only if $\beta ST = \beta$, and in this case the general solution of the equation is the collection of vectors in $\mathfrak{V}$ of the form

$$\beta S + \alpha(I - TS),$$

where α is an arbitrary vector of $\mathfrak{V}$ and I is the identity transformation on $\mathfrak{V}$. For, if $\beta ST = \beta$, then

$$(\beta S + \alpha(I - TS))T = \beta ST + \alpha(T - TST) = \beta,$$

and if $\alpha T = \beta$, then

$$\beta ST = \alpha TST = \alpha T = \beta,$$
$$\alpha = \alpha(TS) + \alpha(I - TS) = \beta S + \alpha(I - TS).$$

Thus, one particular application of (1) is to provide an explicit solution, say, of a consistent algebraic system of n linear equations in m unknowns.

Another consequence of (1) is that the null-space $\mathfrak{N}$ and the range $\mathfrak{R}$ of T are given by

$$\mathfrak{N} = \mathfrak{V}(I - TS), \qquad \mathfrak{R} = \mathfrak{W}(ST). \tag{2}$$

The first equation follows from the preceding paragraph with $\beta = 0$. The second is a consequence of $\alpha T = \alpha(TST) = (\alpha T)(ST)$ and $\beta(ST) = (\beta S)T$, where α and β are arbitrary vectors of $\mathfrak{V}$ and $\mathfrak{W}$ respectively.

Finally, suppose that

$$\mathfrak{M} = \mathfrak{V}(TS), \qquad \mathfrak{S} = \mathfrak{W}(I - ST). \tag{3}$$

The space $\mathfrak{M}$ is isomorphic to the range $\mathfrak{R}$ under the restriction of T to $\mathfrak{M}$ into $\mathfrak{R}$. For $\alpha T = \alpha(TST) = (\alpha(TS))T$ and, if $(\alpha(TS))T = \alpha T = 0$, then $\alpha(TS) = (\alpha T)S = 0S = 0$. Furthermore, since $(TS)^2 = (TST)S = TS$ and $(ST)^2 = S(TST) = ST$, both TS and ST are idempotent. Thus, (see for example [7], sec. 41) TS is the projection on $\mathfrak{M}$ along $\mathfrak{N}$ and ST is the projection on $\mathfrak{R}$ along $\mathfrak{S}$, and the given vector spaces have the direct sum decompositions

$$\mathfrak{V} = \mathfrak{M} \oplus \mathfrak{N}, \qquad \mathfrak{W} = \mathfrak{R} \oplus \mathfrak{S}. \tag{4}$$

Conversely, given any complements $\mathfrak{M}$ and $\mathfrak{S}$ of the null-space $\mathfrak{N}$ and the range $\mathfrak{R}$ of T, respectively, it is shown below that a linear transformation S

exists such that (1), (2), and (3) are satisfied. However, in general such an S is not unique. For example, if $T=0$, then any linear transformation of $\mathfrak{W}$ into $\mathfrak{V}$ satisfies these three equations. But, as is now demonstrated, there is only one such S that also satisfies the equation

$$STS = S, \tag{5}$$

which is symmetrically related to equation (1).

THEOREM. *Let $\mathfrak{V}$ and $\mathfrak{W}$ be finite dimensional vector spaces over a division ring. Let T be a linear transformation of $\mathfrak{V}$ into $\mathfrak{W}$ with null-space $\mathfrak{N}$ and range $\mathfrak{R}$. Finally, let $\mathfrak{M}$ and $\mathfrak{S}$ satisfy* (4). *Then there is one and only one linear transformation S of $\mathfrak{W}$ into $\mathfrak{V}$ such that* (1), (2), (3), *and* (5) *are satisfied.*

We demonstrate this theorem by proving the following equivalent result.

COROLLARY 1. *Let the notation and conditions be as in the theorem above. Furthermore, let E be the projection on $\mathfrak{M}$ along $\mathfrak{N}$ and F be the projection on $\mathfrak{R}$ along $\mathfrak{S}$. Then there is one and only one linear transformation S of $\mathfrak{W}$ into $\mathfrak{V}$ such that*

$$TS = E, \quad ST = F, \quad TST = T, \quad STS = S. \tag{6}$$

Proof. First, if S_1 as well as S satisfies (6), then

$$S_1 = S_1TS_1 = S_1E = S_1TS = FS = STS = S.$$

Second, let β be any vector of $\mathfrak{W}$. It is now shown that there is one and only one μ in $\mathfrak{V}$ such that $\mu E=\mu$ and $\beta F=\mu T$. For, by definition of $\mathfrak{R}$, there exists a vector α in $\mathfrak{V}$ such that $\alpha T=\beta F$. Let $\mu=\alpha E$. Then $\mu E=\alpha E^2=\alpha E=\mu$, and since $\alpha(I-E)$ is in $\mathfrak{N}$, $\mu T=\alpha ET=\alpha T=\beta F$. Suppose further that $\mu_1E=\mu_1$, and $\beta F=\mu_1T$. Since $\mu-\mu_1$ is in both $\mathfrak{M}$ and $\mathfrak{N}$, it follows that $\mu_1=\mu$.

Thus, we define

$$S\colon \beta S = \mu,$$

where $\mu E=\mu$ and $\beta F=\mu T$. The mapping S is clearly a linear transformation of $\mathfrak{W}$ into $\mathfrak{V}$. Furthermore, since $(\alpha E)E=\alpha E$ and $(\alpha T)F=\alpha T=(\alpha E)T$, by the preceding paragraph, $(\alpha T)S=\alpha E$. That is, for any α of $\mathfrak{V}$, $\alpha(TS)=\alpha E$. Thus, $TS=E$ and $TST=ET=T$. Finally, since $\beta(ST)=(\beta S)T=\mu T=\beta F$ and $\beta(STS)=(\beta S)(TS)=(\beta S)E=\mu E=\mu=\beta S$, it follows that $ST=F$ and $STS=S$.

The unique linear transformation S of this theorem is called the *generalized inverse* of T relative to the direct sum decomposition (4). It is evident that $\mathfrak{S}$ is the null-space and $\mathfrak{M}$ is the range of S, and that T is the generalized inverse of S relative to (4). In other words, T and S are symmetrically related in the sense that TS is the projection on the range of S along the null-space of T and ST is the projection on the range of T along the null-space of S. (Compare [1], Lemma 1, p. 178; here $\mathfrak{V}=\mathfrak{W}$.)

If $\mathfrak{V}$ and $\mathfrak{W}$ are real (complex) inner product spaces, then it is natural to choose the complements of $\mathfrak{N}$ and $\mathfrak{R}$ to be their respective orthogonal comple-

ments. The projections E and F are then perpendicular projections, and Corollary 1 may be restated as follows. (See also [12], [15].)

COROLLARY 2. *Let $\mathfrak{V}$ and $\mathfrak{W}$ be finite dimensional real (complex) inner product spaces. Let T be a linear transformation of $\mathfrak{V}$ into $\mathfrak{W}$. Then there exists one and only one linear transformation S of $\mathfrak{W}$ into $\mathfrak{V}$ such that*

$$(7) \qquad (TS)' = TS, \quad (ST)' = ST, \quad TST = T, \quad STS = S,$$

where the prime denotes the (complex conjugate) transpose operation.

Proof. If $\mathfrak{M}$ and $\mathfrak{S}$ are chosen to be the orthogonal complements of the null-space and the range of T, respectively, then since every perpendicular projection is equal to its (complex conjugate) transpose, (7) follows immediately from (6).

Conversely, if S is such that (7) is satisfied, then $I-TS$ and ST are the perpendicular projections on the null-space and the range of T along their respective orthogonal complements. Thus, by the theorem above, S is unique.

Adopting the notation of other writers on this subject, the unique S of (7) is denoted by $T\dagger$.

We conclude this note with an illustration. Let $\mathfrak{V}$ and $\mathfrak{W}$ be real inner product spaces with respective orthonormal bases $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$. Let

$$\begin{aligned} \alpha_1 &\to \beta_1 + \beta_2, \\ \alpha_2 &\to \qquad\quad \beta_3 - \beta_4, \\ \alpha_3 &\to \beta_1 + \beta_2 + \beta_3 - \beta_4, \end{aligned}$$

define a linear transformation T of $\mathfrak{V}$ into $\mathfrak{W}$. Since $\nu = (1/\sqrt{3})(\alpha_1+\alpha_2-\alpha_3)$ and $\rho_1 = (1/\sqrt{2})(\beta_1+\beta_2)$, $\rho_2 = (1/\sqrt{2})(\beta_3-\beta_4)$ are orthonormal bases of the null-space and the range of T, respectively, it follows that $\alpha E = \alpha - (\alpha, \nu)\nu$, $\beta F(\beta, \rho_1)\rho_1 + (\beta, \rho_2)\rho_2$, and the projections E and F are described by

$$\begin{aligned} \alpha_1 &\to \tfrac{1}{3}(2\alpha_1 - \alpha_2 + \alpha_3), & \beta_1, \beta_2 &\to \tfrac{1}{2}(\beta_1 + \beta_2), \\ \alpha_2 &\to \tfrac{1}{3}(-\alpha_1 + 2\alpha_2 + \alpha_3), & \beta_3 &\to \tfrac{1}{2}(\beta_3 - \beta_4), \\ \alpha_3 &\to \tfrac{1}{3}(\alpha_1 + \alpha_2 + 2\alpha_3), & \beta_4 &\to -\tfrac{1}{2}(\beta_3 - \beta_4). \end{aligned}$$

Now, if P is the linear transformation of $\mathfrak{W}$ into $\mathfrak{V}$ such that $\beta_1, \beta_2 \to \frac{1}{2}\alpha_1$, $\beta_3 \to \frac{1}{2}\alpha_2$, and $\beta_4 \to -\frac{1}{2}\alpha_2$, then $PT = F$. Hence $T\dagger = T\dagger TT\dagger = FT\dagger = PTT\dagger = PE$ is given by

$$\begin{aligned} \beta_1 &\to \tfrac{1}{6}(2\alpha_1 - \alpha_2 + \alpha_3) \\ \beta_2 &\to \tfrac{1}{6}(2\alpha_1 - \alpha_2 + \alpha_3), \\ \beta_3 &\to \tfrac{1}{6}(-\alpha_1 + 2\alpha_2 + \alpha_3), \\ \beta_4 &\to \tfrac{1}{6}(\alpha_1 - 2\alpha_2 - \alpha_3). \end{aligned}$$

The method used here to find $T\dagger$ is an adaptation of the familiar method for calculating the inverse of a nonsingular matrix by means of elementary row operations. The reader may find it of interest to rework this example in the language of matrices, using the results to illustrate further this material.

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A PROPERTY OF THE STROPHOID

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If a point X varies on a fixed line l, and if O and T are fixed points, then the circle with center X and passing through T is cut by the line OX (through O and X) in a pair of points which trace a strophoid. For T on l, a special strophoid is obtained. This is a unicursal cubic with a node at T; it is also the pedal curve of a parabola with respect to a pedal point T on the directrix l (O is the midpoint of the segment joining the focus and the pedal point).

A problem which arose in an N.S.F. teachers' institute led to these special strophoids in a quite different way. Given three distinct points, R, S, and T, consider the locus $\mathcal{L}$ of a point P such that the lines PR and PS are equidistant from T. Discarding the trivial part of the locus, namely the line RS, $\mathcal{L}$ can be classified as follows: for T on line RS, $\mathcal{L}$ is the circle having T and the fourth harmonic to R, S, and T as the endpoints of a diameter; for T equidistant

from R and S, $\mathfrak{L}$ consists of the circumcircle of triangle RST and the line perpendicular to segment $\overline{RS}$ at its midpoint; in all other cases $\mathfrak{L}$ is a special strophoid.

To obtain the last result, let R, S, and T have rectangular coordinates $(-1, 0)$, $(1, 0)$, and (a, b) respectively, where $a \cdot b \neq 0$, and let $P: (x, y)$ be a general point of $\mathfrak{L}$. The equation of $\mathfrak{L}$ is then

$$(1)\quad (bx - ay)(x^2 + y^2) - abx^2 + (a^2 - b^2 + 1)xy + aby^2 - bx - ay + ab = 0,$$

and R, S, and T are on $\mathfrak{L}$. Let c be the x-intercept of the line PT and let t be the line perpendicular to PT at P. By the definition of $\mathfrak{L}$, PT and t are the bisectors of the angles formed by the lines PR and PS and so separate these lines harmonically. The x-intercept of t is therefore $1/c$ and the equation of t is

$$(2)\qquad f(x, y, c) = (ac - c^2)x + bcy + c - a = 0.$$

By a straightforward calculation, the envelope of the family of lines t is

$$(3)\qquad (ax + by + 1)^2 = 4ax,$$

which, since $a \cdot b \neq 0$, is a parabola $\mathcal{P}$. The focus of $\mathcal{P}$ is the point

$$[a/(a^2 + b^2),\ -b/(a^2 + b^2)],$$

and the directrix, with equation $bx = ay$, is the line through T and the midpoint of segment $\overline{RS}$. From the way $\mathcal{P}$ was obtained, $\mathfrak{L}$ is the pedal curve of $\mathcal{P}$ with respect to the pedal point T and hence is a special strophoid.

Starting from the parabola $\mathcal{P}$, and with T on the directrix, we do not recover R and S as a unique pair in their original relation to T and the locus $\mathfrak{L}$. Instead, let line r through T be an arbitrary secant of $\mathcal{P}$ intersecting it at points E and F, and let s be the tangent to $\mathcal{P}$ parallel to r and intersecting the directrix at M. The line s' perpendicular to s at M is also tangent to $\mathcal{P}$. The lines t_E and t_F, tangent to $\mathcal{P}$ at E and F respectively, intersect s' in the points R' and S' respectively. From the nature of parabolas, it can be verified that M is the midpoint of R' and S' and that the line perpendicular to t_E at R' and the line perpendicular to t_F at S' both pass through T. Thus S' and R' are on $\mathfrak{L}$. If we begin with R', S', and T, in place of R, S, and T, we obtain a strophoid $\mathfrak{L}^*$ which is the pedal curve with respect to T of a parabola $\mathcal{P}^*$. But the lines t_E, t_F, s, $R'S'$, and the line at infinity are tangent to both $\mathcal{P}$ and $\mathcal{P}^*$, which implies that $\mathcal{P} = \mathcal{P}^*$ and hence that $\mathfrak{L} = \mathfrak{L}^*$. There are thus infinitely many pairs R' and S' on $\mathfrak{L}$ which with T produce the same locus $\mathfrak{L}$ as did R, S, and T.

From the two locus definitions of $\mathfrak{L}$, it follows that each point X on the directrix of $\mathcal{P}$ is the midpoint of two chords of the strophoid. One chord lies on the line passing through X and through the midpoint of the segment joining T and the focus of $\mathcal{P}$. The other chord lies on that tangent line to $\mathcal{P}$ through X which separates T and $\mathcal{P}$. Hence a parallelogram, variable in size and shape, moves so as to remain inscribed in the strophoid. One set of the diagonal lines

envelope a parabola whose directrix passes through the centers of the parallelograms, and the other set of diagonal lines are concurrent.

The R, S, T-definition of $\mathfrak{L}$ also provides a second geometrical construction for the curve. Let C' and C'' be the circles with $\overline{RT}$ and $\overline{ST}$ as respective diameters, and let C be a circle centered at T and cutting C' in a point Y and C'' in a point Z. Then the lines RY and SZ intersect at P on $\mathfrak{L}$.

GRAPHS OF COMPOSITE RELATIONS

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The purpose of this note is to call attention to a graphical procedure for obtaining the graph of the composite of two relations directly from the graphs of the individual relations.

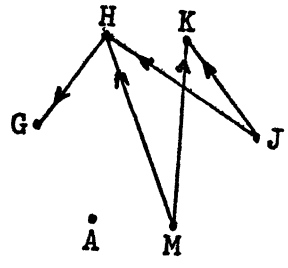
For a given relation R, denote its cartesian graph as a c-graph and its directed graph (in the sense of graph theory) as a d-graph, its incidence matrix by R_m. The composite or relative product of two relations P and Q, in that order, is designated by $P \mid Q$ and defined by

$$P \mid Q = \{(x, y) \mid (\exists z)(x, z) \in Q \quad \text{and} \quad (z, y) \in P\}.$$

Example 1. Let a family unit consist of: George (G), George's son Henry (H), and Henry's wife Kate (K), children of Henry and Kate as follows: Jack (J), Mike (M), Agatha (A).

Let $P = \{(x, y) \mid y \text{ is a son of } x\} = \{(G, H), (H, J), (H, M), (K, J), (K, M)\}$.

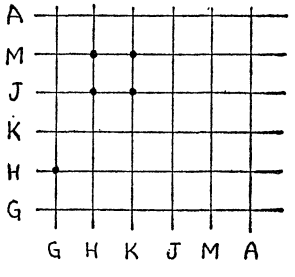
Let $Q = \{(x, y) \mid y \text{ is a child of } x\} = \{(G, H), (H, J), (H, M), (H, A), (K, J), (K, M), (K, A)\}$.



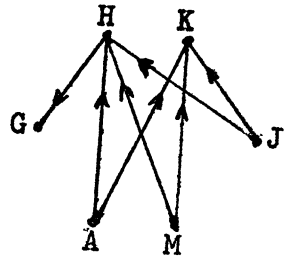
d-graph of P

$P_m =$

	G	H	K	J	M	A
G	0	0	0	0	0	0
H	1	0	0	0	0	0
K	0	0	0	0	0	0
J	0	1	1	0	0	0
M	0	1	1	0	0	0
A	0	0	0	0	0	0



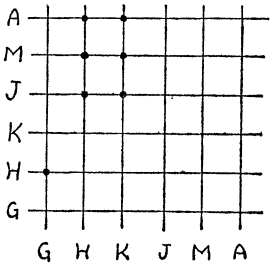
c-graph of P



d-graph of Q

$Q_m =$

	G	H	K	J	M	A
G	0	0	0	0	0	0
H	1	0	0	0	0	0
K	0	0	0	0	0	0
J	0	1	1	0	0	0
M	0	1	1	0	0	0
A	0	1	1	0	0	0



c-graph of Q

$P|Q = \{(x, y) | y \text{ is a son of a child of } x\} = \{(x, y) | y \text{ is a grandson of } x\}$
$= \{(G, J), (G, M)\}.$

We note first that this result is obtainable from the product of the matrices.

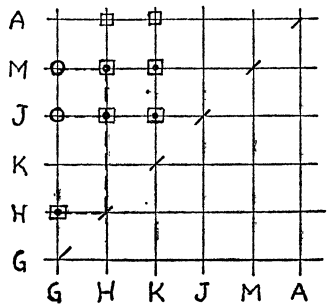
$$P_m \cdot Q_m = \begin{Vmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&0&0&0 \end{Vmatrix} \cdot \begin{Vmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&1&1&0&0&0 \end{Vmatrix} = \begin{Vmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{Vmatrix} = (P|Q)_m.$$

This will be true in general to the extent that $(P|Q)_m$ is obtainable as the associated binary matrix of $P_m \cdot Q_m$ by replacing every nonzero element by one.

The graphical construction of $P|Q$ will be based on the following remarks. For simplicity we consider two relations with domain and range in the same universe U and work with a single rectangular coordinate system for the representation of the cartesian product $U \times U$. On this coordinate system construct three c-graphs, that of relation P, that of relation Q, and that of the identity relation I, using distinct markings to distinguish the elements of the three sets respectively. The elements of $P|Q$ are then determined as follows:

1. Take an element of Q as an *initial point* and construct a horizontal line through this initial point. If (x, z) is this initial element, then the intersection of the horizontal line with I defines an element (z, z) of I.
2. Construct a vertical line through the element of I determined in the first step and note the intersections of this line with P, perhaps vacuous. Such intersections, if any, will be elements of the form (z, y) belonging to P.
3. Through each such element of P determined in the second step construct a horizontal line and note the intersections of such lines with the vertical line through the *initial point*. Such intersections will be of the form (x, y) such that for some z, $(x, z) \in P$ and $(z, y) \in Q$. Hence, by definition of $P|Q$, $(x, y) \in P|Q$. It is evident that each element of $P|Q$ is constructible by this procedure.

Graphical Construction of $P|Q$ for *Example* 1.



P: ·
Q: □
I: /
$P|Q$: o

Construction lines: - - - -

The process works neatly for sketching graphs of composite functions or relations of the type encountered in analytic geometry. Sometimes it anticipates results not immediately obvious from the defining statements.

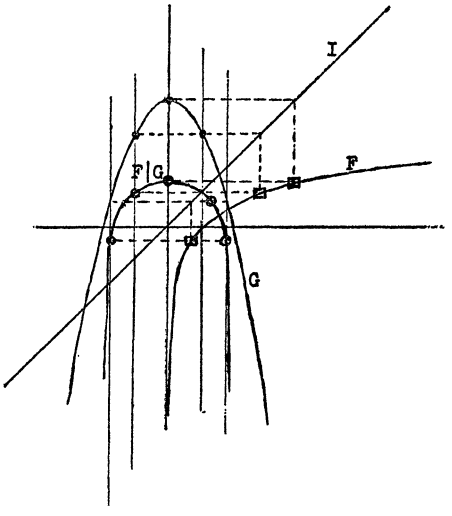
Example 2. Consider F and G to be functions defined over the real numbers where

$$F = \{(x, y) \mid y = \ln x\}, \qquad G = \{(x, y) \mid y = 4 - x^2\};$$

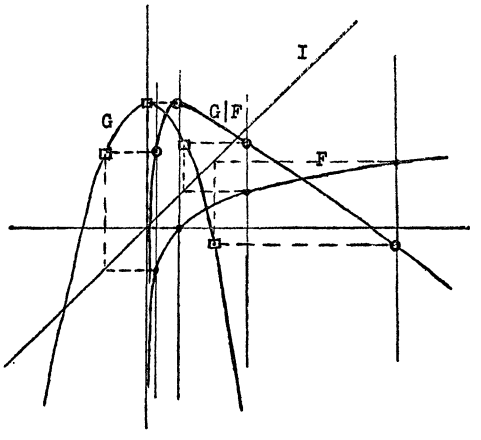
then

$$F \mid G = \{(x, y) \mid y = \ln (4 - x^2)\}, \qquad G \mid F = \{(x, y) \mid y = 4 - \ln^2 x\}.$$

(Construction proceeds from the solid dot to the open dot.)



Graphical Construction of $F|G$



Graphical Construction of $G|F$

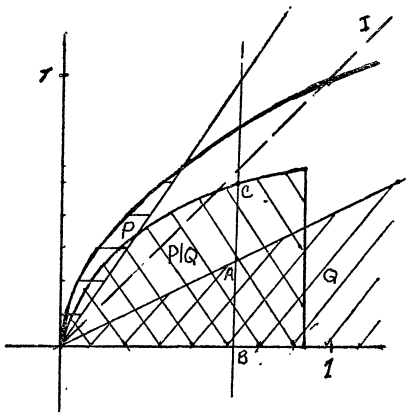
Example 3. Consider P and Q to be relations defined over the real numbers where

$$P = \{(x, y) \mid 3x \leqq 2y \leqq 2\sqrt{x}\}, \qquad Q = \{(x, y) \mid 0 \leqq 2y \leqq x\},$$

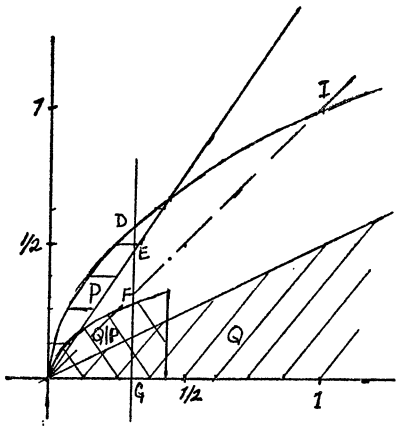
then

$$P \mid Q = \{(x, y) \mid 0 \leqq y \leqq \sqrt{(\tfrac{1}{2}x)} \quad \text{and} \quad 0 \leqq x \leqq 8/9\},$$
$$Q \mid P = \{(x, y) \mid 0 \leqq 2y \leqq \sqrt{x} \quad \text{and} \quad 0 \leqq x \leqq 4/9\}.$$

Note in the graphical constructions that segment AB of relation Q maps into segment CB of $P|Q$, and that point D of relation P maps into segment FG of $Q|P$.



Graphical Construction of $P|Q$



Graphical Construction of $Q|P$

UNIQUENESS THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS

ALLEN D. ZIEBUR, Harpur College

There has recently been an increase in interest in uniqueness theorems for ordinary differential problems of the type of (1) below. The results of many of these current studies (see [1] and [2]) are based on integral inequalities such as Bellman's Lemma and its generalizations. It is the purpose of this note to show that differential inequalities, too, may be used to produce simple uniqueness theorems. Not only do we get more far-reaching results with differential inequalities than we do with integral inequalities, but it also seems more natural to use differential inequalities to study differential problems.

One drawback to presenting many uniqueness theorems based on differential inequalities (such as those in Coddington and Levinson, *Theory of ordinary differential equations*, New York, 1955) to beginning students is the appearance in the proofs of the inequality $|D_x|f(x)|| \leqq |D_x f(x)|$. Although this inequality is a simple consequence of the triangle inequality, the context in which it arises allows us only to assert that it holds almost everywhere or that at some points some of the derivatives involved must be considered as derivatives from the right or from the left. These exceptions, of course, tend to upset students fresh from a course in elementary calculus. The proof below avoids these complications, and it is certainly simple enough to present to anyone to whom one would present any other topic in differential equations.

Let us consider the differential problem

$$y' = f(x, y), \qquad y = y_0 \quad \text{when} \quad x = a. \tag{1}$$

(Here we can think of y and $f(x, y)$ either as numbers or as n-component vectors, depending upon the sophistication of the class. We will indicate multiplication

of such quantities by means of a dot to suggest either ordinary multiplication or inner product.) The most widely used guarantee of the uniqueness of the solution of (1) is the Lipschitz Condition: *There exists a number k such that if (x, y_1) and (x, y_2) are any two points in the domain of f that have the same X-coordinate, then*

$$|f(x, y_2) - f(x, y_1)| \leqq k|y_2 - y_1|. \tag{2}$$

Suppose that the Lipschitz Condition is satisfied and that u and v are two solutions of (1) on an interval $[a, b]$. If we let $w(x) = u(x) - v(x)$, we have, for each $x \in [a, b]$,

$$w'(x) = f(x, u(x)) - f(x, v(x)), \tag{3}$$

and $w(a) = 0$. Now we multiply both sides of (3) by $2w(x)$ and we obtain the equation

$$D_x w(x)^2 = 2w(x) \cdot [f(x, u(x)) - f(x, v(x))]. \tag{4}$$

The right hand side of this equation is not greater than (if we are dealing with vectors, we use the Schwarz Inequality at this point)

$$2|w(x)|\,|f(x, u(x)) - f(x, v(x))|,$$

or in view of (2), $2kw(x)^2$. Thus from (4) we obtain the inequality

$$D_x w(x)^2 \leqq 2kw(x)^2. \tag{5}$$

Now we multiply both sides of (5) by the integrating factor e^{-2kx}, rearrange our terms, and we have the inequality $D_x[w(x)^2 e^{-2kx}] \leqq 0$, which is valid for each $x \in [a, b]$. Therefore, if we integrate from a to an arbitrary point $t \in [a, b]$, taking into account the fact that $w(a) = 0$, we find that $w(t)^2 e^{-2kt} \leqq 0$. Of course this inequality implies that $w(t) = 0$, and since t is arbitrary, we have established the identity of the solutions u and v. (This demonstration applies to intervals to the right of the initial point a, but it is a trivial matter to modify the details so as to take care of intervals to the left of a, too.)

It is perfectly clear that the above argument will also apply if the Lipschitz constant k is replaced by "suitable" functional values. For example, in the uniqueness criterion of Nagumo [3] k is replaced by $1/(x-a)$. Then (5) can be written in the form $D_x[w(x)^2/(x-a)^2] \leqq 0$. Now when we integrate from a to our arbitrary point t, we must exercise a little care because of the indeterminacy at a. But the difficulty is apparent rather than real (when we assume that f is continuous), and it still follows that $w(t) = 0$.

The method of differential inequalities is particularly well-suited to linear differential equations because in that case it is an elementary matter to find a "suitable" replacement for k. Thus suppose that y satisfies the linear differential equation

$$y^{(n)} = p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y. \tag{6}$$

If we write $w(x)^2=y^2+\cdots+(y^{(n-1)})^2$, we find, using (6), that

$$D_x w(x)^2 = 2yy' + \cdots + 2y^{(n-2)}y^{(n-1)} + 2y^{(n-1)}(p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y).$$

Now we use the inequality $2rs \leqq r^2+s^2$ to replace $2yy'$ with $y^2+y'^2$, $\cdots$, $2p_0(x)y^{(n-1)}y$ with $|p_0(x)|\,[(y^{(n-1)})^2+y^2]$ and we obtain the inequality

$$D_x w(x)^2 \leqq [y^2 + y'^2 + \cdots + (y^{(n-2)})^2 + (y^{(n-1)})^2] + |p_{n-1}(x)|\,[2(y^{(n-1)})^2] + \cdots + |p_0(x)|\,[(y^{(n-1)})^2 + y^2].$$

Each term in square brackets is no greater than $2w(x)^2$, so we have the inequality

$$D_x w(x)^2 \leqq 2k(x)w(x)^2, \tag{7}$$

where $k(x)=1+|p_0(x)|+\cdots+|p_{n-1}(x)|$. When we multiply both sides of (7) by the integrating factor $e^{-2K(x)}$, where $K(x)=\int_a^x k(z)dz$, we obtain the inequality $D_x[w(x)^2e^{-2K(x)}]\leqq 0$. Now if we assume that y and its first $n-1$ derivatives are 0 when $x=a$, we have $w(a)=0$. Thus when we integrate from a to an arbitrary point t we find, as before, that $w(t)=0$. In particular, then, $y=0$ when $x=t$, and so we have shown that the only solution of (6) which vanishes, together with its first $n-1$ derivatives, at a is the trivial solution. This result is the uniqueness theorem for linear differential equations of order n.

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1. D. A. Kearns, Elementary uniqueness theorems for differential equations, this MONTHLY, 68 (1961) 275–277.
2. I. Bihari, Acta. Math. Acad. Sci. Hungary, 7 (1956) 81–94.
3. M. Nagumo, Jap. J. Math., 3 (1926) 107–112.

REVERSION OF POWER SERIES

R. M. REDHEFFER, University of California, Los Angeles

Let $f(z)$ be analytic for $|z|<R$, let $|f(z)|\leqq M$ for $|z|\leqq R$, let $f(0)=0$, and $|f'(0)|=a$. Following Landau, Copson's well-known text presents a proof that the equation $w=f(z)$ can be solved for $z=g(w)$ throughout the circle

$$|w| \leqq \tfrac{1}{6}(aR)^2/M.$$

This result is important in practice as well as in theory, because it gives an estimate for the rate of convergence of the reversion series.

It seems worth noting, therefore, that a modification of the Landau-Copson procedure yields

$$|w| \leqq \tfrac{1}{4}(aR)^2/M,$$

under a weaker hypothesis. Although the region finally obtained is even larger than this, the constant $\frac{1}{4}$ cannot be improved. We shall establish the following:

THEOREM. *Let* $f(z)$ *be analytic for* $|z|<R$, *let* $f(0)=0$, *let* $|f'(0)|=a>0$, *let*

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta \leqq M^2 \qquad \text{for } 0<r<R, \tag{1}$$

and define $A=aR/M$. *If* $|w|\leqq w_0$ *is the convergence circle of the reversion series* $z=g(w)$, *then* w_0 *satisfies*

$$w_0 > \frac{(aR)^2}{4M}\,\frac{1}{\sqrt{[1-(3/4)A^2]}} \tag{2}$$

for all $f(z)$ *satisfying the hypothesis, and* w_0 *satisfies*

$$w_0 \leqq \frac{(aR)^2}{4M}\,\frac{1}{\sqrt{(1-A^2)}} \tag{3}$$

for some $f(z)$ *satisfying the hypothesis.*

The proof is simple. If

$$f(z) = a_1z + a_2z^2 + \cdots$$

then (1) yields

$$|a_2|^2R^4 + |a_3|^2R^6 + \cdots \leqq M^2 - a^2R^2 \tag{4}$$

when the term $|a_1|^2R^2$ is transferred to the right. If the terms following a_1z in $f(z)$ are written in the form

$$(a_2R^2)(z/R)^2 + (a_3R^3)(z/R)^3 + \cdots$$

then the Schwarz inequality shows that the magnitude of this part of the series does not exceed

$$(M^2 - a^2R^2)^{1/2}[(r/R)^4 + (r/R)^6 + \cdots]^{1/2}.$$

Here we have used (4), taking $|z|=r<R$. Summing the geometric series gives the estimate

$$|f(z)| \geqq ar - (M^2 - a^2R^2)^{1/2}(r/R)^2[1-(r/R)^2]^{-1/2}.$$

If t is defined by $\sin t=r/R$ the result simplifies to

$$|f(z)| \geqq M \sin t[A - (1-A^2)^{1/2}\tan t]$$

when we factor out M, recalling $A=aR/M$.

This function of t is positive between $t=0$ and $\tan t=A(1-A^2)^{-1/2}$, so that $f(z)\neq 0$ for the corresponding values of r. If such an r is chosen, and if $|w|<|f(z)|$ for all $|z|=r$, then Rouché's theorem shows that the equations

$$f(z) = 0 \quad \text{and} \quad f(z) - w = 0$$

have the same number of roots in $|z|<r$. Since the first has just one root, so

does the second. Hence we see that w_0 satisfies

$$w_0 \geqq \max_{0<t<\pi/2} M \sin t[A - (1 - A^2)^{1/2} \tan t]. \tag{5}$$

This is the main result of the present note. The choice

$$\tan t = \tfrac{1}{2}A(1 - A^2)^{-1/2}$$

yields the inequality (2) of the theorem; location of the true maximum requires solving a cubic for $\tan t$.

The inequality (3) follows by considering the function $w=az+bz^2$, noting that $b^2R^4=M^2-a^2R^2$.

NOTE

It has come to my attention that the final formula in *A method for evaluating* $\int x^n e^{-kx}dx$ by N. C. Scholomiti and R. G. Hill, this MONTHLY, **69** (1962) 45, is a special case of a formula in *An extension of integration by parts* by James W. Brown, this MONTHLY, **67** (1960) 372.

JOHN M. H. OLMSTED

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

REMARKS ON THE MEMORANDUM "ON THE MATHEMATICS CURRICULUM OF THE HIGH SCHOOL"

E. G. BEGLE, Stanford University

I am sure that the large number of individuals who, through SMSG, have worked to strengthen the school mathematics program will be delighted that such distinguished mathematicians as Professors Bers, Kline, Polya, and Schiffer have stated guidelines for curriculum improvement so much in harmony with their own views. (See this MONTHLY **69** (1962) 189–193.)

The primary goal of SMSG is to make *every* student a competent user of the mathematics he can and will use in his life and career. Specifically, SMSG is trying to provide *every* student with greater facility in the basic skills of arithmetic, algebra and geometry; to give him greater facility in the conversion of a practical situation into a form suitable for mathematical treatment; to provide a better foundation for further study of mathematics; and to dispel the general fear of quantitative thinking framed in mathematical language.

The texts prepared by SMSG are designed to illustrate one way of modifying the secondary school curriculum, preserving the substance but changing the point of view, to better achieve these goals. Inspection of these texts will show that Professors Bers, Kline, Polya, and Schiffer have stated formally and explicitly many of the principles used by the SMSG writers, and that agreement with most of these guidelines is implicit in these texts.*

These guidelines, perhaps augmented with a few others to take account of all the goals listed above, could make a very significant contribution to mathematics education by providing a framework for discussion of current efforts at curriculum improvement. Now, given a particular text, new or old, and a particular guideline, we can, after making sure we agree on the meaning and interpretation of the guideline, discuss how well the text follows it. Such a procedure would undoubtedly do much to identify those areas in which there is substantial agreement, and to clarify the nature of our disagreements in other areas. At the same time this procedure would surely provide some badly needed constructive suggestions, which the generalized discussions prevalent in recent years have almost completely neglected.

In particular, those associated with SMSG would be delighted to receive specific suggestions and criticisms, in terms of these guidelines, of the materials they have prepared.

Finally, and regretfully, note must be taken of the authors' unhappiness, expressed in their antepenultimate paragraph and in numerous other comments scattered throughout their article, over current efforts at curriculum improvement. The authors were undoubtedly aware of the great variety of suggestions for curriculum change which have been made in recent years. Yet they fail to distinguish between these many suggestions and in effect disapprove of all of them.

This is a grave and unnecessary discourtesy to the large number of distinguished mathematicians, teachers, educators, and scientists who have devoted thought, time, and effort to the work of SMSG and, in addition, will undoubtedly raise questions in many minds as to how serious the authors really are about wanting any change at all from the status quo.

NOT WHAT CAN BE TAUGHT, BUT WHAT SHOULD BE TAUGHT

I. C. BARKER, Lowell High School, San Francisco Unified District

We do not intend to review the proposals of the major committees established to revitalize the curriculum, for these are well publicized. *The Mathe-*

* A probable exception is the "Genetic Principle" stated in the fifth guideline. This principle, as stated, would, for example, deny to our students the efficiency of using algebra in the first course in geometry and would require children to learn to compute with Egyptian, Babylonian, Greek and Roman numerals before being introduced to the historically later but far more efficient place-value decimal system.

In any case, very few changes from the traditional appear in the order of topics in the SMSG texts.

matics Teacher has featured the new programs, and the summer institutes sponsored by the National Science Foundation have brought thousands of high school teachers into direct contact with the major proposals.

Nearly all of the major curriculum studies recommend that excerpts from analytic geometry be taught at the high school level. Proposed are the slope concept, parallel and perpendicular lines, the distance and midpoint formulas and the simpler analytic proofs. High school sophomores can handle these topics very well, and have been doing so at the author's school for several years. This much analytic geometry requires about ten or twelve lessons. Students also learn about circles and parabolas centered about the origin during their sophomore year; they study the general equations of these two curves and the angle between two lines in the junior year; as seniors they study the ellipse and, if the class is fast, the hyperbola too. In all, this analytic geometry requires about 20 or 25 days.

Many topics other than analytic geometry are proposed for the secondary school level. Among these are elementary set theory, logic, vectors, polar coordinates, some modern abstract algebra, and very much longer units in elementary statistics and matrices. All of these proposals not only can be studied profitably by the college capable senior, but have been taught creditably this year by many high schools throughout the nation. In addition to the above it is frequently recommended that other topics such as inequalities, the Cartesian product $R \times R$, the relation concept and De Moivre's Theorem be taught in the secondary school. All of these, too, are within the capabilities of the better students, and have been successfully taught to many groups.

Nearly all the curriculum committees recommend that solid geometry as a separate course be abolished, and that the basic concepts and chief mensuration formulas be included with appropriate units in plane geometry. If four weeks or so of solid geometry, and two weeks or so of analytic geometry are to be taught concurrently with plane geometry, then about one-third of a semester's material, presently taught in plane geometry, must be dropped. What is to be dropped and what is to be retained? If solid geometry is dropped, of course, one whole semester is gained at the senior level. In this half year a course in elementary probability and statistics may be studied profitably. But it is not always clear where many of other new topics are to be placed. Where and how will such topics as the recommended logic, vectors, probability, number theory and matrices be fitted into the curriculum? Should present work with determinants be dropped? Or theory of equations? Far more new material is being proposed than can be handled in the finite time gained by dropping the obsolete. The problem becomes one of $A \vee B \vee C \vee D$—rather than $A \wedge B \wedge C \wedge D$. . . .

The question today is not what can be taught at the high school level, but rather what should be taught. What is really necessary? Is solid geometry necessary? Approximately one-fourth of the 150 or so schools of engineering in the United States still list solid geometry as an entrance requirement, and about ten percent recommend it. The other two-thirds do not require it. Most

schools of architecture require solid geometry, though when asked by this author what specific parts of solid geometry were really necessary, representatives of these schools replied in effect "The first two chapters." Well, is or is not solid geometry essential? One must assume, from the above, and from the fact that outstanding schools of engineering do not require it, that solid geometry, as a course, is not necessary. Evidently, then, many colleges, too, have deadwood in their curriculum. Are determinants necessary at the high school level? Or should we high school teachers give the week or two spent on determinants to some other topic? In its Field Service Leaflet No. 8, 1958 prepared to help clarify the "muddled" curriculum situation, the University of California at Berkeley, surely one of the great universities of the world, one finds "Simultaneous equations must be retained." In a 1961 bulletin issued by the Massachusetts Institute of Technology one reads "determinants are optional." What conclusion are we teachers at the high school level to draw?

The analytic geometry recommended for secondary schools will be taught universally only as the colleges so demand. Does a college professor proceed on the assumption that all those facing him have this background, or that none have? If so, either he loses the latter group, or gives false impressions of college mathematics to the former. The present need for mathematicians is such that America cannot afford to lose either group. There should also be agreement as to what phases of analytic geometry should be taught in high school, and what phases belong in college. The author suggests for high schools: the midpoint and distance formula and analytic proofs; slopes and linear equations; circles, parabolas and ellipses, including translation, and maybe the hyperbola. The work at the college level would then begin with normals, tangents, rotation, etc.

The time has come to make decisions on content. Far more is being proposed than can be taught adequately, and needless duplication of content is wasteful. If there is not a nationwide committee presently considering this problem, then there should be. The problem is pressing, and an early date for its solution should be set. Education of our youth does not wait.

REPORT OF THE TEXTBOOK EVALUATION PROJECT AT YESHIVA UNIVERSITY

WILLIAM ZLOT, Yeshiva University

This article contains a description of the Textbook Evaluation Project that was carried out in the 1961 Summer Institute for Secondary School Teachers of Mathematics sponsored by the National Science Foundation at Yeshiva University. This course was designed with two main purposes in mind: first, to acquaint the participant with many of the current and standard texts and proposals relating to secondary school mathematics, and, second, to familiarize the participant with various methods of teaching certain key topics.

Each participant was given an assignment with the following items written upon it: a topic from secondary school mathematics; several key points related to the mathematics content and the pedagogical techniques involved in the

presentation of the above topic; the titles of several (about five, on the average) modern and traditional texts; and instructions to prepare both an oral and written report that would compare and evaluate the treatment of the given topic in the cited texts giving special attention to the indicated key points.

The following is a typical assignment sheet:

I. **Topic.** Introduction to Algebra. In particular, attention should be given to the treatment of the transition from arithmetic to algebra; the desirability for, and the technique of, translating English expressions into the language of algebra; the concepts of a variable and of a numeral; the concepts of formula and open sentence.

II. **Textbooks.** Two texts published commercially and two texts associated with current mathematics course content studies were used.

III. **Assignment.** (A) Prepare a 45-minute *oral* report that covers briefly: the presentation of the above topic in each of the texts with particular reference to the items listed in Part I above and your evaluation of the treatments in the several texts. (B) Submit a *written* outline of your analysis.

The following comments seem appropriate:

1) By cutting across textbook lines the participant was able to become familiar with different methods of teaching a given topic.
2) The above approach enabled the participant to become intimately acquainted with, rather than merely "read about," many of the texts that he might be called upon to evaluate and use in the course of his own professional work. (In the process of his study of the presentation on the given topic, the participant usually had to read a good deal of the related material.)
3) The participants were encouraged not to confine their analysis to the "key points" indicated on the assignment sheet. Indeed, they were urged to consider the general mathematical and pedagogical quality of the developments, as well as the inherent features of the book itself (readability, organization, etc.).

The participants were asked to submit brief anonymous comments upon the conduct of the course. For the most part, the students were very satisfied and intrigued with the textbook project. Some of the participants expressed the feeling that they would have liked to have had a "lesson plan" evolve from each of the talks. We had anticipated this opinion, and we still wonder whether it is desirable to give a teacher a "recipe" for the teaching of a topic. We believe that a teacher should be exposed to several different approaches, and merely be guided toward a decision. Thus, the teacher would be encouraged to formulate a lesson plan that would be a personal expression of his own opinion, and that would probably vary with the teaching situation.

Topics covered include: measure of sets, order, betweenness, separation, continuity and continuous variation, function, introduction to algebra, exponents, radicals and logarithms, equations, graphs, algebraic structures, sequences and series, congruence and similarity, the bridge from intuitive to formal geometry, mathematical induction, inequalities, and number systems (naturals, rationals, integers, reals).

Our library for the course contained some 120 items. We will use this textbook evaluation scheme as a nucleus of the "methods course" for our secondary school teachers of mathematics. We believe that such an approach can be the core of methods courses in several other fields.

REMARKS ON "A GENERATION OF HIGH SCHOOL CALCULUS"

E. H. C. HILDEBRANDT, Northwestern University

Professor Neeley's statistics (this MONTHLY 68 (1961) 1004–1005) provide us with information on a very serious problem. They indicate that we may be losing quite a large number of students who showed some promise in science and mathematics while in high school but never survived the first year of college, at least not the first year of college mathematics. Is this the fault of the high school courses or of the persons who taught them or is this due to the failure of the student to adjust to college work?

These figures are especially discouraging to the high school teacher of the advanced placement courses. Most of these teachers have been doing a commendable job of arousing the interests of their students for continuing their studies in mathematics and science. When some were required to teach the advanced placement courses or the course in the calculus, they may have proceeded with their teaching in the same manner in which they remembered that they were taught the calculus several school generations back. Now they may wonder why and where they failed. They would not dare report such a high mortality rate in their own school. A natural reaction on their part (for which you can hardly blame them) is to insist now that their future teaching assignments be limited to courses at the freshman and sophomore level. Perhaps this may partly account for the fact that some teacher-placement bureaus during this past year have had an especially heavy demand for applicants who are willing to teach these advanced placement mathematics courses.

We should keep in mind that a summary such as this one soon gets around to the aspiring college major in mathematics, science, or engineering and presents him with a formidable barrier with which he must deal. He notes that these were "our brightest students," that "only 11 out of 48 completed the first year of college work in 1958–59," and that two years later this ratio was probably the same. He will think more than twice before he registers for courses in which approximately one-fourth pass (and perhaps three-fourths of these get a C grade!). He will wonder also what happened to the 37 failures out of the 1958–59 group of 48 and the more than 37 failures out of the 1960–61 group of 54—

were they permitted to repeat the course?—were they dropped out of college completely? Also he will wonder in what department those nonmathematics majors who survived decided to major? Is the rumor true that most of these do quite well in college physics?

In his first conclusion, Professor Neeley indicates his disappointment that so few college majors in mathematics came from these accelerated groups. Isn't it true, however, that the majority of students in the third and fourth year high school mathematics courses do not intend to major in mathematics at the college level? Their advisers as well as the guidance literature have indicated to them that the best preparation they can get for their studies in the sciences and engineering includes four full years of high school mathematics. Also one can imagine that when the 1959–60 high school seniors heard that a "large" number of the 1958–59 advanced placement group did not "make it," it was quite natural that only 5 out of 54 of the 1960–61 group indicated a desire to major in mathematics.

The second conclusion that "high school calculus is largely a waste of time" does not necessarily follow from the information presented in this paper. For example, we do not know whether the mortality rate in first-year college courses enrolling students with four years of nonadvanced placement courses was just as high or not. Nor do we know how many students in first-year college courses had taken the accelerated high school courses but chose not to enter these special college courses in mathematics. Neither do we know whether the mathematics preparation of both groups is the same.

Some positive suggestions may be in order. Assuming that the high school teacher plays an important part in this picture, would it not be possible to put forth some additional effort which would provide this teacher with strong preparation in advanced mathematics? Here, for example, some slight modifications in summer institute programs could render an excellent service:

1) Arrange an institute whose participants include only teachers of advanced placement or college preparatory courses. Doubtlessly it is true that most of the teachers of advanced placement courses have never attended an institute or certainly have not attended an institute which stressed the background needed for these courses.

2) The program of this institute should deal primarily with the foundations of algebra and the calculus with the aim of providing these teachers with the understanding of basic concepts so they in turn would be able to teach them with real confidence. Lectures alone will not suffice. Graduate assistants can be used effectively in discussion sections. Participants should be encouraged to ask really searching questions about the basic concepts. Each should be required to present some developments at the board so as to demonstrate that he has acquired complete understanding of these basic concepts.

3) Set aside two summers for this preparation instead of one. The second summer could deal with some of the areas which were not properly understood the first summer.

4) Stress quality of understanding rather than quantity of theory covered.

5) Instructors should be persons who have had many years of experience in teaching the calculus as well as in teaching graduate courses and should be sympathetic to the needs of these teachers.

6) As an added incentive perhaps, give these participants a bonus stipend.

Ten such institutes in the summer of 1963 could provide important benefits to students in some three to four hundred high schools.

College teaching could profit too if one could get most of the instructors of our first-year college courses together for a three- to six-week conference or institute on teaching. They could work on the question "How can we develop the fundamental concepts of the calculus so that our students will really understand them?" As their own understanding improves, our problem of enlisting more students to major in our field will be greatly simplified. The tendency in most calculus classes is for students to inquire "how do you work such and such a problem?" instead of asking thought-provoking questions about mathematics. College teachers should also consider whether the greatest asset to passing our present courses in calculus is a good memory; e.g.: Is not the student who has memorized the basic formulas and remembers how to perform typical manipulations the one who "succeeds?" That students do have astonishing memories was demonstrated in several of my classes during this past year. Members were provided with a sheet which contained the value of π to 1000 decimal places. They were required to spend no more than one hour in memorizing as many of these places as possible. In a written "quiz" the following day, one undergraduate student recorded 175 places correctly and one graduate student recorded 250 places correctly, all in less than four minutes. No mnemonics were used.

Perhaps supplementary textbook materials could speed up the learning process. At the present time, most calculus courses use a single textbook for an entire year or a year and half of study. While some of the presentations in the text are excellent examples of condensed developments of the basic concepts, further descriptions and discussions as well as additional specific examples could provide much needed enlightenment for some of the "it is easily seen" passages. There is also a need for many more thought-provoking problems which are often neglected due to the desire to cover the course in a minimum of time. Many helpful suggestions on the teaching and development of the calculus are hidden away in past issues of the MONTHLY and other journals as well as in some out-of-print books whose major content is frowned upon now. Could not this material be collected and sold in paper back editions, thus rendering an added service to a fuller appreciation of the calculus?

Another possible way of reducing the mortality rate would be to develop a succession of short but comprehensive tests which the student could take at periodic intervals (*à la* teaching machine). These tests could provide the student with repeated reminders of his weak spots. By the end of a semester or year of such continuous testing, one might even hope to find that all and not only one fourth of the "brightest" students had completed the course successfully and that perhaps even some of the "not quite so bright" students might have acquired enough interest and courage to major in college mathematics.

SMALL VERSUS LARGE CLASSES

KENNETH O. MAY, Carleton College

Since the teacher shortage is causing many departments to consider large classes in mathematics, our experience with a freshman lecture of 250 may be of interest. The entire group met together for a lecture three times a week. In addition each student had a weekly two-hour laboratory with about 15 or 20 students led by a regular staff member. Detailed assignments in two textbooks, along with sample quizzes and laboratory problems were contained in a syllabus supplied to all students. Daily papers were turned in at the lecture, corrected by a crew of about 10 readers, and returned to the students through campus mail *the same day.*

A laboratory session began with a substantial quiz which was immediately discussed by the instructor. The rest of the session was devoted to student questions and the working of problems with help from the instructor and undergraduate student assistants. Although students were permitted to interrupt the big lecture to ask questions, they were encouraged to turn in questions at the beginning of the lecture on 5×3 cards, with name on one side and question on the other. The lecturer could then answer the question in the lecture if it was of general interest. Otherwise he could answer it on the card and return it through campus mail, or suggest a conference if this seemed desirable.

All staff members had office hours at which participants could get individual attention, and students were invited to visit at any time with whichever professor happened to be available.

Once initial difficulties were ironed out, these arrangements seemed to be satisfactory to the students. Student achievement was as good as or better than with smaller sections. Because of the centralization of lecturing, writing tests, handling papers, and grading, staff time was economized. Although the number of contact hours was about the same for the staff, it was easier to handle a laboratory than a class.

Nevertheless after trying this device two years (during the second year it was modified by having two large lectures instead of one) the staff decided to return to small sections. The majority preferred to have a slightly heavier load accompanied by the satisfaction of managing a small class completely.

Our conclusion is that students may be taught as effectively in large sections (and our methods could have been used just as well with a class of 2000 as with one of 200), provided the work is very carefully organized and provision is made for easy feedback and consultation. The smaller size class group is more enjoyable, however, for both students and staff.

ACTUARIAL SCIENCE PROGRAM

Since 1958–59 the Department of Mathematics at Lebanon Valley College has been developing a program to train beginning actuarial candidates for the life insurance industry. Students take the following courses in the regular college offerings: elementary statistical analysis, probability theory, and mathematical statistics. Each year a special seminar is given, rotating in sequence from the following: life insurance mathematics, finite differences, and life contingencies.

In both courses and seminars, the texts recommended by the Society of Actuaries for basic material to pass their examinations, Parts 2, 3, and 4A, are used. In November and in May these examinations are given on the campus. For further details contact Professor B. H. Bissinger, Department of Mathematics, Lebanon Valley College, Annville, Pennsylvania.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1516. *Proposed by R. J. Oberg, University of California*

Let $N=0.1360518\cdots$, where the kth digit is congruent mod 10 to the kth triangular number. Show that N is rational.

E 1517. *Proposed by R. S. Spira, University of California*

If $u>0$, show that $(u+1/u)\operatorname{arccot} u>1$.

E 1518. *Proposed by R. W. Cottle, University of California*

Let $A=(a_{ij})$ be an m by n matrix whose entries are elements of an ordered set $(S, \geqq)$. Suppose A is column ordered—that is, $a_{1j}\geqq a_{2j}\geqq\cdots\geqq a_{mj}$ for

each $j=1, 2, \cdots, n$. Obtain a row ordered matrix A' by arranging the entries of each row of A so that $a'_{i1} \geqq a'_{i2} \geqq \cdots \geqq a'_{in}$ for each $i=1, 2, \cdots, m$. Is A' column ordered?

E 1519. *Proposed by C. N. Mills, Sioux Falls College*

Find the area of the smallest equilateral triangle inscribed in a right triangle of legs a and b.

E 1520. *Proposed by V. E. Hoggatt, Jr., San Jose State College*

Prove that if the circumcenter O of triangle ABC is inverted with respect to the three Apollonian circles of ABC, the images O_a, O_b, O_c are the vertices of the second Brocard triangle of ABC.

SOLUTIONS

An Application of Pascal's Theorem

E 1486 [1961, 929]. *Proposed by Aboulghassem Zirakzadeh, University of Colorado*

Given a conic E and a line l in the plane of the conic, choose a point K on l and not inside E, and draw the tangents KA and KB to the conic. Consider a point P on the conic and draw the lines PA and PB and find the points R and S, the respective intersections of PA and PB with l. Consider another point P' on the conic and draw the lines $P'R$ and $P'S$ and find the points A' and B', the respective intersections of $P'R$ and $P'S$ with the conic. Prove that $A'B'$ passes through the pole of line l with respect to the conic E.

Solution by Roscoe Woods, State University of Iowa. From the six points A, B, A', B', P, P' consider the arrangements $APBA'P'B'$, $AAB'BBA'$, $A'A'BB'B'A$ and apply Pascal's theorem to them in turn. It is then found that the tangents to the conic E at the points A' and B' intersect at a point on l. It then follows from the fundamental theorem of poles and polars that the line $A'B'$ passes through the pole of l with respect to the conic E.

Also solved by Ragnar Dybvik, L. D. Goldstone, D. C. B. Marsh, Albert Nijenhuis, and Seymour Schuster. Late solution by Hon Pui Lam.

A Representation of a Rational Number

E 1487 [1961, 929]. *Proposed by N. V. Glick and Gregory Sheridan, North American Aviation, Inc.*

Prove that any positive rational number can be expressed as a finite sum of distinct terms of the harmonic series.

I. *Solution by G. S. Cunningham, New Hampshire State Department of Education.* Let a/b be any positive rational. Then $a/b=1/b+\cdots+1/b$, a sum of harmonic terms with $a-1$ duplications. Recursively expand all duplicate elements by the identity $1/n=1/(n+1)+1/n(n+1)$ until all terms are distinct.

II. *Solution by E. P. Starke, Bloomfield College.* Since the harmonic series is divergent we can subtract from any given rational number enough terms from the beginning of the series so that the remainder is less than the next term of the series. Let a/b be this remainder. We have, for some n, $1/n > a/b > 1/(n+1)$. Then $a/b - 1/(n+1) = (an+a-b)/(nb+b)$. From the inequalities we see that $0 < an+a-b < a$, hence the new fraction has a smaller numerator than a/b. Continuing the process, we reach the desired expression in a finite number of steps since there are only a limited number of numerators available.

Also solved by H. D. Arnett, Maurice Brisebois, Brother T. C. Wesselkamper, D. I. A. Cohen, D. M. Danvers, G. C. Dodds, Underwood Dudley, S. J. Einhorn, Mutha Fawker, H. J. Fletcher, Fred Friedman, Michael Goldberg, H. W. Gould, Gregory Howell, Erwin Just and Norman Schaumberger (jointly), Gordon Keller, Israel Krongold and E. H. Weiss (jointly), C. W. Langley, D. L. Linfield, Wallace Manheimer, D. C. B. Marsh, G. J. Michaelides, Albert Nijenhuis, F. D. Parker, C. F. Pinzka, Allen Rubenstein, L. S. Shively, Steven Siegel, D. L. Silverman, W. C. Waterhouse, Dale Woods, and the proposers. Late solutions by Norman Brenner, J. L. Brown, Jr., Allan Gibbard, D. L. Goldsmith, Cornelius Groenewoud, D. L. Jokela and R. I. Snell (jointly), J. B. Muskat, R. J. Oberg, Gertrude D. and M. W. Pownall, and Stephen Ullom.

It was pointed out that the problem is a special case of Problem 227 [1916, 220], which provides an infinite number of solutions. A solution to the problem also readily follows from Problem 4512 [1952, 640]. The problem appeared on the Fourteenth William Lowell Putnam Mathematical Competition, and appears (with solution) in the MONTHLY [1954, 546]. Langley suggested the allied problem: What is the minimum number of terms of the harmonic series whose sum is a given positive rational number?

Matrices of Zero Trace

E 1488 [1961, 929]. *Proposed by R. G. Winter, Pennsylvania State University*

For square matrices of order n, prove that any matrix M: (1) has zero trace if $MQ = -QM$, where Q is some nonsingular matrix, (2) can be written as a matrix of zero trace plus a multiple of the identity.

Solution by W. C. Waterhouse, Harvard University. We have $M + QMQ^{-1} = 0$; since trace M equals trace (QMQ^{-1}) and the trace is additive, trace M is zero. Also, $M - (1/n)(\text{trace } M)I$ has zero trace.

Also solved by H. F. Bechtell, Maurice Brisebois, D. I. A. and S. P. Cohen (jointly), D. M. Danvers, Underwood Dudley, J. C. Eaves, Seymour Geisser, N. V. Glick, J. C. Hickman, G. T. Hogan, Peter Landweber, T. J. Lee, Gerald Leibowitz, Jiang Luh, Marvin Marcus, D. C. B. Marsh, D. A. Moran, Albert Nijenhuis, P. V. O'Neil, F. D. Parker, C. F. Pinzka, D. A. Robinson, E. M. Scheuer, Wayne Shepherdson, Michael Skalsky, E. L. Spitznagel, Jr., H. R. Stevens, E. H. Theil, Olga Taussky Todd, E. H. Weiss, D. S. Yates, David Zeitlin, and the proposer. Late solutions by Norman Brenner, Serge Dubuc, Hon Pui Lam, J. B. Muskat, and H. J. Noble.

Olga Todd noted the slightly stronger result: The characteristic roots $\mu_1, \cdots, \mu_n$ of M have the property that for a suitable permutation

$$\begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$$

the relations $\mu_k+\mu_{i_k}=0$, $k=1, \cdots, n$, hold. If Q is not assumed nonsingular, but merely nonzero, then at least one of these relations holds. This follows from the fact that the linear operator defined by $MX+XM$ on the matrix space of $n\times n$ matrices X has the characteristic roots $\mu_i+\mu_k$ $(i, k=1, \cdots, n)$. See, e.g., Bellman, *Matrix Analysis*, p. 231.

Editorial Note. The above proof assumes that the matrices are taken over a field of characteristic zero. If the field is of characteristic k, then the above proof of part (1) actually holds for all $k\neq 2$, and that of part (2) for all k which are not divisors of n. Robinson exhibited simple examples to show that these restrictions on k are necessary.

A Generalization of the Fermat-Steiner Problem

E 1490 [1961, 930]. *Proposed by Marlow Sholander, Western Reserve University*

Three towns A, B, C are joined by roads AP, BP, CP which cost respectively r, s, t dollars per mile. Locate P that minimizes cost.

Solution by Michael Goldberg, Washington, D. C. The point P corresponds to the equilibrium point of three springs from A, B, C and joined at P where the tensions in these springs are proportional to r, s, t. If a triangle with sides proportional to r, s, t has α, β, γ for the corresponding opposite angles, then $\measuredangle BPC=\beta+\gamma$, $\measuredangle CPA=\gamma+\alpha$, $\measuredangle APB=\alpha+\beta$. The point P is thus easily constructed as a common point of three appropriate arcs constructed on the sides BC, CA, AB of triangle ABC.

If the intersection of these arcs cannot be constructed within the triangle ABC, or the costs r, s, t do not form a triangle, then take P at the vertex requiring the least total cost.

Also solved by D. C. B. Marsh, Albert Nijenhuis, Michael Skalsky, and the proposer. Late solutions by Norman Brenner and Y. E. Yeager.

Editorial Note. If $r=s=t$, and triangle ABC has no angle as great as 120°, then angles BPC, CPA, APB all equal 120° and P is an isogonic center of triangle ABC. This special case, which seeks P so that $PA+PB+PC$ is a minimum, was proposed by Fermat to Torricelli. Torricelli solved the problem, and his solution was published in 1659 by his student Viviani. Later, Steiner gave an elegantly simple analysis of the problem.

The proposer also mentioned three allied problems:

1. Three towns A, B, C are joined by roads AP, BP, CP which are built at the rate of a, b, c miles per day. Locate P that minimizes building time.
2. Three roads form a triangle ABC. From a point P, roads PA', PB', PC' are constructed to the roads BC, CA, AB at a cost of r, s, t dollars per mile. Locate P that minimizes cost.
3. Three roads form a triangle ABC. Three roads $B'C'$, $C'A'$, $A'B'$, where A', B', C' are located on BC, CA, AB, are to be constructed at a cost of r, s, t dollars per mile. Locate A', B', C' that minimize cost.

The first problem can be reduced to the given problem by taking $r=1/a$, $s=1/b$, $t=1/c$. The second problem is easy, with a vertex solution. The third problem is tough and generalizes Schwartz's triangle problem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5025. *Proposed by David Mumford, Cambridge, Mass.*

Given a commutative ring R with unity element. Prove or disprove the statement: If an ideal A is contained in the (set-theoretic) union of a finite set of ideals, then A is contained in one of these ideals.

5026. *Proposed by A. A. Mullin, University of Illinois*

Consider the following Boolean function of four variables

$$f(a, b, c, d) = b'(a + c) + b(ac' + a'd').$$

Show that every switching circuit realization of f contains at least seven contacts and that a circuit exists which contains precisely seven contacts.

5027. *Proposed by A. J. Goldman, National Bureau of Standards*

Let $M_n(F)$ be the set of $n \times n$ matrices over the field F, considered as an n^2-dimensional vector space over F. Call a vector subspace of $M_n(F)$ nonsingular if all its nonzero members are nonsingular matrices. Find maximal nonsingular subspaces of $M_n(F)$.

5028. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

It is known that (a) any two quadrics which have a common enveloping cone intersect in plane curves, and (b) any two enveloping cones of a quadric intersect in plane curves. Does each of these properties characterize quadrics only?

5029. *Proposed by A. C. Zaanen, University of Leiden, and W. A. J. Luxemburg, California Institute of Technology*

Let $I = \{x: 0 \leqq x \leqq 1\}$.

(a) Show that there exists a real function $f(x)$ on I having the following properties: (i) f is continuous and strictly increasing (*i.e.* $x_1 < x_2$ implies $f(x_1) < f(x_2)$), (ii) the derivative $f'(x) = 0$ for all $x \in I$ except on a subset of Lebesgue measure zero.

(b) Let $f(x)$ be continuous and nondecreasing on I and suppose that $f(0) = 0$ and $f(1) = \alpha$. By L we denote the length of the graph of $f(x)$. Show that $L = 1 + \alpha$ if and only if $f'(x) = 0$ for all $x \in I$ except on a subset of Lebesgue measure zero.

(c) Let $f(x)$ be continuous and strictly increasing on I, with $f(0)=0$ and $f(1)=\alpha$, and let $g(y)$ be the inverse function, defined on $I_1=\{y: 0\leqq y\leqq\alpha\}$. Show that $f'(x)=0$ almost everywhere on I if and only if $g'(y)=0$ almost everywhere on I_1.

5030. *Proposed by Hugh Noland, Chico State College, California*

Does there exist a nonterminating decimal, in which the only digits appearing are 1, 2, 3, such that no two adjacent ordered n-tuples of digits are identical for any integer n?

More generally, can three elements generate an infinite set which is closed under an associative multiplication, and has the property that every element is idempotent?

SOLUTIONS

An Elliptic Integral

3352 [1928, 563]. *Proposed by B. C. Keeler*

Evaluate:

$$\int_0^b \left(\frac{a-\cos x}{\cos x-\cos b}\right)^{1/2} dx \qquad (a>1,\ 0<b<\pi).$$

Solution by Stanimir Fempl, Yugoslavia. By the substitution $\cos x=t$, the given integral can be reduced to the elliptic integral

$$\int_{\cos b}^{1} \sqrt{\left[\frac{a-t}{(t-\cos b)(1-t)(t+1)}\right]}\,dt.$$

It can then be reduced by familiar rules of transformation to a canonical form, whereupon tables for such integrals can be consulted. But it is more convenient to use the book, P. Byrd and M. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Berlin, 1954) in which these transformations have already been performed and the results given as formulae for various cases. By use of 254.13 (p. 113) and 414.04 (p. 230) the original integral is easily reduced to

$$K\sqrt{[2(a-\cos b)]} - 2KZ(\beta, k),$$

$$\sin\beta = \sqrt{\left[\frac{a-\cos b}{a+1}\right]}, \qquad k=(\sin\tfrac{1}{2}b)/\sin\beta.$$

The values of K and $KZ(\beta, k)$ have been evaluated in the tables on pp. 322, 336–343.

Matrix with Eigenvalues 0 and 1

4872 [1959, 817; 1961, 1011]. *Proposed by Ky Fan, Northwestern University*

Let A be a matrix (not necessarily square) of rank $r\geqq 1$ and with nonnegative

elements. Let A^* denote the transpose of A. Prove that the square matrix AA^* has no eigenvalue different from 0, 1, if and only if, after deleting all identically vanishing rows and columns, the remaining submatrix of A can be brought by a permutation of the rows and a permutation of the columns into the form

$$B = B_1 + \cdots + B_r, \tag{1}$$

where, for each i, B_i is a rectangular matrix of rank 1, with all its elements positive and such that the sum of squares of all elements of B_i is 1. Equation (1) means that B is obtained by laying out successively the rectangular blocks $B_1, \cdots, B_r$ with the lower right corner of B_i attached to the upper left corner of B_{i+1}, and with zeros filling in the entire matrix outside these blocks.

Correction by Hans Schneider, University of Wisconsin. Refer to former solution [1961, 1011]. (1) Initial remarks preceding paragraph (a): These are correct if H is assumed to be Hermitian as well as nonnegative; in all subsequent applications $H=AA^*$ so that no further modifications are required on this account. The correct conclusion concerning the index k is $k=\text{rank } H$. That $k=r=\text{rank } A$ follows from rank $AA^*=\text{rank } A$.

(2) Paragraph (b) should read: $PAA^*P^*=H_1+\cdots+H_r+0$, if and only if, for a properly chosen permutation matrix Q,

$$B = PAQ = B_1 + \cdots + B_r + 0,$$

where (i) the B_i are rectangular matrices having the same number of rows as H_i, (ii) $B_iB_i^*=H_i$, and (iii) B_i has no zero row or column.

Nonlinear Integral Equation

4951 [1961, 182]. *Proposed by J. F. Heyda, General Electric Company, Philadelphia, Pa.*

Solve the nonlinear integral equation

$$mx + b = \int_0^x \frac{dt}{\sqrt{\{P(t) - P(x)\}}}, \qquad x \geqq 0, m \geqq 0, b \geqq 0,$$

for $P(0)-P(x)$, where $P(x)$ is monotone decreasing.

I. *Solution by R. G. Buschman, Oregon State College.* The substitutions $u=P(0)-P(x)$, $v=P(0)-P(t)$ convert the equation into a convolution type equation

$$mQ(u) + b = \int_0^u (u - v)^{-1/2}Q'(v)dv,$$

where $Q(u)=Q[P(0)-P(x)]=x$. Hence, using the Laplace transformation, we obtain (noting $Q(0)=0$)

$$mq + b/s = \pi^{1/2}s^{1/2}q,$$

$$q = (-b/m)(-m\pi^{-1/2})s^{-1}(s^{1/2} - m\pi^{-1/2})^{-1},$$

and thus

$$Q(u) = (b/m)[\exp(m^2\pi^{-1}u)\ \mathrm{Erfc}(m\pi^{-1/2}u^{1/2}) - 1].$$

Finally $P(0)-P(x)=Q^{-1}(x)$, the inverse function.

II. *Solution by the proposer.* Consider the more general equation

$$(1) \qquad y(x) = \int_0^x \frac{G(t)dt}{\sqrt{\{p(x) - p(t)\}}}, \qquad p(x) = P(0) - P(x),$$

where $y(x)$ and $G(x)$ are positive and continuous for $x \geqq 0$. Denote the inverse of the nonnegative monotone increasing function $p(x)$ by f, and let it take on values t and x at u and v, respectively. Equation (1) can then be written in the form of Abel's equation

$$y[f(v)] = \int_0^v \frac{G[f(u)](df/du)}{\sqrt{(v-u)}}\,du,$$

whose solution for $G[f(v)](df/dv)$ is

$$G[f(v)]\frac{df}{dv} = \frac{1}{\pi}\frac{d}{dv}\int_0^v \frac{y[f(u)]du}{\sqrt{(v-u)}}$$

or, upon simplifying,

$$(2) \qquad G(x) = -\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{y(t)P'(t)dt}{\sqrt{\{P(t) - P(x)\}}}.$$

If in (2) we integrate by parts and perform the indicated differentiation, we obtain

$$(3) \qquad G(x) = -\frac{P'(x)}{\pi}\left[\frac{y(0)}{\sqrt{\{P(0) - P(x)\}}} + \int_0^x \frac{y'(t)dt}{\sqrt{\{P(t) - P(x)\}}}\right].$$

If now we assume that $y'(x)$ is continuous and positive for $x \geqq 0$ and take $G(x)=y'(x)$, then we can rewrite (3) with the aid of (1) as

$$(4) \qquad y'(x) = -\frac{P'(x)}{\pi}\left[\frac{y(0)}{\sqrt{\{P(0) - P(x)\}}} + y(x)\right].$$

Putting $w^2=P(0)-P(x)$, we may write (4) as a linear first order equation in y with w as independent variable, subject to the condition $w=0$, $x=0$, $y=0$:

$$(5) \qquad \frac{dy}{dw} - \frac{2}{\pi}wy = \frac{2}{\pi}y(0).$$

The solution of (5) is readily found to be

$$(6) \qquad y(x) = y(0)[\exp\{(P(0) - P(x))/\pi\}(1 + \mathrm{erf}[(P(0) - P(x))/\pi]^{1/2})].$$

Equation (6) with $y(x)=mx+b$ then furnishes the solution of the given problem for the case $m>0$, $b>0$.

To handle the case $m=0$, $b>0$, we return to the general equation (3) and integrate it from 0 to x:

$$(7)\qquad \int_0^x G(t)dt = \frac{2}{\pi}\left[y(0)\sqrt{\{P(0)-P(x)\}} + \int_0^x y'(t)\sqrt{\{P(t)-P(x)\}}\,dt\right].$$

Using the relations (3) and (7), one easily establishes, using the monotone property of $P(0)-P(x)$, the inequality

$$(8)\qquad \frac{\pi}{2y(x)}\int_0^x G(t)dt \leqq \sqrt{\{P(0)-P(x)\}} \leqq \frac{\pi}{2}\int_0^x \frac{G(t)}{y(t)}\,dt.$$

In (8) we take $G\equiv 1$ and $y(x)\equiv b>0$. Thus we find $P(0)-P(x)=\pi^2x^2/4b^2$.

Finally, when $b=0$, $m>0$, we find upon specializing (4) that $(d/dx)\{P(0)-P(x)\}=\pi/x$, so that $P(0)-P(x)$ becomes logarithmically infinite at $x=0$ and hence the stated problem has no solution for this case.

The writer would be interested in a closed form solution of (1) for arbitrary continuous $G(x)$.

Functional Equation

4956 [1961, 299]. *Proposed by Leonard Carlitz, Duke University*

Find all analytic solutions $f(x)$, $g(x)$, of the functional equation

$$(1)\qquad f(x+y)=f(x)g(y)+f(y)g(x).$$

Solution by Harley Flanders, Purdue University. The given equation is a special case of

$$(2)\qquad f(x+y)=u_1(x)v_1(y)+u_2(x)v_2(y).$$

Assuming the functions involved are twice differentiable, we may differentiate with respect to y twice and set $y=0$, obtaining

$$f(x)=a_0u_1(x)+b_0u_2(x),\qquad f'(x)=a_1u_1(x)+b_1u_2(x),$$
$$f''(x)=a_2u_1(x)+b_2u_2(x).$$

We deduce the existence of three constants, c_0, c_1, c_2, not all zero, satisfying $c_2f''(x)+c_1f'(x)+c_0f(x)=0$, from which we deduce

$$(3)\qquad f(x)=re^{\lambda x}+se^{\mu x}\quad\text{or}\quad f(x)=(mx+n)e^{rx}.$$

If $f(0)\neq 0$, we deduce from (1) that g is a constant multiple of f so that (1) reduces to $f(x+y)=kf(x)f(y)$ which easily yields $f(x)=ce^{\lambda x}$, $g(x)=\frac{1}{2}e^{\lambda x}$.

Otherwise $f(0)=0$, so from (3) we have $f(x)=r(e^{\lambda x}-e^{\mu x})$ or $f(x)=mxe^{rx}$, and we find $g(x)=\frac{1}{2}(e^{\lambda x}+e^{\mu x})$ or $g(x)=e^{rx}$, respectively. In the last case, if $m=0$, $g(x)$ is arbitrary.

Also solved by Robert Breusch, W. G. Brown, F. P. Callahan, Jr., E. W. Cheney and C. C. Farrington, Paul Frederickson, G. Gilbert and P. G. Heyda, W. L. Jones, R. W. Means and A. P. Hillman, Herman Meyer, Marlow Sholander, W. G. Spohn, R. L. Van de Wetering and D. Saltz, J. H. van Lint, W. C. Waterhouse, David Zeitlin, and the proposer.

Minimum Integral in the Complex Plane

4957 [1961, 300]. *Proposed by H. S. Shapiro, New York University*

Let D be a bounded domain in the complex plane, and let $f(z)$, $\phi_1(z)$, $\cdots$, $\phi_n(z)$ be linearly independent functions analytic in and on the boundary of D. Then there exists a unique set of complex numbers $\lambda_1, \cdots, \lambda_n$ such that the following integral is a minimum:

$$\int\int_D \left| f(z) - \sum_{k=1}^{n} \lambda_k \phi_k(z) \right| dS.$$

Solution by the proposer. The existence of an extremal g^* (we use g to denote linear combinations of the ϕ_k) follows by a routine compactness argument. Suppose now g_1^* and g_2^* are two extremals; then, writing $g_3 = \frac{1}{2}(g_1^* + g_2^*)$ we have

$$\int\int_D |f(z) - g_3(z)|\, dS = \int\int_D \left| \tfrac{1}{2}(f - g_1^*) + \tfrac{1}{2}(f - g_2^*) \right| dS$$

$$\leqq \frac{1}{2}\left[\int\int_D |f - g_1^*|\, dS + \int\int_D |f - g_2^*|\, dS \right] = m,$$

where m denotes the minimum of the integral. Hence g_3 is also extremal and, since equality must hold throughout, we have $\operatorname{sgn}[f(z) - g_1^*(z)] = \operatorname{sgn}[f(z) - g_2^*(z)]$ almost everywhere in D. Hence the meromorphic function

$$[f(z) - g_1^*(z)]/[f(z) - g_2^*(z)]$$

is real (in fact, positive) at all its regular points in D, hence is constant. But this implies that g_1^* and g_2^* are linearly dependent, hence $g_1^* \equiv g_2^*$ which is the proposed theorem.

Also solved by Robert Vermes.

Primitive, Symmetric Polynomial

4958 [1961, 300]. *Proposed by A. G. Konheim, IBM Research, Yorktown Heights, N. Y.*

A polynomial $P(x) = \sum_{i=0}^{n} a_{n-i} x^{n-i}$ will be called symmetric if $a_{n-i} = a_i$, $i = 0, 1, \cdots, n$. Prove that a symmetric polynomial over $GF(2)$ is not primitive.

Solution by Leonard Carlitz, Duke University. Since it is no more difficult we consider polynomials with coefficients in $GF(q)$ where q is prime. An irreducible polynomial $P(x) \in GF[q, x]$ is primitive provided the least r such that $x^r \equiv 1 \pmod{P(x)}$ is $r = q^n - 1$, where n is the degree of $P(x)$. A symmetric poly-

nomial of odd degree is divisible by $x+1$: we accordingly assume $n=2m$. It follows from the symmetry of $P(x)$ that $P(x)=x^mQ(y)$, $y=x+1/x$, where $Q(y)$ is an irreducible polynomial in $GF[q, y]$ of degree m. Hence $y^{q^m}\equiv y \pmod{Q(y)}$. This implies $2^{2q^m}+1\equiv x^{q^m+1}+x^{q^m-1} \pmod{P(x)}$, which gives

$$(x^{q^m+1}-1)(x^{q^m-1}-1)\equiv 0 \pmod{P(x)}.$$

Therefore $P(x)$ belongs at most to the exponent q^m+1. But $q^m+1<q^{2m}-1$ except when $q=2$, $m=1$. Hence, except for the quadratic $x^2+x+1\in GF[2, x]$, $P(x)$ cannot be primitive.

Also solved by H. F. Mattson, and the proposer.

Editorial Note. Some confusion arose regarding the definition of "primitive." The definition used in the above solution was intended rather than either (a) irreducible, or (b) having coefficients which are in G and have no common divisors except units of G (see Birkhoff-Mac Lane, *A Survey of Modern Algebra*). The intended definition should have been stated in the proposal.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

New Mathematical Library, published under the auspices of the School Mathematics Study Group by Random House, New York, at $1.95 per volume, available to High School students and teachers for only $.90 per volume. These six books are the first of a proposed series of 30 to 40 volumes. All of the books have numerous exercises, with answers in the back.

1. *Numbers: Rational and Irrational.* By Ivan Niven, 1961.

Ivan Niven has presented here as a living vital topic what can be a deadly bore to beginning mathematicians. Even the treatment of odd and even numbers is interesting. He makes the periodic decimals live for the reader. The formal logic may be beyond most high school students but it is well done. A similar remark might apply to his treatment of the incommensurable cases of geometry. But the student can read a chapter with interest up to his limit and pass on to the next chapter with adequate understanding. The book is developed primarily on the intuitive basis for the elementary reader with a suggestion of rigor for the more advanced student.

2. *What Is Calculus About?* By W. W. Sawyer, 1961.

The writer answers the question of the title clearly and develops the ideas

of the calculus using speed or velocity as an example of the derivative, and uses this analogy in developing the idea of the slope of the tangent. He does it well and with charming informality. His development of the "reverse problem" is, of course, skimpy but well done. The only criticism that might be directed at this book is that he moves too slowly, but the probabilities are that most high school students will welcome this feature of the book.

3. *An Introduction to Inequalities.* By Edwin Beckenbach and Richard Bellman, 1961.

This book will be a real challenge to the superior high school student, as well as to the better college underclassman. The reader will start in with what seems elementary, almost trivial, mathematics and see it developed into a complicated and fascinating presentation of the same ideas in more intricate form. Students at different ability levels may undertake the study of this book, stopping off at different points, but all profiting by the exposure. The writers initiate the study with the discussion of the ordinary "less than" and "greater than" symbols, take up the various definitions of absolute value with its triangular inequality, and then follow with an elementary but difficult deduction of the Cauchy, Hölder, Triangle, and Minkowski inequalities. These inequalities are used in solving certain maximum and minimum problems. The book ends in a discussion of different definitions of distance that satisfy the distance axioms.

4. *Geometric Inequalities.* By Nicholas D. Kazarinoff, 1961.

As with No. 3, this book provides a challenge to the exceptional high school student. The less able student will be frustrated almost immediately, but the superior student will find much that is interesting. There is some minor overlap with No. 3 at the beginning where the treatment is arithmetic and algebraic rather than geometric. The author deals with the isoperimetric problem in a most interesting fashion, and solves many problems by the method of reflection. The hints and solutions for the numerous exercises will be of considerable value to the student reader as they were to your reviewer.

5. *The Contest Problem Book.* By Charles T. Salkind, 1961.

Of the books in this set, this one should be the most useful to the teacher of high school students, as a source of extra problems for tests and for occasional assignment to the better members of the class. The book includes the problems from the M.A.A. annual high school mathematics contest from 1950 to 1960, with a compendium of answers and categorization of problems. The solutions are well presented.

6. *The Lore of Large Numbers.* By Philip J. Davis, 1961.

Once in a while a man can take a topic that is basically dull to most of us and turn out an interesting and useful book. Philip Davis has done that. Naturally he runs all over the field of numbers, from number theory to residue theory,

from the relaxation method of solving the heat equation to the fact that man is the geometric mean of the sun and the hydrogen atom. It is a book of which all high school students can read a considerable part, and those in the upper quartile can possibly read it all.

J. L. BOTSFORD
University of Idaho

Stability in Nonlinear Control Systems. By A. M. Letov. (Translated from the Russian by J. George Adashko.) Princeton University Press, 1961. xiv+316 pp. $8.50.

The ideas of A. M. Lyapunov (see, e.g., a recent edition of a French translation of the original work, *Problème Général de la Stabilité du Mouvement*, Princeton, 1949) form the cornerstone of Soviet work in linear and in nonlinear mechanics. Especially since 1945 mathematicians and engineers in the USSR have collaborated in extending Lyapunov's methods to stability questions in a variety of fields—dynamical systems, differential equations (ordinary and partial), difference-differential equations, control systems. The successes of Soviet missilery have stimulated much non-Soviet interest in all this work. To the "pure" mathematician the wonder is that a method with so many practical applications is at the same time intrinsically elegant. On the other hand, many engineers are surprised that so "theoretical" a method can be used so effectively in practice. Letov's book is intended primarily for the engineer or applied mathematician, although the "pure" mathematician who already knows the essentials of the Lyapunov theory will enjoy seeing how it all can be applied to controls. The reader who wants to work through the details of the Lyapunov theory for the first time would do better to read Lyapunov's treatise itself (see above) or else *Stability by Lyapunov's Direct Method with Applications* by S. Lefschetz and J. La Salle (Academic Press, New York, 1961). Letov's book should be easily accessible to all who know calculus and some of the elementary theory of matrices and determinants.

An outline of the direct method of Lyapunov and of the two main questions treated by Letov is in order before giving a chapter by chapter summary of the book. Let $x=0$ be a steady state solution of the system, (*) $dx/dt=X(x)$,—i.e., $X(0)=0$. This solution is said to be stable if for any positive ϵ there is a positive η such that, if $\|x_0\|<\eta$, then the solution $x=x(t)$ of (*) through x_0 at $t=0$ satisfies the inequality $\|x(t)\|<\epsilon$ for all $t>0$. The stability is said to be asymptotic if $\|x(t)\|\to 0$ as $t\to\infty$. Lyapunov's direct method is based on the following theorem. The steady state solution, $x=0$ of (*) is [asymptotically] stable if there is a positive definite function $V(x)$—called a Lyapunov function—whose total time derivative computed with respect to the solutions of (*) is nonpositive [negative definite]. It is this theorem and its consequences that underlie most of Letov's work.

As given by Letov the system of differential equations of a control system with one regulator is,

$$\dot{\eta}_k = \sum_{\alpha=1}^{m} b_{k\alpha}\eta_\alpha + n_k \qquad (k = 1, \cdots, m)$$

$$(**) \qquad V^2\ddot{\mu} + W\dot{\mu} + S\mu = f^*(\sigma)$$

$$\sigma = \sum_{\alpha=1}^{m} p_\alpha\eta_\alpha - r\mu$$

where the η_k and the $b_{k\alpha}$ are respectively the generalized coordinates and the constant parameters of the regulated object, μ is the coordinate of the regulator, the constants n_k characterize the effect of the regulator on η_k, V^2, W, and S are constants of the device actuating the regulator and the p_α and r are constants of the regulator. A steady state solution of (**) is a solution of the algebraic system obtained by equating the right hand side of the first m equations of (**) to zero and setting S equal to $f^*(\sigma)$, where σ is given by the last equation of (**). The first fundamental problem in the theory of automatic control is to determine the set B of parameter values of the regulator for which the steady state solution is stable. The type of stability considered is called absolute stability—i.e., the steady state must be stable for all functions $f^*(\sigma)$ belonging to a certain class of functions. The second fundamental problem is connected with what is called control quality. Let B be the region of absolute stability defined above for a steady state solution. Let P be a point of the phase space of the variables of the regulator and the regulated object, and let $\rho(t)$ denote the phase space distance at time t between the steady state solution and the solution of (**) passing through P at $t=0$. The conditional damping time is the time during which $\rho(t)$ decreases by a prescribed factor. The direct problem of control quality consists of determining the conditional damping time at every point of B. The indirect problem consists of finding a subregion of B throughout which the conditional damping time does not exceed a prescribed quantity.

The introduction to Letov's book consists of a brief discussion of stability problems, Lyapunov's direct method, and related topics. Chapter 1 presents the equations of regulated objects and regulators, the definition of absolute stability, and the statement of the first fundamental problem. In Chapters 2 and 3 the original equations are transformed into various canonical forms. Everything is done in the greatest of detail here—apparently for the benefit of the practicing engineer. In Chapter 4 a method due to Lur'e [*Nekotorye neleineinye zadachi teorii avtomaticheskovo regulirovania*, Gostekhizdat, 1951] is used to construct Lyapunov functions for certain types of canonical forms of the control system equations. The stability criteria for the systems treated in Chapter 4 are simplified in various ways in Chapter 5. The guiding principle is the obtaining of criteria that are practical and easy to apply. Chapter 6 is devoted to an analysis of inherently unstable control systems and Chapter 7 to programmed control.

The second fundamental problem is taken up in Chapter 8 and methods are given for estimating the quality of control. In Chapter 9 the analysis of the preceding chapters is extended to the case that the regulated object has two

regulators. Two special, but extremely important topics, are treated in Chapter 10—stability in the first approximation and stability in the case of constantly acting forces. In Chapter 11, the stability of nonsteady state solutions is considered, both over a finite and over an infinite time interval. Here the parameters of the system may be time dependent. Chapter 12, which is not in the Russian edition, contains a treatment of regulated objects whose regulators contain tachometers to measure $\dot{\mu}$. The right hand side of the last equation of (**) is augmented by the term—$N\dot{\mu}$. The analysis of the preceding chapters is then applied and the book concludes with the example of an automatically controlled bicycle rolling over a horizontal plane.

The translation is adequate, though rather awkward in places. The exposition throughout is careful and detailed—at all times the author lets the reader in on what he is doing and why. As a model mathematical engineering treatise, as a practical guide to control stability and as a useful application of the direct method of Lyapunov, the book is to be highly recommended.

C. S. COLEMAN
Harvey Mudd College

Homology Theory, An Introduction to Algebraic Topology. By P. J. Hilton and S. Wylie. Cambridge University Press, 1960. xv+484 pp. $14.50.

Fulfilling a real need, this book attempts to reach the frontiers of algebraic topology while beginning with introductory material "intelligible to the mathematician inexperienced in the techniques and problems described." As a nonspecialist, this reviewer feels that the authors have done a remarkable job of writing for those without a specialist's insight. The reader is carried along to a real grasp of results and an overall picture of topics by the commentary at every stage. Each section, in fact almost every idea, is preceded or followed by an explanatory statement (e.g. "We offer the following result as an interesting application . . . , many of the ideas used in the proof occur again in the next section" p. 182; "The idea of the proof is attractive and easy . . . (idea stated) For completeness we now give the technical details of this proof" p. 299). This style together with many details, pertinent examples and good sets of exercises give the reader both understanding and confidence.

The book is divided into two parts: the first, simplicial homology theory and the second, singular homology theory. The first part is by its content the easiest to comprehend in that it is geometrically satisfying. The background summary for part two, unlike that for part one, is material (homotopy theory) that is less likely to be known to the reader, or to be readily usable after only this brief summary. This fact, together with the nature of the material of the second part, makes it less readily accessible to the reader without outside background.

Considering the authors' announced aim "to provide the links which we believe the student might find difficulty in providing for himself from a study of the available literature," it is indeed regrettable that the Bibliography is limited

to a brief list of standard books, and that references to the literature are found only in occasional footnotes rather than in a more systematic fashion.

The authors have made a real contribution in providing a much needed readable comprehensive algebraic topology text.

LIDA K. BARRETT
The University of Tennessee

Introduction to Partial Differential Equations. By Donald Greenspan. McGraw-Hill, New York, 1961. viii+195 pp. $7.50.

This book is concerned with second-order equations in two variables. Chapter 1, Basic Concepts, concerns itself largely with the pertinent parts of advanced calculus. The amassing of such a phalanx of definitions as presented here is perhaps justified since most of them should be familiar to properly prepared students. Chapter 2 is an adequate discussion of Fourier series. In Chapter 3 the Cauchy problem is defined, characteristics discussed briefly, and linear equations with constant coefficients reduced to canonical forms. Chapter 4 discusses the wave equation. D'Alembert's solution of the Cauchy problem is obtained for both the homogeneous and the nonhomogeneous equations. A uniqueness proof for the initial-boundary value problem is given, and the Fourier series solution is given in the case of zero initial derivative and zero boundary values.

A discussion of Laplace's equation constitutes Chapter 5. A weak maximum principle is established, the Dirichlet problem is defined, a uniqueness proof given, and the solution by Perron's method is carried out for regions bounded by contours. Chapter 6 contains a short discussion of the heat equation. Again a weak maximum principle is presented, and is used to prove uniqueness for the initial-boundary value problem for a rectangle. A Fourier series solution of this problem is given in the case of zero boundary data. Chapter 7 contains a discussion of numerical methods, and Chapter 8 mentions the application of the Laplace transform and states the Cauchy-Kowalewski theorem.

The book is well written with copious examples and problems. I am somewhat concerned about the absence of any discussion of the Neumann problem, even in so short an introduction. (It is mentioned in an exercise.) Also, since the Perron method is essentially an interior method, the author's standard restriction to regions bounded by contours may be misleading. Such a condition could well be postponed until the assumption of the boundary values is discussed.

WATSON FULKS
Oregon State University

Calculus, Volume I. By Tom M. Apostol. Blaisdell Publishing Co., New York, 1961. xvii 515 pp. $8.50.

This is Volume I of a two-volume first course in calculus and analytic geometry. It attracts attention not only as the first venture of a young and ambitious publishing firm, but also by its intrinsic qualities, being venturesome, carefully planned and written, rigorous, readable, lively, and the opposite of banal. De-

signed, according to the preface, to accommodate a variety of backgrounds and interests, its level and tone in fact aim it most naturally at a class of students whose background (or ability) is rather stronger and whose interest is rather more purely mathematical than most colleges are accustomed to, but which we hope accurately anticipates the wave of the future. This book is perhaps most nearly a 1961 counterpart of R. Courant's still classic text of 1934. Indeed, it approaches the latter in soundness, clarity and elegance of exposition, and surpasses it in several valuable respects, especially exercises. These are numerous and varied, many of them theoretical, all well-chosen, interesting and instructive, some highly original and challenging. ("Prove that a triangle whose vertices are lattice points cannot be equilateral.") Other striking features are copious and careful historical comments, clean but unlabored use of notation, various neat and modern details of technique (such as using step functions in defining the integral), and a nonstandard order of topics.

This being 1962, "rigor" requires that calculus be treated deductively, starting from the real-number axioms. To be in step with the times and still strike a sensible balance between rigor and intuition, the axioms and discussion of the real-number system are given in a starred supplement, as are also proofs of the harder function-theoretic theorems. The author is lenient about using a needed theorem long before its proof, and treats some things, notably trigonometry, in a frankly intuitive way. In short, a deductive theory based on the axioms is indicated and endorsed, but is not allowed to push the ideas out of a pedagogically natural arrangement. Rather, the organization of the book is governed by a shrewd sense of the important applications of calculus. Differential equations, coming as early as Chapter 4, is the first goal, for which the initial chapters on integral calculus, differential calculus, and the elementary transcendental functions are preparation. Then two chapters on analytic geometry and vectors lead up to the solution of the vector differential equation of planetary motion. The rest of the book builds from the Mean Value Theorem through its generalizations, and applications to extremum problems and indeterminate forms, winding up with infinite series, power series, and differential equations again. Like a concerto! A pretty detail is a collection of six questions on page 1, each of which reappears later as an exercise; each, that is, except one involving integration, whose reincarnation the reviewer could not find.

The author's way into the subject at the beginning is interesting and perhaps controversial. Like Courant he treats the definite integral first, before mentioning derivatives, but unlike Courant he uses the definite integral as a vehicle for emphasizing various formal and preliminary techniques and ideas (function and summation notations, inequalities, mathematical induction, etc.) and delaying the central ideas of calculus, namely limits and continuity. He does this by exploiting two observations: 1) that the definite integral, if defined by upper and lower approximations, does not explicitly use the limit idea, and 2) that piecewise monotonicity can substitute for continuity as a sufficient condition for existence of the integral. The result is a rather large first chapter (98 pages, of

which 24 are starred) containing much mathematics, if not much calculus. The second chapter, Differential Calculus, but perhaps better titled simply "Calculus," complements the first, containing as it does integration in the sense of finding primitives, the fundamental theorem for evaluating definite integrals, and in a starred supplement a thorough discussion of continuity, including integrability of continuous functions.

All this is good. Still, the reviewer would favor even more emphasis on applications, and a lighter touch on some questions of rigor. While the historical remarks often mention physical problems as motivation, the applicability of the ensuing theory to these problems is not consistently enough followed up. Purely mathematical exercises predominate, and the art of numerical computation, which has pedagogic as well as practical value, is rather neglected. The definition of area (first for ordinate sets, then for more general sets by decomposition) is so awkward, the reviewer would rather see area left undefined, but governed by axioms such as additivity, positivity and invariance.

One feels that the publisher has allowed the author as much space as he wanted, with happy results. Local faults of workmanship are few, and arise from mannerisms and errors of judgment rather than carelessness. Here are some: The wording most frequently used to define a function $f(x)$ for all x in an interval is "choose an x satisfying $a<x<b$ and let $f(x)=\cdots$". The graphs given on page 76 to familiarize the reader with the trigonometric functions are skimpy and inaccurate and lack vertical scales, which would be different from the horizontal scales. On page 101 we read about the highest point of a projectile's free flight, "At the midpoint of the motion the influence of gravity just balances the upward forces, and the projectile is momentarily at rest." In an exercise on page 104 "f" is confusingly used for two different functions. A footnote on page 346 about hyperbolic and parabolic orbits seems to suggest that the planets could have originated by entering the solar system from somewhere with insufficient velocity to escape again.

F. CUNNINGHAM, JR.
Bryn Mawr College

Lectures on Differential and Integral Equations. By Kôsaku Yosida. Interscience, New York, 1960. ix+220 pp. $7.00.

This excellent little book, a translation from the Japanese, is essentially a self-contained introduction to the theory of ordinary differential equations and integral equations. Since both of these subjects are considered, and since the author chooses to give proofs of many of the theorems which he uses, the material covered is not as extensive as that usually found in textbooks on these subjects individually. In particular, applications are not dealt with, except insofar as one may regard examples involving the Legendre, Bessel, Laguerre, and Hermite functions as such. The author's pace is steady, however, and rather brisk, so that he covers sufficient material for a viable introductory survey of his chosen topics.

The treatment is deliberately elementary in the sense that the Lebesgue integral is not used nor is the familiar panoply of linear operator theory referred to. The discussion of integral equations thus lacks the generality and elegance which is found in modern treatises such as Reisz-Nagy, for example. The book is perhaps most suitable for the engineer and/or physicist with a thorough grounding in classical analysis, who wishes to obtain a survey of the elements of the two disciplines which are his principal tools. It is not, however, a suitable tool for the *solving* of boundary value problems.

The contents are approximately as follows: Chapter 1 is on initial value problems. Existence and uniqueness, approximate solution, perturbation theory, the linear equation of order n, and the second order equation of Fuchs type are taken up. Chapter 2 considers the Sturm-Liouville boundary value problem for the second order equation. The Green's function, reduction to integral equations, Hilbert-Schmidt theory for symmetric kernels, Hermite, Laguerre, and Legendre polynomials, and asymptotic expressions for eigenvalues and eigenfunctions are discussed. In Chapter 3 Fredholm integral equations are treated; the Fredholm alternative and the Schmidt and Mercer expansion theorems are discussed. Chapter 4 has to do with Volterra equations of first and second kinds. Chapter 5, which is the heart of the book, has to do with the proof of the Weyl-Stone expansion theorem for the Sturm-Liouville problem, and more particularly with the development of the Titchmarsh-Kodaira formula for the density matrix. Illustrative examples are: the Fourier, Hermite, and Fourier-Bessel expansions. Chapter 6 has merely a few remarks on nonlinear integral equations.

There are no problems. A short bibliography is included and *en passant* references are sometimes made to more general or extensive treatments.

WILLIAM H. PELL,
National Bureau of Standards

Error-Correcting Codes. By W. W. Peterson. M.I.T. Press and Wiley (jointly), New York, 1961. 285 pp. $7.75.

An error-correcting or detecting code in this book is, roughly, a subset or subgroup of a vector space or algebra of n-tuples, where errors made in a code vector are correctable or detectable if no more than a given number of the coordinates of the code vector have been altered. Practical implementation through linear switching circuits, systematic techniques of this type of coding and decoding, burst error-correcting and detecting, and parity-checking are discussed. The approach is unified within abstract algebra, with mathematical concepts essential to this type of coding from groups through Galois fields clearly presented.

Anyone interested in Marshall Stone's article "The Revolution in Mathematics" (This MONTHLY, vol. 68, 1961, pp. 715–734) should take a good look at this book. *Error-Correcting Codes* is written by an engineer for engineers, but could be appropriately entitled "Applied Abstract Algebra."

J. H. ABBOTT
University of New Mexico

Set Theory, The Structure of Arithmetic. By Norman T. Hamilton and Joseph Landin. Allyn and Bacon, Inc., Boston, 1961. xi+264 pp. $7.75.

Directors of Academic Year Institutes for high school mathematics teachers have combed the book market in search of a suitable text for an introductory course in modern mathematics. General dissatisfaction with available standard material first led to a series of brochures, pamphlets, etc. on isolated topics in modern mathematics such as Set Theory, Number Bases, Symbolic Logic, etc. The present text (the first of three volumes) is a more detailed introduction to modern mathematics specifically for high school teachers and is a result of 9 semesters of experimental work with a series of lecture notes tried with teachers in several Academic Year Institutes and mathematics teachers in the School of Education at the University of Illinois.

The book contains five chapters: Elements of the Theory of Sets, Natural Numbers, Integers and Rational Numbers, Real Numbers (two chapters). The chapter on the theory of sets includes a very brief (in some cases too brief) treatment of many of the basic concepts and definitions of sets and set terminology, Venn diagrams, symbolic logic, variables, Cartesian sets, relations and functions. This chapter would require additional reading by the student or considerable expansion by the lecturer in the class room. The chapter on Natural Numbers is well done and perhaps will appeal most to high school teachers. An alternate definition of finite and infinite sets in terms of subsets should have been stated in this chapter. The remaining chapters on Integers and especially on the Real Numbers are well organized but at present would prove somewhat difficult for the general high school mathematics teacher. The proofs are stated symbolically, briefly, and with very little explanation. This objection is not too serious since more and more teachers are coming to Institutes with better preparation (from Summer and In-Service Institutes).

There are only minor omissions in a few statements of theorems. The latter are rather obvious and can be supplied by the reader.

As far as the reviewer is aware, this is the first major attempt to produce a text that is suitable for the retraining of high school teachers in the elements of modern mathematics at Academic Year Institutes by writers who know the particular problems of such Institutes. The text has benefited from the experience of the writers with teachers in and out of the class room. This experience is reflected in many sections which emphasize for the teacher the sources of difficulties and common mistakes in interpretation of definitions and theorems. It is doubtful that at present all directors of Academic Year Institutes will have the caliber of teachers who can readily master the content of the text in one semester (four hours per week). Those fortunate to have the right class will find the book an excellent text for the teachers. Others, by judiciously selecting sections from the text, will find the treatment better than that currently available in the pamphlet and brochure line.

STANLEY J. BEZUSZKA, S.J.
Boston College

Analytical Elements of Mechanics. Volume 2. Dynamics. By Thomas R. Kane. Academic Press, New York 1961. vii+337 pp. $6.25.

New textbooks in recognized courses often seem to be merely modifications of their predecessors. This book, however, is novel in organization, in emphasis, and particularly in tone. It is a deliberately austere, tightly constructed, treatment of dynamics for seniors or beginning graduate students. Descriptive and intuitive materials are left for the lecturer. The book covers methods in dynamics associated with the names of d'Alembert and Newton but not Lagrange and Hamilton. The reader is assumed to be familiar with *Volume 1, Statics.* The reviewer was unable to qualify in this respect.

One is continually aware of the structure of this book. There are two hundred sections ranging in length from a single sentence to a dozen pages. Titles for these sections appear in the *Contents* but not in the text proper. The section numbers are used repeatedly for cross references. In proofs, for example, an equals sign is often embellished by one or more section numbers in justification of the equality. The sections are collected into twenty-eight supersections which in turn constitute the parts of the four chapters.

Chapter 1 treats differentiation of vectors, emphasizing from the start the dependence of derivatives upon reference frames. The intricate nomenclature indicates the reference frame by a left superscript (usually R for rectangle or rigid body). Right superscripts denote the vector or body or particle under consideration. Such explicitness seems at first to clutter proofs. For example the proof of the Serret-Frenet formulas contains eighteen cross-references while superscript R appears forty-two times emphasizing the fact that only one reference frame is involved. As one becomes accustomed to these trimmings one tends to ignore them just as one ignores billboards beside a highway, but sometimes the information is useful. In Chapter 1 a vector form of Taylor's Theorem is developed and employed as a principal tool for deriving formulas concerning the osculating plane.

Chapter 2, "Kinematics," systematically approaches and applies the idea of angular velocity. In the sections concerned with relative accelerations the advantages of an explicit notation are apparent. In Chapter 3, "Second Moments," a rather general study of radius of gyration is given. The subject is approached by vector second moments rather than by scalar moments and products of inertia. This is particularly advantageous for handling principal directions without tensors.

The last and longest chapter is entitled "Laws of Motion." Inertia forces and inertia couples are introduced immediately. A form of d'Alembert's principle is the basic tool. The derived proposition relating force and time-derivative of linear momentum appears considerably later. The dependence of such principles on the nature of the reference frame is emphasized. The concepts of work and potential energy do not occur, although the idea of power (under the alias *activity*) is used along with kinetic energy. The conservation principles for linear momentum and for angular momentum are deduced.

A generous number of worked examples add greatly to the value of the book. Many of the examples concern useful engineering situations: linkages, mechanisms, satellites, gyroscopes, etc. At the end of the book are over a hundred problems, with answers, arranged in a dozen problem sets. These problems require systematic analysis and should serve to demonstrate the effectiveness of the uncompromising program followed in this text.

The typography generally is good although better matching between text type and figure type would be desirable. The figures, mostly line drawings, are clear but not in the slick style to which teachers of mathematics and mechanics are increasingly becoming accustomed.

A textbook so singular will not be adopted casually. A teacher eager to follow diligently the path laid out with such dogmatic care will probably find the book to be an effective instrument for imparting the analytical habits essential to engineering mechanics.

DAN E. CHRISTIE
Bowdoin College

The Second Scientific American Book of Mathematical Puzzles and Diversions. By Martin Gardner, Simon and Schuster, New York, 1961. 253 pp. $3.95.

This is a second volume of selections of Mr. Gardner's contributions to the Scientific American. Those who have followed these contributions for years will welcome this selection containing further elaborations, commentaries and developments on the subjects presented in this book. A major portion of these vignettes represents material off-the-beaten-paths of subject matter in the field of mathematical recreations. One might be so bold as to suspect and then suggest that here (including the first book on the same subject by the same author) we are confronted with a leap into the field of mathematical recreations starting from where the classical authorities have stopped.

The twenty chapters of this book cover and treat topics ranging from the five Platonic Solids to the Golden Ratio, from excursions into logical twists and convolutions to Magic Squares, from topological curios to numerical and computational curiosities. The material is well developed and presented by the author while he skillfully skirts the pitfalls of getting involved in complicated and complex mathematical excursions. The purists will frown upon these omissions, while the adherents of the more popular approach might deplore the insufficiency of more explanations. These are the days of mathematical confusion when an author is damned when he does not do something and he is no less damned when he does it. The author should be commended, however, for creating a pleasant diversion from infantile Bourbaki'itis or, perhaps, senile Bourbaki'osis.

The book would be even more useful if it contained a well organized and annotated bibliography; the author is well known for his erudition in his field of endeavor. An index would also be welcome. Many of the answers would be more effective if they were treated in a more systematic manner. The mathematically trained reader will succumb to this book as the starting point of a

series of intriguing explorations, while the no less fortunate nonmathematically instructed reader will become enthralled by the interesting subjects and their treatment.

Paraphrasing Robert Schumann: Gentlemen! A masterpiece!

AARON BAKST
New York University

Probability and Experimental Errors in Science. By Lyman G. Parratt. Wiley, New York, 1961. 255 pp. $6.00.

The author's stated objective is "to introduce the undergraduate student to the unifying concepts of probability and statistics as they apply in science," the central theme of the book being that "no scientific knowledge is complete, no measurement exact." The process of constructing a probability model which idealizes a physical experiment, comparing theoretical conclusions with actual measurements, consequently modifying the model and then repeating the cycle, is stressed throughout. The style is clear and concise, yet gives one the impression of participating in a give-and-take discussion. Numerous footnotes add to this illusion by providing interesting sidelights and background material.

The first three chapters contain the elements of classical probability theory and some statistical methods such as maximum likelihood estimation, curve fitting, significance tests, and a discussion of the propagation of errors. The last two chapters discuss the normal and Poisson probability models and their use in the analysis of errors, illustrating with problems drawn from a variety of situations. In presenting this material, although many derivations are given, the point of view is that of providing the science student with a working knowledge of probability and statistical methods, based on an understanding of the general principles involved and an awareness of the suitability or unsuitability of a model to a particular situation.

The generous number of problems included in each chapter provide reasonable and interesting settings in which the student can secure practice in handling the methods treated in the text; several offer opportunity for group discussion.

This is a very readable book and merits the attention of students of probability theory as well as the group for which it was written.

GRACE E. BATES,
Mount Holyoke College

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Mr. G. E. Abrahms, Michigan State University has accepted a position as Research Engineer-Statistician with the Lockheed Aircraft Company, Burbank, California.

Mr. R. F. Appel, Dime Savings Bank, Brooklyn, New York, has accepted a position as Programmer with the Shell Oil Company, New York, New York.

Associate Professor Emeritus Ralph Beatley, Harvard University, has been appointed part-time Professor at Boston University.

Mr. W. R. Becker, Service Bureau Corporation, New York, New York, has accepted a position with the Atomic Energy Commission Computing Center, New York, New York.

Dr. R. K. Brown, U. S. Army Research and Development Laboratories, Red Bank, New Jersey, has accepted a position as a Member of the Technical Staff of the David Sarnoff Research Center, Radio Corporation of America, Princeton, New Jersey.

Mrs. Dorothy L. Chesnut, Waco Public Schools, Waco, Texas, has been appointed Instructor at Texas Wesleyan College.

Mr. C. L. Conner, Radio Corporation of America, Moorestown, New Jersey, has accepted a position with the Planning Research Corporation, Washington, D. C.

Miss Rita Dy, University of Michigan, has been appointed Instructor at Mary Manse College.

Mr. R. P. Everett, University of California, Berkeley, has accepted a position as Engineer with the Lockheed Missiles and Space Company, Sunnyvale, California.

Mr. R. T. Fallon, Michigan State University, has accepted a position in the United Engineering Center of the Junior Engineering Technical Society, New York, New York.

Mr. D. F. Flanagan, Jr., Newfield High School, Centereach, New York, has been appointed a Teacher at the James Wilson Young High School, Bayport, New York.

Assistant Professor J. W. Frankowsky, Lincoln University, has been promoted to Associate Professor.

Mr. M. L. Glasser, Carnegie Institute of Technology, has accepted a position with the Battelle Memorial Institute, Columbus, Ohio.

Dr. P. W. Healy, Aerojet-General Nucleonics, San Ramon, California, has accepted a position as Head of the Scientific Systems and Programming Branch of the Quality Evaluation Laboratory, U. S. Naval Ammunition Depot, Concord, California.

Mr. B. R. Henry, University of Wisconsin, has accepted a position as Engineer in Navigation Development at the General Electric Company, Pittsfield, Massachusetts.

Mr. P. G. Heyda, de Havilland Aircraft Company, Ltd., Hatfield, England, has accepted a position with British Insulated Callenders Cables, Ltd., London, England.

Mr. David Horwitz, Armour Research Foundation, Chicago, Illinois, has accepted a position as a Member of the Technical Staff in the Materials Technology Department of the Hughes Aircraft Company, Culver City, California.

Mr. L. L. Israel, Worcester Polytechnic Institute, has accepted a position as Associate Engineer with the Minneapolis-Honeywell Electronic Data Products Division, Newton, Massachusetts.

Professor Irving Kaplansky, Acting Chairman of the Department of Mathematics at the University of Chicago, has been appointed Chairman.

Mr. L. A. Kenna, Radio Corporation of America, Tucson, Arizona, has accepted a position as Mathematical Supervisor of the Avionics Department of the Electronic Proving Ground, Fort Huachuca, Arizona.

Mr. L. I. Larson, Douglas Aircraft Company, Santa Monica, California, has been promoted to Computing Engineer in the Missiles and Space Systems Division.

Mr. William Lindorfer, Grumman Aircraft Engineering Corporation, Bethpage, New York, has accepted a position with the Astro-Electronics Division of the Radio Corporation of America, Princeton, New Jersey.

Mr. J. L. Linnstaedter, Agricultural and Mechanical College of Texas, has been appointed Instructor at Northeast Louisiana State College.

Mr. B. K. McGuire, University of Missouri, has accepted a position as Mathematical Analyst in the Florida Research and Development Division of Pratt and Whitney Aircraft, West Palm Beach, Florida.

Assistant Professor J. H. Manheim, Montclair State College, has been appointed Associate Professor at Wagner College.

Mr. G. J. Marks, General Electric Company, Evandale, Ohio, and the University of Cincinnati, has accepted a position as Propulsion Design Specialist in the Aerospace Division of the Martin-Marietta Corporation, Orlando, Florida, and has been appointed Instructor at Rollins College.

Dr. J. A. Meier, Michigan State University, has accepted a position as Section Head of the Applied Mathematics Department in the Research and Development Center of the Armstrong Cork Company, Lancaster, Pennsylvania.

Mr. F. F. Morrison, Wesleyan University, has accepted a position as Astronomer with the Army Map Service, Washington, D. C.

President C. V. Newsom, New York University, has accepted a position as Senior Vice-President of Prentice-Hall, Englewood Cliffs, New Jersey.

Professor Rufus Oldenburger of the School of Mechanical Engineering, Purdue University, received the Automatic Control Division Award for 1961 from the American Society of Mechanical Engineers.

Mr. E. D. O'Neil, State University of New York, College of Education at Albany, has been appointed Teacher at Linton High School, Schenectady, New York.

Mr. J. L. Peterson, Case Institute of Technology, has accepted a position with Technical Operations Inc.-OMEGA, Washington, D. C.

Mr. Eugene Price, Wayne State University, has accepted a position as Scientist with the Edgerton, Germeshausen, and Grier Corporation, Santa Barbara, California.

Mr. Arthur Reetz, Convair, Fort Worth, Texas, has accepted a position with the General Dynamics Corporation, San Diego, California.

Associate Professor B. E. Rhoades, Lafayette College, has been promoted to Professor.

Mr. J. A. Roth, Bradley University, has accepted a position as Numerical Analyst in the Manned Spacecraft Center of the National Aeronautics and Space Administration, Houston, Texas.

Mr. D. M. Saxe, Bausch and Lomb Optical Company, Rochester, New York, has accepted a position in the Digital Computer Programming Department of Aircraft Armaments, Inc., Cockeysville, Maryland.

Dr. E. M. Scheuer, Space Technology Laboratories, Los Angeles, California, has accepted a position as Mathematician with The Rand Corporation, Santa Monica, California.

Mr. F. G. Stockton, Shell Development Company, Emeryville, California, has been promoted to Supervisor.

Assistant Professor Betty Thomas, Wesleyan College, has been appointed Associate Professor at Erskine College.

Mr. N. W. Torgerson, Montrose Chemical Company, Torrance, California, has accepted a position as Junior Civil Engineer, Division of Highways, State of California, Department of Public Works, San Luis Obispo, California.

Miss Shirley Weihe, San Diego State College, has accepted a position as Programmer with General Dynamics-Astronautics, San Diego, California.

Mrs. Mary K. Tulock, U. S. Office of Education, Washington, D. C., died December 31, 1961. She was a member of the Association for six years.

Assistant Professor Emeritus A. R. Williams, University of California, Berkeley, died December 6, 1961. He was a Charter Member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-fourth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Davis, January 13, 1962. Professor D. W. Blakeslee, Chairman of the Section, presided at the general sessions. Professor Stanley Hughart, Program Chairman of the Section, and Professor Charles Hayes, Chairman of the Mathematics Department of the host institution, presided at other sessions on teaching mathematics and on research. There were 160 persons registered in attendance including 110 members of the Association.

At the business meeting Professor Max Kramer, San Jose State College, was elected Vice-Chairman of the section. The Chairman for next year is Professor G. E. Latta, Stanford University; the Secretary-Treasurer is Professor B. J. Lockhart, U. S. Naval Postgraduate School; the Program Chairman is Professor D. W. Blakeslee, San Francisco State College.

A panel presentation entitled *The panels of CUPM* was given by Professor R. J. Wisner, Michigan State University Oakland, Executive Director of the Committee on the Undergraduate Program in Mathematics and Professor Murray Protter, University of California, Berkeley. Two additional invited addresses were given. Professor P. W. Berg, Stanford University spoke on *The impact of SMSG on the colleges*, and Professor Jacob Korevaar, University of Wisconsin and Stanford University spoke on *What should be true, but isn't—how distribution calculus helps.*

The following additional papers were presented:

1. *Cylindrical projections and double integrals*, by Professor John Hwang, Sacramento State College.
2. *Area bounded by a simple closed curve*, by Professor A. E. Labarre, Fresno State College.
3. *Hexagonal geometry*, by Professor Verner Hoggatt, San Jose State College.
4. *Directed distance geometry*, by Professor Curtis Fulton, University of California, Davis.
5. *Extremum problems associated with triangles*, by Professor Lester Lange, San Jose State College.
6. *Some mathematics the children did not learn*, by Professor Colbert Purvis, Alameda State College.
7. *Some innovations in mathematical education in British universities due to the impact of digital computers*, by Professor D. G. Williams, University of Glasgow and U. S. Naval Postgraduate School.
8. *The bottleneck in the preparation of mathematics teachers—a way out*, by Professor Stephen Bryant, University of California, Berkeley.

9. *Bounds for pairs of quadratic and cubic residues*, by Professor Marguerite Dunton, Sacramento State College.
10. *A method of matrix inversion*, by Professor Thomas Kipps, Fresno State College.
11. *A simple interpretation of a degenerate Euler equation arising in the ship routing problem*, by Professor Frank Faulkner, U. S. Naval Postgraduate School.
12. *Chebyshev type inequalities for distributions with monotone hazard rate*, by Professor R. E. Barlow, San Jose State College.

B. J. LOCKHART, *Secretary*

FILM BY HEWITT

The Association now has available for loan and for sale the film: "What Is An Integral?" by Edwin Hewitt.

This is a kinescope in two reels with sound. The total running time is one hour. The film is particularly suitable for college juniors and seniors and beginning graduate students.

If you wish to borrow this film for showing at your institution, write to the Buffalo office of the Association. The only cost is return postage. The film may also be purchased from the Association.

VISITING LECTURERS PROGRAM FOR 1962–1963

The annual announcement describing the Visiting Lecturers Program will not be available in May as has been the custom in the past, but will be printed and distributed in September. It will be sent automatically to all college and university department chairmen. Others who wish to have a copy should write to the Buffalo office of the Association.

CALENDAR OF FUTURE MEETINGS

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 16–17, 1963.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.
NORTHEASTERN, November 24, 1962.
NORTHERN CALIFORNIA, University of California, Berkeley, January 1963.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.
PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

MANUALS BY HENKIN AND McSHANE

FM #1, MATHEMATICAL INDUCTION, by Leon Henkin

FM #2, THEORY OF LIMITS, by E. J. McShane

These books were originally prepared as supplements to the films of the same name sponsored by the M.A.A. Committee on Production of Films. They contain complete discussions of the topics listed, approximately as spoken in the films together with supplementary material to amplify the treatment of the subject and a number of problems.

Copies may be purchased at $1.00 each or $.50 per copy for orders of ten or more. Orders accompanied by payment should be sent to:

Harry M. Gehman, *Executive Director*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

MAA STUDIES IN MATHEMATICS

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Each member of the Association may purchase one copy of each volume of the Studies. The price of Volume 1 is $2.00. Orders with remittance should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies and copies for non-members may be purchased at $4.00 from Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

THE MATHEMATICAL ASSOCIATION OF AMERICA

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The Association is a national organization of persons interested in mathematics at the college level. It was organized at Columbus, Ohio, in December 1915 with 1045 individual charter members and was incorporated in the State of Illinois on September 8, 1920. Its present membership is over 12,000, including more than 500 members residing in foreign countries.

Any person interested in the field of mathematics is eligible for election to membership. Annual dues of $5.00 includes a subscription to the American Mathematical Monthly. Members are also entitled to reduced rates for purchases of the Carus Mathematical Monographs and the MAA Studies, and for subscriptions to Mathematics Magazine.

Further information about the Association, its publications and its activities may be obtained by writing to:

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On behalf of the Mathematical Association of America and other mathematical organizations, the American Mathematical Society is assembling and maintaining a register of mathematicians, statisticians, and mathematical scientists. The mathematics register is a section of the National Register of Scientific and Technical Personnel, which is an official responsibility of the National Science Foundation.

The main objective of the Register is to provide up-to-date information on the scientific manpower resources of the United States. It is also increasingly valuable to our profession as a source of statistical information.

At the request of the National Science Foundation, the American Mathematical Society has mailed the 1962 reporting forms to the mathematics section of the National Register. We are counting again on the splendid cooperation accorded to the previous cycle of the Register by mathematicians and mathematical scientists.

If you have received a National Register questionnaire, please fill it in now and send it to the Headquarters Offices of the American Mathematical Society at 190 Hope Street, Providence 6, Rhode Island.

If you have never received a questionnaire and feel that you are qualified for inclusion in the Register, please send a note to that effect to the above address.

Harry M. Gehman, Executive Director
Mathematical Association of America



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THE CALCULUS of VARIATIONS



N. I. AKHIEZER, *University of Kharkov*

Translated from the Russian by Aline H. Frink, *Pennsylvania State University*

This book presents a concise and lucid treatment of the calculus of variations, striking a nice balance between theory and application. Presented so that it can be covered in a one-semester course, the book includes necessary conditions for an extremum, various sufficient conditions, direct methods for establishing the existence of solutions of extremum problems and approximating these solutions practically, and also applications to problems in the physical sciences. The extensive collection of illustrative examples of the original text has been suplemented by the translator with lists of interesting problems. Finally, this book gives the scientist and engineer a full discussion of this very useful tool presented in a way that makes it easily accessible. The mathematician will enjoy the clarity and elegance of the development of the basic theory.

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Supplementary Topics and Exercises

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c. 128 pp., Summer 1962—Price to be announced

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By Fuchs and Levin, *Moscow Forestry Institute*

Each chapter contains many examples which serve as illustrations of conclusions, and as models for problems which the reader may encounter. (Volume I, by Fuchs and Shabat, will be published Summer 1962.)

Vol. II 286 pp., $7.00
Vol. I c. 300 pp., Spring 1962—probably $7.50

INTRODUCTION TO CALCULUS
By K. Kuratowski, *University of Warsaw*

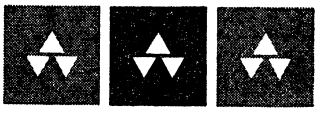
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315 pp., Spring 1962—probably $5.00

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By V. I. Smirnov, *U.S.S.R. Academy of Sciences*

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JUNE-JULY 1962

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THE NARROW MATHEMATICIAN

CARL B. ALLENDOERFER,* University of Washington

1. Introduction. My object today is to review some basic problems facing all of us in connection with the improvement of the teaching of mathematics in our schools and colleges. Since a title closely related to this purpose would have been rather dull in appearance I have chosen to speak about the "Narrow Mathematician" because I believe that narrowness in our approach to these matters is one of the greatest obstacles in the path of the reform movement. This narrowness is an understandable phenomenon, for mathematics and teaching are very complex subjects. When they are combined it is not surprising that individuals become specialists, each believing that he has isolated the key to the problem, and hence that disagreements and controversies arise. The sad part of the affair is that teachers at all levels who are new to the movement are likely to become confused and possibly to say "a plague on both your houses" and consequently justify their failure to become involved in what we are advocating.

The first steps in the revolution have now been firmly taken. Sample textbooks are available for secondary schools and the higher elementary grades. Outlines of curricula and certain courses are now available at the college level and modern textbooks are rapidly becoming available. Yet, we have hardly scratched the surface in getting these into the hands of the students in the country. Those of us who attend meetings like this may well get the idea that the battle has been won. But if you go out into the field and see what is going on in many school districts, or if you examine the knowledge of recent graduates of good colleges who are now entering graduate schools, you will be sadly disabused. The revolution has just begun in earnest, and before we push it farther it is time that we take a look and examine what we are really attempting to do. In doing so I hope that each of us can eliminate his own special breed of narrowness so that we can proceed together on a broad front.

If I appear to take pot shots at various prominent points of view which may have been expressed by friends of mine, I hope that I shall not lose their friendship; in any case I shall not identify them. As a matter of fact I, too, am guilty of my own narrowness and so far as possible am recanting in this talk. It was inevitable that our experiments have not all turned out as we had hoped, and an honest reexamination of our own positions would do each of us a lot of good. I hope that I may stimulate such a general reexamination.

2. The nature of mathematics. Since such a discussion must proceed from a set of premises, I had better state mine at the outset. I shall do so by presenting my own outline of the nature of mathematics as I see it. Later I shall examine how the reform movement is related to the various portions of this outline.

Let me begin by describing the structure of a mathematical theory in a mature form; later I shall discuss the differences between this picture and that of a theory in a stage of development. (Fig. 1) The process begins with "Nature,"

* Retiring Presidential Address, January 26, 1962.

a word which I wish to use in the broadest possible sense. Under this heading I shall include the physical and biological aspects of nature as well as human disciplines such as economics, psychology, anthropology, business, and warfare. The scholars in these various fields have initially gone to very great trouble to describe their subjects as best they can in words, and as their scholarship matures they begin to investigate the quantitative aspects of their domains. Then they join with mathematicians to build what is currently called a "mathematical model" of their portion of nature. This model is incomplete in many ways, for first it contains only the quantitative and geometric aspects of nature; and moreover like all models it is only an imperfect copy and not the real thing. Extent models differ in their degree of excellence; they are remarkably exact for the physical sciences, but are just emerging in hesitant form in the social sciences. But whatever the value of the model, it is the beginning of a mathematical theory.

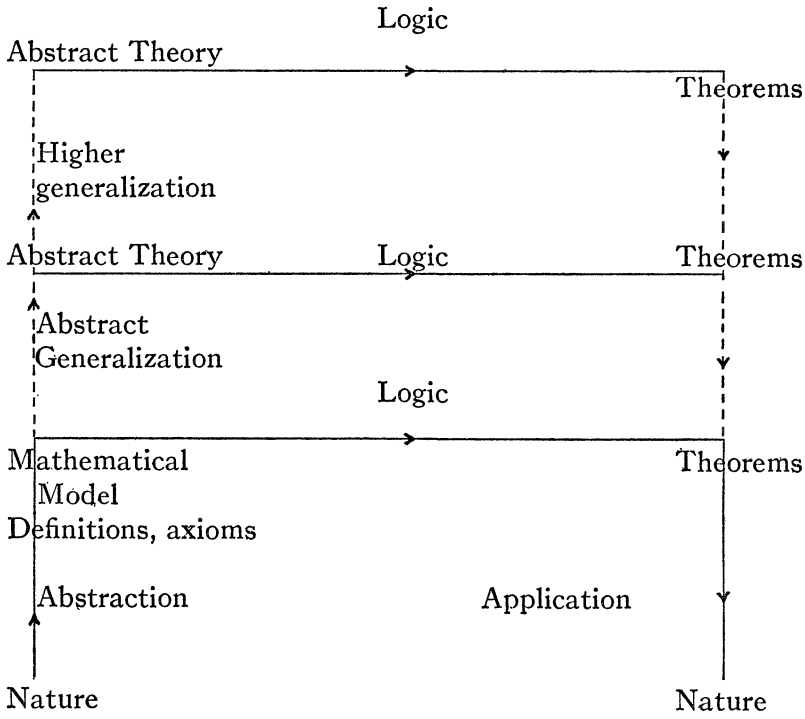


FIG. 1

In a mature mathematical subject such as we are discussing, these models must meet rigid criteria. They must contain undefined terms, definitions, and axioms and must meet the established standards of consistency, completeness, and elegance. As you well know, these standards have changed from century to century so that models such as Euclidean Geometry have required periodic

restudy and reformulation. Nevertheless the mathematics of any particular generation must be built upon a model which meets the current standards rather than upon a model constructed in a by-gone age.

On the basis of such an abstract model, the subject is now developed through the proof of theorems. These proofs require the use of logic which again must meet contemporary standards of rigor. Finally the theorems are applied back to nature as educated guesses of relationships which finally need to be verified experimentally.

This description of mathematics would have been reasonably accurate up until one hundred years ago, but it omits the most significant development in our subject during the past century. I am referring to the consequences of the discovery of non-Euclidean Geometry, which was one of the major "breakthroughs" in the history of mathematics. As you well know, this showed us that we can build a perfectly sound mathematical theory upon a model which is not derived from an examination of nature, and indeed upon one which may seem to contradict nature. And so now we have a very large portion of contemporary mathematics which is built upon abstract foundations having no apparent motivation as models of nature and having in turn no applications to nature. I fear, however, that this is something of an overstatement. In each case with which I am aware, these higher abstract systems have been developed as generalizations or mutations of systems that were invented as models of nature. Our geometry has gone from three dimensions, to n dimensions, to abstract spaces of great variety; we have investigated all the mutations of the axioms of elementary algebra; and there is a continuous line of development from elementary calculus to harmonic analysis on semi-groups. I believe that the most abstract modern mathematics is more "nature-bound" than its advocates would care to admit, and that by the same process in reverse it will find its applications to nature.

This outline of mathematics, however, is accurate only for a mature theory which has essentially been embalmed in textbooks, or in our own jargon is "classical." Surely, this is not the way in which it was invented, for we have not followed through these three basic steps in order. Indeed, we usually perform the magical feat of building the fifth story of our structure without first establishing a foundation or even thinking about the first four stories. We start at both ends and the middle and only after decades of effort bring forth the beautiful, logical and polished gems which we dangle before our students. It is no wonder that students are baffled by the very idea of doing research in mathematics.

Let me then describe briefly the process of mathematical discovery. (Fig. 2) Beginning with nature, as before, we seek to find as many relationships within it as we can. If we can systematize these we do so, but a lack of organization of our material does not keep us from pushing forward. On the basis of what we have observed, we guess theorems and use these to derive other theorems. Immediately we rush to apply these back again to nature and proceed headlong if

our predictions are successful. Axioms, logic, and rigor are thrown to the winds, and we become intoxicated with our success and open to dreadful errors.

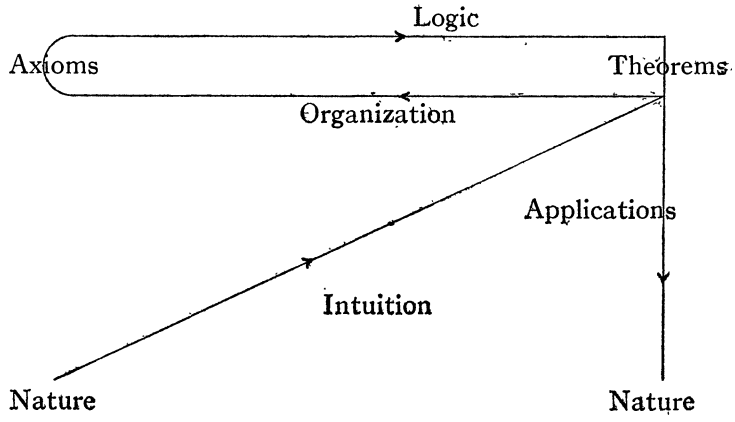


FIG. 2

This process is called "intuition" and its nature is a matter of the greatest conjecture in spite of the writings of several of our most distinguished colleagues. The successful unraveling of this process would be a major contribution to the understanding of the human mind. But, it is by this means, explained or not, that the great majority of mathematical theorems are first discovered. The products of this intuitive discovery are frequently wrong, usually unorganized, and always speculative. And so there follows the task of sorting them out, weaving them into a proper theory, and proving them on the basis of a set of axioms. It is at this stage that the mathematical model is likely to be constructed. The details of this process go on in our seminars and in our discussions in the corridors of meetings like this, but almost never appear in print. Hence the inner circle of creative mathematicians have the well-kept trade secret that in a great many cases theorems come first and axioms second. This process of justifying a belief by finding premises from which it can be deduced is shockingly similar to much reasoning in our daily lives, and it is somewhat embarrassing to me to realize that mathematicians are experts at this art.

3. The weakness of the traditional approach. Let us now turn to the teaching of mathematics and see how its problems can be related to the outline just presented. My chief criticism of the traditional textbooks is that they are too strongly oriented toward the routine aspects of mathematics. Theorems and rules of operation are stated with only the slightest amount of motivation or theoretical justification, and students are expected to spend their time learning how to follow these rules in a limited number of relatively obsolete areas of application. It is no wonder that mathematics seemed dull to so many of our brighter students. We have followed this path for so long that the ruts are deep

and a veritable fleet of tow trucks is needed to lift us all out of the mire.

4. Intuition. As I turn now to the reform movement in the teaching of mathematics, let me first discuss intuition. It is here that the learning process must begin, for in some sense the student must follow the path by means of which mathematics was developed in the first place. We are very familiar with the argument that the content of a good course should be chosen with the aid of a research mathematician, but it is not so commonly understood that our methods of teaching as well must be patterned on the personal experiences of those who themselves have created mathematics. We need to be more articulate about this in educational circles, and the text in our arguments should be "intuition."

One of the strongest tenets of reform movement has been that intuition be developed before rules of operation are formalized. We observe this in the recommendation that informal geometry be taught in the junior high school (or even at a lower level) as preparation for the formal treatment to follow in the high school. It has also been recommended that arithmetic be used as a motivation for algebra; that algebra be taught informally at the outset, but then presented as a mathematical system before the conclusion of a four-year high school course. I fear, however, that many of us have overlooked the application of this well-established principle when we begin to introduce calculus in the colleges. I note a deplorable tendency for the first course in calculus to be taught without a proper intuitive background, and suggest to those responsible for this course that they apply the same good judgment about intuition to their college teaching that they are recommending to high school teachers. Although we may believe that the so-called "discovery method" is an extreme variant of this trend which is applicable only in very special circumstances, we should all listen attentively to Professor Polya when he urges us "to learn by guessing."

5. Mathematical structures. As the products of our intuition become organized we then proceed to build the model, or as it is often called, a "mathematical structure." Instead of arbitrary rules and special tricks, we are now teaching general principles which are based on logical reasoning from well-constructed abstract models. Although this is not only good but essential, the formalistic approach to mathematics can be overdone. Let me speak of several of its excesses.

My first topic here is that of mathematical language and definitions. It is absolutely shocking to read some of the traditional textbooks where we find that "zero is not a number," that a "variable stands for a number which varies from problem to problem," or that a "function is an expression containing a variable." Modern writers have tightened up on this terminology but I fear that they are instituting a new kind of pedantry. Mathematics is full of "abuses of language", and we would be very foolish to try to eliminate many of these. We cannot consistently distinguish between a function and its value at x, even though we fully understand the difference between these related ideas; we cannot

always put quantifiers before every equation even though these are logically necessary; and we cannot handle two kinds of minus signs (one for negative numbers and one for subtraction) beyond the first few pages of an elementary algebra book.

The problem here, however, is deeper than this. In a number of quarters I find teachers who have identified the reform movement with a fancy language which only they can speak and who have missed all the important ideas in the new materials. The villain of this piece is the language of sets, and we might just as well get it on the table. I cannot teach mathematics without talking about collections of points, numbers, or the like, and there is no reason for not using the proper word for these collections, namely "sets". The issue before us is not *whether* we use the language of sets, but *when* we use it and when we develop the formal algebra of sets. Let us recognize two ridiculous extremes: the first is the young child who cannot add fractions properly but who can draw Venn diagrams all over the wall; and the other is the graduate student in mathematics who does not have a firm grasp of set theory. It is certainly appropriate to talk about sets and to use their simple properties, *where there is a pay-off*, in the junior high school and the early grades of the senior high school. It is equally clear that the algebra of sets must be understood before the student is introduced to probability theory or to the sophisticated portions of advanced calculus or modern algebra. Judgment is needed to settle the exact emphasis to be given to sets at any level, but too much too early or too little too late are equally to be deplored.

Even when a proper foundation has been laid I am seriously concerned about the replacement of the universally understood phrase "the solution of an equation" by its modern counterpart "the truth set of an open sentence" which is clear only to those on the inside of the magic circle. Have we really improved the clarity of the idea of a function when we define it as a "set of ordered pairs, etc." when we might with equal rigor define it in terms of its domain, range, and rule? I fear that in cases like these we are doing little more than confusing clarity with pedantry.

Another aspect of the subject of mathematical structures is the sets of axioms on which they are based. When one already understands almost everything about a subject, its axiomatic formulation is a thing of beauty and is essential for rigor. But to the novice axioms are frequently no help at all. Time and time again I have found groups of research mathematicians repelled by talks on topics outside their own specialty in which the speaker spent his time developing unmotivated foundations which apparently led to no new theorems. Are not our students similarly minded?

Axioms serve two purposes in high school and college mathematical education. The first is the statement of general principles which have wide applicability, like the distributive law, or which introduce general methods of procedure, such as the congruence axioms of geometry. The second purpose is teaching the deductive method and the true nature of mathematics. In neither of these

cases, however, is it necessary to present a complete and categorical set of axioms such as that for plane geometry, and indeed it is foolish to do so if we wish most of our students to stay with us. As a matter of consistency how can we college professors urge this kind of thing on high school teachers when we are content to teach calculus to young people who will never truly understand the real number system until they are graduate students, if ever?

It is my suggestion, therefore, that we avoid complicated axiom systems but choose from these the most important ideas and relationships and push these strongly in our teaching. We can also examine a simple axiom system like that of a group in considerable detail as an illustration of the deductive method. As the student's maturity grows, we can eventually merge these two approaches and begin to treat him like a mathematician, but this is not likely to occur until his later undergraduate years.

6. Logic. I find substantial disagreement among mathematicians regarding the importance of teaching elementary logic in a mathematics course. The objections are two-fold: first that one does not need to know logic in order to learn mathematics, for are we not all counterexamples to such a positive assertion; and second, that any logic which can be taught in a reasonable time is bound to be bad logic on the ground that essential difficulties must be slurred over. I am not personally convinced by these arguments and am willing to start a battle on this at any time. It is true that the elementary reasoning needed in mathematics can be learned without any formal training in logic if the learner is sufficiently intelligent (as all of you are), but repeatedly I find students who are completely baffled by contrapositives and indirect proof, who do not know how to form the negation of a statement, and who do not understand the proper use of counterexamples. In almost every case a few days instruction in simple logic has straightened them out completely. I grant that this approach to logic is imperfect and indeed sloppy, yet it serves the desired purpose and this is enough for me. The more sophisticated logic can best be put off until it can be used as a motivation for the algebra of sets and as an important application of this theory.

7. Applications. The last major feature of the reform movement is its attitude toward the applications of mathematics. Although there are some "ivory tower" mathematicians who cannot stand the mere thought of an application, almost all of us are interested in seeing our efforts pay off for the good of society. I was subjected to more classical applied mathematics than most of you when I worked through the undergraduate curriculum at Oxford University in England, and I can speak from first hand knowledge of its deficiencies. The so-called applications were actually very far removed from the real world of physics and had become pretty much a sophisticated intellectual game. Everyone knew it and in one student skit a mathematician proposed the problem of finding the terminal velocity of an infinitely smooth elephant of negligible weight who was supposedly sliding down a mountain. Although we have not gone to such extremes, the applied problems in our textbooks are unrealistic and for the

most part obsolete. It is again no wonder that students and faculty alike have shrunk from this kind of thing.

Insofar as possible the reform movement has tried to replace applications of this kind with relevant applications to matters of current importance. This is perhaps the most difficult aspect of our problem, and much more work needs to be done. We have been more successful in modern applications in the fields of social science and in statistics than in classical mechanics and we would do well to repair our deficiencies. For example, an excellent case can be made for the introduction of the trigonometric functions of a real number and their applications to alternating current, electromagnetic waves, radio frequencies, and amplitude and frequency modulation. Let us be as imaginative about these matters as we have been about the other areas of the movement. The applied mathematicians who are opposing the current trends would be in a sounder position if they diverted their energies from captious criticism to the development of good illustrations of applications which are suitable for young students. In any case, however, let it be said for the record that although there have been some widely publicized and purposely misleading statements to the contrary, no curriculum group in the country has advocated the elimination or even the soft-pedalling of manipulative or applied mathematics.

8. New customers for mathematics. Finally I should like to say something about the increasing interest in mathematics on the part of the social scientists and the biologists. I am still staggering from the shock of being informed that our College of Business Administration at the University of Washington has voted to require a year of solid mathematics, including calculus, for all their students. It is our obligation to provide these new students with material which is mathematically sound and which still springs from the intellectual content of their major field. Entirely new courses will have to be prepared, and there is much work ahead. I fear that some mathematicians may be inclined to tell these people to go away with their compound interest and sand-pile calculus and to leave us alone for more serious work. If so, we are again narrow in a different way, and we cannot afford to succumb to this luxury. By the same token we have obligations in the fields of biostatistics and mathematical biology, and we would be well advised to train young men who are competent in these fields so that we can meet our obligations there.

9. Conclusion. In summary I wish to emphasize that the main feature of the reform movement is its emancipation from the narrowness of the traditional curriculum. There has been much constructive work on the encouragement of intuition, on the organization of material into comprehensible mathematical structures, and on the modernization of the applications of mathematics. None of this is really new, for good teachers of every generation have emphasized these same ideas. Although mathematical ideas of the twentieth century have been incorporated into our courses, the term "The New Mathematics" is a very misleading description of the present approach. In essence what we are seeking

is sound teaching of each of the distinct, but closely related, aspects of mathematics which I have discussed in this paper. If I have been critical of some of the present trends, I do so as a supporter of the movement and not as its enemy. There is still much more work that needs to be done, and I hope that by these efforts we can bring about the extinction of the genus of "Narrow Mathematicians."

METRIC ENTROPY, WIDTHS, AND SUPERPOSITIONS OF FUNCTIONS*

G. G. LORENTZ, Syracuse University

1. Introduction. The purpose of this paper is to give an introduction to some recent developments connected with properties of compact sets of continuous functions. These developments, because of their importance on one hand and their simple and basic character on the other deserve to be more widely known. It is impossible in this paper to give complete proofs, but we will at least explain their main ideas. Our bibliography is not complete; it is restricted to papers most useful for the first reading and to a few papers of historical interest.

It is convenient to begin with the well-known theorems of D. Jackson and S. Bernstein (compare [1] and [9]). We wish to approximate continuous 2π-periodic functions $f(x)$, defined on the real line, by trigonometric polynomials of degree n

$$T_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx);$$

the degree of approximation is measured in the uniform norm

$$\|f - T_n\| = \max_x |f(x) - T_n(x)|.$$

We want this deviation of f from T_n to be as small as possible; an elementary fact is that there is always a polynomial $\tilde{T}_n$ of best approximation, for which $\|f - T_n\|$ attains its minimum. The quantity

$$E_n(f) = \min_{T_n} \|f - T_n\| = \|f - \tilde{T}_n\|$$

is called the degree of approximation of f (by the trigonometric polynomials).

Let $\Lambda(p+\alpha, M)$, where $p=0, 1, \cdots, 0<\alpha\leqq 1, M>0$ be the set of all

* Written with the support of the United States Air Force through the AFOSR, under Contract no. AF 49(638)-619. This paper is based on an invited address delivered at the Annual Meeting of the Mathematical Association of America in Washington, D. C., January 1961.

periodic continuous functions f, p times differentiable, and such that the pth derivative satisfies a Lipschitz condition of order α: $|f^{(p)}(x)-f^{(p)}(x')| \leqq M|x-x'|^{\alpha}$. Then Jackson's theorem asserts that if $f \in \Lambda(p+\alpha, M)$, then, as a reward for this smoothness, f has a small degree of approximation:

$$E_n(f) \leqq \text{Const. } n^{-(p+\alpha)}. \tag{1}$$

The theorem of Bernstein asserts that inversely, if (1) is satisfied for all n, then f is smooth: $f \in \Lambda(p+\alpha, M)$.

We shall look upon these theorems as dealing with classes of functions. For example, Bernstein's theorem can be explained by saying that the class $\Lambda(p+\alpha, M)$ has such a great thickness, width, massivity, that it is not possible to approximate all its functions too well by trigonometric polynomials of a given degree n. One will notice that this formulation carries with it the conjecture that the degree of approximation will not improve substantially if we replace the trigonometric polynomials by other means of approximation. And such is indeed the case.

The dealing with classes of functions rather than with individual functions is also justified, I am sure, from the point of view of the modern computer. He has often to compute many functions at once, functions whose properties he does not know well, or has no time to investigate. He will want to base his approximation methods on properties which are common to all functions of his class, without being interested to know that for some of them a better approximation exists.

2. Entropy and capacity.

2.1. Let A be a compact metric space with a metric ρ. The following definitions are well known. A finite set of points $x_1, \cdots, x_p$ of A is called an *ϵ-net in A* if $\epsilon>0$ and if for each $x \in A$ there is at least one point x_i of the net at a distance from x not exceeding ϵ: $\rho(x, x_i) \leqq \epsilon$. A family $U_1, \cdots, U_n$ of sets is an *ϵ-covering of A* if $A \subset \cup U_i$ and if the diameter of each set U_i does not exceed 2ϵ. It is convenient to take here 2ϵ and not ϵ in order to simplify the relation between the two definitions. In fact, if $x_1, \cdots, x_p$ is an ϵ-net for A, then the balls with centers x_i and radius ϵ form an ϵ-covering of A. A standard theorem of topology guarantees that a compact metric space A has a (finite) ϵ-net for each $\epsilon>0$. Hence A has also a finite ϵ-covering for each $\epsilon>0$.

Of course, the number n of sets U_i in a covering family depends on its choice, but the minimal value of n, $N_\epsilon(A)=\min n$ is an invariant of the set A which depends on $\epsilon>0$. The logarithm

$$H_\epsilon(A) = \log N_\epsilon(A) \tag{1}$$

is the entropy of the set A.* It was Kolmogorov's idea [11, 13] to characterize the "massiveness" of a set A by means of this function. We are interested, of

* There is a relation of analogy between $H_\epsilon(A)$ and the *probabilistic entropy* [7]. To underline the distinction, we propose to call $H_\epsilon(A)$ the *metric entropy* of A.

course, in the asymptotic behavior of $H_\epsilon(A)$ for $\epsilon \to 0$; in general, $H_\epsilon(A)$ will increase rapidly to infinity. We take logarithms in (1) partly because $N_\epsilon(A)$ is often unwieldily large.

Points $y_1, \cdots, y_m$ of A are called *ϵ-distinguishable* if the distance between any two of them exceeds ϵ. Again, their number m depends on the choice of the points, but the maximal value of m (which exists, as we shall see in a moment), $M_\epsilon(A) = \max m$ is an invariant of A. We put

$$C_\epsilon(A) = \log M_\epsilon(A); \tag{2}$$

this is the *capacity* of A.

The general theory of entropy and capacity is not rich; the main result is the following simple

THEOREM 1. *For each compact metric set A and each $\epsilon > 0$,*

$$C_{2\epsilon}(A) \leqq H_\epsilon(A) \leqq C_\epsilon(A). \tag{3}$$

The reasons for these inequalities are easy to see. Let $y_1, \cdots, y_m$ be a 2ϵ-distinguishable set of points of A, and let $U_1, \cdots, U_n$ be an ϵ-covering of A. Then $m \leqq n$, for otherwise at least one set U_i would contain two points y and consequently would have a diameter greater than 2ϵ. We have therefore $M_{2\epsilon}(A) \leqq N_\epsilon(A)$. (It follows also that $M_{2\epsilon}(A)$ is finite for each 2ϵ.) On the other hand, if $z_1, \cdots, z_{m'}$ is a maximal set of ϵ-distinguishable points (with $m' = M_\epsilon(A)$), then it is also an ϵ-net in A, for otherwise there would exist a point z in A with $\rho(z, z_i) > \epsilon$ for all i, and this would contradict the maximality of the set of the z_i's. As we know, there is then an ϵ-covering of A which consists of m' sets. Hence $N_\epsilon(A) \leqq M_\epsilon(A)$, and taking logarithms in the established relations, we obtain (3).

The exact determination of the entropy and the capacity of a set A is in most cases very difficult. Often one is satisfied to compute them only up to a strong or a weak equivalence; we write $f(\epsilon) \sim g(\epsilon)$ or $f(\epsilon) \approx g(\epsilon)$ for $\epsilon \to 0$ if $f(\epsilon)/g(\epsilon) \to 1$ or if respectively this quotient remains between two positive constants for $\epsilon \to 0$. In this connection Theorem 1 is used in the following way: one tries to find an upper bound for $H_\epsilon(A)$ (often also for the logarithm of the number of points of an ϵ-net in A) and a lower bound for $C_{2\epsilon}(A)$; if the bounds are close to each other, a good estimation of both entropy and capacity will result.

We shall discuss the most important sets A for which entropies are known (see [13, 18]). Of course, each time not only the set A, but also its metric must be given.

2.2. *s-dimensional spaces.* If A is a bounded closed set with interior points in the s-dimensional euclidean space R^s, then

$$N_\epsilon(A) \approx \epsilon^{-s}, \quad \text{hence} \quad H_\epsilon(A) = s \log \frac{1}{\epsilon} + \mathcal{O}(1). \tag{4}$$

If A has some regularity, for instance if it is an s-dimensional parallelopiped with measure $|A|$, then more exactly

$$N_\epsilon(A) \sim \nu_s |A| \epsilon^{-s}, \qquad M_\epsilon(A) \sim \mu_s |A| \epsilon^{-s} \tag{5}$$

with some constants ν_s, μ_s. All this is quite elementary. But the calculation of ν_s, μ_s for $s>2$ is a difficult (unsolved) problem of number theory; it is connected with the determination of the tightest packing of balls into R^s, and of the most economical covering of R^s by balls.

2.3. *Smooth functions.* Let B be an s-dimensional parallelopiped in R^s. A function $f(x)=f(x_1, \cdots, x_s)$ on B is smooth with the degree of smoothness $p+\alpha$, where $p=0, 1, \cdots, 0<\alpha\leqq 1$, if it has on B all partial derivatives of orders less than or equal to p. In addition, each partial derivative of order p, $D^p f(x_1, \cdots, x_s)$ must satisfy a Lipschitz condition with exponent α:

$$|D^p f(x_1, \cdots, x_s) - D^p f(x_1', \cdots, x_s')| \leqq M \max_{i=1,\ldots,s} |x_i - x_i'|^\alpha. \tag{6}$$

For example, if $p=0$, we obtain the class Lip α.

The metric will be determined by the uniform norm on B: $\|f\| = \max_{x\in B} |f(x)|$. But the classes just defined are not compact in this metric, and have no entropy. We consider therefore the smaller sets

$$\Lambda = \Lambda(p+\alpha, s, C_0, \cdots, C_p, M).$$

The set Λ consists of all functions f on B, smooth of degree $p+\alpha$, for which all kth derivatives $D^k f$, $k=0, \cdots, p$ satisfy $|D^k f| \leqq C_k$, while all pth derivatives satisfy in addition (6).

THEOREM 2 [13, 18]. *With the metric given by the uniform norm, one has*

$$H_\epsilon(\Lambda(p+\alpha, s, C_0, \cdots, C_p, M)) \approx \epsilon^{-s/(p+\alpha)} \tag{7}$$

the same is true for the capacity $C_\epsilon(\Lambda)$.

Consider first the case $s=1$, when B is a segment $[a, b]$. Let $\Delta>0$ be a small number. The points $x_1=a+\Delta, \cdots, x_n=a+n\Delta$, where n is so chosen that $a+n\Delta\leqq b<a+(n+1)\Delta$, form not only a Δ-net for B, but also a Δ-*chain*: any two consecutive points x_k, x_{k+1} are at a distance $\leqq\Delta$ from each other. The only computational device available for functions $f\in\Lambda$ is Taylor's formula; it is clear therefore that we must use it. This formula gives for instance the value $f(x_{k+1})$, if the values of the function f and of its derivatives at the preceding point x_k are known:

$$f(x_{k+1}) = f(x_k) + \frac{\Delta}{1} f'(x_k) + \cdots + \frac{\Delta^p}{p!} f^{(p)}(x_k) + \frac{\Delta^p}{p!}[f^{(p)}(\xi) - f^{(p)}(x_k)], \tag{8}$$

$$x_k < \xi < x_{k+1}.$$

Let us first try to estimate $H_\epsilon(\Lambda)$ from above; for this purpose we must find a fairly economical covering of Λ by sets U of diameter $\leqq 2\epsilon$. The formula (8) gives us the idea to let each U consist of functions $f\in\Lambda$ for which the derivatives $f^{(i)}(x_k), i=0, 1, \cdots, p; k=1, \cdots, n$, are given with an error not exceeding small fixed numbers $\epsilon_i>0$. More formally, each U will be given by the inequalities

$$m_{ik}\epsilon_i \leqq f^{(i)}(x_k) < (m_{ik}+1)\epsilon_i. \tag{9}$$

Thus each set of integers m_{ik} determines a set U; but some of the U may be empty.

How should we select the ϵ_i? Surely in such a way that if we substitute $m_{ik}\epsilon_i$ for $f^{(i)}(x_k)$ into (8), and zero for the last term, the errors of all terms would be approximately equal: it does not make sense to compute some terms of a sum very carefully if large errors are committed in others. This leads to the formula $\epsilon_i = C\epsilon\Delta^{-i}$. An easy calculation shows that the diameter of each U indeed does not exceed 2ϵ if one selects the constant C properly and puts $\Delta=\epsilon^{1/(p+\alpha)}$.

It remains to estimate the number N of sets U which contain functions from Λ. For each k, for instance for $k=1$, it is easy by means of $|f^{(i)}(x_1)| \leqq C_i$ to estimate the number of the possible integers m_{i1} from above; one obtains that the number of m_{i1} with a given i does not exceed $2C_i/\epsilon_i+2$. This gives us an upper bound for all possible combinations of the m_{i1}, $i=0, \cdots, p$ the number const. $\prod_{i=0}^{p}\epsilon_i^{-1}$. If done for each k, this would give an unsatisfactorily large upper bound for N. Instead, we do this only for $k=1$; for the following values of k, the smoothness of our function and Taylor's formula give a much more precise information about the number of possible integers m_{ik} once the $m_{i,k-1}$ are known. Thus one obtains $N\leqq C^n$, hence $H_\epsilon(\Lambda)\leqq$ const. $n \leqq$ const. Δ^{-1} $\leqq$ const. $\epsilon^{-1/(p+\alpha)}$. We do not discuss the estimation of $C_{2\epsilon}(A)$ from below: this is much simpler.

In this way we obtain (7) for $s=1$. For larger values of s, the computation remains the same; the only difference is that now a Δ-chain in B consists of $\approx\Delta^{-s}$ points.

2.4. *Functions of bounded variation.* The set V of all functions f of bounded variation on $[a, b]$, restricted by $|f(x)| \leqq 1$, $\operatorname{Var} f \leqq 1$ is not compact in the uniform norm. But if the distance $\rho(f_1, f_2)$ is defined as the Hausdorff distance [8, p. 166] between the curves $y=f_1(x)$, $y=f_2(x)$, G. F. Clements (Syracuse University) obtains $H_\epsilon(V)\approx\epsilon^{-1}$. If V_α is the intersection of V with $\operatorname{Lip}_1\alpha$, $0<\alpha<1$, then in the uniform norm, $H_\epsilon(V_\alpha)\approx\epsilon^{-1}\log 1/\epsilon$.

2.5. *Analytic functions.* Classes of analytic functions have smaller entropy than the classes Λ. We begin with $A_r(C)$, $r>1$, this consists of all functions $f(z)$ analytic in $|z|<r$ and continuous in $|z|\leqq r$, with the absolute value in $|z|\leqq r$ bounded by C, the metric being given by the uniform norm on $|z|\leqq 1$.

THEOREM [13, 18]. *Both the entropy $H_\epsilon(A_r)$ and the capacity $C_\epsilon(A_r)$ are given by*

$$\frac{\log^2 \frac{1}{\epsilon}}{\log r} + \mathcal{O}\left(\log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon}\right). \tag{10}$$

To obtain this result, we use the Taylor series representations

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \qquad |z| < r \tag{11}$$

of functions $f \in A_r$ and the following facts about their Taylor coefficients c_k:

$$\|f\| \leqq \sum_0^\infty |c_k| ; \tag{12}$$

$$\max_{|z| \leqq r} |f(z)| \leqq \sum_0^\infty |c_k| r^k; \tag{12a}$$

$$|c_k| \leqq \|f\| ; \tag{13}$$

$$|c_k| \leqq Cr^{-k}. \tag{13a}$$

The first two relations are obvious, while the last two follow from the Cauchy estimates of Taylor coefficients applied to circles $|z| \leqq 1$ and $|z| \leqq r$, respectively.

The entropy $H_\epsilon(A_r)$ can be estimated as follows. For an $f \in A_r$ given by (11), we put $r_n(z) = \sum_{k=n+1}^{\infty} c_k z^k$, and begin by finding by means of (13a) an n for which $\|r_n\| \leqq \frac{1}{2}\epsilon$ for all $f \in A_r$. One gets $n \approx \log 1/\epsilon$. Now consider the Taylor coefficients $c_k = \alpha_k + i\beta_k$ of functions f for $k = 0, 1, \cdots, n$. Let α_k', β_k' denote the closest approximations from below to α_k, β_k by means of integral multiples of $\epsilon/2(n+1)$:

$$\alpha_k' = \epsilon l_k/2(n+1), \qquad \beta_k' = \epsilon m_k/2(n+1), \qquad k = 0, \cdots, n. \tag{14}$$

Let U be the set of all $f \in A_r$ for which α_k', β_k' have some given set of values. It is obvious from (14) that the diameter of U does not exceed 2ϵ, and it is clear that the U cover A_r. The number N of the sets can be estimated from above if we note that by (13a), each of the integers l_k, m_k in (14) takes $\leqq$const. $r^{-k}n\epsilon^{-1}$ values. This allows us to show that $\log N$ does not exceed (10).

In an equally simple way we can estimate the capacity $C_{2\epsilon}(A_r)$ from below. One has merely to find constants a_k such that

$$|\alpha_k| \leqq a_k, \qquad |\beta_k| \leqq a_k, \qquad k = 0, 1, \cdots \tag{15}$$

will imply $f(z) = \sum(\alpha_k + i\beta_k)z^k \in A_r$. For instance $a_k = C/(4kr^k)$ will do. Then we allow α_k, β_k to take all integral multiples of 3ϵ for some $\epsilon > 0$ which satisfy (15). The number of all possible selections is easily estimated, and (13) insures that the corresponding functions are 2ϵ-distinguishable. In this way (10) proves to be a lower bound for $C_{2\epsilon}(A_r)$.

More general sets A of analytic functions are the following. Let K be a bounded continuum in the plane, and G a bounded open set containing K. Let A consist of all functions analytic in G continuous, and bounded by a constant on $\overline{G}$, with the distance given by the uniform norm on K. Then [5, 13]

$$H_\epsilon(A) \sim \tau(K, G) \log^2 \frac{1}{\epsilon} \cdot \tag{16}$$

The constant $\tau(K, G)$ can be determined in many cases (for instance by means of the "conformal radius" of the pair K, G). The main difference with the proof sketched above is that one uses other representations instead of the Taylor series: expansions in series of Tchebyshev polynomials (if K is a segment and G an ellipse), of Faber polynomials, etc. In the general case, expansions suggested by Erohin [4] can be used. If A consists of functions $f(z)$, $z=x+iy$ analytic in a strip $-l \leqq y \leqq l$, periodic with period 2π, then Fourier series expansions will be used. Very similar computations are possible for analytic functions of several variables. Let for example $A=A_{r_1,\ldots,r_s}(C)$ consist of functions $f(z_1, \cdots, z_s)$ analytic in the region $|z_1| < r_1, \cdots, |z_s| < r_s$, $r_i > 1$, $i=1, \cdots, s$, and continuous and bounded in absolute value by C on its closure; then for the uniform norm on $|z_1| \leqq 1, \cdots, |z_s| \leqq 1$ one has [18]:

$$H_\epsilon(A) \sim \frac{2}{(s+1)! \log r_1 \cdots \log r_s} \log^{s+1} \frac{1}{\epsilon} \cdot$$

Also the entropies of sets of entire functions with some restrictions on their growth at infinity have been determined [13, 18].

3. Widths.

3.1. We shall now discuss another function associated with a set A, which characterizes its "massivity" in a different way, the n-dimensional width $d_n(A)$ of A. Also this notion is due to Kolmogorov [10], who computed for the first time the exact widths of some sets. Let us first agree on some notations. Let A be a bounded subset of a Banach space X, $G=\{g_n\}$ a sequence of elements of X. Then

$$E_n^G(f) = \inf_{a_i} \left\| f - \sum_{i=1}^{n} a_i g_i \right\|, \tag{1}$$

where the infimum is taken for all selections of real scalars a_i, is the degree of approximation of f by G;

$$E_n^G(A) = \sup_{f \in A} E_n^G(f) \tag{2}$$

is the degree of approximation of the set A by G; finally

$$d_n(A) = \inf_{G} E_n^G(A), \tag{3}$$

is the optimal degree of approximation, the *n-th width of* A. The infimum in (3) is taken for all possible selections of sequences G in X. (Actually, only the first n elements $g_i \in G$ play a role.) The name width is justified geometrically, because in order to obtain $d_n(A)$, we take the maximal distance of A to an n-dimensional

plane in X passing through the origin, and then vary the plane, and take the minimum of the distances, for all possible planes.

The width $d_n(A)$ depends not only on the set A, but also on the norm selected, or more exactly upon the space X; one writes $d_n^X(A)$ in order to emphasize this dependence.

Examples: if A is the closed unit ball in X, then one shows that $d_n(A)=1$ for all n, which are smaller than the dimension of X; and if A is compact, then $d_n(A)\to 0$ for $n\to\infty$.

It is the case of a compact set A which shall occupy us in the following. For a few sets A of functions, the widths are exactly known; in most cases one has only their estimates. It is the asymptotic behavior of $d_n(A)$ for $n\to\infty$ that shall be of primary interest to us. The inequality $d_n(A)\leqq E_n^G(A)$ is often useful as an estimate from above; in fact, the degree of approximation $E_n^G(A)$ is known in cases when the linear combinations $\sum_1^n a_i g_i$ are all polynomials, or all trigonometric polynomials, etc. And it is to be expected that this selection of G will make $E_n^G(A)$ its minimum, or at least close to it.

3.2. *Estimation of $d_n(A)$ from below.* In many simple cases, the following lemma, given by Kolmogorov and the author [14, 16] is useful. Let A be a compact subset of the space $C(B)$ of continuous functions which are defined on a compact infinite metric set B. Consider all numbers $\delta\geqq 0$ with the following property. There exist $n+1$ points of B, $x_0, x_1, \cdots, x_n$ such that for each distribution of signs, $\epsilon_k=\pm 1$, $k=0, 1, \cdots, n$, one can find a function $f\in A$ with

$$\operatorname{sign} f(x_k)=\epsilon_k, \qquad |f(x_k)|\geqq\delta, \qquad k=0,\cdots,n. \tag{4}$$

Then one has:

LEMMA 1. *For each $A\subset C(B)$ and each δ of the described kind,*

$$d_n(A)\geqq\delta. \tag{5}$$

Thus the main problem in application of this lemma is to find the best distribution of points x_i in B.

The proof is simple. Let $G=\{g_n\}$ be arbitrary. The n linear equations $\sum_{k=0}^n c_k g_i(x_k)=0$, $i=1,\cdots,n$, with $n+1$ unknowns c_k have a nonzero solution, which we may assume to be normed by $\sum_{k=0}^n |c_k|=1$. Select a function of $f\in A$ satisfying (4) with $\epsilon_k=\operatorname{sign} c_k$. Then.

$$\left\|f-\sum_{i=1}^n a_i g_i\right\| \geqq \left|\sum_{k=0}^n c_k\left[f(x_k)-\sum_{i=1}^n a_i g_i(x_k)\right]\right|$$

$$= \left|\sum_{k=0}^n c_k f(x_k)\right| \geqq \sum |c_k|\,\delta = \delta,$$

hence $E_n^G(f)\geqq\delta$, and (5) follows.

Also for other metrics similar lemmas exist. If A is a subset of the space $L^1(B)$ of integrable functions on B with respect to a measure μ, then one has:

LEMMA 2 [14]. *Assume that n, p are positive integers, $\delta \geqq 0$ and that $B = \bigcup_1^{n+p} B_k$ is a partition of B into disjoint measurable sets with the property that for each distribution of signs $\epsilon_k = \pm 1$, $k = 1, \cdots, n+p$, and for each selection of p sets B_{k_i}, $i = 1, \cdots, p$, there is a function $f \in A$ with*

$$\operatorname{sign} \int_{B_k} f d\mu = \epsilon_k, \quad k = 1, \cdots, n+p; \qquad \left| \int_{B_{k_i}} f d\mu \right| \geqq \delta, \qquad i = 1, \cdots, p.$$

Then $d_n(A) \geqq p\delta$.

These and similar tools for the estimation of $d_n(A)$ from below, and classical results about approximation of functions by polynomials for the estimation of $d_n(A)$ from above, lead for example to the following estimates [14]:

THEOREM 4.

$$d_n(\Lambda(p+\alpha, s, C_0, \cdots, C_p, M)) \approx n^{-(p+\alpha)/s} \tag{6}$$

$$d_n(A) \approx \rho^{-n}, \qquad \rho > 1. \tag{7}$$

Here A is the set of all functions which are analytic inside the ellipse E_ρ, with foci ± 1 and the sum of half-axes equal to ρ, and bounded by 1 and continuous in the closed region; the metric in (6) or (7) is the C- or the L^1-metric (or any intermediate metric).

Thus (6) and (7) mean roughly that the known degrees of approximation by polynomials of the functions of classes Λ, A cannot be essentially improved if polynomials are replaced by other linear means of approximation.

3.3. *Cases where the widths can be exactly determined.* Several such cases have been found by Tihomirov [16]. To explain his method, let us start with the following remarks. From the definition of the width of a set A in a Banach space X, $d_n^X(A)$ it follows immediately that

$$d_n^X(A) \leqq d_n^{X'}(A) \qquad \text{if } X' \text{ is a subspace of } X. \tag{8}$$

It is more striking to know [16, p. 119] that inequality can occur here even if X and X' are finite dimensional. Tihomirov bases his method on the following

THEOREM 5 [6, p. 47]. *Let A be the closed unit ball of an $(n+1)$-dimensional subspace X' of a Banach space X. Then*

$$d_n^X(A) = 1.$$

Since $d_n^{X'}(A) = 1$, this means, that the nth width of A does not change when A is imbedded into a larger space X.

This theorem can be applied as follows. Let $\tilde{W}_p$, $p \geqq 1$ be the set of all 2π-periodic functions f with the property that $f^{(p)}(x)$ exists almost everywhere

and satisfies $|f^{(p)}(x)| \leqq 1$, with $f^{(p-1)}$ being an indefinite integral of $f^{(p)}$. We consider $\tilde{W}_p$ in the uniform norm, as a subspace of the space $\tilde{C}$ of periodic continuous functions.

A classical result of Favard, and Akhiezer and Kreĭn (see [1]), gives exactly the degree of approximation of $\tilde{W}_p$ by trigonometrical polynomials of degree $n-1$:

$$E^{\sigma}_{2n-1}(\tilde{W}_p) = K_p n^{-p} = \rho, \qquad K_p = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(p+1)}}{(2k+1)^{p+1}}; \tag{9}$$

moreover, a function $f_{pn} \in \tilde{W}_p$ is known for which the polynomial of best approximation is zero, and $\|f_{pn}\| = \rho$. This f_{pn} is the pth periodic indefinite integral of sign sin nx (if p is odd) or of sign cos nx (if p is even). Tihomirov considers the $2n$-dimensional subspace X' of C which consists of all pth periodic integrals of the functions of the form $g(x) = c_k$, $x \in \Delta_k$, $k = 0, 1, \cdots, 2n-1$, where the c_k are constants, and Δ_k are the intervals of constancy of sign sin nx (or of sign cos nx). Let U be the subset of X' corresponding to the values $|c_k| \leqq 1$, $k = 0, \cdots, 2n-1$. It is clear that $U \subset \tilde{W}_p$, on the other hand one can prove, using Rolle's theorem and the special form of f_{pn}, that if $f \in X'$ and $\|f\| \leqq \|f_{pn}\| = \rho$ then $f \in U$.

Consider the ball U_ρ of X' with the center in the origin and the radius ρ. What has been just said implies that $U_\rho \subset \tilde{W}_p$; on the other hand, by Theorem 5, $d_{2n-1}(U_\rho) = \rho$. Hence $d_{2n-1}(\tilde{W}_p) \geqq \rho$, and in view of (9) we obtain

$$d_{2n-1}(\tilde{W}_p) = K_p n^{-p}. \tag{10}$$

Tihomirov obtains further similar results, for example

$$d_{2n-1}(W_p) = K_p n^{-p} + \mathcal{O}(n^{-(p+1)}) \tag{11}$$

in the uniform norm; W_p consists of functions on $[0, 2\pi]$ with the same restrictions as for $\tilde{W}_p$, except that the condition of periodicity is dropped.

3.4. *A result of Kolmogorov.* We consider the subspace Y_p, $p \geqq 1$ of $L^2[0, 1]$, which consists of all functions f such that $f^{(p)}(x)$ exists almost everywhere and belongs to L^2, and that $f^{(p-1)}$ is an indefinite integral of $f^{(p)}$. Let V_p be the subset of Y_p with $\|f^{(p)}\| \leqq 1$, in the L^2-norm.

THEOREM 6 [10]. *Let* λ_n, $n = 1, 2, \cdots$ *denote all nonzero eigenvalues of the differential equation*

$$\begin{cases} y^{(2p)} = (-1)^p \lambda y \\ y^{(p)}(0) = y^{(p)}(1) = \cdots = y^{(2p-1)}(0) = y^{(2p-1)}(1) = 0. \end{cases} \tag{12}$$

All λ_n *are positive; if they are arranged in increasing order one has*

$$d_n(V_p) = +\infty \qquad \text{for } n = 0, 1, \cdots, p-1; \tag{13}$$

$$d_n(V_p) = \lambda_{n-p+1}^{-1/2} \qquad \text{for } n = p, p+1, \cdots. \tag{14}$$

To explain the proof, consider the equation

$$\begin{cases} z^{(2p)} = (-1)^p \lambda z, \\ z(0) = z(1) = \cdots = z^{(p-1)}(0) = z^{(p-1)}(1) = 0. \end{cases} \tag{15}$$

It is easy to see that for $\lambda \neq 0$ there is a one-to-one correspondence between the solutions of (12) and (15) given by $y^{(p)} = z$. For $\lambda = 0$, (15) has no solutions, while (12) has as solutions all polynomials of degree $p-1$.

Two sources of information are available for the study of equations (12) and (15). One is the general theory of eigenvalues of differential equations, the other the elementary formula, obtained by partial integration:

$$\int_0^1 z\overline{f^{(p)}}dx = \lambda \int yfdx; \qquad z = y^{(p)}. \tag{16}$$

Here y is a solution of (12) corresponding to an eigenvalue λ, and $f \in Y_p$.

From the first source we obtain that the solutions ϕ_n, $n = 1, 2, \cdots$ of (15), corresponding to the eigenvalues λ_n, form an orthogonal system, which is complete in L^2. Then it follows from (16) very easily: all eigenvalues of (12) except $\lambda = 0$ are greater than zero; if ψ_n is the solution of (12) corresponding to λ_n and given by $\psi_n^{(p)} = \phi_n$, then also the ψ_n form an orthogonal system; each ψ_n is orthogonal to each polynomial P_{p-1} of degree $p-1$. We shall assume $\|\psi_n\| = 1$ and obtain then

$$\|\phi_n\| = \lambda_n^{1/2}. \tag{17}$$

It follows also that for each $f \in Y_p$ one has the expansions, convergent in the L^2-norm,

$$f = P_{p-1} + \sum_{n=1}^{\infty} a_n \psi_n, \qquad f^{(p)} = \sum_{n=1}^{\infty} a_n \phi_n, \tag{18}$$

where P_{p-1} is properly chosen. This, together with (17), implies that $f \in V_p$ if and only if $\sum_1^\infty a_n^2 \lambda_n \leqq 1$.

From this fact the theorem follows easily. On one hand, V_p contains the linear subspace X_p of the P_{p-1} of dimension p (hence (13)) and for each m contains all f in (18) with $\sum_1^m a_k^2 \leqq \lambda_m^{-1}$. Thus for $n \geqq p$, V_p contains an $n+1$ dimensional ball with the center in the origin and the radius $\lambda_{n-p+1}^{-1/2}$. By means of Theorem 5 we obtain the relation

$$d_n(V_p) \geqq \lambda_{n-p+1}^{-1/2}, \qquad n \geqq p.$$

On the other hand, the distance from an $f \in V_p$ given by (18) to the n-dimensional subspace of L^2 spanned by X_p and the first $n-p$ vectors ψ_k is equal to

$$\left\{ \sum_{k=n-p+1}^{\infty} a_k^2 \right\}^{1/2} \leqq \lambda_{n-p+1}^{-1/2} \left\{ \sum_1^\infty \lambda_k a_k^2 \right\}^{1/2} \leqq \lambda_{n-p+1}^{-1/2},$$

hence the theorem.

The reader will be interested to learn that all eigenvalues λ_n are simple and that

$$\lambda_n = (\pi n)^{2p} + \mathcal{O}(n^{2p-1}).$$

If he would ask how one could arrive at the equation (12) before knowing the solution of our problem, the following answer could be offered. In finding the widths of V_p, we try to find more and more functions $y_1, \cdots, y_n$ in V_p, orthogonal to each other. This means that $y=y_n$ is the solution of the variational problem $\int_0^1 y^2dx = \max.$ with the restraint $\int_0^1 y^{(p)2}dx = 1$. The Euler equation of this problem is exactly (12).

3.5. *Relations between widths and entropy. Nonlinear approximation.* Both the entropy $H_\epsilon(A)$ and the widths $d_n(A)$ of a compact set A measure its "size." The widths measure the amount of deviation of A from linear subspaces, while the entropy depends much less upon the shape of A. Thus it is not astonishing that inequalities connecting these two functions exist, but are not very conclusive. Mitiagin ([15]; see also [13]) proves the following:

$$\int_0^{1/2\epsilon} \frac{l(t)}{t}\,dt \leqq H_\epsilon(A) \leqq \left[m\left(\frac{2}{\epsilon}\right)+1\right]\log\frac{8(d_0+\epsilon)}{\epsilon}. \tag{19}$$

Here A is a compact convex subset of a Banach space, symmetric with respect to the origin, $d_0=d_0(A)=\sup_{x\in A}\|x\|$, and the functions $m(t)$, $l(t)$ for $t>0$ are defined as the largest integers m or l satisfying $d_m \geqq t^{-1}$ or respectively $l^{-1}d_{l-1} \geqq t^{-1}$. The left hand side of (19) can be obtained in the following way. From the definition of widths and by induction one can find in A points x_n such that the distance from x_n to the linear space spanned by $x_0, \cdots, x_{n-1}$ is not less than $d_n(A)=d_n$. This implies that A contains for each n the octahedron Ω_n formed by the points $\sum_0^n \xi_k x_k$ with $\sum|\xi_k| \leqq 1$. If we consider the points of the form $y=\sum_0^n \eta_k y_k$ with $y_k=\epsilon x_k/d_k$ and integral η_k belonging to Ω_n, then it is easy to see that different y's are at a distance $\geqq\epsilon$ from each other and to estimate their number in Ω_n. This leads to an estimate from below of $M_{2\epsilon}(A)$ and hence of $H_\epsilon(A)$ (see 2.1 and Theorem 1).

We must make at least a passing mention of the nonlinear approximation. The notion of width is based upon the approximation of a function $f(x)$ by expressions $P(x)=a_1\phi_1(x)+\cdots+a_n\phi_n(x)$, linear in the parameters $a_1, \cdots, a_n$. In the book [18], Vituškin presents a theory which gives estimates from below of the degree of approximation of f by much more general algorithms P. For example, P may be the quotient of two polynomials in $a_1, \cdots, a_n$ with coefficients which are functions of x; also the operations max and min are allowed. This theory uses essentially new means (multidimensional variations of sets, topological properties of level surfaces of polynomials in Euclidean spaces) and cannot be reviewed here.

4. Representation of functions by sums and superpositions of functions of fewer variables.

4.1. Superpositions are functions of functions. For example,

$$F(x, y, z) = f(g(x, y), h(x, k(y, z))) \tag{1}$$

is a superposition of functions of two variables. Also the sum

$$x_1 + x_2 + \cdots + x_n = x_1 + (x_2 + \cdots + (x_{n-1} + x_n) \cdots)$$

is a superposition of functions of two variables. Studying the roots of equations of the seventh degree, Hilbert conjectured that not all continuous functions of three variables are superpositions of continuous functions of two variables. This is the 13th problem of Hilbert of his famous address at the International Congress of Mathematicians in Paris 1900. Recently Kolmogorov [12] and Arnol'd [2] disproved Hilbert's conjecture. The remarkable result of Kolmogorov is that each continuous function of s variables can be represented by sums and superpositions of functions of one variable! For example, if $s=2$, Kolmogorov's result can be formulated as follows:

THEOREM 7. *There exist ten continuous monotone increasing functions* $\phi_q(x)$, $\psi_q(x)$, $0 \leqq x \leqq 1$, $q=1, \cdots, 5$ *with values in the interval* $[0, 1]$ *which have the property that each continuous function* $f(x, y)$ *on the two-dimensional unit square* B *is representable in the form*

$$f(x, y) = \sum_{q=1}^{5} g(\phi_q(x) + \psi_q(y)), \tag{2}$$

where $g(u)$ *is some continuous function defined for* $0 \leqq u \leqq 2$.

Thus, the functions ϕ_q, ψ_q are independent of f, while g depends upon it. A similar statement holds for function of s variables, which are representable in the form

$$f(x_1, \cdots, x_s) = \sum_{q=1}^{2s+1} g\left(\sum_{p=1}^{s} \phi_{pq}(x_p) \right). \tag{3}$$

We restrict ourselves to $s=2$, because the general case is treated in exactly the same way.

The idea of the proof, roughly speaking, is to construct a pair of functions $\phi(x), \psi(y)$ which have the property that on disjoint subsets B', B'' of B, the sum $\phi(x)+\psi(y)$ takes values belonging to two disjoint subsets of the real line. Of course, this cannot be achieved for all pairs B', B'' of subsets of B, but we want it to be true for "sufficiently many" pairs.

Kolmogorov begins by constructing two increasing continuous functions $\phi_1(x), \psi_1(x)$ defined on $[0, 1]$ and with values in this interval with the following

* Actually, Kolmogorov [12] writes $2s+1$ functions g_q instead of g in (3), five functions g_q in (2). Only apparently weaker, this is in fact equivalent to our formulation.

property. For each $k=1, 2, \cdots$ there are finitely many closed subsets B_{1i}^k, $i=1, 2, \cdots, m_k$ of B and corresponding closed disjoint intervals I_{1i}^k of the real line such that if a point (x, y) belongs to B_{1i}^k, then $\phi_1(x)+\psi_1(y)$ belongs to I_{1i}^k. The diameter of B_{1i}^k tends to zero for $k\to\infty$ (uniformly in i; the sets B_{1i}^k are actually squares). For the proof this is not enough: the sets B_{1i}^k are disjoint for each fixed k, and therefore cannot cover B completely.

Therefore Kolmogorov repeats this construction five times and obtains ten increasing continuous functions $\phi_q(x)$, $\psi_q(x)$, $q=1, \cdots, 5$ and corresponding closed subsets $B_{qi}^k \subset B$ and intervals $I_{qi}^k \subset [0, 2]$ with $k=1, 2, \cdots$; $q=1, \cdots, 5$; $i=1, \cdots$; m_k such that $\phi_q(x)+\psi_q(y)\in I_{qi}^k$ if $(x, y)\in B_{qi}^k$. All intervals I_{qi}^k are disjoint for a fixed k; therefore the B_{qi}^k are disjoint for each fixed q and k. It is very essential that the B_{qi}^k cover B very well; precisely, for each k, each point (x, y) in B is covered at least three times by the B_{qi}^k (there are at least "three hits out of five shots"). The following lemma concerning the functions ϕ_q, ψ_q is useful:

LEMMA 3. *Let C be the space of continuous functions on B, and let C' consist of all functions of the form*

$$h(x, y) = \sum_{q=1}^{5} g(\phi_q(x) + \psi_q(y)) \tag{4}$$

with an arbitrary continuous function g. Assume that there is a constant $0<\lambda<1$ such that for each $f\in C$ there is an $h\in C'$ given by (4) *with $\|f-h\| \leqq \lambda\|f\|$; $\|g\| \leqq \|f\|$. Then $C'=C$.*

The proof is easy: one first selects an h_1 with the corresponding g_1 so that $\|f-h_1\| \leqq \lambda\|f\|$, $\|g_1\| \leqq \|f\|$, then an h_2 with g_2 so that $\|(f-h_1)-h_2\| \leqq \lambda\|f-h_1\| \leqq \lambda^2\|f\|$, $\|g_2\| \leqq \lambda\|f\|$, and so on. Then the series $g=\sum_{n=1}^{\infty} g_n$ converges uniformly, and with h connected with this g by (4) one has $f=h=\sum_1^{\infty} h_n$.

To complete the proof of the theorem, we take $\frac{2}{3}<\lambda<1$. We can fix k so large that the oscillation of the given function f is not more than $\epsilon\|f\|$ on each B_{qi}^k, and that $\epsilon>0$ is so small that $\frac{2}{3}+\epsilon<\lambda$. Let z_{qi}^k denote some point of B_{qi}^k. We determine $g(u)$, $0\leqq u\leqq 2$ in the following way. On each I_{qi}^k we let g be a constant: $g(u)=\frac{1}{3}f(z_{qi}^k)$. Onto the rest of $[0, 2]$ we extend g in an arbitrary fashion, but so that $g(u)$ remains continuous and that $|g(u)| \leqq \frac{1}{3}\|f\|$. Now consider the sum (4) for some $(x, y)\in B$. Three of its terms $g(\phi_q(x)+\psi_q(y))$ have the property that $(x, y)\in B_{qi}^k$ for some i; each of these terms is equal to $\frac{1}{3}f(x, y)$ with an error not exceeding $\frac{1}{3}\epsilon\|f\|$. The remaining two terms are each $\leqq\frac{1}{3}\|f\|$. Hence

$$|f(x, y) - h(x, y)| \leqq \tfrac{2}{3}\|f\| + 3\cdot\tfrac{1}{3}\epsilon\|f\| < \lambda\|f\|,$$

and the proof is completed by applying Lemma 3.

The functions ϕ_q, ψ_q in Kolmogorov's construction seem not to be as wild as one could expect a priori. For example, they prove to belong to the class Lip α for $\alpha=\frac{1}{5}$. Theorem 7 is very important in principle. Will it have useful applications? Theoretically, one could hope to derive by means of it results

concerning functions of several variables from corresponding results about functions of one variable. For example, from the Weierstrass theorem about the approximation of functions of one variable by polynomials at once follows the corresponding theorem for functions of s variables. This is not very astonishing, since most proofs of the theorem of Weierstrass generalize immediately to higher dimensions. One wonders whether Kolmogorov's theorem can be used to obtain positive results of greater depth.

4.2. Let us return to Hilbert's conjecture. Despite its disproof, it originated from a sound idea: that "bad" functions in general cannot be represented by superpositions of "good" functions. From this point of view, Theorem 7 makes it clear, that the characteristic of badness $\chi=s$ of $f(x_1, \cdots, x_s)$, equal to the number of variables of f, has failed. The reason is that general continuous functions are so bad, that the number of variables is not particularly important. Are there other characteristics of this kind which will work? That this is the case has been found by Vituškin [17] and Kolmogorov [13, p. 80]. Let a function of s variables $f(x_1, \cdots, x_s)$ on the s-dimensional unit square have all partial derivatives of orders not exceeding p, with the derivatives of order p satisfying a Lipschitz condition with exponent $\alpha(p=0, 1, \cdots, 0<\alpha\leqq 1)$. We shall then say that f belongs to the characteristic $\chi=s/(p+\alpha)$.

THEOREM 8. *Not all continuous functions with a given characteristic χ_0 can be represented by superpositions of functions of characteristics $\chi=s/(p+\alpha)<\chi_0$ and with $p+\alpha\geqq 1$.*

In order to sketch the proof, we first introduce the notion of a *type* T of superpositions. A type T is a set of functions F given by a formula such as (1) and by an indication of classes of type $\Lambda(p+\alpha, s, C_0, \cdots, C_p, M)$ (see 2.3) to which the functions $f, g, \cdots$ contained in the formula are allowed to belong. We shall assume that all $C_0, \cdots, C_p, M$ are positive integers and that the classes have characteristics $\chi=s/(p+\alpha)<\chi_0$. Then T consists of all functions F which can be obtained by varying the functions $f, g, \cdots$, but leaving the structure of the formula and the given classes fixed. One sees immediately that there are only countably many different types T.

The entropy (in the uniform norm) of a class $\Lambda(p+\alpha, s, C_0, \cdots, C_p, M)$ is of the order $\epsilon^{-\chi}$, $\chi=s/(p+\alpha)$. What is the entropy of a type T? Consider the superpositions

$$F(x_1, \cdots, x_s) = f(g_1(x_1, \cdots, x_s), \cdots, g_r(x_1, \cdots, x_s)), \tag{5}$$

where the f satisfy the Lipschitz condition

$$|f(y_1, \cdots, y_r) - f(y_1', \cdots, y_r')| \leqq M \max_{i=1,\ldots,r} |y_i - y_i'|,$$

and let $f, g_1, \cdots, g_r$ run through sets $A, B_1, \cdots, B_r$ with known entropies (with respect to the uniform norm). We shall show that the entropy of the set T of the F's satisfies

$$H_{(M+1)\epsilon}(T) \leqq H_\epsilon(A) + \sum_1^r H_\epsilon(B_i). \tag{6}$$

Let the functions $\bar{f}$, $\bar{g}_i$ run through ϵ-nets in A, B_i respectively, and let N_ϵ, N_ϵ^i be the numbers of elements in these sets. Consider the functions

$$\overline{F} = \bar{f}(\bar{g}_1(x_1, \cdots, x_s), \cdots, \bar{g}_r(x_1, \cdots, x_s)), \tag{7}$$

$N_\epsilon \prod_{i=1}^r N_\epsilon^i$ in number. For each function (5) there exist $\bar{f}$, $\bar{g}_i$ with

$$\|f - \bar{f}\| < \epsilon \qquad \|g_i - \bar{g}_i\| < \epsilon, \qquad i = 1, \cdots, r,$$

and then

$$\begin{aligned} |F - \overline{F}| &\leqq |f(g_1, \cdots, g_r) - f(g_1, \cdots, g_r)| + |f(g_1, \cdots, g_r) - \bar{f}(\bar{g}_1, \cdots, \bar{g}_r)| \\ &\leqq \epsilon + M\epsilon = (M+1)\epsilon. \end{aligned}$$

Hence the $\overline{F}$ form an $(M+1)\epsilon$-net in T, so that $N_{(M+1)\epsilon}(T) \leqq N_\epsilon \prod_{i=1}^r N_\epsilon^i$. Relation (6) follows from this by taking logarithms. In particular, if A, B_i are classes Λ with characteristics $\chi = s/(p+\alpha) \geqq m$, then

$$H_\epsilon(T) \leqq \text{const. } \epsilon^{-m}. \tag{8}$$

The formula defining a type T may have a more complicated structure than (5), but it reduces to this case by finite iteration, and by induction we obtain: the entropy of each type T satisfies (8), where $m = \max \chi < \chi_0$.

The proof of the Theorem 8 is completed by a category argument. Let $\chi_0 = s_0/(p_0+\alpha_0)$; we consider the set X of all functions f on the s_0-dimensional unit square with the smoothness $p_0+\alpha_0$(2.3). A norm can be defined for $f \epsilon X$ in a natural way: $\|f\|_X$ is the maximum of the absolute values of all partial derivatives of f of orders $\leqq p$ and of the smallest constants with which the derivatives of order p satisfy the Lipschitz α_0-condition. In a standard way one proves that X is a Banach space; its completeness in particular is essential. A ball $U_r(0)$ in X of center 0 and radius r is a class $\Lambda(p_0+\alpha_0, s_0, r, \cdots, r, r)$ and has (2.3) the entropy $\approx \epsilon^{-\chi_0}$. Any other ball in X is a translation of a ball $U_r(0)$ and has the same entropy. From this and (8) it is clear: a type T cannot contain any ball of X.

On the other hand, the functions f, g, $\cdots$ in a formula (1) determining a class T, as well as their derivatives are equicontinuous. This allows us to apply Arzela's theorem and prove that T is closed under uniform convergence, and hence under convergence in the norm of the space X. Thus, sets $T \cap X$ are nowhere dense in X. Since there are countably many types T, the union $\bigcup_T (T \cap X)$ is a set of the first category in X. By Baire's theorem, its complement in X is a nonempty set. It consists of functions not representable by superpositions of functions with characteristics $\chi < \chi_0$.

We have proved the theorem by establishing not only that there are functions in X not representable by superpositions of our type, but even that "almost all" functions (in the category sense) in X are not so representable. A fine

point of the proof will be noted: it uses two norms; the uniform norm to compute the entropies, and the norm $\|f\|_X$, in which the space X is complete.

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N-GON ROTORS MAKING $N+1$ CONTACTS WITH FIXED SIMPLE CURVES

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1. Introduction. In a paper by the author [1], closed curves were derived which have the property of remaining in contact with a set of fixed circles while the curve is rotated through all orientations. Among these curves were some which closely approximated regular polygons of n sides. These could be rotated while remaining in contact with $n+1$ fixed equal circles. This relation suggests the possibility of the existence of $n+1$ noncircular symmetric arcs which may replace the circles so that the nearly-rectilinear polygon may be replaced by a perfectly rectilinear polygon. The construction of a stator made of such a set of arcs is demonstrated here for polygons of an odd number of sides. Other generalizations and related problems will be discussed.

2. Stators for regular polygon rotors. In Figures 1 and 2, the point O is the origin. Let the midpoints of the symmetric arcs of the stator be placed at the ends of $n+1$ uniformly spaced radii of length a centered on O. Let A and H be the ends of a pair of opposite radii. Place the regular n-gon of altitude $2a$ so that a vertex coincides with A while the midpoint of the opposite side coincides with H. Let B and C be the ends of the opposite side. Then, the length e of the side of the n-gon is given by $e=(4a \sin \pi/n)/(1+\cos \pi/n)$.

Now place the polygon symmetrically about a stator diagonal through O making the angle $\phi=\pi/(n+1)$ with OH so that a vertex lies at the point D which is symmetric with respect to the point of tangency G of the opposite side of the stator. If the coordinates of G are (x, y), then the coordinates of D will be $(x, -y)$. These conditions define the locations of D and G. The angle between a line of the polygon in this position and the corresponding line in the original position is $\theta=\pi/n(n+1)$. Hence,

$$\cos \theta = \cos \pi/n(n+1) = 2a/2y, \quad \text{or} \quad y = a \sec \pi/n(n+1).$$

Draw a parallel to the principal diagonal and at a distance $\frac{1}{2}e$. This line intersects the line $y=-a \sec \pi/n(n+1)$ in the point D. Then, $x=\frac{1}{2}e \cos \phi - y \tan \phi$. Draw $DJ=2a$ so that DJ makes the angle θ with DG. Draw EF through J and normal to DJ so that $EJ=JF=\frac{1}{2}e$. Then EF passes through G. Reflect the points A, B, F and G about the axes of symmetry to produce the corresponding points on the other arcs of the sought contour.

Join H and G by an arbitrary curve tangent to BC and EF. To realize a working model of the rotor, the tangent of the selected curve should be constantly turning in the same direction. Duplicate this curve at AD. Although the curve HG is arbitrary, the remainder of this arc will be prescribed by geometric considerations in accordance with the following kinematic procedure. Rotate the polygon, three of whose vertices are A, B, C, through the angle θ in a counterclockwise direction so that the vertex A traverses the curve from A to D while the side remains tangent to the curve HG. Then the vertex B traces the curve

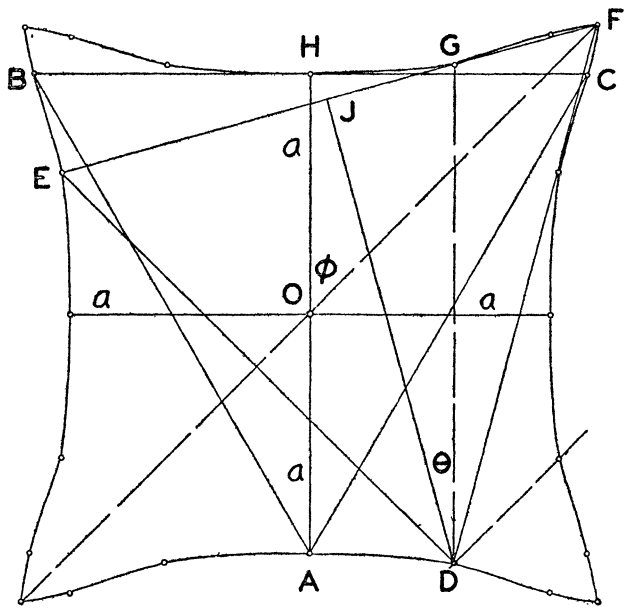


Fig. 1. Triangle rotor in four-lobed stator.

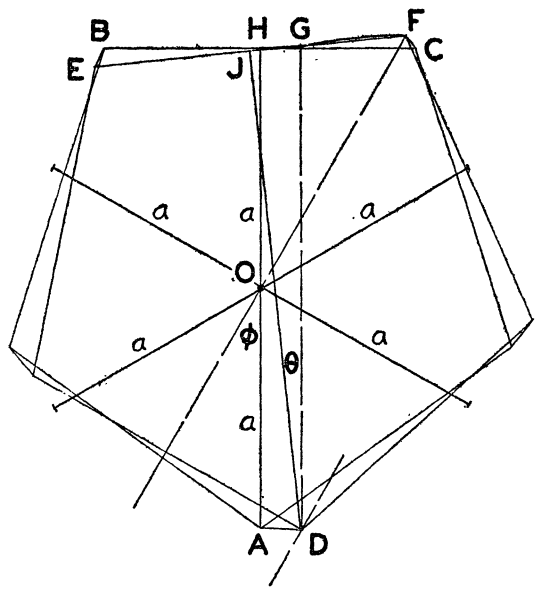


Fig. 2. Pentagon rotor in six-lobed stator (only portion made by rotation θ is shown).

BE while the vertex C traces the curve CF. Also, the other vertices of the polygon trace portions of the contour. These curves are then reflected to complete the contour of the sought simple curve. The repetition of the rotation θ through $2(n+1)$ cycles will rotate the polygon through the angle $2\pi/n$ so that it returns to its initial position and the vertices trace the complete contour. However, tangency occurs only along the original arbitrary portions of the contour.

It is conceivable that similar stators exist for the polygons of an even number of sides, but these are not derivable in the foregoing manner.

3. Minimal problems. In the famous Kakeya minimal problem, a straight line segment is to be rotated through all orientations to sweep out the least area. The three-cusped hypocycloid was suggested as the likely solution. In this case, the ends of the line traverse the curve while the line remains tangent to the curve. However, this is not the least area. Besicovitch [2] showed that the area can be made smaller than any selected value. This suggests similar problems with rotating polygons. In the foregoing development of a stator for the rotating polygon, it would be of interest to determine the choice of the arbitrary curve which would minimize the area within the complete contour. Also, the possibility of other contours of lesser area should be considered.

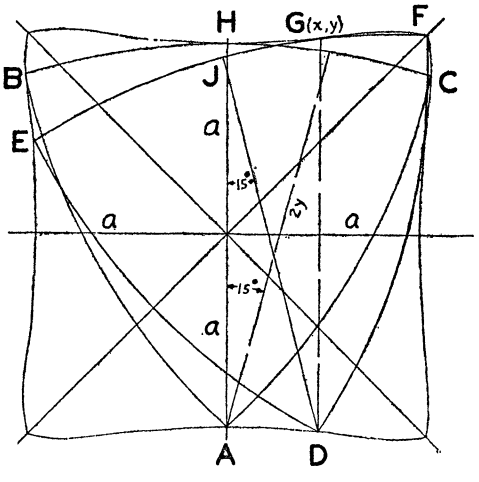


Fig. 3. Convex three-arc rotor in four-lobed stator.

4. Intermittent rotors. In an earlier paper by the author [9], various methods were shown for generating intermittent rotors. These are rotors which make contact with a number of fixed elements, but not always with the same elements. The stators in the present paper may be considered as another family of intermittent rotors since, in this case, the fixed elements are not alike as they were in the earlier paper. If the polygon is held fixed, and the curve rotated about it, the rotating curve always makes contact with the fixed vertices. However, contact

with the straight lines is intermittent. Contact is made with only one line at a time, except at a transition position where contact is made momentarily with two straight lines. The fixed polygon may be replaced by a parallel body which is made of n circles and n straight lines, while the rotor is replaced by a parallel rotor.

5. Stators for other regular rotors. The regular rotor of straight lines may be replaced by a regular rotor of equal circular arcs of arbitrary radius. The method of generating the corresponding stator is shown for the three-arc rotor in Figure 3. The value $2y$ is the length of the chord through a vertex making an angle of 15° with the axis of symmetry. As in the case of the rectilinear polygon, the curve HG of the stator is arbitrary, provided it is tangent to the circular arcs at these positions.

The convex rotor of circular arcs may be replaced by a nonconvex rotor of equal concave circular arcs. An example is shown in Figure 4.

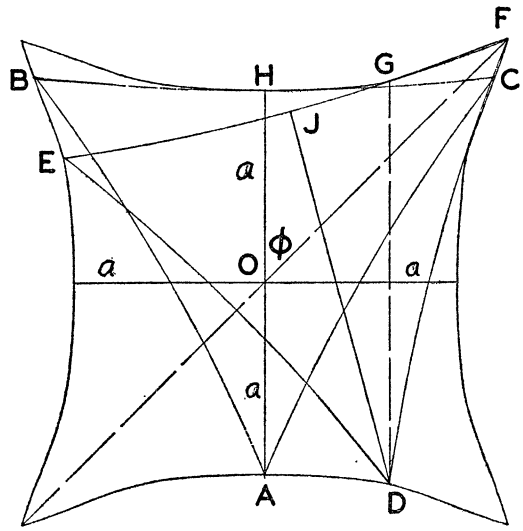


Fig. 4. Concave three-arc rotor in four-lobed stator.

A stator can be constructed for a regular rotor with an *even* number of vertices provided that the arcs are sufficiently concave. In Figure 5, the sides are arcs of circles. The points A and D have interchanged the roles they played in the odd case of Figure 1. The point D is now the midpoint of the stator arc, whereas, in the odd case, it was the point A. If one attempts to construct a stator for a rotor of straight sides, the length y becomes less than the length a. Therefore, a working model cannot be constructed since the rotor and stator contours intersect for portions of the motion.

A further generalization for both even and odd cases is obtained by using any symmetric curve in place of the circular arcs of the rotors [7]. This general method of deriving rotor-stator combinations in which the stators make $n+1$ contacts with rotors of n-fold symmetry includes, as special interesting cases, the following previously derived cases. Steiner discovered that a straight line segment can be rotated within a three-cusped hypocycloid and remain tangent to the curve while its ends are constrained to move along the curve [8]. Morley

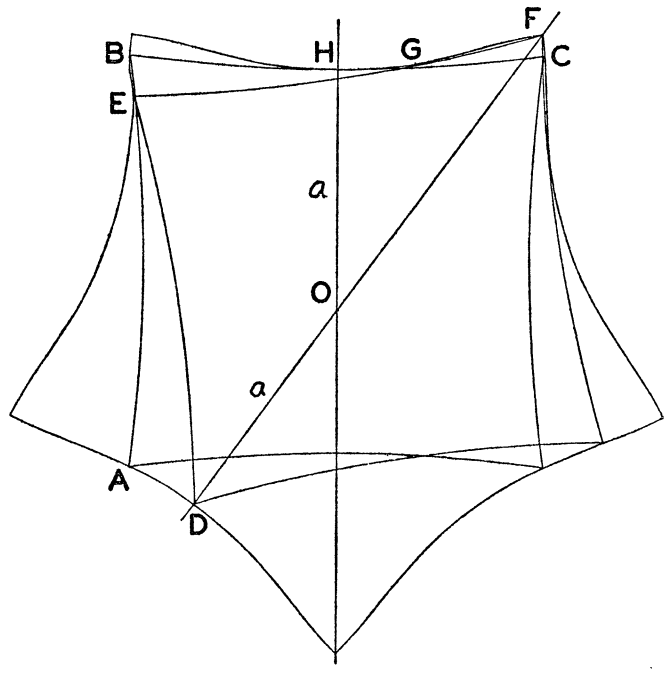


Fig. 5. Concave four-arc rotor in five-lobed stator.

generalized this theorem by showing that hypocycloids of n cusps can rotate within hypocycloids of $n+1$ cusps [3]. Fréchet showed that the rotating straight line segment (an hypocycloid of two cusps) may be replaced by an ellipse [4]. The author showed that a two-lobed rotor in a triangle (the curve of Ribaucour) is also a rotor making contact with the three arches of a convex involute of the three-cusped hypocycloid [5]. Wunderlich generalized the foregoing result of Fréchet to obtain a series of n-lobed trochoid rotors which make $n+1$ contacts with the arches of an $(n+1)$-cusped hypocycloid [6]. Another series of stators may be obtained by the use of polygon rotors tangent to $n-1$ arcs instead of $n+1$ arcs. However, the number of contacts is still $n+1$. Examples are shown in Figures 6 and 7.

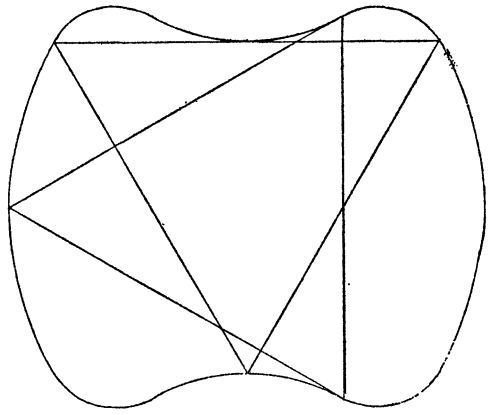


Fig. 6. Triangle rotor in two-lobed stator.

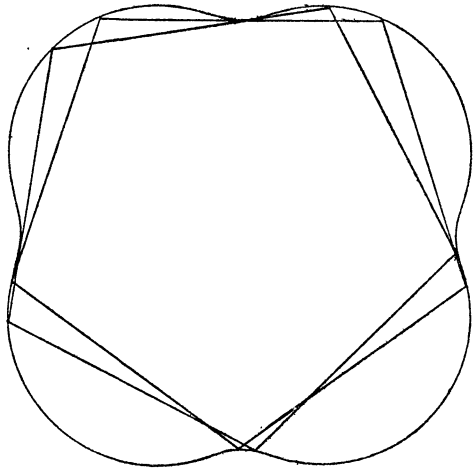


Fig. 7. Pentagon rotor in four-lobed stator.

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RIEMANN INTEGRATION ON SUBSPACES OF THE REALS

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1. Introduction. In defining the Lebesgue integral, Riesz and Nagy [4] define a class C_1 of functions which are limits, almost everywhere, of step functions. They show that if f is Riemann integrable then $f\in C_1$ and $-f\in C_1$, and state that conversely the latter implies that f is equal, almost everywhere, to a Riemann integrable function. To prove the latter it is natural to consider a "pseudo-Riemann" integral as defined below, in which one essentially ignores the points of the interval at which the function is undefined. A similar and not unrelated method of integration was investigated by Carr and Hill [1]. Pierpont [3] uses the same type of definition as we do, but only for multiple integrals over measurable sets.

2. Definition of the integral, conditions for integrability and some properties of the integral.

DEFINITION 2.1. *Let $E\subset[0, 1]$ and let f be a bounded function defined on E. Let $P=\{x_i\}$ be a partition of $[0, 1]$ and δ be the norm of P. If F is a function defined for all such P whose values are sets of real numbers, $\lim_{\delta\to 0} F(P)=L$ if and only if for any $\epsilon>0$ there is a $\Delta>0$ such that $|r-L|<\epsilon$ for all $r\in F(P)$, provided $\delta<\Delta$. Define the E-integral of f to be*

$$(E)\int f(x)dx = \lim_{\delta\to 0} (E)\sum_i f(\bar{x}_i)\Delta x_i.$$

The symbol $(E)\sum_i$ will mean the sum is to be taken only over the i for which $E\cap[x_{i-1}, x_i]\neq\varnothing$, and $\bar{x}_i\in E\cap[x_{i-1}, x_i]$.

As an immediate consequence of Definition 2.1 and the definition of Jordan content, if a bounded function f is defined on a set E whose Jordan content is zero, then $(E)\int f(x)dx=0$.

As in the case of Riemann integration we define upper and lower sums. Let $M=\sup f$ on E and $m=\inf f$ on E. Then form the sums $S=(E)\sum_i M_i\Delta x_i$ and $s=(E)\sum_i m_i\Delta x_i$, where M_i and m_i are the supremum and infimum, respectively, of f on $[x_{i-1}, x_i]\cap E$. Then $m\leqq m_i\leqq M_i\leqq M$ for all i and since $m_i\leqq f(\bar{x}_i)\leqq M_i$ for any $\bar{x}_i\in E\cap[x_{i-1}, x_i]$, $s\leqq(E)\sum_i f(\bar{x}_i)\Delta x_i\leqq S$.

We would like to define upper and lower integrals relative to the upper sums S and lower sums s in a manner similar to that for the Riemann integral. However, for the E-integral the usual definitions for upper and lower integrals are unsatisfactory. The following example indicates the difficulty which arises:

Let $E=\{\frac{1}{2}\}$, $P=\{0, 1\}$ and $f(\frac{1}{2})=1$. Then the sums s and S relative to P are both 1. If we add the points $\frac{1}{4}$ and $\frac{3}{4}$ to P so that $P'=\{0, \frac{1}{4}, \frac{3}{4}, 1\}$, then $s'=S'=\frac{1}{2}$. Thus $s>S'$, so a lower sum can be greater than an upper sum. If P' is refined further, the upper and lower sums are both decreasing. Considering all possible partitions of $[0, 1]$, $\inf S=0$ and $\sup s=1$. Hence if $J=\inf S$ and $I=\sup s$, $J<I$. However, since the Jordan content of E is zero, $(E)\int f(x)dx=0$.

In order to obtain conditions for integrability, it is desirable to have the equality of upper and lower integrals as a necessary and sufficient condition for integrability. To this end we define the upper and lower integrals of f over E as follows.

DEFINITION 2.2. *The upper integral J is the supremum for all $\Delta>0$ of the infima of S over all P with norm $\delta<\Delta$, i.e.,*

$$J = \lim_{\delta\to 0}\inf S = \sup_{\Delta>0}\left[\inf_{\delta<\Delta} S\right].$$

The lower integral I is the infimum for all $\Delta>0$ of the suprema of s over all P with norm $\delta<\Delta$, i.e.,

$$I = \lim_{\delta\to 0}\sup s = \inf_{\Delta>0}\left[\sup_{\delta<\Delta} s\right].$$

Each of the theorems of this section is similar to a result from Riemann integration. The proofs of all but Lemma 2.7 and Theorem 2.8 are either similar to the corresponding theorems or follow readily. Hence we will prove only Lemma 2.7 and Theorem 2.8.

We first list two lemmas which reduce the proof of Theorem 2.5 to the proof of the corresponding theorem of Riemann integration. We state them in terms of lower sums but similar statements may be made in terms of upper sums. The lemmas indicate that while the sums are not necessarily monotone under refinement, they could be called "almost monotone."

LEMMA 2.3. *Let f be defined on E. Let $P=\{x_i\}$ be a partition of $[0, 1]$ with norm δ, having lower sum s, and let P' be a refinement of P with sum s' and norm δ'. Then given any $\epsilon>0$ there exists a $\Delta>0$ such that, if $0<\delta'\leqq\delta\leqq\Delta$, then $s\leqq s'+\epsilon$.*

LEMMA 2.4. *Given any $\epsilon>0$ and any $\Delta>0$, there exists a partition P such that its norm $\delta<\Delta$ and $|s-I|<\epsilon$.*

THEOREM 2.5. *If s and S are the lower and upper sums corresponding to the partition P of $[0, 1]$ with norm δ, then $\lim_{\delta\to 0} s=I$ and $\lim_{\delta\to 0} S=J$.*

THEOREM 2.6. *Let f be defined on E. Then f is E-integrable if and only if $I=J$.*

LEMMA 2.7. *Let f be defined on E and let $P=\{x_i\}$ be a partition of $[0, 1]$ having upper sum S and lower sum s. Then if $P_1=\{x_{ij}\}$ is any refinement of P having upper sum S_1 and lower sum s_1, then $S_1-s_1\leqq S-s$.*

Proof. Let $M_i=\sup f$ and $m_i=\inf f$ on $E\cap[x_{i-1}, x_i]$ and $M_{ij}=\sup f$ and $m_{ij}=\inf f$ on $E\cap[x_{i-1,j-1}, x_{i-1,j}]\subset[x_{i-1}, x_i]$. Then $M_i\geqq M_{ij}$ and $m_i\leqq m_{ij}$ for all j for which M_{ij} and m_{ij} exist. (If $E\cap[x_{i-1,j-1}, x_{i-1,j}]=\varnothing$, then M_{ij} and m_{ij} do not exist.) Thus $M_i-m_i\geqq M_{ij}-m_{ij}$ and

$$(M_i - m_i)\Delta x_{ij} \geqq (M_{ij} - m_{ij})\Delta x_{ij}. \tag{1}$$

Summing (1) over j while holding i fixed, we obtain

$$(2) \qquad (M_i - m_i)\Delta x_i \geqq (E_i) \sum_j (M_i - m_i)\Delta x_{ij} \geqq (E_i) \sum_j (M_{ij} - m_{ij})\Delta x_{ij},$$

where $E_i = E \cap [x_{i-1}, x_i]$. Summing (2) over i gives

$$(E) \sum_i (M_i - m_i)\Delta x_i \geqq (E) \sum_i (E_i) \sum_j (M_{ij} - m_{ij})\Delta x_{ij} = (E) \sum_{i,j} (M_{ij} - m_{ij})\Delta x_{ij}.$$

Therefore $S - s \geqq S_1 - s_1$.

THEOREM 2.8. *Let f be defined on E. Then f is E-integrable if and only if, given any $\epsilon > 0$ there is a partition P with sums S and s such that $S - s < \epsilon$.*

Proof. The condition is necessary, for suppose f is E-integrable. Given any $\epsilon > 0$, by Theorem 2.5, $|J - S| < \frac{1}{2}\epsilon$ and $|s - I| < \frac{1}{2}\epsilon$ for all partitions P with sufficiently small norm δ. Then

$$\epsilon > |J - S| + |s - I| \geqq |J - S + s - I| = |(J - I) + (S - s)|.$$

But since f is E-integrable, by Theorem 2.6, $I = J$. Hence $S - s = |S - s| < \epsilon$ and the necessity of the condition is proved.

To prove the condition is sufficient, let $\epsilon > 0$ be given and let P be a partition with sums S and s such that $S - s < \frac{1}{3}\epsilon$. Since $\lim_{\delta \to 0} S = J$ and $\lim_{\delta \to 0} s = I$, there is a partition P_1 with sufficiently small norm δ_1, P_1 a refinement of P, such that $|J - S_1| < \frac{1}{3}\epsilon$ and $|s_1 - I| < \frac{1}{3}\epsilon$, where ϵ is as above. Since P_1 is a refinement of P, by Lemma 2.7, $S_1 - s_1 \leqq S - s$. Thus

$$\begin{aligned} |J - I| &= |J - S_1 + S_1 - s_1 + s_1 - I| \\ &\leqq |J - S_1| + |S_1 - s_1| + |s_1 - I| \\ &< \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon. \end{aligned}$$

Since ϵ is arbitrary $I = J$ and f is E-integrable by Theorem 2.5.

THEOREM 2.9. *If f is uniformly continuous on E, then f is E-integrable.*

We let $\overline{E}$ denote the closure of E.

DEFINITION 2.10. *Let f be defined on E and let N_a denote a neighborhood of $a \in \overline{E}$. Then the limit superior and limit inferior of f at a are given by*

$$\limsup_{x=a} f(x) = \operatorname*{infimum}_{\text{over all } N_a} \left[\operatorname*{supremum}_{x \in N_a \cap E} f(x) \right],$$

$$\liminf_{x=a} f(x) = \operatorname*{supremum}_{\text{over all } N_a} \left[\operatorname*{infimum}_{x \in N_a \cap E} f(x) \right].$$

DEFINITION 2.11. *The oscillation of f at $x = a$ is*

$$\limsup_{x=a} f(x) - \liminf_{x=a} f(x)$$

if each of these limits is finite. If at least one of them is infinite, the oscillation is plus infinity.

THEOREM 2.12. *Let f be defined on E and let O_k denote the set of points of $\overline{E}$ at which the oscillation of f is greater than or equal to k. Then the E-integral of f exists if and only if the Jordan content of O_k is zero for every positive number k.*

DEFINITION 2.13. *The function f is discontinuous at $a \in \overline{E}$ if and only if*

$$\limsup_{x=a} f(x) \neq \liminf_{x=a} f(x).$$

THEOREM 2.14. *Let f be defined on E and let D denote the set of points of $\overline{E}$ at which f is discontinuous. Then f is E-integrable if and only if the Lebesgue measure of D is zero.*

Many of the properties of the Riemann integral are also valid for the E-integral. Two which may be slightly less obvious than others we list below.

THEOREM 2.15. *Let $A \subset E$. Then if f is E-integrable, it is also A-integrable.*

THEOREM 2.16. *Let A and B be sets and let f be defined and integrable on $A \cup B$. Then if the Jordan content of $\overline{A} \cap \overline{B}$ is zero,*

$$(A \cup B) \int f(x)dx = (A) \int f(x)dx + (B) \int f(x)dx.$$

We emphasize at this point that the sets under consideration are quite arbitrary. In fact they may even be nonmeasurable. It should be noted that in all but Theorem 2.16, the only restrictions we have stated have been with respect to the function, not the set. To illustrate this fact further we list below some examples. These examples indicate that while the E-integral is in some respects more general than the Lebesgue integral, in others it is more restrictive. For, while a function may be defined on a nonmeasurable set in such a way that it is E-integrable, it is also the case that a function defined on a set E having Lebesgue measure zero need not be E-integrable. Furthermore, continuity of a function on an arbitrary set E is not a sufficient condition for E-integrability.

Example 2.17. Consider a Cantor set of measure one half obtained by deleting the middle $(1/4)^n$ from each remaining subinterval of $[0, 1]$ at the nth stage of the procedure; i.e. delete $(3/8, 5/8)$, $(5/32, 7/32)$, $(25/32, 27/32)$, and so on. Let E be the union of the open intervals deleted in the construction of this Cantor set. Let $f(x) = 1$ on $(3/8, 5/8)$ and $f(x) = 0$ on $(5/32, 7/32)$ and $(25/32, 27/32)$. In general let $f(x) = 1$ on the intervals at the nth stage of the construction if n is odd, and $f(x) = 0$ if n is even. Then f is continuous on E since E is the union of open intervals of $[0, 1]$ on each of which f is constant. $\overline{E} = [0, 1]$ and $\overline{E} - E$ is nowhere dense. Thus every neighborhood of every point of $\overline{E} - E$ contains an interval of E on which $f(x) = 1$ and also an interval of E on which

$f(x)=0$. Hence the Jordan content of the set of points of $\overline{E}$ at which the oscillation of f is greater than or equal to one is not zero, and f is not E-integrable by Theorem 2.12.

Example 2.18. Let $E=\{p/q \mid p \text{ and } q \text{ are relatively prime integers, } q\neq 0, 0\leqq p\leqq q\}$. Let $f(x)=1$ on the rational numbers with even numerators, and $f(x)=0$ on the remaining rationals. It is easy to show that both subsets of E are everywhere dense in $[0, 1]$. Hence $m_i=0$ and $M_i=1$ for every i, so $s=0$ and $S=1$ for every partition P. Hence by Theorem 2.8 f is not E-integrable. Thus although E has Lebesgue measure zero, f is not E-integrable.

Example 2.19. Although $(E)\int f(x)dx=0$ if the Jordan content of E is zero, having Lebesgue measure zero does not suffice even if f is E-integrable. For if $f(x)=1$ on the set E of rationals in $[0, 1]$, then $(E)\int f(x)dx=1$.

The authors wish to thank the referee for suggesting the inclusion of this example and other improvements.

Example 2.20. Let E be any nonmeasurable set whose closure has Jordan content. Then a constant function f is E-integrable. In McShane's *Integration* [2], there is an example of a nonmeasurable everywhere dense subset of $[0, 1]$. Thus if E is such a set and $f=c$ on E, then

$$(E)\int f(x)dx = c.$$

3. Extension of an E-integrable function to a Riemann integrable function. Let E be a set and let f be an E-integrable function. We first extend $f\mid E$ to a function $\bar{f}$ on $\overline{E}$ so that $\bar{f}\mid E=f\mid E$, and $\bar{f}$ is $\overline{E}$-integrable. In fact the extension will be such that the E-integral of f and the $\overline{E}$-integral of $\bar{f}$ are equal.

LEMMA 3.1. *If* $\bar{f}\mid E=f\mid E$ *and* $\liminf_{x=x'} f(x)\leqq\bar{f}(x')\leqq\limsup_{x=x'} f(x)$ *for all* $x'\in\overline{E}-E$, *then* $\bar{f}$ *is discontinuous at* $a\in\overline{E}$ *if and only if* f *is.*

From Lemma 3.1 it follows that if the Lebesgue measure of the set of discontinuities of $f\mid E$ is zero then so is the measure of the set of discontinuities of $\bar{f}\mid\overline{E}$. Thus if f is E-integrable, $\bar{f}$ is $\overline{E}$-integrable.

THEOREM 3.2. *If* $\bar{f}$ *satisfies the hypotheses of Lemma* 3.1, *then*

$$(E)\int f(x)dx = (\overline{E})\int f(x)dx.$$

Proof. Given any $\epsilon>0$ there exists a $\Delta>0$ such that

$$\left|(E)\sum_i f(\bar{x}_i)\Delta x_i - L\right| < \tfrac{1}{2}\epsilon \text{ and } \left|(\overline{E})\sum_i \bar{f}(\bar{x}_i')\Delta x_i - L'\right| < \tfrac{1}{2}\epsilon \text{ when } \delta<\Delta,$$

and L and L' are the values of the respective integrals. The limits are independent of the partition P since the integrals exist. Hence let P be a fixed parti-

tion with norm $\delta<\Delta$.

The sum $(\overline{E})\sum_i \bar{f}(\bar{x}_i')\Delta x_i$ consists of $(E)\sum_i f(\bar{x}_i')\Delta x_i$ plus contribution from points of $\overline{E}\cap P$, which are end points of intervals containing no points of E. Let A be this particular subset of P. Then replace each $x_i\in A$ by x_i' which is within $\frac{1}{2}(\Delta-\delta)$ of x_i in the interval which contains no points of E. If some interval of this type has both end points in A, then replace both of them by new points in the same manner as above. Then the new partition $P'=\{x_i'\}$ thus formed has norm $\delta'<\Delta$, and $\overline{E}\cap[x_{i-1}', x_i']=\varnothing$ if and only if $E\cap[x_{i-1}', x_i']=\varnothing$. Since the inequalities in the first sentence of the proof hold for every choice of $\bar{x}_i$ and $\bar{x}_i'$, let $\bar{x}_i=\bar{x}_i'$. Hence for the partition P',

$$(\overline{E})\sum_i \bar{f}(\bar{x}_i')\Delta x_i' = (E)\sum_i f(x_i')\Delta x_i'.$$

Then

$$\begin{aligned}|L-L'| &= \left|L-(E)\sum_i f(\bar{x}_i')\Delta x_i' + (\overline{E})\sum_i f(\bar{x}_i')\Delta x_i' - L'\right| \\ &\leqq \left|L-(E)\sum_i f(\bar{x}_i')\Delta x_i'\right| + \left|(\overline{E})\sum_i f(\bar{x}_i')\Delta x_i' - L'\right| \\ &< \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon.\end{aligned}$$

Since ϵ is arbitrary, $L=L'$.

We now let I represent the closed interval $[0, 1]$ and extend an E-integrable function to a Riemann integrable function on I. We will assume that E, the domain of f, is closed; for if not, then we need only apply Lemma 3.1 first. Since E is closed, $I-E$ is open and hence the union of a countable collection of open intervals. We will define $\bar{f}$ on I in such a way that it is continuous on these open intervals and at every point of E where f is continuous.

LEMMA 3.3. *Let $\bar{f}$ be defined on I such that $\bar{f}|E=f$ and $\bar{f}$ is monotone and continuous on the closure of each component of $I-E$. Then if f is continuous at $a\in E$, so is $\bar{f}$.*

Proof. Let f be continuous at a. Then $\liminf_{x=a} f(x)=\limsup_{x=a} f(x)=f(a)$. Thus from Definition 2.10 given $\epsilon>0$ there exists a neighborhood N_a of a such that

$$\sup_{x\in N_a\cap E} f(x) - \inf_{x\in N_a\cap E} f(x) < \epsilon.$$

If N_a contains points b, $c\in E$ such that $b<a<c$, then the open interval $(b, c)=N_a'$ is a neighborhood of a for which the above inequality is satisfied and $\bar{f}(N_a')\subset(f(a)-\epsilon, f(a)+\epsilon)$. If there are no points b and c as above, then a is the endpoint of intervals of $I-E$ on either or both sides of a. In this case the continuity and monotonicity of $\bar{f}$ guarantee the existence of a neighborhood N_a' such that $\bar{f}(N_a')\subset(f(a)-\epsilon, f(a)+\epsilon)$. Therefore $\bar{f}$ is continuous at a.

THEOREM 3.4. *If $\bar{f}$ satisfies the conditions of Lemma 3.3, then $\bar{f}$ is Riemann integrable if and only if f is E-integrable.*

If $I-\overline{E}$ is the union of a finite collection of open intervals, we can let $\bar{f}$ be zero on $I-\overline{E}$. Then the E-integral of f and the Riemann integral of $\bar{f}$ are equal. That $\bar{f}$ cannot in general be defined in this manner is illustrated by the following example.

Example 3.5. Let E be the Cantor set of Example 2.17, and let $f(x)=1$ on E. E is closed and $I-E$ is the union of the intervals (3/8, 5/8), (5/32, 7/32) and (25/32, 27/32), and so on. If we let $\bar{f}=0$ on $I-E$ and $\bar{f}\,|\,E=f$, then $\bar{f}$ is not Riemann integrable. For E is nowhere dense in I, and every neighborhood of every point of E contains an interval of $I-E$. Thus $\bar{f}$ is discontinuous at every point of E by definition 2.13 and the Lebesgue measure of E is one half.

We note that if E is everywhere dense in I (i.e. $\overline{E}=I$), f is E-integrable and $\bar{f}$ is the extension of f as in Lemma 3.1, then the E-integral of f and the Riemann integral of $\bar{f}$ are equal by Theorem 3.2. Under the extension of Lemma 3.3, it is clear that in general the two integrals are not equal. However, the definition of $\bar{f}$ on one of the intervals of $I-E$ in Lemma 3.3 could be modified so that $(I)\int\bar{f}(x)dx=(E)\int f(x)dx$. Although the integrals over I and E, respectively, may thus be made equal, in some cases, e.g. the function of Example 3.5, it is impossible to get an extension such that for every interval $[a, b]=I'\subset I$,

$$(I')\int \bar{f}(x)dx = (E\cap I')\int f(x)dx.$$

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A CHARACTERIZATION OF RIEMANN-INTEGRABILITY

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1. Introduction. In 1823 Cauchy [1], restricting himself to functions f continuous over the interval $I=[0, 1]$, gave a definition of what may be called the *left integral* of f: If $P: 0=x_0<x_1<\cdots<x_n=1$ is an arbitrary partition of I into

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nonoverlapping closed intervals $I_i=[x_{i-1}, x_i]$ of lengths $|I_i| = x_i - x_{i-1}$, then the sum

$$S_L(P;f) = \sum_{i=1}^{n} f(x_{i-1})\,|I_i|$$

has a limit $\int_L f(x)dx$ when $\|P\| = \max_i |I_i|$ approaches 0.

In 1854 Riemann [2] proposed a definition of what is now called the *Riemann integral of* f (not necessarily continuous). *The function* f *is called Riemann integrable with the Riemann integral* $\int f(x)dx$ *if and only if, to every* $\epsilon>0$ *there exists* $\delta>0$ *such that*

$$\left| \int f(x)dx - \sum_{i=1}^{n} f(\xi_i)\,|I_i| \right| < \epsilon$$

for all P *with* $\|P\|<\delta$ *no matter how* ξ_i *is chosen in* I_i.

In presentations of the theory of the Riemann integral, the impression is often given that there is an essential difference between Riemann's definition and the older Cauchy definition. We shall show that this is not the case: *Every bounded left integrable function is also Riemann integrable.* (Clearly every Riemann integrable function is left integrable with its left integral equal to its Riemann integral.)

Since the left integral corresponds to a specific choice of the point ξ_i in I_i, it seems natural in this connection to consider other ways of choosing ξ_i. Thus, if ψ is a function of two variables with $\psi(a, b)$ defined for $0 \leqq a < b \leqq 1$ and satisfying $a \leqq \psi(a, b) \leqq b$, let us consider, for each partition P of I, the sum

$$S_\psi(P;f) = \sum_{i=1}^{n} f(\psi(x_{i-1}, x_i))\,|I_i|\,.$$

We shall say that a function f *is* ψ*-integrable with* ψ*-integral* $\int_\psi f(x)dx$*, if and only if, to every* $\epsilon>0$ *there exists* $\delta>0$ *such that*

$$\left| \int_\psi f(x)dx - S_\psi(P;f) \right| < \epsilon$$

for all P *with* $\|P\|<\delta$.

In particular the Cauchy definition corresponds to the choice function $\psi(a, b)=L(a, b)=a$.

Of course every Riemann integrable function is ψ-integrable. We shall prove that, conversely for a large class of functions ψ it is true that every bounded ψ-integrable function is Riemann integrable. This is a useful fact in certain applications (for instance in the theory of Markov processes in which x represents *time*, and where the L-integral arises in a more natural fashion than the Riemann integral).

It is easy to see that for certain ψ there exist ψ-integrable functions f which are *not* Riemann integrable. For example, if $\psi_1(a, b)$ is a rational number in $[a, b]$

and $\psi_2(a, b)$ an irrational number in $[a, b]$, and if f_0 is the function

$$f_0(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ irrational,} \end{cases}$$

then f_0 is ψ_1-integrable and ψ_2-integrable with

$$\int_{\psi_1} f_0(x)dx = 0, \qquad \int_{\psi_2} f_0(x)dx = 1;$$

hence, f_0 is not Riemann integrable. (It is even possible to construct a function ψ such that, for every interval $[a, b]$, the range of ψ on intervals $[\alpha, \beta] \subset [a, b]$ coincides with $[a, b]$, and such that f_0 is ψ-integrable. This can be done by modifying the definition of ψ_2).

2. Left integrals. The theorem proved in this section is actually a special case of the theorem proved in the next section. We believe this redundancy to be worthwhile, however, because of the more elementary method which is available in this special case.

THEOREM 1. *Suppose that f is bounded and $\int_L f(x)dx$ exists. Then f is Riemann integrable.*

Proof. It is enough to prove that when P is a subdivision of I into a sufficiently large even number of subintervals of equal length then

$$\left| \sum f(t_i)(x_i - x_{i-1}) - \int_L f(x)dx \right| < \epsilon$$

for every set $\{t_i\}$, $t_i \in [x_{i-1}, x_i]$.

Let ϵ be given. We determine δ such that

$$\|P\| < \delta \Rightarrow \left| S_L(P;f) - \int_L f(x)dx \right| < \frac{\epsilon}{12}$$

and we determine N such that $1/N < \delta$ and consider the subdivision

$$P_0: 0 = x_0 < x_1 < x_2 < \cdots < x_{2N} = 1$$

of I into $2N$ equal parts.

Let $t_1, t_2, \cdots, t_{2N}$ be arbitrary numbers such that

$$t_i \in [x_{i-1}, x_i], \qquad i = 1, 2, \cdots, 2N.$$

In the following subdivisions:

$$P_1: x_0, t_1, t_2, x_2, t_3, t_4, x_4, \cdots, x_{2N}$$

$$P_2: x_0, t_1, \quad x_2, t_3, \quad x_4, \cdots, x_{2N},$$

the subintervals are shorter than $1/N$. Consequently

$$(2.1) \qquad S_L(P_2;f) - S_L(P_1;f) = \sum_{i=1}^{N} (f(t_{2i-1}) - f(t_{2i}))(x_{2i} - t_{2i}) = \rho_1,$$

with $|\rho_1| < 2\epsilon/12$.

Similarly, by considering the subdivisions

$$P_3\colon x_0, x_1, t_2, t_3, x_3, t_4, t_5, x_5, \cdots, x_{2N-1}, x_{2N},$$

$$P_4\colon x_0, x_1, t_2, \quad x_3, t_4, \quad x_5, \cdots, x_{2N-1}, x_{2N},$$

we find:

$$(2.2) \qquad \sum_{i=1}^{N-1} (f(t_{2i}) - f(t_{2i+1}))\cdot(x_{2i+1} - t_{2i+1}) = \rho_2,$$

with $|\rho_2| < 2\epsilon/12$.

Adding (2.1) and (2.2) we find

$$(2.3) \qquad \sum_{i=2}^{2N} (f(t_{i-1}) - f(t_i))\cdot(x_i - t_i) = \rho_3$$

with $|\rho_3| < 4\epsilon/12$.

The formula (2.3) can be written:

$$(2.3') \qquad \sum_{i=1}^{2N-1} f(t_i)(x_{i+1} - t_{i+1}) - \sum_{i=2}^{2N} f(t_i)(x_i - t_i) = \rho_3.$$

Now we consider the subdivision $P_5\colon x_0, t_1, t_2, t_3, \cdots, t_{2N}, x_{2N}$, and we find

$$(2.4) \quad f(x_0)\cdot(t_1 - x_0) + \sum_{i=1}^{2N-1} f(t_i)\cdot(t_{i+1} - t_i) + f(t_{2N})\cdot(x_{2N} - t_{2N}) = \int_L f(x)dx + \rho_4,$$

with $|\rho_4| < \epsilon/12$.

Adding (2.3') and (2.4) we obtain

$$\sum_{i=2}^{2N-1} f(t_i)(x_{i+1} - x_i) + \rho_5 = \int_L f(x)dx + \rho_4 + \rho_3,$$

where ρ_5 is the sum of five terms which all have absolute values not exceeding M/N, $M = \sup_{x\in I} |f(x)|$.

Assuming that N is chosen so large that $M/N < \epsilon/12$ we find

$$(2.5) \qquad \sum_{i=2}^{2N-1} f(t_i)(x_{i+1} - x_i) = \int_L f(x)dx + \rho_6$$

with $|\rho_6| < 10\epsilon/12$.

Since P_0 is a subdivision of I into equal parts we find

$$\sum_{i=2}^{2N-1} f(t_i)(x_i - x_{i-1}) = \int_L f(x)dx + \rho_6 \tag{2.5'}$$

and consequently

$$\sum_{i=1}^{2N} f(t_i)(x_i - x_{i-1}) = \int_L f(x)dx + \rho_7$$

with $|\rho_7| < 12\epsilon/12 = \epsilon$.

This proves that f is Riemann integrable.

3. ψ-integrals. The theorem proved in this section shows that for a large class of ψ's only Riemann integrable functions are ψ-integrable.

A function F is said to have DP (the Darboux property) if whenever $F(x_1) < z < F(x_2)$, $x_1 < x_2$, there exists an x with $x_1 < x < x_2$ such that $F(x) = z$.

THEOREM 2. *Suppose that for each fixed a and b, $0 \leqq a < b \leqq 1$, both $\psi(a, x)$ and $\psi(x, b)$, $a < x < b$, have DP. Then every bounded ψ-integrable function is Riemann integrable.*

Proof. The proof is by contradiction. Namely, we assume that f is bounded and not Riemann integrable and prove that then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there exist partitions P and P' with $\|P\| < \delta$, $\|P'\| < \delta$ for which

$$|S_\psi(P; f) - S_\psi(P'; f)| > \epsilon_0.$$

It is well known that a bounded function f is Riemann integrable if and only if the (outer) content of the set

$$E_\eta = \left\{ x_0 \in I \,\middle|\, \eta < \lim_{h \to 0} \sup_{x_0 - h < x < x_0 + h} |f(x) - f(x_0)| \right\}$$

is 0 for every $\eta > 0$ (See any good standard textbook on advanced calculus). Hence, since f is not Riemann integrable, there exists $\eta > 0$ such that $|E_\eta| > 0$, where $|S|$ denotes the outer content of S.

By considering, if necessary, $-f$ instead of f, we can actually assume without loss of generality that $|A_\eta| > 0$, where

$$A_\eta = \left\{ x_0 \in I \,\middle|\, \eta < \lim_{h \to 0} \sup_{x_0 - h < x < x_0 + h} (f(x) - f(x_0)) \right\}.$$

Now, let $\delta > 0$ be given. In the first part of the proof we shall find two finite sets $\{J_i\}$ and $\{J_i'\}$, $i = 1, 2, \cdots, \nu$, each consisting of nonoverlapping intervals of length $< \delta$ such that J_i and J_i' differ by "very little" while

$$f(\psi(J_i')) - f(\psi(J_i)) > \eta \tag{3.1}$$

(for convenience, if $J = [\alpha, \beta]$, we write $\psi(J) = \psi(\alpha, \beta)$). These two sets of inter-

vals will be chosen so that the length-sum

$$\sum_{i=1}^{\nu} |J_i| > \frac{|A_\eta|}{16}. \tag{3.2}$$

In the second part of the proof we then construct partitions P and P' with $\|P\| < \delta$, $\|P'\| < \delta$ such that

$$S_\psi(P'; f) - S_\psi(P; f) \geqq \epsilon_0 = \epsilon_0(\eta, |A_\eta|).$$

We begin by finding a finite set of auxiliary nonoverlapping intervals $S_i = [\alpha_i, \beta_i]$, $|S_i| < \delta$, such that

$$A_\eta \subset \bigcup_i S_i, \qquad \sum_i |S_i| < (1 + \tfrac{1}{6})|A_\eta|.$$

Let us call i *good* if

$$|A_\eta \cap S_i| > \tfrac{3}{4}|S_i|. \tag{3.3}$$

Then

$$\begin{aligned} |A_\eta| &= \sum_i |A_\eta \cap S_i| \\ &= \sum_{i \text{ good}} |A_\eta \cap S_i| + \sum_{i \text{ not good}} |A_\eta \cap S_i| \\ &\leqq \sum_{i \text{ good}} |S_i| + \tfrac{3}{4} \sum_{i \text{ not good}} |S_i| \\ &= \tfrac{3}{4} \sum_i |S_i| + \tfrac{1}{4} \sum_{i \text{ good}} |S_i|, \end{aligned}$$

and hence

$$\sum_{i \text{ good}} |S_i| > \frac{|A_\eta|}{2}.$$

If i is good, we shall call i *left* or *right* depending on whether

$$\psi(S_i) \leqq \mu_i = \tfrac{1}{2}(\alpha_i + \beta_i)$$

or

$$\psi(S_i) \geqq \mu_i.$$

Since

$$\sum_{i \text{ left}} |S_i| + \sum_{i \text{ right}} |S_i| \geqq \sum_{i \text{ good}} |S_i| \geqq \frac{|A_\eta|}{2},$$

there is no loss of generality in assuming

$$\sum_{i \text{ left}} |S_i| \geqq \frac{|A_\eta|}{4}. \tag{3.4}$$

(If necessary, we can achieve this by the transformation $x \to 1-x$.) From now on, the index i will be assumed to run only over left values, say $i=1, 2, \cdots, \nu$.

Let

$$\gamma_i = \mu_i + \frac{|S_i|}{4} = \frac{\alpha_i + 3\beta_i}{4}.$$

By (3.3) it follows that the open interval (μ_i, γ_i) contains some point $y_i \in A_\eta$, $i=1, 2, \cdots, \nu$, as well as a point y_i' arbitrarily close to y_i such that

$$f(y_i') - f(y_i) > \eta. \tag{3.5}$$

Put $g(x)=\psi(x, \beta_i)$ (i fixed). Note that

$$g(\alpha_i) = \psi(S_i) \leqq \mu_i, \qquad g(\gamma_i) \geqq \gamma_i.$$

Then, since g has DP, there exist points t_i and t_i' with

$$\alpha_i < t_i < \gamma_i, \qquad \alpha_i < t_i' < \gamma_i$$

such that

$$g(t_i) = y_i, \qquad g(t_i') = y_i'.$$

Furthermore, since y_i' can be chosen arbitrarily close to y_i, it follows from a somewhat more refined, but straightforward exploitation of DP that $|t_i' - t_i|$ can be made arbitrarily small, say,

$$|t_i' - t_i| \leqq h = \frac{\eta |A_\eta|}{48\nu M}, \tag{3.6}$$

where $M=\sup_{0\leq x\leq 1} |f(x)|$.

Let now J_i be the interval $[t_i, \beta_i]$ and J_i' the interval $[t_i', \beta_i]$. Since both are contained in S_i, their lengths are $<\delta$. Since $\psi(J_i)=y_i, \psi(J_i')=y_i'$, it follows from (3.5) that (3.1) holds. Since $|J_i| > \frac{1}{4}|S_i|$, it follows from (3.4) that (3.2) holds.

We now proceed to construct two partitions P and P' such that

$$S_\psi(P'; f) - S_\psi(P; f) \geqq \epsilon_0(\eta, |A_\eta|, M).$$

Let $\{K_j\}$ be a partition of

$$I - \bigcup_{i=1}^{\nu} (J_i \cup J_i'),$$

with all $|K_j| < \delta$. We then define P to be the partition consisting of the closures of the non-empty intervals among

$$\{J_i\}, \{J_i' - J_i\}, \{K_j\},$$

and P' to be the partition consisting of the nonempty intervals among

$$\{J_i'\}, \{J_i - J_i'\}, \{K_j\}.$$

Then

$$S_\psi(P';f) - S_\psi(P;f) = \sum_{i=1}^{\nu} f(y_i')\,|\,J_i'\,| - \sum_{i=1}^{\nu} f(y_i)\,|\,J_i\,| + \Delta_1,$$

where Δ_1, the contribution due to the intervals $\{J_i - J_i'\}$ and $\{J_i' - J_i\}$, satisfies $|\Delta_1| \leqq \nu Mh$, by (3.6).

On the other hand,

$$\sum_{i=1}^{\nu} f(y_i')\,|\,J_i'\,| - \sum_{i=1}^{\nu} f(y_i)\,|\,J_i\,| = \sum_{i=1}^{\nu} \{f(y_i') - f(y_i)\}\,|\,J_i\,| + \Delta_2,$$

where

$$|\,\Delta_2\,| \leqq \sum_{i=1}^{\nu} |\,f(y_i')\,|\,|\,t_i' - t_i\,| \leqq \nu Mh.$$

It follows that

$$S_\psi(P';f) - S_\psi(P;f) = \sum_{i=1}^{\nu} \{f(y_i') - f(y_i)\}\,|\,J_i\,| + \Delta_1 + \Delta_2$$

$$\geqq \eta \sum_{i=1}^{\nu} |\,J_i\,| - 2\nu Mh$$

$$\geqq \frac{\eta\,|\,A_\eta\,|}{16} - 2\nu Mh \geqq \frac{\eta\,|\,A_\eta\,|}{48} = \epsilon_0.$$

Remarks. The hypothesis on ψ is satisfied, in particular, if ψ is continuous in each variable. For instance, if we take $\psi(a, b) = L(a, b) = a$, we obtain Theorem 1 as a special case of Theorem 2. By taking $\psi(a, b) = M(a, b) = \frac{1}{2}(a+b)$ we obtain the following: *If the "midpoint" integral $\int_M f(x)dx$ of the bounded function f exists, then f is Riemann integrable.*

It is of course possible, for certain ψ, that unbounded functions may be ψ-integrable. For example, if f is monotone in some open interval $(1-\epsilon, 1)$, then existence of the left integral, $\int_L f(x)dx$, is equivalent to existence of the improper Riemann integral $\lim_{\epsilon \to 0} \int_0^{1-\epsilon} f(x)dx$.

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CARD-GUESSING WITH INFORMATION— A PROBLEM IN PROBABILITY

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Let us consider the following card-guessing experiment. A deck of cards is placed face downwards on a table. This deck has n^2 cards, each of which bears one of n different symbols, and each symbol occurs exactly n times. (In telepathy experiments $n=5$ is common). One person (the percipient) makes a guess at the symbol on the top card of the deck. Another person (the experimenter) looks at this card (the *target* card) and records whether the percipient's guess was correct (a *hit*) or incorrect (a *miss*). The target card is then put on one side, and the procedure is repeated with the remainder of the deck. The experiment continues until the deck is exhausted.

It is clear that if the percipient has no information about the cards that have gone, and does not know whether his guesses were right or wrong, then in the absence of telepathy or clairvoyance the expected number of hits that the percipient will make is n. We now consider how this expectation is increased if the percipient is given complete information about each card after he has guessed its symbol; for the sake of concreteness we may imagine that after each guess the target card is placed face upward on the table.

At each stage of this modified experiment the percipient will know how many symbols of each kind are left in what remains of the deck. His best strategy is then to guess at the symbol which occurs most often (or at one of them if there is a tie for the most frequent symbol). Suppose that after r guesses the remainder of the deck contains α_1 cards bearing the first symbol, α_2 bearing the second symbol, and so on, so that $\alpha_1+\alpha_2+\cdots+\alpha_n=n^2-r$. Let $E(\alpha_1, \alpha_2, \cdots, \alpha_n)$ denote the expected number of hits to be made from this stage onwards.

Let us suppose that $\max(\alpha_1, \alpha_2, \cdots, \alpha_n)=\alpha_1$, so that the percipient will make the first symbol his next guess. The probability of a hit is $\alpha_1/(n^2-r)$ in which case the expectation is $1+E(\alpha_1-1, \alpha_2, \cdots, \alpha_n)$; if the card bears not the first symbol but the ith symbol (probability $\alpha_i/(n^2-r)$) then the expectation is $E(\alpha_1, \alpha_2, \cdots, \alpha_i-1, \cdots, \alpha_n)$. Hence, if E_j denotes the expected number of hits when there are α_i cards bearing the ith symbol ($i\neq j$) and α_j-1 cards bearing the jth symbol, then

$$E(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{\alpha_1}{n^2-r}[1+E_1] + \frac{\alpha_2}{n^2-r}E_2 + \cdots + \frac{\alpha_n}{n^2-r}E_n.$$

Thus in the general case we have

$$E(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{1}{n^2-r}\left[\max_{1\leq k\leq n}\alpha_k + \sum_{j=1}^{n}\alpha_j E_j\right]. \tag{1}$$

By way of example, when $n=3$ we have

$$E(3,3,3) = \frac{1}{9}[3 + 3E(2,3,3) + 3E(3,2,3) + 3E(3,3,2)].$$

Since, by symmetry,

$$E(2,3,3) = E(3,2,3) = E(3,3,2), \tag{2}$$

we may simplify this to

$$E(3,3,3) = \frac{1}{3} + E(3,3,2).$$

Similarly

$$E(3,3,2) = \frac{1}{8}[3 + 3E(2,3,2) + 3E(3,2,2) + 2E(3,3,1)]$$

$$= \frac{3}{8} + \frac{3}{4}E(3,2,2) + \frac{1}{4}E(3,3,1),$$

and so on. In this way the expectations after r guesses can be expressed in terms of those after $r+1$ guesses. Since $E(1, 0, 0, \cdots, 0)=1$ (the last card can always be guessed correctly), the required expectation $E(n, n, \cdots, n)$ can be calculated by this step by step process. This would be laborious, however, and we look for some way of lightening the labour.

Let us replace each expression $E(\alpha_1, \alpha_2, \cdots, \alpha_n)$ by the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x^{\alpha_n}$, where $x_1, x_2, \cdots, x_n$ are indeterminates. The expression

$$\sum_{i=1}^{n} \alpha_i E(\alpha_1, \alpha_2, \cdots, \alpha_i - 1, \cdots, \alpha_n)$$

will then be replaced by

$$\sum_{i=1}^{n} \alpha_i x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n} = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n}\right) x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Using the monomials the recursive relation (1) can be reduced go the following rule: act on the monomial with the operator $(\partial/\partial x_1+\partial/\partial x_2+ \cdots +\partial/\partial x_n)$; to the result add the highest power that occurs in the monomial, and finally divide the result by n^2-r. For $n=3$ we start with $x_1^3x_2^3x_3^3$ and derive first

$$\frac{1}{9}[3x_1^2x_2^3x_3^3 + 3x_1^3x_2^2x_3^3 + 3x_1^3x_2^3x_3^2 + 3]. \tag{3}$$

By virtue of (2) the monomials $x_1^2x_2^3x_3^3$, $x_1^3x_2^2x_3^3$ and $x_1^2x_2^3x_3^3$ represent the same number. Hence we shall allow monomials differing only by a permutation of the variables to be lumped together. We shall call this process *condensation*. The *condensed* version of (3) is then $\frac{1}{9}[9x_1^2x_2^3x_3^3+3]$.

It is instructive to carry out the complete procedure for $n=3$. At each stage we obtain a constant term representing the expected number of hits just made. These play no further part in the calculation, so we put them on one side and add them up at the end. We obtain the following polynomials

$$
\begin{array}{|l|l}
x_1^3x_2^3x_3^3 & \\
\frac{1}{9}[9x_1^2x_2^3x_3^3] & +\frac{1}{3} \\
\frac{1}{9.8}[18x_1x_2^3x_3^3+54x_1^2x_2^2x_3^3] & +\frac{3}{8} \\
\frac{1}{9.8.7}[18x_2^3x_3^3+324x_1x_2^2x_3^3+162x_1^2x_2^2x_3^2] & +\frac{3}{7} \\
\frac{5!}{9!}[432x_2^2x_3^3+648x_1x_2x_3^3+1944x_1^2x_2^2x_3] & +\frac{25}{56} \\
\frac{4!}{9!}[2160x_2x_3^3+3240x_2^2x_3^2+9720x_1x_2^2x_3^2] & +\frac{33}{70} \\
\frac{3!}{9!}[2160x_3^3+38880x_2x_3^2+19440x_1x_2x_3] & +\frac{15}{28} \\
\frac{2!}{9!}[45360x_3^2+136080x_2x_3] & +\frac{4}{7} \\
\frac{1!}{9!}[362880x_3] & +\frac{5}{8} \\
\frac{0!}{9!}[362880] & +1
\end{array} \tag{4}
$$

The bottom term to the left of the vertical dotted line represents $E(0, 0, \cdots, 0)$ which is clearly zero. The constant terms which have been put to the right of the dotted line are the expectations, and their sum (which is $4\frac{661}{840}$) is the expected number of hits.

It will be seen that the constants, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{3}{7}$, etc., to the right of the dotted line can be found from the polynomials to the left and one line above. If we denote these polynomials successively by P_0, P_1, P_2, $\cdots$, then

$$P_r=\frac{(n^2-r)!}{(n^2)!}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\cdots+\frac{\partial}{\partial x_n}\right)^r x_1^n x_2^n\cdots x_n^n \tag{5}$$

followed by condensation as described above.

Let us take a typical term in P_r, and suppose that in it p_0 indeterminates occur with zero power, that p_1 of them occur to the first power, p_2 of them are

squared, and so on. Then

$$\begin{aligned} p_0 + p_1 + p_2 + \cdots + p_n &= n, \\ p_1 + 2p_2 + \cdots + np_n &= n^2 - r. \end{aligned} \tag{6}$$

Such a term will arise from the condensation of $n!/p_0!p_1! \cdots p_n!$ terms of (5) in which the variables occur in different orders. If a variable x_ν occurs to the power i, then x_ν^n must have been operated on by $(\partial/\partial x_\nu)^{n-i}$, and this operation will introduce a factor $n(n-1) \cdots (n-n+i+1) = n!/i!$. Hence, applying the multinomial theorem to the operator in (5), we see that a term in (5) which is of the type described by (6) will have coefficient

$$\frac{(n^2-r)!}{(n^2)!} \cdot \frac{r!}{[n!]^{p_0}[(n-1)!]^{p_1}[(n-2)!]^{p_2} \cdots} \left(\frac{n!}{0!}\right)^{p_0} \left(\frac{n!}{1!}\right)^{p_1} \left(\frac{n!}{2!}\right)^{p_2} \cdots$$

or

$$\frac{(n^2-r)!r!}{(n^2)!} \binom{n}{0}^{p_0} \binom{n}{1}^{p_1} \binom{n}{2}^{p_2} \cdots .$$

On condensation the $n!/p_0!p_1! \cdots p_n!$ such terms will give a single term for which the coefficient is

$$\frac{(n^2-r)!r!}{(n^2)!} \cdot \frac{n!}{p_0!p_1! \cdots p_n!} \binom{n}{0}^{p_0} \binom{n}{1}^{p_1} \cdots \binom{n}{n}^{p_n}. \tag{7}$$

Each such term will contribute to the expectations occurring on the right of the dotted line in (4), and it is easily seen that this contribution is found by multiplying the coefficient by the highest power occurring in the term, i.e. by the largest integer k for which $p_k > 0$, and dividing by $n^2 - r$. For a given value of k, the terms for which this value applies will therefore contribute

$$\sum_{r=0}^{n-1} \left\{ \frac{k(n^2-r)!r!}{(n^2)!(n^2-r)} \sum \frac{n!}{p_0!p_1! \cdots p_k!} \binom{n}{0}^{p_0} \binom{n}{1}^{p_1} \cdots \binom{n}{k}^{p_k} \right\} \tag{8}$$

where the second summation is for

$$\begin{aligned} p_0 + p_1 + p_2 + \cdots + p_k &= n, \\ p_1 + 2p_2 + \cdots + kp_k &= n^2 - r, \\ p_k &\geqq 1. \end{aligned}$$

This second summation is the coefficient of t^{n^2-r} in

$$\left[\binom{n}{0} + \binom{n}{1}t + \cdots + \binom{n}{k}t^k\right]^n - \left[\binom{n}{0} + \binom{n}{1}t + \cdots + \binom{n}{k-1}t^{k-1}\right]^n$$

in which the second expression subtracts the contributions for $p_k=0$, which are not to be included. If we write

$$\phi_k(t) = \left\{\binom{n}{0} + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{k}t^k\right\}^n,$$

then (8) can be written

$$\sum_{r=0}^{n^2-1}\left\{\frac{k}{n^2\cdot\binom{n^2-1}{r}} \times \text{coefficient of } t^{n^2-r} \text{ in } \phi_k(t) - \phi_{k-1}(t)\right\}.$$

Summing over all values of k, we have a total contribution

$$\frac{1}{n^2}\sum_{r=0}^{n^2-1}\frac{A_{n^2-r}}{\binom{n^2-1}{r}},$$

where A_m stands for the coefficient of t^m in

$$\begin{aligned} &n\{\phi_n(t) - \phi_{n-1}(t)\} + (n-1)\{\phi_{n-1}(t) - \phi_{n-2}(t)\} \\ &\quad + \cdots + 2\{\phi_2(t) - \phi_1(t)\} + \phi_1(t) - \phi_0(t) \\ &= n\phi_n(t) - \sum_{s=0}^{n-1}\phi_s(t) \qquad (9) \\ &= n(1+t)^{n^2} - \sum_{s=0}^{n-1}\phi_s(t). \end{aligned}$$

Hence the expected number of hits will be

$$\frac{1}{n^2}\sum_{m=1}^{n^2}\frac{A_m}{\binom{n^2-1}{m-1}} \qquad (m = n^2 - r) \qquad (10)$$

with A_m defined by (9).

For $n=3$, A_m is the coefficient of t^m in

$$3(1+t)^9 - 1 - (1+3t)^3 - (1+3t+3t^2)^3$$
$$= 9t + 45t^2 + 144t^3 + 270t^4 + 297t^5 + 225t^6 + 108t^7 + 27t^8 + 3t^9.$$

Hence (10) becomes

$$\frac{1}{9}\left\{\frac{9}{1} + \frac{45}{8} + \frac{144}{28} + \frac{270}{56} + \frac{297}{70} + \frac{225}{56} + \frac{108}{28} + \frac{27}{8} + \frac{3}{1}\right\}$$
$$= 1 + \frac{5}{8} + \frac{4}{7} + \frac{15}{28} + \frac{33}{70} + \frac{25}{56} + \frac{3}{7} + \frac{3}{8} + \frac{1}{3}$$

as before.

It is of interest to record the values (correct to two decimal places) of these expectations for a few low values of n. They are as follows.

n	1	2	3	4	5
Expectation	1	2.83	4.79	6.73	8.65

THE SLACKNESS OF FINITE PACKINGS IN E_2†

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In a recent address [1] before the American Mathematical Society, Professor H. Zassenhaus reviewed the concept of the density of a packing. Taking particular note of the limitations which this concept has in comparing different finite packings he introduced what he has termed the slackness function.

Let K^* be a centrally symmetric convex domain with boundary K and let μ be the Minkowski distance function which is determined thereby. A packing with respect to K is defined as a pair (Π, E) where Π is a Jordan polygon and E is a finite set of points $\{P_1, P_2, \cdots, P_{N-n}, P_{N-n+1}, \cdots, P_N\}$ such that $P_1, P_2, \cdots, P_{N-n}$ are the vertices of Π and $P_{N-n+1}, \cdots, P_N$ are contained in its interior Π^*. Further the Minkowski distance $\mu(P_i, P_j)$ $(i \neq j)$ shall be not less than 1 whenever the segment P_iP_j lies in $\Pi^* \cup \Pi$.

The slackness function of a packing (Π, E) with respect to K is defined by:

$$\sigma(\Pi, E, K) = \frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} + 1 - N$$

where $A(\Pi^*)$ is the area of Π^*, $M(\Pi)$ is the μ-length of Π, N is the number of points in E and Δ is the critical determinant of K.

We have shown [2] that the function $\sigma(\Pi, E, K)$ as defined above is nonnegative.

Our purpose here is to demonstrate that *if K is other than a parallelogram then a necessary condition that $\sigma(\Pi, E, K) = 0$ is that E consist of points of a lattice critical with respect to K and that sides of Π each have μ-length equal to* 1.

The proof of this assertion which is to follow is by means of induction over $N+n$ where N is the number of points in E and n is the number of points of E which are in the interior of Π. It rests upon the possibility of decomposing a packing (Π, E) for which $\sigma(\Pi, E, K) = 0$ into two others to each of which the inductive hypothesis is applicable.

Such a decomposition is to be achieved by means of a *linkage*, i.e. a simple

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polygonal path joining a pair of vertices of Π which is either a diagonal of Π of μ-length 1 or it has vertices which are points of E in Π^* and sides each of μ-length 1.

We claim that *for* $N+n>3$ *if* $\sigma(\Pi, E, K)=0$ *then there exists a linkage in* (Π, E).

With $\{P_1, P_2, \cdots, P_{N-n}\}=E^0$ the vertices of Π and $\{P_{N-n+1}, \cdots, P_N\}=E^I$ the points of E in Π^* we let $L(P_i)(P_i \in E^0)$ denote the set of points consisting of P_i itself and those of E each of which can be joined to P_i by a polygonal path lying, except for P_i and possibly its other end point, in Π^*, which has its vertices in E^I and sides of unit μ-length.

It should be observed that such paths may indeed be assumed to be simple. For suppose that $P_{j_0}P_{j_1}P_{j_2}\cdots P_{j_r}$, a path such as we have just described, has a self-intersection, say at Q. Then Q belongs to each of two segments, say $P_{j_s}P_{j_{s+1}}$ and $P_{j_t}P_{j_{t+1}}$. Q cannot, however, be an interior point of either of these. For then

$$\mu(P_{j_s}, P_{j_{t+1}}) + \mu(P_{j_t}, P_{j_{s+1}}) < \mu(P_{j_s}, Q) + \mu(P_{j_{t+1}}, Q) + \mu(P_{j_t}, Q) + \mu(P_{j_{s+1}}, Q) = 2$$

and at least one of the terms $\mu(P_{j_s}, P_{j_{t+1}})$ and $\mu(P_{j_t}, P_{j_{s+1}})$ would be less than 1. Suppose $\mu(P_{j_s}, P_{j_{t+1}})$ is less than 1. This could occur only if $P_{j_s}P_{j_{t+1}}$ intersected Π requiring that there be a vertex, P_a, of Π in the interior of the triangle $P_{j_s}QP_{j_{t+1}}$. P_a in turn would have μ-distance from P_{j_s}, in particular, less than 1 requiring a second vertex of Π in the interior of $P_{j_s}QP_{j_{t+1}}P_a$. Repeating the argument leads finally to a vertex, P_b, of Π such that $\mu(P_b, P_{j_s})<1$ and $P_bP_{j_s}$ lies in $\Pi^* \cup \Pi$ which is a contradiction. Thus the two segments $P_{j_s}P_{j_{s+1}}$ and $P_{j_t}P_{j_{t+1}}$ have an end point in common. That is, self intersections of $P_{j_0}P_{j_1}\cdots P_{j_r}$ can occur only at its vertices. Suppose $P_{j_l} \equiv P_{j_m}$ $(l<m)$ then we replace $P_{j_0}P_{j_1}\cdots P_{j_r}$ by $P_{j_0}\cdots P_{j_l}P_{j_{m+1}}\cdots P_{j_r}$ and proceeding similarly at each self intersection arrive thereby at a simple path.

Let us assume now that there is no linkage in (Π, E). This implies that

$$L(P_i) \cap L(P_j) = \phi \qquad (1 \leqq i < j \leqq N - n).$$

Moreover there exist neighborhoods of the points of $L(P_i)$ and in particular a neighborhood of P_i which does not intersect the sides of Π other than those having P_i as an end point with the following property. If $L(P_i)$ is varied in such a way that the vectors joining P_i to the other points of $L(P_i)$ are preserved and each point of $L(P_i)$ remains within its neighborhood while all other points of E remain fixed then (Π, E) remains a packing with respect to K provided only that the sides of Π remain of μ-length not less than 1. Furthermore, two vertices P_i and P_j may be subject simultaneously to local variation the corresponding sets $L(P_i)$ and $L(P_j)$ being varied as above and all other points of E held fixed provided that the sides of Π remain of μ-length not less than 1.

We claim that such a local variation of (Π, E) must exist which leads to a decrease in $\sigma(\Pi, E, K)$. For the analytical details relating to the local variations

to be considered we refer to the proof of the inequality: $\sigma(\Pi, E, K) \geqq 0$ [3] and provide here only an outline of the argument.

If Π has only three vertices the steps are the same as must be taken in the particular case $N=3$ which we consider later. So we suppose that Π has at least four vertices and that P_2 (if need be, after re-enumeration) is one at which the interior angle is less than π. Were $\mu(P_1, P_2)$ and $\mu(P_2, P_3)$ each greater than 1, then varying P_2 along P_2P_3 towards P_3 or along P_2P_1 towards P_1 would in either case yield a decrease in σ. So let us suppose that $\mu(P_2, P_3)$, say, is equal to 1 and consider next the three sides P_1P_2, P_2P_3 and P_3P_4.

Two cases are distinguished according as the interior angle at P_3, $\angle P_3 \geqq \pi$ or $\angle P_3 < \pi$.

In the case $\angle P_3 \geqq \pi$ it is found that a variation of P_2 and P_3 under which the μ-lengths of P_1P_2, P_2P_3 and P_3P_4 are preserved and $\angle P_4$ is decreased leads to a decrease in σ.

The case $\angle P_3 < \pi$ is treated as follows. If $\mu(P_1, P_2)$ and $\mu(P_3, P_4)$ were both greater than 1 then σ could be decreased by moving P_2 along P_1P_2 towards P_1 while P_3 varies in such a way as to preserve the vector from P_2 to P_3. So let us suppose that $\mu(P_1, P_2) = 1$ as well as $\mu(P_2, P_3) = 1$. Varying P_3 along P_3P_4 and P_2 in such a way that $\mu(P_1, P_2)$ and $\mu(P_2, P_3)$ remain equal to 1 we find that σ decreases by suitably choosing the direction in which P_3 is moved: towards P_4 or away from P_4. Such a choice is not available if $\mu(P_3, P_4) = 1$ also and this is the final possibility to be considered. The vertices P_2 and P_3 are now varied in such a way that $\mu(P_1, P_2)$, $\mu(P_2, P_3)$ and $\mu(P_3, P_4)$ each remain equal to 1. Again we find that σ is a strictly monotone function of the variation parameter or has a local maximum so that a decrease in σ is always possible.

Thus the absence of a linkage in (Π, E) implies the existence of a packing (Π', E') for which $\sigma(\Pi', E', K) < \sigma(\Pi, E, K)$. Since, however, $\sigma(\Pi', E', K) \geqq 0$ it follows that if $\sigma(\Pi, E, K) = 0$ then there must exist a linkage in (Π, E). Let $\lambda = P_iP_{k_1} \cdots P_{k_r}P_j$ be such a linkage.

λ divides (Π, E) into two packings (Π_1, E_1) and (Π_2, E_2) where Π_1 has vertices $P_i, P_{i+1}, \cdots, P_{N-n}, P_1, \cdots, P_j, P_{k_1}, P_{k_2}, \cdots, P_{k_r}$ and the vertices of Π_2 are $P_j, P_{j+1}, \cdots, P_{i-1}, P_i, P_{k_1}, \cdots, P_{k_r}$ (where, we suppose, as we may, that $i > j$). Also E_1 is the subset of E in the interior of Π_1 and E_2 the subset of E in the interior of Π_2.

Denote by $N(\Pi_i, E_i)$ the number of points in E_i $(i=1, 2)$ and by $n(\Pi_i, E_i)$ the number of points of E_i in the interior of Π_i $(i=1, 2)$. If λ is a diagonal then P_i and P_j are not consecutive so that $N(\Pi_i, E_i) < N$ $(i=1, 2)$ while $n(\Pi_i, E_i) \leqq n$ $(i=1, 2)$. If λ is not a diagonal then $n(\Pi_i, E_i) < n$ $(i=1, 2)$ while $N(\Pi_i, E_i) \leqq N$ $(i=1, 2)$. Thus in each case

$$N(\Pi_i, E_i) + n(\Pi_i, E_i) < N + n \qquad (i = 1, 2).$$

Assume that the proposition is true for packings (Π', E') for which $N(\Pi', E') + n(\Pi', E') < N + n$. We have

$$\sigma(\Pi_1, E_1, K) = \frac{A(\Pi_1^*)}{\Delta} + \frac{M(\Pi_1)}{2} + 1 - N(\Pi_1, E_1),$$

$$\sigma(\Pi_2, E_2, K) = \frac{A(\Pi_2^*)}{\Delta} + \frac{M(\Pi_2)}{2} + 1 - N(\Pi_2, E_2).$$

We note that

$$A(\Pi_1^*) + A(\Pi_2^*) = A(\Pi^*),$$
$$M(\Pi_1) + M(\Pi_2) = M(\Pi) + 2M(\lambda) = M(\Pi) + 2(r+1),$$
$$N(\Pi_1, E_1) + N(\Pi_2, E_2) = N(\Pi, E) + r + 2.$$

Hence

$$\sigma(\Pi_1, E_1, K) + \sigma(\Pi_2, E_2, K) = \frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} + 1 - N(\Pi, E)$$
$$= \sigma(\Pi. E, K).$$

But $\sigma(\Pi, E, K) = 0$ and $\sigma(\Pi_i, E_i, K) \geqq 0$ $(i = 1, 2)$. Thus $\sigma(\Pi_i, E_i, K) = 0$ $(i = 1, 2)$ and the inductional assumption says that E_1 and E_2 are each subsets of a critical lattice. Since, however, E_1 and E_2 have at least two points in common and we are supposing K not a parallelogram, it follows ([4], Lemma 5) that $E = E_1 \cup E_2$ is a subset of a critical lattice. Moreover, Π_1 and Π_2 each having sides of μ-length 1, this is true of Π also.

The proof of the proposition is thus reduced to considering the case $N = 3$, namely that of a triangle T with E consisting only of the vertices P_1, P_2 and P_3 of T.

Arguing as above, if T has a pair of sides each of μ-length greater than 1 then $\sigma(T, E)$ can be decreased in the manner described. Assume therefore that T has just one side, say P_2P_3 with μ-length greater than 1. Holding P_1 fixed we vary P_2 and P_3 in such a way that the direction of P_2P_3 is preserved and $\mu(P_1, P_2)$ and $\mu(P_1, P_3)$ remain equal to 1. By suitably choosing the sense in which the line through P_2 and P_3 is translated, namely nearer to or further from P_1, again σ can be decreased. It follows that $\sigma(T, E) = 0$ implies that $\mu(P_1, P_2) = \mu(P_2, P_3) = \mu(P_3, P_1) = 1$. We know therefore [5] that P_1, P_2 and P_3 generate an admissible lattice. Furthermore $A(T^*)/\Delta + \frac{3}{2} + 1 - 3 = 0$ so that $A(T^*) = \frac{1}{2}\Delta$ and P_1, P_2 and P_3 belong in fact to a critical lattice. Thus the proposition is valid for $N = 3$ and the proof is complete.

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ANALYSIS OF THE MATHEMATICAL TRAINING OF GRADUATE STUDENTS IN THE BIOLOGICAL, MANAGEMENT, AND SOCIAL SCIENCES

The Committee on the Undergraduate Program in Mathematics (CUPM)* is an action committee of the Mathematical Association of America, charged with formulating recommendations for revising and upgrading the undergraduate mathematics curriculum and with making efforts to effect these recommendations. With support from the National Science Foundation and the Ford Foundation, CUPM operates through four Panels, each one being responsible for a particular segment of undergraduate education which is involved with mathematics. One of these is the Panel on Mathematics for the Biological, Management, and Social Sciences (BMSS Panel).† The present report is the result of a study made by the Panel for the purpose of determining some facts about the status of mathematics in the curriculum of students who are in graduate work in one of several fields of interest to the Panel. The results are helping the Panel proceed with its work of designing curricula, and we hope they will be of interest to others.

The study proceeded in the following way. In ten of the leading graduate schools in the country we analyzed (i) the mathematical training as undergraduates of the first-year graduate students who entered graduate school during the academic year, 1960–61, (ii) the undergraduate statistical training of these graduate students, (iii) the level of mathematics used in dissertations accepted since September 1, 1958, (iv) courses in mathematics and statistics taken by the writers of these dissertations as graduate students, and (v) the subject matter graduate courses offered that used mathematics at the level of calculus or beyond. The analysis of (i) and (ii) was made from the undergraduate transcripts of the students; the analysis of (iv) from the graduate transcripts of the pertinent students; the analysis of (iii) from an actual examination of the dissertations; and the analysis of (v) from an examination of the Announcement of Courses and some informal consultations.

Data from the following ten universities were used: Chicago, Columbia, Cornell, Harvard, Michigan, Minnesota, Pennsylvania, Stanford, Wisconsin, and Yale.‡ Within a given university, analysis was made for the following eight de-

* R. C. Buck, *Chairman*, E. G. Begle, L. W. Cohen, W. T. Guy, Jr., R. D. James, J. L. Kelley, J. G. Kemeny, E. E. Moise, J. C. Moore, Frederick Mosteller, H. O. Pollak, G. B. Price, Patrick Suppes, Henry Van Engen, R. J. Walker, A. D. Wallace, R. J. Wisner, *Executive Director*, A. W. Tucker, *ex officio*.

† Patrick Suppes, *Chairman*, D. G. Chapman, W. K. Estes, R. D. James, Harold Kuhn, Frederick Mosteller, Howard Raiffa, G. S. Watson, R. C. Buck, *ex officio*, R. J. Wisner, *ex officio*.

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partments or schools: anthropology (physical and cultural), biochemistry, business administration, economics, political science, psychology (excluding clinical psychology), sociology, and zoology.†

Detailed instructions were given for handling the problems of administrative classifications of these departments in different universities, in order to achieve a relatively high homogeniety in the student bodies studied. For example, in universities which included zoology or biochemistry as part of a larger biological sciences department, it was recommended that an attempt be made to consider only the students who would be majoring in zoology or, as the case may be, majoring in biochemistry. We also asked that students concentrating in genetics not be included under zoology, although in some universities they fall within the department of zoology. In some universities anthropology and sociology are one administrative unit, but an attempt was made to make a reasonably exact classification of graduate students into one of the two categories.

For each of the eight departments or schools in a given university, we asked that the entire population of students be tabulated (for example, all entering first-year graduate students in economics) unless the size of the total population exceeded forty. When the number of students within a given category exceeded forty, we asked that it be sampled either randomly or systematically alphabetically. We left the determination of the exact sample size when it was greater than forty to the individual analyst. The summarizing tables below indicate what data are based on a sample rather than an entire population.

TABLE IA

UNDERGRADUATE TRAINING IN MATHEMATICS OF FIRST-YEAR GRADUATE STUDENTS, 1960–61

	Anthropology	Biochemistry	Bus. Adm.	Economics	Poli. Sci.	Psychology	Sociology	Zoology
No. mathematics	33	1	78	19	112	25	34	25
Precalculus	28	10	184	48	69	69	31	77
Calculus	15	25	138	45	28	49	11	37
Math. beyond calculus	4	21	117	32	16	33	7	20
Totals	80	57	517*	144	225**	176	83	159

* Random samples rather than entire populations used for Chicago, Columbia, Harvard, Pennsylvania, and Stanford in Tables IA, IB, and IC.

** Random sample rather than entire population used for Columbia in Tables IA, IB, and IC.

In Table IA the undergraduate training in mathematics of first-year graduate students (1960–61) is classified into one of four categories: no mathematics; precalculus mathematics; calculus but not beyond; and mathematics beyond calculus. The data in Table IA are incomplete in two major respects. First, because of the difficulty of evaluating the undergraduate transcripts of foreign

† No data on biochemistry are reported from Harvard.

students, they are not included under this heading. Secondly, because first-year graduate students are often accepted during their senior year, many of the undergraduate transcripts are incomplete. The data reported are from the actual transcripts. No attempt was made to get the supplementary data for the last half of the senior year.

TABLE IB

NUMBER OF SEMESTER HOURS OF COLLEGE MATHEMATICS ON UNDERGRADUATE TRANSCRIPTS OF FIRST-YEAR GRADUATE STUDENTS, 1960–61

No. of Hours	Anthro-pology	Biochem-istry	Bus. Adm.	Econom-ics	Poli. Sci.	Psychol-ogy	Sociology	Zoology
0	33	1	78	19	112	25	34	25
1–3	13	1	43	10	27	22	12	23
4–6	12	8	95	28	35	35	22	39
7–9	11	8	89	28	21	36	6	30
10–12	3	14	59	18	12	23	2	17
13–15	1	5	33	10	5	11	2	8
16–20	3	16	54	15	7	11	1	9
21–25	2	4	38	6	1	7	2	4
26–35	1	0	15	7	4	5	1	2
≥ 36	1	0	10	3	2	0	1	2
Totals	80	57	514	144	226	175	83	159

TABLE IC

UNDERGRADUATE TRAINING IN STATISTICS OF FIRST-YEAR GRADUATE STUDENTS, 1960–61

	Anthro-pology	Biochem-istry	Bus. Adm.	Econom-ics	Poli. Sci.	Psychol-ogy	Sociology	Zoology
No. statistics	68	55	267	60	201	47	42	137
Precalculus statistics	12	2	208	71	22	121	39	17
Postcalculus statistics	0	0	43	13	2	8	2	5
Totals	80	57	518	144	225	176	83	159

The number of semester hours of college mathematics on the undergraduate transcripts of the first-year graduate students (1960–61) are shown in Table IB. The two remarks made about Table IA also apply to IB. Minor discrepancies in the totals of the two tables are due to inability to classify certain students in terms of the courses listed on their transcripts, etc. The same remark applies to the minor discrepancies in totals for Table IC above, and Tables IIA, IIB, IIC, and III as well. None of these discrepancies is sufficient to disturb in any way the interpretation of the results.

Table IC contains data on the undergraduate training in statistics of the first-year graduate students in 1960–61. On the basis of the undergraduate transcript of the student, he was placed in exactly one of the following classes: no statistics; statistics not requiring calculus; statistics requiring calculus. We recognize that there are probably several errors of observation in these data because of the difficulty in classifying certain courses in economics, psychology, and other subjects as primarily courses in statistics or not. It should be mentioned that courses in mathematical statistics which require the calculus were also counted as courses in mathematics beyond the calculus in Tables IA and IB above, and Tables IIA and IIB below.

TABLE IIA

MATHEMATICAL TRAINING, AS GRADUATE STUDENTS, OF GRADUATE STUDENTS WHOSE DISSERTATIONS HAVE BEEN ACCEPTED, 1958–1961

	Anthro-pology	Biochem-istry	Bus. Adm.	Econom-ics	Poli. Sci.	Psychol-ogy	Sociology	Zoology
No. mathematics	78	84	115	150	185	167	139	136
Precalculus	0	0	2	32	1	12	8	10
Calculus	0	12	2	18	0	25	3	9
Math. beyond calculus	1	5	5	30	1	32	10	9
Totals	79	101	124	230	187	236	160	164

TABLE IIB

NUMBER OF SEMESTER HOURS OF MATHEMATICS OF GRADUATE STUDENTS WHOSE DISSERTATIONS HAVE BEEN ACCEPTED, 1958–1961

No. of Hours	Anthro-pology	Biochem-istry	Bus. Adm.	Econom-ics	Poli. Sci.	Psychol-ogy	Sociology	Zoology
0	78	84	115	160	185	167	139	137
1–3	0	7	3	31	1	22	9	12
4–6	1	8	2	13	1	21	6	7
7–9	0	2	1	5	0	9	3	2
10–12	0	0	2	11	0	7	0	3
13–15	0	0	1	3	0	5	0	0
16–20	0	0	0	4	0	3	2	3
21–25	0	0	0	1	0	0	1	0
26–35	0	0	0	1	0	0	0	0
≥ 36	0	0	0	1	0	2	0	0
Totals	79	101	124	230	187	236	160	164

We now turn to the second category of students, those whose dissertations were accepted by the department between September 1, 1958, and March 1,

1961. Tables IIA, IIB, and IIC list the same data for these students as were presented in Tables IA, IB, and IC for the first-year graduate students. However, it is to be emphasized that the data in IIA, IIB and IIC are based on graduate transcripts only. Thus a student who obtained a good undergraduate training in mathematics might show no courses whatsoever as a graduate student and would then be listed as having only a very slight training.

TABLE IIC

GRADUATE TRAINING IN STATISTICS OF GRADUATE STUDENTS WHOSE DISSERTATIONS HAVE BEEN ACCEPTED, 1958–1961

	Anthro-pology	Biochem-istry	Bus. Adm.	Econom-ics	Poli. Sci.	Psychol-ogy	Sociology	Zoology
No. statistics	60	92	45	30	167	30	25	129
Precalculus statistics	19	5	74	134	19	185	126	33
Postcalculus statistics	0	4	4	66	1	21	9	2
Totals	79	101	123	230	187	236	160	164

TABLE III

LEVEL OF MATHEMATICS USED IN PH.D. DISSERTATIONS ACCEPTED BY THE DEPARTMENT BETWEEN SEPTEMBER 1, 1958, AND MARCH 1, 1961

	Anthro-pology	Biochem-istry	Bus. Adm.	Econom-ics	Poli. Sci.	Psychol-ogy	Sociology	Zoology
No. mathematics	74	43	75	114	178	75	94	102
Precalculus	10	33	42	66	12	142	63	52
Calculus	0	20	5	21	0	9	4	8
Math. beyond calculus	0	1	2	26	1	11	2	1
Totals	84	97	124	227	191	237	163	163

In Table III each of the Ph.D. dissertations accepted by the department between September 1, 1958, and March 1, 1961, has been classified into one of the four categories used in Tables IA and IIA, regarding the level of mathematics actually used in the Ph.D. dissertation. Computation of means, graphing, and displaying tables of results were not considered to be mathematics or statistics. Computation of standard errors, chi square and t tests, determination of probability levels, and curve fitting were considered to be mathematics at the precalculus level. Standard analysis of variance was also considered to be at the precalculus level. The basic attitude was that the routine use of statistical tools whose theory requires advanced mathematics was not to be counted as other than precalculus mathematics.

We also obtained information on graduate courses using calculus and mathematics beyond calculus during the academic years 1959–60 and 1960–61. We briefly summarize these data in nontabular form.

Anthropology: 2/10 universities list such courses; 3 courses in all, 2 of which are apparently related to genetics.

Biochemistry: 7/9 universities list such courses; due primarily to physical chemistry, crystallography, enzymology.

Business Administration: 7/9 universities list such courses; some 50 percent of courses are, loosely speaking, statistical; the others involve computation, decision making, accounting and insurance (1/2 use calculus, 1/3 linear algebra, 6% advanced calculus, 12% computing).

Economics: 10/10 universities list such courses; the breakdown by subjects is here well known (about 1/2 make very slight use of calculus, about 1/6 use linear algebra, 1/3 use statistics).

Political Science: 0/10 universities list such courses.

Psychology: 9/10 universities list such courses; a sizeable fraction of the courses are primarily statistical; courses in mathematical models account for the rest (about 1/2 of all courses use at least calculus).

Sociology: 5/9 universities list such courses; in some schools, courses seem entirely statistical and in others entirely nonstatistical, i.e., theoretical sociology.

Zoology: 6/10 universities list such courses; of 23 courses in all, 5 are genetical, 3 are pure statistics, 6 are physiological, 4 are biochemical, 5 are miscellaneous.

Interpretation of results. Frequency data are summarized in Tables IA to III. The levels of undergraduate preparation in mathematics in the various fields can be most readily compared in Table IA. Considering first the percentages of entering graduate students who have had at least elementary calculus, biochemistry ranks first, as might have been expected, with approximately 75%; next economics, business administration, and psychology are closely grouped with a surprisingly robust 50%; then zoology drops to 30%; and anthropology, political science, and sociology are clustered at the bottom with 20–25%. This ranking is correlated almost perfectly with one based on the percentages of students having had at least one college mathematics course, the values running in this case from 98% for biochemistry down to 50% for political science.

Levels of preparation mean little except in relation to facts concerning the extent to which mathematics is actually used in graduate work. Some information on this point is available in Tables IIA and III. It would seem reasonable to assume that for the most part students in BMSS fields will take mathematics during their period of graduate study only if their courses or research require more mathematical background than they have achieved as undergraduates. One could not, of course, argue the converse; students with inadequate preparation may simply avoid courses or lines of research requiring additional mathe-

matical training rather than take courses in relatively elementary mathematics for which they frequently cannot receive graduate credit. It is apparent from Table IIA that students in the fields considered typical add little to their mathematical backgrounds (except in statistics, which will be discussed separately below) after entering graduate school. The only substantial exceptions to this generalization are psychology, where an imposing 25% of graduate students take mathematics involving calculus or beyond, and economics and biochemistry with corresponding percentages of 22 and 20. Comparing Table IIA with Table IA, we note a rather striking correspondence between the 13% of graduate students in economics who had no undergraduate mathematics and the 14% of students in this field who took precalculus courses in mathematics after entering graduate school. The two percentages are not based on the same students, of course; nevertheless, it appears a fair inference that some college mathematics is absolutely essential for graduate work in economics. The situation is not very different in psychology and zoology, where apparently no more than 10% of students in the population examined get through graduate work without at least precalculus college mathematics.

Turning to Table III, a relatively close relationship is apparent between the use of mathematics in doctoral research and the level of preparation characteristic of the field. Mathematics at the level of calculus or beyond is used in more than 20% of dissertations in the samples from economics and biochemistry, 8% in psychology, and about 5% in business administration, zoology, and sociology. At least precalculus mathematics is used in 70% of theses in psychology, 50–60% in biochemistry and economics, 30–40% in business administration, sociology, and zoology, less than 12% in anthropology and political science. Information concerning graduate courses in various departments which require relatively advanced mathematical preparation, summarized above, is difficult to interpret because of the many local idiosyncrasies in course organization. It is of some interest to note, however, that the proportion of departments offering at least some courses in this category correlates quite closely with the other indices of use of mathematics in graduate work. Three-fourths or more of departments in biochemistry, business administration, economics (100%), and psychology offer such courses, but only 2 out of 10 in anthropology and none in political science.

Taking together all of the indices of use, it appears reasonable to conclude that the extent to which mathematics is required for graduate courses and used in doctoral research is sufficiently high in biochemistry, psychology, economics, and business administration that students entering graduate work in these fields without some undergraduate mathematics must be seriously handicapped. More advanced mathematics is needed to an important (and, although this study contains no evidence on the point, probably increasing) extent in biochemistry, economics, and psychology. In anthropology and political science, preparation is relatively low, but so is use of mathematics in graduate study. Only in business administration does there seem to be a substantial proportion of students

with more advanced mathematical training than they find active use for in later work.

Undergraduate training in statistics, summarized in Table IC, follows two main patterns. Anthropology, biochemistry, political science, and zoology are alike in that few students take even an elementary statistics course as undergraduates; evidently they are not greatly handicapped in later study, for the percentages of students in these fields taking statistics in graduate work are also small. In the other four fields, the level of statistical training is much higher. Students of psychology tend to begin this training earliest, with a heavy majority taking both undergraduate and graduate (Table IIC) courses. In business administration, economics, and sociology the percentage of students taking statistics increases from the undergraduate to the graduate level. On the assumption that the graduate courses are taken partly because training in statistics is needed in other concurrent graduate work, the suggestion arises that more preparation at the undergraduate level would be advantageous for students in these areas. It cannot be determined from our data whether any students in psychology, business administration, economics, or sociology get through without at least one course in statistics, but certainly the number must be very small. As graduate students, a substantial proportion take mathematical statistics (i.e., statistics courses with calculus prerequisite) only in economics (25%); this is followed by a drop to 10% in psychology. It is interesting to note that in all fields the level of mathematical preparation is adequate to enable a much larger proportion of students to take mathematical statistics than actually do so. Further information is needed in order to determine why so many students, even in the BMSS areas where statistical training is clearly important, stop with courses which do not utilize much of their mathematical background. It is one matter if mathematical statistics is known to be inessential background for research in these areas, quite another if it is potentially valuable but relatively unavailable to the students. The BMSS Panel plans to make recommendations on the undergraduate curriculum in mathematical statistics in the near future.

MATHEMATICAL NOTES

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SOME TOTALLY EQUIVALENT MATRICES*

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Let $A=(a_{nk})$ denote an infinite matrix and x a sequence. Then A is called regular if A is limit-preserving; i.e. (i) $x_n\to l$, l finite implies $A_n(x)\to l$, where $A_n(x)=\sum_k a_{nk}x_k$. Let $\|A\|=\sup_n \sum_k |a_{nk}|$, $a_k=\lim_n a_{nk}$, and $t=\lim_n \sum_k a_{nk}$. Then the necessary and sufficient conditions for A to be *regular* are: $\|A\|<\infty$, $a_k=0$ for each k, and $t=1$. A matrix A is called *totally regular* if A satisfies condition (i) for l both finite and infinite. Let $c_A=\{x\,|\,A_n(x) \text{ converges}\}$. Then c_A is called the *convergence domain* of x. Given two regular matrices A and B, we say that B is stronger than A if $c_A\subset c_B$. We say that B is *totally stronger* than A, written B t.s. A, if $c_A\subset c_B$ and, for every sequence x for which $A_n(x)\to+\infty$, it follows that $B_n(x)\to+\infty$. If A t.s. B and B t.s. A we say that A and B are *totally equivalent.*

It is a well-known result (see, e.g., [1, Lemma 2]) that two Hausdorff matrices with nonvanishing moment sequences cannot be totally equivalent unless they are identical. Miss Debi in [2] proved the analogous result for Nörlund matrices.

The purpose of this note is to give examples of matrices which are not identical and yet are totally equivalent.

All of the matrices used in the examples are triangles. A matrix A is called a *triangle* if $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$, $n=1, 2, 3, \cdots$.

If A and B are two triangles, then B t.s. A if and only if BA^{-1} is totally regular. Therefore, to prove two regular triangles totally equivalent it is necessary and sufficient to show that AB^{-1} and BA^{-1} are both totally regular.

A regular triangle will be totally regular if, in addition to satisfying the conditions for regularity, there exists an integer q such that the a_{nk} are nonnegative for all $k\geq q$; i.e., except for at most a finite number of columns, all of the elements of A are nonnegative.

Let $C=AB^{-1}$ and $D=BA^{-1}$.

Example 1. Let A and B be diagonal matrices; i.e., $a_{nk}=b_{nk}=0$ for $n\neq k$; $a_{11}=b_{11}=1$, $a_{nn}=1-(1/n)$, and $b_{nn}=1-(1/n^2)$, $n>1$. Then $c_{11}=d_{11}=1$, $c_{nn}=n/(n+1)$, $d_{nn}=(n+1)/n$, $n>1$, and C and D are both totally regular.

Example 2. Let $a_{11}=b_{11}=1$, $a_{n1}=1/n$, $a_{nn}=1-(1/n)$, $b_{n1}=1/n^2$, $b_{nn}=1-(1/n^2)$, $n>1$, $a_{nk}=b_{nk}=0$ otherwise. Without difficulty we find that $c_{11}=d_{11}=1$, $c_{n1}=1/(n+1)$, $c_{nn}=n/(n+1)$, $d_{n1}=-1/n$, $d_{nn}=(n+1)/n$, $n>1$. Hence C and D are totally regular.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

In the above examples, the matrices A and B are equivalent to convergence; i.e., $x \in c_A$ implies x is a convergent sequence.

It is natural to ask whether it is possible to have two nonidentical matrices A and B totally equivalent with A and B both regular and both stronger than convergence; i.e., each matrix sums at least one divergent sequence.

Wilansky in [3] has shown that a necessary and sufficient condition for a triangle A to sum a divergent sequence is that $\|A^{-1}\|$ be infinite.

Before answering the above question I shall make the following definition.

DEFINITION. *A and B are said to be almost identical if there exists an integer q such that $a_{nk}=b_{nk}$ for $k \geqq q$.*

Example 3. Define A by $a_{jj}=1/j$, $a_{nk}=1$ for $1+\frac{1}{2}k(k-1) \leqq n \leqq \frac{1}{2}k(k+1)$, $k=3, 4, 5, \cdots$, $a_{nk}=0$ otherwise. In other words, except for the elements on the main diagonal, A has three 1's in column 3 starting with row 4, four 1's in column 4, starting with row 7, etc. Then $A^{-1}=(a_{nk}^{-1})$ is defined by $a_{jj}^{-1}=j$, $a_{nk}^{-1}=-kn$, $1+\frac{1}{2}k(k-1) \leqq n \leqq \frac{1}{2}k(k+1)$, $a_{nk}^{-1}=0$ otherwise. Let B be any matrix almost identical with A. For example, let B agree with A for all $k>9$, and let $b_{nn}=a_{nn}$, $1 \leqq n \leqq 9$, and $b_{nk}=0$, $1 \leqq k \leqq 9$, $n \neq k$. Then, for all $k>9$, $c_{nn}=d_{nn}=1$, $c_{nk}=d_{nk}=0$, $n \neq k$. Hence C and D are totally regular.

Let A and B be regular and strictly stronger than convergence. The following is an interesting open question. Is A almost identical with B a necessary or sufficient condition for A and B to be totally equivalent?

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A FURTHER NOTE ON CLOSURE AND INTERIOR OPERATORS

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Norman Levine [3] has investigated conditions under which closure and interior operators commute in topological spaces. His result may be stated in the following manner:

Let A be a subset of a topological space X. Then $c\,\mathrm{Int}\,A = \mathrm{Int}\,cA$ (*c denotes the closure operator and* Int *denotes the interior operator*) *if and only if* $A=(M-P)\cup(P-M)$ *where M is both open and closed and P is nowhere dense.*

Actually, this result may be considered as a special case of the study of the possible equalities that may occur when the closure and interior operators are applied successively to a subset of a topological space.

It is the purpose of this paper to give characterizations (similar to the characterization given by Levine) of all such possible equalities. These possible

equalities are somewhat limited by the following lemma which, in a more general form, is due to Kuratowski.*

LEMMA 1. *Let A be a subset of a topological space X. Then at most seven distinct sets can be constructed from A by successive application of the closure and interior operators.*

Proof. Since $c \operatorname{Int} c \operatorname{Int} A = c \operatorname{Int} A$ and $\operatorname{Int} c \operatorname{Int} cA = \operatorname{Int} cA$, the seven possible distinct sets are A, $\operatorname{Int} A$, $c \operatorname{Int} A$, $\operatorname{Int} c \operatorname{Int} A$, cA, $\operatorname{Int} cA$ and $c \operatorname{Int} cA$.

We notice that if A is the subset in the space of reals that consists of all points in $(0, 1) \cup (1, 2)$, all rationals in $(2, 3)$ and the point 4, then the seven sets listed in the preceding lemma are all distinct. Therefore there are no necessary equalities that have to exist among these sets.

DEFINITION. *Number the seven sets given in Lemma 1 as follows:*

1. A, 2. $\operatorname{Int} A$, 3. $c \operatorname{Int} A$, 4. $\operatorname{Int} c \operatorname{Int} A$, 5. cA, 6. $\operatorname{Int} cA$, 7. $c \operatorname{Int} cA$.

Then a subset A of a topological space X is said to satisfy property $(n-m)$ if and only if set number n is equal to set number m.

The following lemma contains some useful relations that exist between the properties $(n-m)$.

LEMMA 2. *Let A be a subset of a topological space X. Then*

(i) *A has property $(n-m)$ if and only if A has property $(p-q)$, where $(n-m)$ and $(p-q)$ are paired as ((1–3), (1–7)), ((1–6), (1–4)), ((3–7), (4–6)) and ((3, 6) (4–7)).*

(ii) *A has property $(n-m)$ if and only if CA (C denotes the complement operator) has property $(p-q)$, where $(n-m)$ and $(p-q)$ are paired as ((1–3), (1–6)), ((1–4), (1–7)), ((2–3), (5–6)), ((2, 4) (5–7)), ((2–6), (3–5)), ((2–7), (4–5)) and ((3–4), (6–7)).*

Proof. (i). If $A = c \operatorname{Int} A$ then A is closed and $A = c \operatorname{Int} cA$. If $A = c \operatorname{Int} cA$ then again A is closed and $A = c \operatorname{Int} A$.

If $A = \operatorname{Int} cA$ then A is open and $A = \operatorname{Int} c \operatorname{Int} A$. If $A = \operatorname{Int} c \operatorname{Int} A$ then again A is open and $A = \operatorname{Int} cA$.

If $c \operatorname{Int} A = c \operatorname{Int} cA$ then $\operatorname{Int} c \operatorname{Int} A = \operatorname{Int} c \operatorname{Int} cA = \operatorname{Int} cA$. If $\operatorname{Int} cA = \operatorname{Int} c \operatorname{Int} A$ then $c \operatorname{Int} cA = c \operatorname{Int} c \operatorname{Int} A = c \operatorname{Int} A$.

If $c \operatorname{Int} A = \operatorname{Int} cA$ then obviously $\operatorname{Int} c \operatorname{Int} A = c \operatorname{Int} cA$. Now suppose that $\operatorname{Int} c \operatorname{Int} A = c \operatorname{Int} cA$. Then we have $\operatorname{Int} cA \subset \operatorname{Int} c \operatorname{Int} A \subset c \operatorname{Int} A$. But $\operatorname{Int} A \subset \operatorname{Int} c \operatorname{Int} A$ implies that $c \operatorname{Int} A \subset \operatorname{Int} c \operatorname{Int} A$ and $\operatorname{Int} c \operatorname{Int} A \subset cA$ implies that $\operatorname{Int} c \operatorname{Int} A \subset \operatorname{Int} cA$. Thus $c \operatorname{Int} A \subset \operatorname{Int} cA$, and finally we have $c \operatorname{Int} A = \operatorname{Int} cA$.

(ii). These properties are easily verified by using the familiar relations that exist between the closure, complement and interior operators [1].

* Kuratowski [2] has shown that at most 14 distinct sets can be constructed from a subset of a topological space by application of the closure, complement and interior operators in any order.

Dismissing the trivial properties (1–2), (1–5) and (2–5), we begin by characterizing property (1–3).

THEOREM 1. *A subset A of a topological space X satisfies property* (1–3) *if and only if A is closed and $A = M \cup N$ where M is open and N has a void interior in the subspace A of X.*

Proof. If the conditions are satisfied, let $q \in A$ and let U be any open neighborhood of q. Now $U \cap A$ is open in A and therefore must intersect M since N has a void interior in A. This implies that $q \in cM$ which in turn implies that $A \subset cM$. $M \subset \text{Int } A$ implies that $cM \subset c \text{ Int } A$ and thus $A \subset cM \subset c \text{ Int } A$. Therefore, since $c \text{ Int } A \subset A$, $A = c \text{ Int } A$.

If $A = c \text{ Int } A$, define $M = \text{Int } A$ and $N = A - \text{Int } A$. We see that A is closed and M is open and, denoting by Int/A, c/A and C/A the interior, closure and complement operators in the subspace A of X, we have

$$\begin{aligned}(\text{Int}/A)N &= (C/A)(c/A)(C/A)(A - \text{Int } A) \\ &= (C/A)(c/A) \text{ Int } A = (C/A)c \text{ Int } A = (C/A)A = \phi.\end{aligned}$$

Thus N has a void interior in A. Finally, $M \cup N = \text{Int } A \cup (A - \text{Int } A) = A$.

COROLLARY 1. *A subset A of a topological space X satisfies property* (1–6) *if and only if A is open and $A = M - N$ where M is closed and N has a void interior in the subspace $X - A$ of X.*

Proof. This corollary, as all succeeding corollaries, follows immediately from (ii) of Lemma 2.

THEOREM 2. *A subset A of a topological space X satisfies property* (3–5) *if and only if $A = M \cup N$ where M is open and N has a void interior in the subspace A of X.*

Proof. If the conditions are satisfied, let $q \in cA$ and let U be any open neighborhood of q. Now $U \cap A$ is open in A and is nonempty since every open neighborhood of q intersects $A = M \cup N$. Thus $U \cap A$ must intersect M and consequently $cA \subset cM$. Since $cM \subset cA$ we have $cM = cA$. Now $c \text{ Int } A \subset cA$, and $M \subset \text{Int } A$ implies that $cA = cM \subset c \text{ Int } A$. Therefore $cA = c \text{ Int } A$, which is precisely property (3–5).

If $cA = c \text{ Int } A$ then define $M = \text{Int } A$ and $N = A - \text{Int } A$. We see that M is open and also

$$\begin{aligned}(\text{Int}/A)N &= (C/A)(c/A)(C/A)(A - \text{Int } A) \\ &= (C/A)(c/A) \text{ Int } A = (C/A)A = \phi.\end{aligned}$$

Thus N has a void interior in A. Finally, $M \cup N = \text{Int } A \cup (A - \text{Int } A) = A$.

COROLLARY 2. *A subset A of a topological space X satisfies property* (2–6) *if and only if $A = M - N$ where M is closed and N has a void interior in the subspace $X - A$ of X.*

LEMMA 3. *Let M and N be subsets of a topological space X such that M is open and N has a void interior. Then $c(M-N)=cM$.*

Proof. Let $q\in cM$ and let U be any open neighborhood of q. Then $U\cap M$ is open and must intersect $M-N$. If not, then it lies in N, a contradiction of the fact that N has a void interior. Therefore $q\in c(M-N)$ which implies that $cM\subset c(M-N)$. Now since $c(M-N)\subset cM$ we have the desired equality.

THEOREM 3. *A subset A of a topological space X satisfies property* (5–7) *if and only if $A=(M-P)\cup Q$ where M is open, P has a void interior and Q has a void interior in the subspace A of X.*

Proof. If the conditions are satisfied then $cA=c(M-P)\cup cQ=cM\cup cQ$, from Lemma 3. Now $q\in cQ$ implies that every open neighborhood of q intersects Q. Let U be any open neighborhood of q. Then $U\cap A$ is open in A and must intersect M, which implies that $q\in cM$. Therefore $cA=cM\cup cQ=cM$. But $c\operatorname{Int} cA=c\operatorname{Int} cM=c\operatorname{Int} c\operatorname{Int} M=c\operatorname{Int} M=cM=cA$. Thus $cA=c\operatorname{Int} cA$.

If $cA=c\operatorname{Int} cA$ then define $M=\operatorname{Int} cA$, $P=\operatorname{Int} cA-A$ and $Q=A-\operatorname{Int} cA$. We see that M is open and

$$\begin{aligned}\operatorname{Int} P &= \operatorname{Int}(\operatorname{Int} cA - A) = \operatorname{Int} cA \cap \operatorname{Int} CA\\ &= \operatorname{Int} cA \cap CcA \subset c\operatorname{Int} cA \cap Cc\operatorname{Int} cA = \phi.\end{aligned}$$

Thus P has a void interior. Also

$$\begin{aligned}(\operatorname{Int}/A)Q &= (C/A)(c/A)(C/A)(A - \operatorname{Int} cA) = (C/A)(c/A)(A \cap \operatorname{Int} cA)\\ &\subset (C/A)\{(c/A)A \cap (c/A)\operatorname{In} cA\} = (C/A)A = \phi.\end{aligned}$$

Therefore Q has a void interior in A. Finally we have

$$\begin{aligned}(M - P)\cup Q &= \{\operatorname{Int} cA - (\operatorname{Int} cA - A)\}\cup(A - \operatorname{Int} cA)\\ &= \{\operatorname{Int} cA \cap (C\operatorname{Int} cA \cup A)\}\cup(A \cap C\operatorname{Int} cA)\\ &= (A \cap \operatorname{Int} cA)\cup(A \cap C\operatorname{Int} cA)\\ &= (\operatorname{Int} cA \cup C\operatorname{Int} cA)\cap A\\ &= X \cap A = A.\end{aligned}$$

COROLLARY 3. *A subset A of a topological space X satisfies property* (2–4) *if and only if $A=(M-Q)\cup P$ where M is closed, P has a void interior and Q has a void interior in the subspace $X-A$ of X.*

LEMMA 4. *Let M be closed and P have a void interior in a topological space X. Then* $\operatorname{Int}(M\cup P)=\operatorname{Int} M$.

Proof. From Lemma 3 we have

$$\begin{aligned}c(CM - P) &= cCM\\ c(CM \cap CP) &= cCM\end{aligned}$$

$$C \operatorname{Int} C(CM \cap CP) = C \operatorname{Int} CCM$$
$$C \operatorname{Int} (M \cup P) = C \operatorname{Int} M$$
$$\operatorname{Int} (M \cup P) = \operatorname{Int} M,$$

which is the desired equality.

THEOREM 4. *A subset A of a topological space X satisfies property* (4–6) *if and only if $A = (N-P) \cup (P-N)$ where $N = \operatorname{Int} cN$ and P is nowhere dense.*

Proof. If the conditions are true then

$$\begin{aligned} \operatorname{Int} cA &= \operatorname{Int}\{c(N - P) \cup c(P - N)\} \\ &= \operatorname{Int}\{cN \cup c(P - N)\} \subset \operatorname{Int}\{cN \cup (cP \cap cCN)\} \\ &= \operatorname{Int}(cN \cup cP) = \operatorname{Int} cN = N. \end{aligned}$$

Now obviously $\operatorname{Int} cA \supset N$ which implies that $\operatorname{Int} cA = N$. We have

$$\begin{aligned} c \operatorname{Int} A &= c \operatorname{Int}\{(P \cup N) \cap (CN \cup CP)\} = c\{\operatorname{Int}(P \cup N) \cap \operatorname{Int}(CN \cup CP)\} \\ &= c\{N \cap \operatorname{Int}(CP \cup CN)\} = c \operatorname{Int}(N \cap CP) = c(N \cap CcP) \\ &= C \operatorname{Int}(CN \cup cP) = C \operatorname{Int} CN = cN. \end{aligned}$$

Now $\operatorname{Int} c \operatorname{Int} A = \operatorname{Int} cN = N = \operatorname{Int} cA$, which is precisely property (4–6).

If $\operatorname{Int} cA = \operatorname{Int} c \operatorname{Int} A$ then define $N = \operatorname{Int} cA$ and $P = (A - \operatorname{Int} cA) \cup (\operatorname{Int} cA - A)$. Now obviously $\operatorname{Int} cN = \operatorname{Int} c \operatorname{Int} cA = \operatorname{Int} cA = N$. We have $\operatorname{Int} c(A - \operatorname{Int} cA) \subset \operatorname{Int} cA \cap \operatorname{Int} cC \operatorname{Int} cA = \operatorname{Int} cA \cap Cc \operatorname{Int} A = \phi$. Thus $(A - \operatorname{Int} cA)$ is nowhere dense. Also $\operatorname{Int} c(\operatorname{Int} cA - A) \subset \operatorname{Int} cA \cap \operatorname{Int} cCA = \operatorname{Int} cA \cap Cc \operatorname{Int} A = \phi$ which implies that $(\operatorname{Int} cA - A)$ is nowhere dense. Therefore P is nowhere dense. Finally we have

$$\begin{aligned} (N - P) \cup (P - N) &= (P \cup N) \cap (CN \cup CP) \\ &= \{(A - \operatorname{Int} cA) \cup (\operatorname{Int} cA - A) \cup \operatorname{Int} cA\} \cap (CN \cup CP) \\ &= (A \cup \operatorname{Int} cA) \cap \{C \operatorname{Int} cA \cup [(C \operatorname{Int} cA \cup A) \cap (CA \cup \operatorname{Int} cA)]\} \\ &= (A \cup \operatorname{Int} cA) \cap (A \cup C \operatorname{Int} cA) \\ &= A \cup (\operatorname{Int} cA \cap C \operatorname{Int} cA) \\ &= A \cup \phi = A. \end{aligned}$$

THEOREM 5. *A subset A of a topological space X satisfies property* (2–3) *if and only if $A = M \cup N$ where M is both open and closed, N has a void interior and $M \cap N = \phi$.*

Proof. If the conditions are true then $\operatorname{Int} A = \operatorname{Int}(M \cup N) = \operatorname{Int} M \cup \operatorname{Int} N = \operatorname{Int} M = M$ and $c \operatorname{Int} A = cM = M$. Thus A satisfies property (2–3).

If $\operatorname{Int} A = c \operatorname{Int} A$ then define $M = \operatorname{Int} A$ and $N = A - \operatorname{Int} A$. Now M is both open and closed and $M \cap N = \phi$. Now N obviously has a void interior and $M \cup N = \operatorname{Int} A \cup (A - \operatorname{Int} A) = A$.

COROLLARY 4. *A subset A of a topological space X satisfies property* (5–6) *if and only if $A = M - N$ where M is both open and closed, N has a void interior and $N \subset M$.*

THEOREM 6. *A subset A of a topological space X satisfies property* (2–7) *if and only if $A = M \cup P$ where M is both open and closed, P is nowhere dense and $M \cap P = \phi$.*

Proof. If the conditions are true then $\operatorname{Int} A = \operatorname{Int}(M \cup P) = \operatorname{Int} M \cup \operatorname{Int} P = \operatorname{Int} M = M$ and $c \operatorname{Int} cA = c \operatorname{Int} cM \cup c \operatorname{Int} cP = c \operatorname{Int} cM = M$. Thus A satisfies property (2–7).

If $\operatorname{Int} A = c \operatorname{Int} cA$ then define $M = \operatorname{Int} A$ and $P = A - \operatorname{Int} A$. Now M is both open and closed and $M \cap P = \phi$. Also $\operatorname{Int} cP = \operatorname{Int} c(A - \operatorname{Int} A) \subset \operatorname{Int} cA \cap \operatorname{Int} cC \operatorname{Int} A = \operatorname{Int} cA \cap C \operatorname{Int} A \subset c \operatorname{Int} cA \cap C \operatorname{Int} A = \phi$. Thus P is nowhere dense. Finally we have $M \cup P = \operatorname{Int} A \cup (A - \operatorname{Int} A) = A$.

COROLLARY 5. *A subset A of a topological space X has property* (4–5) *if and only if $A = M - P$ where M is both open and closed, P is nowhere dense and $P \subset M$.*

THEOREM 7. *A subset A of a topological space X satisfies property* (6–7) *if and only if $A = (M - N) \cup P$ where M is both open and closed, N has a void interior, P is nowhere dense and $P \cap M = \phi$.*

Proof. If the conditions are true then $\operatorname{Int} cA = \operatorname{Int} c\{(M - N) \cup P\} = \operatorname{Int} c(M - N) \cup \operatorname{Int} cP = \operatorname{Int} c(M - N) = M$. Also $c \operatorname{Int} cA = cM = M$. Thus A satisfies property (6–7).

If $\operatorname{Int} cA = c \operatorname{Int} cA$ then define $M = \operatorname{Int} cA$, $N = (\operatorname{Int} cA - A)$ and $P = (A - \operatorname{Int} cA)$. Then M is both open and closed and $P \cap M = \phi$. We see that

$$\operatorname{Int} cP = \operatorname{Int} c(A - \operatorname{Int} cA) \subset \operatorname{Int} cA \cap \operatorname{Int} cC \operatorname{Int} cA = \operatorname{Int} cA \cap C \operatorname{Int} cA = \phi.$$

Thus P is nowhere dense. Also

$$\operatorname{Int} N = \operatorname{Int}(\operatorname{Int} cA - A) = \operatorname{Int} cA \cap \operatorname{Int} CA = \operatorname{Int} cA \cap CcA = \phi.$$

Therefore N has a void interior. Finally we have

$$\begin{aligned}(M - N) \cup P &= \{\operatorname{Int} cA \cap C(\operatorname{Int} cA - A)\} \cup (A - \operatorname{Int} cA) \\ &= \{\operatorname{Int} cA \cap (C \operatorname{Int} cA \cup A)\} \cup (A \cap C \operatorname{Int} cA) \\ &= A \cap (\operatorname{Int} cA \cup C \operatorname{Int} A) = A \cap X = A.\end{aligned}$$

COROLLARY 6. *A subset A of a topological space X satisfies property* (3–4) *if and only if $A = (M - P) \cup N$ where M is both open and closed, P is nowhere dense, N has a void interior and $P \subset M$.*

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TRANSFORMATION OF SERIES BY A VARIANT OF STIRLING'S NUMBERS

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Consider the generating functions of the powers t^k:

$$A_k(z) = \sum_{t=0}^{\infty} t^k z^t \qquad (|z| < 1,\ k = 0, 1, \cdots).$$

Because these series satisfy the differential-difference equation

$$A_{k+1}(z) = z\ dA_k(z)/dz,$$

and the initial condition $A_0(z) = (1-z)^{-1}$, it is easily seen that they have a representation of the form

$$A_k(z) = \sum_{j=0}^{k} {}^{k}c_j B_j(z), \qquad B_j(z) = \frac{1}{j!}(1-z)^{-j-1} \quad (j = 0, 1, \cdots),$$

the coefficients ${}^{k}c_j$ being real constants. This system is triangular, the coefficients ${}^{j}c_j$ are clearly nonzero, and so there exist real constants ${}^{j}d_i$ such that

$$B_j(z) = \sum_{i=0}^{j} {}^{j}d_i A_i(z).$$

How does one find ${}^{k}c_j$ and ${}^{j}d_i$?

The answer makes use of a variant of Stirling's numbers, which arise from the expansion of a generalized factorial in powers of z and the inverse expansion. In place of Stirling's formulation [1], it is more convenient to use modern techniques on special functions [2] employing the generalized factorial

$$(x+1)_n = (x+1)(x+2)\cdots(x+n), \qquad (x+1)_0 = 1.$$

In terms of this function, "bracket symbols" $\left[{n \atop j}\right]$ and "brace symbols" $\left\{{n \atop i}\right\}$ may be defined by the formulas

$$(x+1)_n = \sum_{j=0}^{n} \left[{n \atop j}\right] x_j, \qquad (x+1)_0 = \left[{0 \atop 0}\right] = 1, \qquad \left[{n \atop j}\right] = 0 \quad \text{for } j > n,$$

and

$$x^n = \sum_{i=0}^{n} \left\{{n \atop i}\right\} (x+1)_i, \qquad x^0 = \left\{{0 \atop 0}\right\} = 1, \qquad \left\{{n \atop i}\right\} = 0 \qquad \text{for } i > n.$$

The bracket (brace) symbols are analogs of Stirling's numbers of the first (second) kind. The symbolism is intended to suggest that these numbers have similarities with the binomial coefficients $\binom{n}{k}$; indeed, formulas similar to those known for the binomial coefficients are easily established by methods such as are used in [1] to discuss Stirling's numbers. Here is a sampling.

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For $n=0, 1, 2, \cdots$, one has

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ n \end{bmatrix} = n!, \quad \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} = (n+1)!, \quad \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} = 0;$$

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = (-1)^n, \quad \begin{Bmatrix} n \\ n \end{Bmatrix} = 1, \quad \operatorname{sgn} \begin{Bmatrix} n \\ i \end{Bmatrix} = (-1)^{n+i} \qquad (i \leqq n),$$

$$\sum_{i=0}^{n} \begin{Bmatrix} n \\ i \end{Bmatrix} = 0, \quad \sum_{i=0}^{n} (i+1)! \begin{Bmatrix} n \\ i \end{Bmatrix} = 1.$$

Both sets of symbols satisfy simple partial difference equations:

$$\begin{bmatrix} n \\ h \end{bmatrix} = \begin{bmatrix} n-1 \\ h-1 \end{bmatrix} + n \begin{bmatrix} n-1 \\ h \end{bmatrix}, \quad \begin{Bmatrix} n \\ h \end{Bmatrix} = \begin{Bmatrix} n-1 \\ h-1 \end{Bmatrix} - (h+1) \begin{Bmatrix} n-1 \\ h \end{Bmatrix}$$

$$(1 \leqq h \leqq n).$$

Finally, the inverse nature of the formulas defining the symbols implies:

$$\sum_{j=0}^{\infty} \begin{bmatrix} k \\ j \end{bmatrix} \begin{Bmatrix} j \\ h \end{Bmatrix} = \sum_{j=h}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \begin{Bmatrix} j \\ h \end{Bmatrix} = \delta_h^k,$$

$$\sum_{h=0}^{\infty} \begin{Bmatrix} i \\ h \end{Bmatrix} \begin{bmatrix} h \\ j \end{bmatrix} = \sum_{h=j}^{i} \begin{Bmatrix} i \\ h \end{Bmatrix} \begin{bmatrix} h \\ j \end{bmatrix} = \delta_j^i$$

(the Kronecker symbol).

From this last pair of identities it follows that an infinite system of series whose coefficients are bracket or brace symbols is easily inverted. In the present case one may rewrite

$$B_j(z) = (1-z)^{-j-1}/j! = \sum_{t=0}^{\infty} (t+1)_j z^t = \sum_{t=0}^{\infty} \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} t^i z^t,$$

for $|z| < 1$. The right-hand member is an absolutely convergent double series, so that the order of summation may be reversed [3],

$$B_j(z) = \sum_{t=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} \sum_{t=0}^{\infty} t^i z^t = \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} A_i(z):$$

that is, $d_t^j = \left[{j \atop i}\right]$. Multiplication by $\left\{{k \atop j}\right\}$ and summation on j from 0 to k gives, by virtue of the preceding identity

$$\sum_{j=0}^{k} \begin{Bmatrix} k \\ j \end{Bmatrix} B_j(z) = \sum_{j=0}^{k} \sum_{i=0}^{j} \begin{Bmatrix} k \\ j \end{Bmatrix} \begin{bmatrix} j \\ i \end{bmatrix} A_i(z) = A_k(z),$$

that is, $c_j^k = \left\{{k \atop j}\right\}$. The final result is

$$\sum_{t=0}^{\infty} t^k z^t = \sum_{j=0}^{k} \begin{Bmatrix} k \\ j \end{Bmatrix} (1-z)^{-j-1}/j! \qquad (|z| < 1).$$

Incidentally, the right member furnishes an analytic continuation for the function A_k—defined in the unit disk by the left member—to the entire plane punctured at the pole $z=1$.

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A SET CONTAINING ALL INTEGERS

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THEOREM. *If* $1/a+1/b=1$, *a and b positive irrationals, then every positive integer occurs exactly once in the set* $\{[ka], [kb]\}$, $k=1, 2, 3, \cdots [1]$.

A simple proof by lattice points follows. From $1/a+1/b=1$, $(a-1)(b-1)=1$. Since a and b are irrational, they cannot here be equal; assume $a<b$ and let $r=a-1$. Then $0<r=a-1<1<b-1=1/r$, and $1<1+r=a<2<b=1+1/r$.

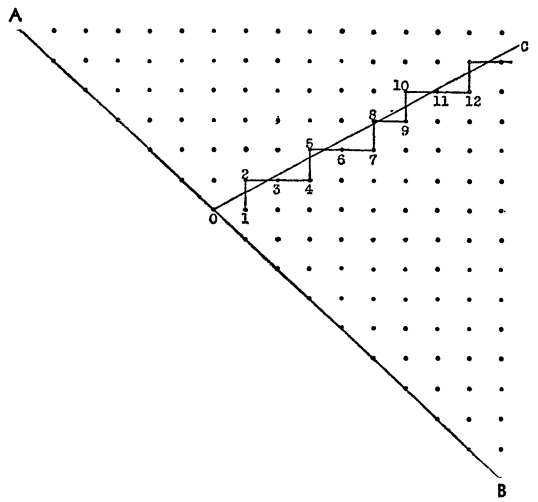


FIG. 1

Through the origin O draw lines $AOB: x+y=0$, and $OC: y=rx$; $0 \leqq x, y$. On the kth vertical lattice line to the right of the origin, the number of interior lattice points between $OB: y=-x$, and $OC: y=rx$, is $[(r+1)k]=[ka]$. On the kth horizontal lattice line above the origin, the number of interior lattice points between $OA: x=-y$, and $OC: x=y/r$, is $[(1/r+1)k]=[kb]$.

In what order must we choose these $[ka]$, $[kb]$ in order to get all positive integers without exception or repetition? Go from O to C, observing successive vertical and horizontal "streets" crossed by OC. Let the nth such line intersect OC in nonlattice point E_n. If this line is vertical, let it intersect OB in lattice point D_n; if horizontal, let it intersect OA in lattice point D_n.

ASSERTION. *D_nE_n has exactly n interior lattice points, not counting the boundary point D_n.*

Proof by induction. Since $1<a<2<b$, D_1E_1 is vertical and has exactly one interior lattice point, corresponding to $[1\cdot a]=1$.

Suppose D_nE_n vertical. (Analogous argument for horizontal D_nE_n.) There are two cases.

(1) $D_{n+1}E_{n+1}$ is also vertical. Then along horizontal lattice lines, interior lattice points on D_nE_n correspond one-one to those on $D_{n+1}E_{n+1}$, except for the additional lattice point on $D_{n+1}E_{n+1}$ just above D_{n+1} and to the right of D_n.

(2) $D_{n+1}E_{n+1}$ is horizontal. Then along lattice lines parallel to AOB, interior lattice points on D_nE_n correspond one-one to those on $D_{n+1}E_{n+1}$, except for the additional lattice point on $D_{n+1}E_{n+1}$ immediately to the left of E_{n+1} and above E_n, where D_nE_n extended would meet $D_{n+1}E_{n+1}$.

Summing up, the first vertical lattice line crossed by OC, bears one lattice point inside region BOC; and the $(n+1)$th horizontal or vertical lattice line crossed, bears one more lattice point inside region AOC or BOC than the nth line crossed. Then if we take these lines in their order of crossing OC, we get successive positive integers starting from unity, among $[ka]$, $[kb]$, $k=1,2,3,\cdots$.

The figure is drawn for $a=3-\sqrt{2}=1.586$, $b=\frac{1}{2}(4+\sqrt{2})=2.707$, $1/a+1/b=\frac{1}{7}(3+\sqrt{2})+\frac{1}{7}(4-\sqrt{2})=1$. It shows:

$[1a]=1$	$[1b]=2$
$[2a]=3,\ [3a]=4$	$[2b]=5$
$[4a]=6,\ [5a]=7$	$[3b]=8$
$[6a]=9$	$[4b]=10$
...	...

The numbered, zigzag route, island-hopping from one lattice point to another, tends to the right, following OC. Whenever this motion would carry it more than a lattice unit below OC, it rises one unit across OC before resuming direction to the right. On the zigzag path, the numbers are at terminal (topmost or rightmost) lattice points on successive lines crossed by OC. They count the order of these crossings, and also on each line crossed, the number of interior lattice points in region AOC or BOC.

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A NEW PROOF OF A THEOREM OF JARDEN

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The Fibonacci sequence is defined by the recurrence relation $u_{n+2}=u_{n+1}+u_n$ and the initial values $u_1=u_2=1$. The rank of apparition of a prime p, denoted by $\alpha(p)$, is the rank of the first member of the sequence which is divisible by p.

Jarden [2] pointed out that $\alpha(p)$ divides $p-(5\,|\,p)$, where $(5\,|\,p)$ is a Legendre symbol, and proved the

THEOREM. *$[p-(5\,|\,p)]/\alpha(p)$ is an unbounded function of p.*

The proof depended upon Dirichlet's famous theorem on the number of primes in arithmetic progressions. Our proof is based upon

1) Carmichael's theorem [1, pp. 61–62] that for every positive integer n except $n=1$, 2, 6, and 12, there is some prime, p, such that $\alpha(p)=n$; and

2) the existence (established by Chebyshev) of an absolute constant $c>0$ for which $p_n>cn\log n$, where p_n is the nth member of the sequence of prime numbers.

Proof of the theorem. It follows from 1) that there are at least $n-4$ different primes which divide the first n Fibonacci numbers, hence for any n there exists p_m such that $\alpha(p_m)\leqq n$ and $m\geqq n-4$. Let c be a positive constant such that $p_n-1>cn\log n$ for all n (the existence of such a constant follows from 2)). Then for every $n>4$ there exists p_m such that

$$\frac{p_m-(5\,|\,p_m)}{\alpha(p_m)}\geqq\frac{p_m-1}{n}>\frac{cm\log m}{n}\geqq\frac{c(n-4)\log(n-4)}{n},$$

and the theorem follows.

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FURTHER EXAMPLES OF EXACT INTEGRATION USING THE TRAPEZOIDAL RULE

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In a previous note [1] the value of the integral $\int_0^\infty(\sin^m t/t^m)dt$ together with certain relations involving powers of m and binomial coefficients, was derived by making use of the fact that the trapezoidal rule gives an exact value of an integral over the range $(0,\ \infty)$ when the Fourier cosine transform of the integrand is zero. Two further examples, namely

$$I_c(k,m)=\int_0^\infty\left[\left\{\cos t-\sum_{r=0}^{k}(-1)^r t^{2r}/(2r)!\right\}\Big/\,t^{2k+2}\right]^m dt,$$

$$I_s(k, m) = \int_0^\infty \left[\left\{ \sin t + \sum_{r=1}^{k} (-1)^r t^{2r-1}/(2r-1)! \right\} \Big/ t^{2k+1} \right]^m dt,$$

where m is a positive integer, will now be evaluated in a similar way.

Consider $I_c(k, m)$ and let

$$f(t) = \left[\left\{ \cos t - \sum_{r=0}^{k} (-1)^r t^{2r}/(2r)! \right\} \Big/ t^{2k+2} \right]^m,$$

so that $f(t)$ is an even function. Then the Fourier cosine transform of $f(t)$ may be determined by evaluating $\int_C f(z)e^{i\phi z}dz$ where C is the contour formed by the real axis and the upper semicircle. For, along the real axis the contribution to the integral is

$$\int_0^\infty f(z)e^{i\phi z}dz + \int_{-\infty}^0 f(z)e^{i\phi z}dz = \int_0^\infty f(z)e^{i\phi z}dz + \int_0^\infty f(z)e^{-i\phi z}dz,$$

which is twice the Fourier cosine transform of $f(t)$. Also, using Jordan's lemma, we see that, provided $\phi \geqq m$, there is no contribution along the upper semicircle. Thus, as $f(t)$ has no poles in the upper half plane, it follows that the Fourier cosine transform of $f(t)$ is zero, provided $\phi \geqq m$. A similar result holds for the transform of $I_s(k, m)$.

For general k and m the algebraic summations resulting from applying the trapezoidal rule are complicated and lead to elaborate results. Five special cases are of interest, however, and these are treated in turn below.

Case 1.

$$\begin{aligned} I_c(0, m) &= \int_0^\infty \{(\cos t - 1)^m/t^{2m}\}dt \\ &= (-1)^m 2^{1-m} \int_0^\infty (\sin^{2m}\theta/\theta^{2m})d\theta, \quad \text{where} \quad 2\theta = t, \\ &= [(-1)^m \pi/\{2^m(2m-1)!\}]\left[m^{2m-1} - \binom{2m}{1}(m-1)^{2m-1} + \cdots \right. \\ &\qquad \left. + (-1)^{m-1}\binom{2m}{m-1} \right], \end{aligned} \tag{1}$$

from [1] equation (5), on replacing m by $2m$ and simplifying.

Case 2.

$$I_c(k, 1) = \int_0^\infty \left[\left\{ \cos t - \sum_{r=0}^{k} (-1)^r t^{2r}/(2r)! \right\} \Big/ t^{2k+2} \right] dt.$$

Applying equation (1) of [1] we find, as $G(\phi) = 0$ for $\phi \geqq 1$ and provided $h \leqq 2\pi$,

$$I_c(k,1) = h\left[\frac{1}{2}\frac{(-1)^{k+1}}{(2k+2)!} + \sum_{r+1}^{\infty}\frac{\cos rh}{(rh)^{2k+2}} - \sum_{s=0}^{k}\sum_{r=1}^{\infty}\frac{(-1)^{k-s}}{(2k-2s)!(rh)^{2s+2}}\right]$$

$$= h\left[\frac{1}{2}\frac{(-1)^{k+1}}{(2k+2)!} + \frac{(-1)^k 2^{2k+1}\pi^{2k+2}}{h^{2k+2}(2k+2)!}\left\{\phi_{2k+2}\left(\frac{h}{2\pi}\right) + (-1)^k B_{k+1}\right\}\right. \tag{2}$$

$$\left. - \sum_{s=0}^{k}\frac{(-1)^{k-s}2^{2s+1}\pi^{2s+2}B_{s+1}}{h^{2s+2}(2s+2)!(2k-2s)!}\right]$$

on using the summation formulae, valid for $h<2\pi$, given by Bromwich ([2] p. 298 and p. 370, No. 5.6). $\phi_{2k+2}(h/2\pi)$ is a polynomial of degree $2k+2$ containing all the even powers of h and h^{2k+1} (see [2] p. 300), whence the above expression, (2), becomes a series in powers of h from 1 to $-(2k+2)$. The constant term originates from the term in h in $\phi_{2k+2}(h/2\pi)$ and is of amount

$$\frac{(-1)^k 2^{2k+1}\pi^{2k+2}}{(2k+2)!}\cdot\frac{-(2k+2)}{2(2\pi)^{2k+1}}$$

so that

$$I_c(k,1) = (-1)^{k+1}\pi/\{2(2k+1)!\}\cdot \tag{3}$$

The coefficients of all the other powers of h must be zero, either by direct cancelling, as is the case here, or through known results, or by yielding new identities.

Case 3.

$$I_s(k,1) = \int_0^{\infty}\left[\left\{\sin t + \sum_{r=1}^{k}(-1)^r t^{2r-1}/(2r-1)!\right\}\Big/ t^{2k+1}\right]dt.$$

The expansion is similar to that of Case 2 and gives

$$I_s(k,1) = (-1)^k\pi/\{2(2k)!\} \tag{4}$$

Case 4.

$$I_c(k,2) = \int_0^{\infty}\left[\left\{\cos t - \sum_{r=0}^{k}(-1)^r t^{2r}/(2r)!\right\}\Big/ t^{2k+2}\right]^2 dt.$$

The integrand may be expanded to give the form

$$I_c(k,2) = \int_0^{\infty}\left[(1+\cos 2t)/(2t^{4k+4}) - 2\cos t\sum_{r=0}^{k}\frac{(-1)^{k-r}}{(2k-2r)!t^{2r+2k+4}}\right.$$

$$\left. + \left\{\sum_{r=0}^{k}\frac{(-1)^{k-r}}{(2k-2r)!t^{2r+2}}\right\}^2\right]dt.$$

Again, provided $h<2\pi$, the integral may be replaced by sums giving

$$I_c(k,2) = h\Big[\frac{1}{2}\Big\{\frac{1}{(2k+2)!}\Big\}^2 + \frac{1}{2}\sum_{r=1}^{\infty}\frac{\cos 2rh}{(rh)^{4k+4}} - 2\sum_{s=1}^{\infty}\sum_{r=0}^{k}\frac{(-1)^{k-r}\cos sh}{(2k-2r)!(sh)^{2r+2k+4}} + 4k \text{ sums with coefficients } h^{-(4r+4)},\ r = 0, 1, \cdots, k\Big]. \tag{5}$$

The final sums do not yield a term independent of h and so need not be detailed in the evaluation of the integral. Again the sums involving the cosines may be expressed in terms of Bernoulli polynomials (according to the formula in [2] p. 300 No. 5.6), provided h is now restricted to be less than π. The expression then derived involves various powers of h from $h^{-(4k+3)}$ to h. The term independent of h arises from the second term of each Bernoulli polynomial and after some reduction gives for the value of the integral

$$I_c(k,2) = \frac{\pi}{(4k+3)!}\Big[2^{4k+1} - \Big\{\binom{4k+3}{0} + \binom{4k+3}{2} + \cdots + \binom{4k+3}{2k}\Big\}\Big]. \tag{6}$$

The coefficients of all other powers of h in the power series derived from (5) must be zero and this follows in every case (including those requiring the final terms in (5)) on performing the necessary algebraic reduction using one or other of the standard results

$$\binom{4k+4}{0} + \binom{4k+4}{2} + \cdots + \binom{4k+4}{2k} + \frac{1}{2}\binom{4k+4}{2k+2} = 2^{4k+2},$$
$$\binom{4k+2}{0} + \binom{4k+2}{2} + \cdots + \binom{4k+2}{2k} = 2^{4k}. \tag{7}$$

Case 5.

$$I_s(k,2) = \int_0^{\infty}\Big[\Big\{\sin t + \sum_{r=1}^{k}(-1)^r t^{2r-1}/(2r-1)!\Big\} \Big/ t^{2k+1}\Big]^2 dt.$$

The working follows closely that of Case 4 and gives

$$I_s(k,2) = \frac{\pi}{(4k+1)!}\Big[2^{4k-1} - \Big\{\binom{4k+1}{1} + \binom{4k+1}{3} + \cdots + \binom{4k+1}{2k-1}\Big\}\Big]. \tag{8}$$

The expressions corresponding to (7) are here

$$\binom{4k+2}{1} + \binom{4k+2}{3} + \cdots + \binom{4k+2}{2k-1} + \frac{1}{2}\binom{4k+2}{2k+1} = 2^{4k},$$
$$\binom{4k}{1} + \binom{4k}{3} + \cdots + \binom{4k}{2k-1} = 2^{4k-2}, \tag{9}$$

which are again standard results.

It is interesting to consider the similarity of the sums of binomial coefficients in expressions (6) to (9). The four results in (7) and (9) may readily be obtained by elementary means, but not so the sums in (6) and (8) which may, therefore, on rewriting, be regarded as expressions for the partial sums of the binomial coefficients of odd exponent, namely

$$(10)\quad \begin{aligned} \binom{4k+1}{1}+\binom{4k+1}{3}+\cdots+\binom{4k+1}{2k-1} &= 2^{4k-1}+\frac{(4k+1)!}{\pi}I_s(k,2),\\ \binom{4k+3}{0}+\binom{4k+3}{2}+\cdots+\binom{4k+3}{2k} &= 2^{4k+1}+\frac{(4k+3)!}{\pi}I_c(k,2). \end{aligned}$$

But for *any* odd exponent $2m+1$ we have

$$\binom{2m+1}{0}+\binom{2m+1}{1}+\cdots+\binom{2m+1}{m}=2^{2m}$$

whence, combining this with each of (10), there is also obtained

$$(11)\quad \begin{aligned} \binom{4k+1}{0}+\binom{4k+1}{2}+\cdots+\binom{4k+1}{2k} &= 2^{4k-1}-\frac{(4k+1)!}{\pi}I_s(k,2),\\ \binom{4k+3}{1}+\binom{4k+3}{3}+\cdots+\binom{4k+3}{2k+1} &= 2^{4k+1}-\frac{(4k+3)!}{\pi}I_c(k,2). \end{aligned}$$

The four expressions (10) and (11) may be regarded as complimentary to the simple results (7) and (9) in the case of the binomial coefficients corresponding to an odd exponent.

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A CONJECTURE CONCERNING THE EULER NUMBERS

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The Euler numbers may be defined by means of

$$\frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_{2n}\frac{x^{2n}}{(2n)!}\qquad (E_{2n+1}=0)$$

or if we prefer

$$(E+1)^n+(E-1)^n=\begin{cases}2 & (n=0)\\ 0 & (n>0)\end{cases}.$$

The first few values are

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385.$$

It follows easily from the definition that the E_{2n} are odd integers; also it is familiar that they are alternately positive and negative, that is

$$E_{4n} > 0, \quad E_{4n+2} < 0. \tag{1}$$

In this note we examine the following

CONJECTURE. *The number* $|E_{2n}|$ *is never a square for* $n > 1$.

It is easy to prove the truth of the conjecture for even n. Indeed by a special case of Kummer's congruence (see [1, Ch. 14]) we have

$$E_{2n} \equiv E_2 \equiv -1 \pmod 3 \qquad (n \geqq 1). \tag{2}$$

Thus $|E_{4n}| = E_{4n} \equiv -1 \pmod 3$ and is therefore not a square.

For odd n we proceed differently. We shall require the following formulas which can be found in [2, Ch. 2]:

$$C_{n-1} = 2^n(1 - 2^n)\frac{B_n}{n}, \qquad E_n = (C + 1)^n, \tag{3}$$

where B_n is the Bernoulli number. It follows from (3) that if $f(x)$ is an arbitrary polynomial then $f'(C) = f(2B) - f(4B)$. In particular for $f(x) = (x+1)^{2n+1}$

$$(2n + 1)E_{2n} = (2B + 1)^{2n+1} - (4B + 1)^{2n+1} = \sum_{r=1}^{2n+1} \binom{2n+1}{r} (2^r - 2^{2r})B_r.$$

Since $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$ and $2B_{2r}$ is integral (mod 2) we get

$$(2n + 1)E_{2n} \equiv 2n + 1 + \binom{2n+1}{2}\frac{4 - 16}{6} \pmod 8,$$

and therefore

$$E_{2n} \equiv 1 - 2n \pmod 8. \tag{4}$$

Thus by (1) and (4) we have

$$|E_{4n+2}| \equiv 4n + 1 \pmod 8. \tag{5}$$

Replacing n by $2n+1$, (5) becomes $|E_{8n+6}| \equiv 5 \pmod 8$ and therefore $|E_{8n+6}|$ is not a square.

If in place of (2) we use

$$E_{2n+6} \equiv E_{2n} \pmod 7 \qquad (n \geqq 1), \tag{6}$$

then we have

$$E_{6m} \equiv E_6 \equiv -61 \equiv 2 \pmod 7. \tag{7}$$

Now $8n+2=6m$ provided $n\equiv 2 \pmod 3$. Thus (7) yields $|E_{24n+18}| \equiv -2 \pmod 7$. Since -2 is a quadratic nonresidue (mod 7) we conclude that $|E_{24n+18}|$ is not a square.

We next take

$$E_{2n+10} \equiv E_{2n} \pmod{11} \qquad (n \geqq 1). \tag{8}$$

In particular $E_{10n+4}\equiv E_4\equiv 5 \pmod{11}$, $E_{10n+6}\equiv E_6\equiv -61\equiv 5 \pmod{11}$. Since $8n+2\equiv 4 \pmod{10}$ provided $n\equiv 4 \pmod 5$ and $8n+2\equiv 6 \pmod{10}$ provided $n\equiv 3 \pmod 5$ we get

$$|E_{40n+34}| \equiv |E_{40n+26}| \equiv -5 \pmod{11};$$

since -5 is a quadratic nonresidue (mod 11) it follows that $|E_{40n+34}|$ and $|E_{40n+26}|$ are not squares.

No further information is provided by (8). For example if $8n+2\equiv 8 \pmod{10}$ we get $E_{8n+2}\equiv E_8\equiv 1385\equiv -1 \pmod{11}$ so that $|E_{8n+2}| \equiv 1 \pmod{11}$.

By using larger moduli possibly additional information can be obtained but it seems doubtful that the general conjecture can be proved in this way.

To summarize, we have proved that $|E_{4n}|$ and $|E_{8n+6}|$ are not squares. The numbers $|E_{8n+2}|$ remain in doubt; however $|E_{24n+18}|$, $|E_{40n+34}|$, $|E_{40n+26}|$ are not squares.

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A THEOREM ON PSEUDO-PRIMES

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Definition. *By a pseudo-prime with respect to a positive integer a is meant a composite number s such that $s \mid a^{s-1}-1$; that is to say an s which though composite satisfies the conclusion of Fermat's well-known theorem for a given a.*

Theorem. *If $a(>0)$ is even but $\neq 2^{2^r}$ (r a nonnegative integer) then for every integer $n\geqq 1$, $a^{a^n}+1$ is a pseudo-prime with respect to a.*

Proof. $a^{a^n}+1 \mid a^{2a^n}-1$. Since a is even, $a^n>n$ and $2 \mid a^{(a^n-n)}$. Hence $2a^n \mid a^{a^n}$. Thus $a^{2a^n}-1 \mid a^{a^{a^n}}-1$ and it follows that, for any even a, $a^{a^n}+1 \mid a^{a^{a^n}}-1$.

As is well known and easily seen, $a^{a^n}+1$ is composite for $a\neq 2^{2^r}$ if r is a nonnegative integer. Hence the conditions of the definition are satisfied and the theorem follows.

This theorem shows constructively and in a very elementary way the existence of an infinitude of pseudo-primes with respect to almost every even integer. Also this construction, unlike some others, is not in terms of the primes, which are very difficult to construct, but in terms of the positive integers.

ELEMENTARY PROOFS OF THE INFINITE PRODUCT FOR SIN Z AND ALLIED FORMULAE

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I. In a recent note, Yoshio Matsuoka [2] has given a simple and elementary proof of the formula $\sum_{k=1}^{\infty} 1/k^2 = \frac{1}{6}\pi^2$. We wish to point out a similar elementary method of obtaining the classical infinite products for sin z and cos z and the partial fraction expansions of cot z and tan z.

II. Consider the integral

$$I_n(z) = \int_0^{\pi/2} \cos zt \cos^n t\,dt.$$

Let us write for brevity $c = \cos t$, $s = \sin t$. Taking $n \geqq 2$ and integrating by parts, we have:

$$zI_n(z) = \sin zt \cdot c^n \Big|_0^{\pi/2} + n\int_0^{\pi/2} \sin zt \cdot c^{n-1} s\,dt,$$

$$z^2 I_n(z) = -n \cos zt \cdot c^{n-1} s \Big|_0^{\pi/2} + n\int_0^{\pi/2} \cos zt[c^n - (n-1)c^{n-2}(1-c^2)]\,dt.$$

We thus have the reduction formula

$$n(n-1)I_{n-2}(z) = (n^2 - z^2)I_n(z).$$

Since $I_n(0) > 0$ for each $n \geqq 0$, this yields:

$$\frac{I_{n-2}(z)}{I_{n-2}(0)} = \left(1 - \frac{z^2}{n^2}\right)\frac{I_n(z)}{I_n(0)}, \qquad n \geqq 2. \tag{1}$$

Referring back to the above reduction of $I_n(z)$ $(n < 2)$ and observing that $I_0(0) = \frac{1}{2}\pi$ and $I_1(0) = 1$, we have

$$\sin\frac{\pi z}{2} = \frac{\pi z}{2}\frac{I_0(z)}{I_0(0)}, \qquad \cos\frac{\pi z}{2} = (1 - z^2)\frac{I_1(z)}{I_1(0)}. \tag{2}$$

We now prove that

$$\lim_{n\to\infty} \frac{I_n(z)}{I_n(0)} = 1. \tag{3}$$

For the proof, consider

$$I_n(0) - I_n(z) = \int_0^{\pi/2} (1 - \cos zt)\cos^n t\,dt.$$

Let $t \in [0, \frac{1}{2}\pi]$ and set $z = x + iy$. It is easily seen that

$$|1 - \cos zt| = 2(\sin^2 \tfrac{1}{2}xt + \sinh^2 \tfrac{1}{2}yt) \leqq \tfrac{1}{2}(x^2 + \eta^2)t^2 = \tfrac{1}{2}|\zeta|^2 t^2,$$

where $\frac{1}{4}\pi\eta = \sinh \frac{1}{4}\pi y$, $\zeta = x + i\eta$. Hence

$$|I_n(0) - I_n(z)| \leqq \tfrac{1}{2}|\zeta|^2 \int_0^{\pi/2} t^2 \cos^n t\,dt, \tag{4}$$

an estimate that is obvious when z is real in which case $\zeta = z$. Now, since $t < \tan t$ in $(0, \frac{1}{2}\pi)$, we have

$$\int_0^{\pi/2} t^2 \cos^n t\,dt < \int_0^{\pi/2} t \cos^{n-1} t \sin t\,dt = \frac{1}{n} I_n(0),$$

by an integration by parts. This together with (4) yields the estimate

$$\left|1 - \frac{I_n(z)}{I_n(0)}\right| \leqq \frac{|\zeta|^2}{2n}, \qquad n \geqq 1. \tag{5}$$

This proves (3).

From (1), (2) and (3) we immediately deduce

$$\sin\frac{\pi z}{2} = \frac{\pi z}{2}\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{(2k)^2}\right), \qquad \cos\frac{\pi z}{2} = \prod_{k=0}^{\infty}\left(1 - \frac{z^2}{(2k+1)^2}\right). \tag{6}$$

III. Consider the integral

$$J_n(z) = \int_0^{\pi/2} t \sin zt \cos^n t\,dt.$$

Taking $n \geqq 2$ and integrating by parts twice as before, we obtain

$$n(n-1)J_{n-2}(z) = 2zI_n(z) + (n^2 - z^2)J_n(z).$$

Using the reduction formula for $I_n(z)$, we obtain, by division,

$$\frac{J_{n-2}(z)}{I_{n-2}(z)} = \frac{2z}{n^2 - z^2} + \frac{J_n(z)}{I_n(z)}, \qquad n \geqq 2. \tag{7}$$

A simple direct calculation shows that

$$\frac{\pi}{2}\cot\frac{\pi z}{2} = \frac{1}{z} - \frac{J_0(z)}{I_0(z)}, \qquad \frac{\pi}{2}\tan\frac{\pi z}{2} = \frac{2z}{1 - z^2} + \frac{J_1(z)}{I_1(z)}. \tag{8}$$

Of course, (7) and (8) follow from (1) and (2) by logarithmic differentiation which one may not consider to be strictly elementary.

Now let $t \in [0, \frac{1}{2}\pi]$ and observe that

$$|\sin zt|^2 = \sin^2 xt + \sinh^2 yt \leqq (x^2 + Y^2)t^2 = |Z|^2 t^2,$$

where $\frac{1}{2}\pi Y = \sinh \frac{1}{2}\pi y$, $Z = x + iY$. Hence

$$|J_n(z)| \leqq |Z| \int_0^{\pi/2} t^2 \cos^n t\,dt \leqq \frac{|Z|}{n} I_n(0),$$

so that $\lim_{n\to\infty} J_n(z)/I_n(0) = 0$. This combined with (3) yields

$$\lim_{n\to\infty} \frac{J_n(z)}{I_n(z)} = 0. \tag{9}$$

From (7), (8) and (9), we see at once that

$$\frac{\pi}{2}\cot\frac{\pi z}{2} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - (2k)^2}, \qquad \frac{\pi}{2}\tan\frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{2z}{(2k+1)^2 - z^2}. \tag{10}$$

IV. It may be relevant here and of interest to point out an elementary derivation of the classic formulas

$$\frac{1}{\alpha} + 2\alpha \sum_{r=1}^{\infty} \frac{\cos rx}{\alpha^2 - r^2} = \frac{\pi \cos \alpha(\pi - x)}{\sin \pi\alpha}, \qquad 0 \leqq x < 2\pi, \tag{*}$$

$$\sum_{r=1}^{\infty} \frac{2r \sin rx}{\alpha^2 - r^2} = -\frac{\pi \sin \alpha(\pi - x)}{\sin \pi\alpha}, \qquad 0 < x < 2\pi, \tag{**}$$

which are generalisations of the partial fraction expansions of $\cot \pi\alpha$ and $\csc \pi\alpha$. In fact our procedure is motivated by one that has been employed by Cesàro [1] to prove the formula

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}, \qquad 0 < x < 2\pi.$$

Let α be nonintegral, and consider

$$\phi_n(x) + i\psi_n(x) = \sum_{r=-n}^{n} \frac{e^{i(r+\alpha)x}}{r + \alpha}, \qquad 0 < x < 2\pi.$$

Differentiating we obtain

$$\phi_n'(x) + i\psi_n'(x) = ie^{i\alpha x} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} = ie^{i\alpha x} \frac{\sin (n + \frac{1}{2})x}{\sin \frac{1}{2}x}.$$

Thus the derivatives of the functions

$$\phi_n(x) = \sum_{r=-n}^{n} \frac{\cos (r + \alpha)x}{r + \alpha}, \qquad \psi_n(x) = \sum_{r=-n}^{n} \frac{\sin (r + \alpha)x}{r + \alpha}$$

are given by

$$\phi_n'(x) = -\frac{\sin (n + \frac{1}{2})x}{\sin \frac{1}{2}x} \sin \alpha x, \qquad \psi_n'(x) = \frac{\sin (n + \frac{1}{2})x}{\sin \frac{1}{2}x} \cos \alpha x.$$

Following Cesàro, let

$$A_n(x) = \frac{\cos (n + \frac{1}{2})x}{n + \frac{1}{2}} \frac{\sin \alpha x}{\sin \frac{1}{2}x}.$$

Then

$$\phi_n'(x) - A_n'(x) = -\frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}} \frac{d}{dx}\left(\frac{\sin \alpha x}{\sin \frac{1}{2}x}\right). \tag{11}$$

Now it is easily verified that

$$2\sin^2 \tfrac{1}{2}x \cdot \frac{d}{dx}\left(\frac{\sin \alpha x}{\sin \frac{1}{2}x}\right) = (\alpha - \tfrac{1}{2})\sin(\alpha + \tfrac{1}{2})x - (\alpha + \tfrac{1}{2})\sin(\alpha - \tfrac{1}{2})x.$$

Observing that $\sin z = z + O(|z|^3)$ for $|z| \to 0$, we find that

$$\frac{d}{dx}\left(\frac{\sin \alpha x}{\sin \frac{1}{2}x}\right) = O(|x|), \qquad x \to 0.$$

Hence (11) shows that $\phi_n'(x) - A_n'(x)$, as well as $A_n(x)$, is of the order of $1/n$ in the interval $0 \leqq x < 2\pi$. By Rolle's theorem it follows that in this interval

$$\phi_n(x) - \phi_n(0) - [A_n(x) - A_n(0)] = O\left(\frac{1}{n}\right),$$

so that $\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \phi_n(0)$, or

$$\lim_{n\to\infty} \sum_{r=-n}^{n} \frac{\cos(r+\alpha)x}{r+\alpha} = A(\alpha), \tag{12}$$

where

$$A(\alpha) = \lim_{n\to\infty} \sum_{r=-n}^{n} \frac{1}{r+\alpha} = \frac{1}{\alpha} + 2\alpha \sum_{r=1}^{\infty} \frac{1}{\alpha^2 - r^2}. \tag{12a}$$

Let

$$B_n(x) = \frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}} \frac{\cos \alpha x}{\sin \frac{1}{2}x}.$$

Then in any interval of the form $0 < \delta \leqq x \leqq 2\pi - \delta$, we find that $\psi_n'(x) + B_n'(x)$, as well as $B_n(x)$, is of the order of $1/n$. It follows on using Rolle's theorem as before that $\lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} \psi_n(\pi)$, or

$$\lim_{n\to\infty} \sum_{r=-n}^{n} \frac{\sin(r+\alpha)x}{r+\alpha} = B(\alpha), \tag{13}$$

where

$$B(\alpha) = C(\alpha) \sin \pi\alpha, \tag{13a}$$

$$C(\alpha) = \lim_{n\to\infty} \sum_{r=-n}^{n} \frac{(-1)^r}{r+\alpha} = \frac{1}{\alpha} + 2\alpha \sum_{r=1}^{\infty} \frac{(-1)^r}{\alpha^2 - r^2}. \tag{13b}$$

Taking $x=\pi$ in (12), we have

$$A(\alpha) = C(\alpha)\cos\pi\alpha. \tag{14}$$

Taking suitable linear combinations of (12) and (13) and using (14) and (13a), we obtain:

$$\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{\cos rx}{r+\alpha} = C(\alpha)\cos\alpha(\pi-x), \qquad 0<\delta\leqq x\leqq 2\pi-\delta, \tag{15}$$

$$\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{\sin rx}{r+\alpha} = C(\alpha)\sin\alpha(\pi-x), \qquad 0<\delta\leqq x\leqq 2\pi-\delta. \tag{16}$$

On putting $x=\frac{1}{2}\pi$ in (15), we find that

$$\frac{1}{2}C\left(\frac{\alpha}{2}\right) = C(\alpha)\cos\frac{\pi\alpha}{2}\cdot \tag{17}$$

Multiplying on both sides of (17) by $2\sin\frac{1}{2}\pi\alpha$ and using (13a), we have

$$B(\alpha) = B\left(\frac{\alpha}{2}\right) = B\left(\frac{\alpha}{2^2}\right) = \cdots. \tag{18}$$

This together with (13a) and (13b) gives

$$B(\alpha) = \lim_{\alpha\to 0}\frac{\sin\pi\alpha}{\alpha} = \pi, \tag{19}$$

so that we have:

$$C(\alpha) = \pi\csc\pi\alpha, \qquad A(\alpha) = \pi\cot\pi\alpha. \tag{20}$$

Inserting (20) in (12a) and (13b), we have the well-known partial fraction expansions of $\cot\pi\alpha$ and $\csc\pi\alpha$. At the same time, (15) and (16) prove the more general formulae (*) and (**).

References

1. E. Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung, Leipzig, 1904, p. 305.

2. Yoshio Matsuoka, this MONTHLY, **68** (1961) 485–486.

Editorial Note. In this MONTHLY **68** (1961), 906–907, Harry Lass and Carleton B. Solloway proved that if A is an $m\times n$ matrix and B is an $n\times m$ matrix then $\lambda^n|\lambda I_m - AB| = \lambda^m|\lambda I_n - BA|$. After publication Mr. Solloway learned of a simpler proof given by S. N. Afriat in Proc. Camb. Phil. Soc. **53** (1957), 802 and by H. Scheffé on p. 405 of "An Analysis of Variance," Wiley, New York, 1959. The result follows, on taking determinants, from the identity

$$\begin{bmatrix}\lambda I_m - AB & A\\ 0 & \lambda I_n\end{bmatrix}\begin{bmatrix}I_m & 0\\ B & I_n\end{bmatrix} = \begin{bmatrix}I_m & 0\\ B & I_n\end{bmatrix}\begin{bmatrix}\lambda I_m & A\\ 0 & \lambda I_n - BA\end{bmatrix}.$$

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

AN ISOMETRY OF THE PLANE

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It is well known that any distance-preserving map of the plane into the plane that fixes the origin is either a rotation or a rotation followed by reflection about the x-axis. The purpose of this note is to give a short rigorous proof of this theorem from elementary principles with no recourse to any notions from geometry.

Suppose f is such a map. Since f is distance preserving and $f(0)=0$, f preserves moduli. We have

$$|f(z)-f(1)|^2 = |f(z)|^2 + |f(1)|^2 - 2\,\mathrm{Re}\,f(z)\overline{f(1)}$$
$$|z-1|^2 = |z|^2 + |1|^2 - 2\,\mathrm{Re}\,z.$$

Thus we see for any z,

$$\mathrm{Re}\,f(z)\overline{f(1)} = \mathrm{Re}\,z. \tag{1}$$

For $z=i$, this gives $\mathrm{Re}\,f(i)\overline{f(1)} = \mathrm{Re}\,i = 0$, hence $\mathrm{Im}\,f(i)\overline{f(1)} = \pm 1$ since $|f(i)| = |f(1)| = 1$. Consequently, $f(i) = \pm i f(1)$. Similarly,

$$|f(z)-f(i)|^2 = |f(z)|^2 + |f(i)|^2 - 2\,\mathrm{Re}\,f(z)\overline{f(i)},$$
$$|z-i|^2 = |z|^2 + |i|^2 - 2\,\mathrm{Re}\,z\bar{i}.$$

Now

$$\pm\,\mathrm{Im}\,f(z)\overline{f(1)} = \pm\,\mathrm{Re}\,f(z)\bar{i}\,\overline{f(1)} \tag{2}$$
$$= \mathrm{Re}\,f(z)\,\overline{f(i)} = \mathrm{Re}\,z\,\bar{i} = \mathrm{Im}\,z.$$

Combining (1) and (2), we obtain

$$f(z)\overline{f(1)} = z \text{ or } \bar{z},$$

hence

$$f(z) = f(1)z \text{ or } f(1)\bar{z}.$$

A RECURSION FORMULA FOR A CLASS OF DEFINITE INTEGRALS

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Recently, Levenson [1] and Narasimhan [2], using a similar method, derived a recursion formula for a particular definite integral. In this note, we will derive a recursion formula for a class of definite integrals which yields the results of [1] and [2] as special cases.

THEOREM. *Let* $s_0 \leqq 0$ *be a constant such that*

$$y(s) \equiv \int_\alpha^\beta f(x)[g(x)]^{as+b}dx = H(s), \qquad as + b \geqq s_0, \tag{1}$$

where s *is a real parameter,* $a \neq 0$, b, α, *and* β *are constants, and* $g(x) \geqq 0$ *on* $[\alpha, \beta)$, *which may be finite or infinite. We assume that* $D^n y(s) = D^n H(s)$, $n = 0, 1, \cdots$, *where* $D^n \equiv d^n/ds^n$. *(If* $[\alpha, \beta) = [0, \infty)$, *we would require uniform convergence in* s *for the integrals involved). Let* $h(s) = DH(s)/H(s)$. *Then, for* $s = -b/a$, $n = 0, 1, \cdots$,

$$\int_\alpha^\beta f(x) \log^{n+1} g(x)dx = \sum_{r=0}^{n} a^{-(r+1)} \binom{n}{r} [D^r h(s)]_{s=-b/a} \int_\alpha^\beta f(x) \log^{n-r} g(x)dx. \tag{2}$$

A special case of (2) *occurs when* $h(s) \equiv$ *constant, say* c. *Then*

$$\int_\alpha^\beta f(x) \log^n g(x)dx \equiv \left(\frac{c}{a}\right)^n \int_\alpha^\beta f(x)dx, \qquad n = 0, 1, \cdots. \tag{3}$$

Proof. (2) is obtained by applying to $Dy(s) = y(s)h(s)$ Leibnitz's rule for the nth derivative, evaluated for $as + b = 0$. If $h(s) \equiv c$, then (2) yields $u_{n+1} = (c/a)u_n$, where $u_n = \int_\alpha^\beta f(x) \log^n g(x)dx$, and (3) follows by induction on n.

Remarks. The Mellin transform (see [3], [4], and [5])

$$y(s) \equiv \int_0^\infty f(x)x^{s-1}dx, \qquad s - 1 \geqq s_0, \tag{4}$$

is an important example of (1), where $g(x) \equiv x$, $a = 1$, $b = -1$, $[\alpha, \beta) = [0, \infty)$. If $f(x) \equiv e^{-x}$, (4) yields $y(s) \equiv \Gamma(s)$, $s > 0$, where $s_0 = -1$. If we use the Mellin transforms, $y(s) \equiv \Gamma(s)$ and $y(s) \equiv \int_0^1 (1 - x^2)^{-\frac{1}{2}} x^{2s-1}dx$, $s > 0$, in (1), then (2) yields the results of [1] and [2], respectively.

After this paper was submitted for publication, I recalled that the paper by Beumer [6] was concerned with the evaluation of integrals of the type $I_n = \int_0^{\pi/2} (\log \sin x)^n dx$, $n = 1, 2, \cdots$. A recursion formula for I_n is given by (11) in [6, p. 647], and we note that (11) is a special case of the general formula, (2). Since

$$\int_0^1 \frac{x^{2s-1}dx}{\sqrt{(1 - x^2)}} \equiv \int_0^{\pi/2} \sin^{2s-1} v dv, \qquad s > 0,$$

the recursion formulas in [2] and [6, (11)] are identical.

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EQUATIONS OF POLYGONAL PATHS

CHARLES P. SEGUIN, Beloit College

In most introductory courses in mathematical analysis students are asked to graph functions like $g(x)=|x-1|+x$ or $h(x)=|x-1|+|x-2|$. If we restrict the domains of such functions, their graphs are polygonal paths joining certain points. For example, if $h(x)$ above is graphed for $0\leqq x\leqq 3$, we obtain the polygonal path joining the points (0, 3), (1, 1), (2, 1), and (3, 3) in that order.

This suggests the following problem. Given the points $P_i(x_i, y_i)$, $i=0, 1, \cdots, k$ with $x_0<x_1<\cdots<x_k$, find the equation of the polygonal path joining these points.

Functions like $g(x)$ above suggest for the case $k=2$ a solution of the form

$$f(x) = A|x-x_1| + Bx + C, \qquad x_0 \leqq x \leqq x_2.$$

If we impose the conditions $f(x_i)=y_i$, $i=0, 1, 2$ and solve the resulting system of equations for the undetermined constants, we obtain

$$f(x) = \tfrac{1}{2}[(m_2-m_1)|x-x_1| + (m_2+m_1)x + (x_2y_1-x_1y_2)/(x_2-x_1) + (x_1y_0-x_0y_1)/(x_1-x_0)], \qquad x_0 \leqq x \leqq x_2$$

where $m_i=(y_i-y_{i-1})/(x_i-x_{i-1})$, $i=1, 2$. This can be written more simply as

$$f(x) = \tfrac{1}{2}[(m_2-m_1)|x-x_1| + m_1(x-x_1) + m_2(x-x_2) + (y_1+y_2)], \qquad x_0 \leqq x \leqq x_2.$$

Functions like $h(x)$ lead us to assume a solution of the form

$$f(x) = A|x-x_1| + B|x-x_2| + Cx + D, \qquad x_0 \leqq x \leqq x_3$$

for the case $k=3$ and here we find that

$$f(x) = \tfrac{1}{2}[(m_2-m_1)|x-x_1| + (m_3-m_2)|x-x_2| + m_1(x-x_1) + m_3(x-x_3) + (y_1+y_3)], \qquad x_0 \leqq x \leqq x_3,$$

where $m_i=(y_i-y_{i-1})/(x_i-x_{i-1})$, $i=1, 2, 3$.

These two cases point to the following conjecture for the general case:

$$f(x) = \frac{1}{2}\left[\sum_{i=1}^{k-1}(m_{i+1} - m_i)\,|\,x - x_i\,| + m_1(x - x_1) + m_k(x - x_k) + (y_1 + y_k)\right],$$

$x_0 \leqq x \leqq x_k$, where $m_i = (y_i - y_{i-1})/(x_i - x_{i-1})$, $i = 1, 2, \cdots, k$.

To establish this it is sufficient to show that for $x_j \leqq x \leqq x_{j+1}$, $0 \leqq j \leqq k-1$, $f(x)$ reduces to the equation of the line through P_j and P_{j+1}. For such an x we have

$$\begin{aligned}
2f(x) &= \sum_{i=1}^{j}(m_{i+1} - m_i)(x - x_i) + \sum_{i=j+1}^{k-1}(m_{i+1} - m_i)(x_i - x) \\
&\quad + m_1(x - x_1) + m_k(x - x_k) + (y_1 + y_k) \\
&= \sum_{i=1}^{j}(m_{i+1} - m_i)x - \sum_{i=j+1}^{k-1}(m_{i+1} - m_i)x + \sum_{i=1}^{j} m_i x_i - \sum_{i=1}^{j} m_{i+1}x_i \\
&\quad + \sum_{i=j+1}^{k-1} m_{i+1}x_i - \sum_{i=j+1}^{k-1} m_i x_i + m_1(x - x_1) + m_k(x - x_k) + (y_1 + y_k) \\
&= (m_{j+1} - m_1)x - (m_k - m_{j+1})x + m_1 x_1 + \sum_{i=2}^{j} m_i x_i - \sum_{i=1}^{j-1} m_{i+1}x_i - m_{j+1}x_j \\
&\quad + \sum_{i=j+1}^{k-2} m_{i+1}x_i + m_k x_{k-1} - m_{j+1}x_{j+1} - \sum_{i=j+2}^{k-1} m_i x_i + m_1(x - x_1) \\
&\quad + m_k(x - x_k) + (y_1 + y_k) \\
&= m_{j+1}(x - x_j) + m_{j+1}(x - x_{j+1}) + \sum_{i=1}^{j-1} m_{i+1}x_{i+1} - \sum_{i=1}^{j-1} m_{i+1}x_i \\
&\quad + \sum_{i=j+1}^{k-2} m_{i+1}x_i - \sum_{i=j+1}^{k-2} m_{i+1}x_{i+1} - m_k(x_k - x_{k-1}) + (y_1 + y_k) \\
&= m_{j+1}(x - x_j) + m_{j+1}(x - x_{j+1}) + \sum_{i=1}^{j-1} m_{i+1}(x_{i+1} - x_i) \\
&\quad - \sum_{i=j+1}^{k-2} m_{i+1}(x_{i+1} - x_i) - (y_k - y_{k-1}) + (y_1 + y_k) \\
&= m_{j+1}(x - x_j) + m_{j+1}(x - x_{j+1}) + \sum_{i=1}^{j-1}(y_{i+1} - y_i) \\
&\quad - \sum_{i=j+1}^{k-2}(y_{i+1} - y_i) + (y_1 + y_{k-1}) \\
&= m_{j+1}(x - x_j) + m_{j+1}(x - x_{j+1}) + (y_j - y_1) - (y_{k-1} - y_{j+1}) + (y_1 + y_{k-1}) \\
&= m_{j+1}(x - x_j) + y_j + m_{j+1}(x - x_{j+1}) + y_{j+1} = 2(m_{j+1}(x - x_j) + y_j)
\end{aligned}$$

since $m_{j+1}(x - x_j) + y_j = m_{j+1}(x - x_{j+1}) + y_{j+1}$ for all x. Hence $f(x) = m_{j+1}(x - x_j) + y_j$ which is the equation of the line through P_j and P_{j+1}.

AN IMPOSSIBLE DISSECTION

FREE JAMISON, San Jose State College

The following question recently appeared in this MONTHLY (68 (1961), 912–913). Are eight isosceles acute triangles necessary for the (dissection of the) general obtuse triangle? The answer is affirmative. It is not possible to dissect certain triangles with two small unequal angles into seven isosceles acute triangles. As an example take the case of $\triangle ABC$ with $\measuredangle A$ and C of four-trillionths and eight-trillionths of a radian respectively. In the seven piece dissection these two angles would be vertex angles of isosceles triangles. The other five triangles would have a common vertex as shown in Figure 1. There are no more than 81 cases to investigate, three choices for the vertex angle in each of four of the five triangles.

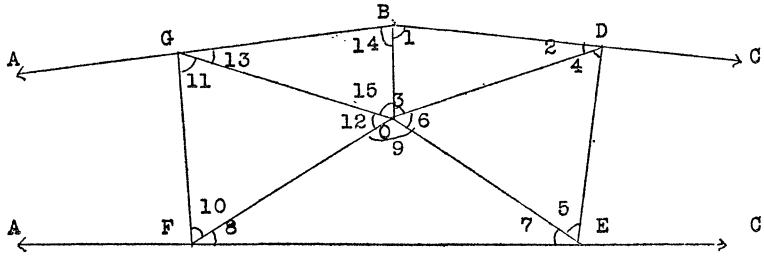


FIG. 1

However, this number of cases is quickly reduced to the fifteen shown in Table I. It is noted that $\measuredangle 1$ and 14, the sums of $\measuredangle 2$ and 4, $\measuredangle 5$ and 7, $\measuredangle 8$ and 10, and $\measuredangle 11$ and 13, are each approximately 90°. The approximation for any angle of $\measuredangle OBD$ and OGB is 0°, 45°, or 90°. Starting with the possibilities for $\triangle OBD$ and working in a clockwise direction about point 0, we also note that a 45° approximation for $\measuredangle 4$, 7, or 10 can not be used as a vertex angle for this gives a base angle of approximately $67\frac{1}{2}°$ from which there is no recovery to obtain a multiple of 45° for the angles of $\triangle OGB$.

Using the law of sines on each of the five triangles, we see that the product of the sines of $\measuredangle 1$, 4, 7, 10, and 13 must equal the product of the sines of $\measuredangle 2$, 5, 8, 11, and 14. This eliminates cases II, III, IV, V, VII, VIII, XII, and XV.

Computation of the angles for each of the other seven cases is easily made. In Table II the corrections in trillionths of radians to be applied to the approximations in Table I are given for the vertex angles. In cases I, VI, XIII, and XIV the underlined values indicate obtuse angles, and these cases are eliminated as possibilities. This leaves only cases IX, X, and XI in which each triangle has angles of approximately 0°, 90°, and 90°. In each of these cases it is

easily seen that the relation of products of sines given in the preceding paragraph, necessary for possibility, is not met. The underlined values in these three cases are the actual values of the angles in trillionths and are good approximations for the sines; the sine of each of the other of the ten angles for any case is near unity.

TABLE I

APPROXIMATE VALUES OF ANGLES IN MULTIPLES OF 45°

Case	Angle														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I	2	1	1	1	1	2	1	1	2	1	1	2	1	2	1
II	2	1	1	1	1	2	1	1	2	1	2	1	0	2	2
III	2	1	1	1	1	2	1	2	1	0	2	2	0	2	2
IV	2	1	1	1	2	1	0	2	2	0	2	2	0	2	2
V	2	0	2	2	0	2	2	1	1	1	1	2	1	2	1
VI	2	0	2	2	0	2	2	1	1	1	2	1	0	2	2
VII	2	0	2	2	0	2	2	0	2	2	1	1	1	2	1
VIII	2	0	2	2	0	2	2	0	2	2	0	2	2	2	0
IX	2	0	2	2	0	2	2	0	2	2	2	0	0	2	2
X	2	0	2	2	0	2	2	2	0	0	2	2	0	2	2
XI	2	0	2	2	2	0	0	2	2	0	2	2	0	2	2
XII	2	0	2	2	1	1	1	1	2	1	1	2	1	2	1
XIII	2	0	2	2	1	1	1	1	2	1	2	1	0	2	2
XIV	2	0	2	2	1	1	1	2	1	0	2	2	0	2	2
XV	2	2	0	0	2	2	0	2	2	0	2	2	0	2	2

TABLE II

CORRECTIONS IN TRILLIONTHS OF RADIANS FOR VERTEX ANGLES

Case	Angle														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I	−6					−2			−6			2		−6	
VI		−6			−20		24				−28		30		
IX		9.6			11.2			14.4				24.8	14.4		
X		13.5			19				30	17			10.5		
XI		16.8				25.6	16.8			10.4			7.2		
XIII		15		−11					3		−7		9		
XIV		30		−26				18		−16			−6		

All cases having been eliminated, it is impossible to dissect into seven isosceles acute triangles the triangle two of whose angles are four-trillionths and eight-trillionths of a radian. Many other triangles with two small unequal angles could have been used to demonstrate the impossibility.

However, every isosceles triangle whose vertex angle is greater than 144° can be dissected into seven isosceles acute triangles; case X will lead to the solution which is shown in Figure 2.

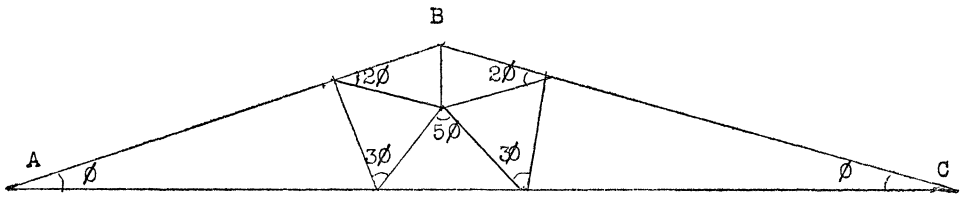


FIG. 2. Case X for equal base angles. $0<\phi<18°$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware,
AND
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TEACHING MACHINES AND MATHEMATICS PROGRAMS

A symposium, "Teaching Machines and Mathematics Programs: The Interaction of Content and Programming Specialists in Developing Self-instructional Programs," was held at the annual AAAS meeting in Denver on December 29. The program was sponsored by the AAAS Cooperative Committee on the Teaching of Science and Mathematics and AAAS Sections A—Mathematics, and I—Psychology. The symposium was arranged by Joseph Hammock, Bell Telephone Laboratories, with the assistance of J. R. Mayor. Dr. Hammock presided.

Below are several of the papers presented at the symposium and the comments of two of the discussants, Robert M. Gagné and R. C. Buck. Additional papers were presented by: Lloyd E. Homme, Teaching Machines Incorporated, Albuquerque, New Mexico; and Jack E. Forbes, Britannica Center for Studies in Learning and Motivation, Palo Alto, California. A third discussant was Max Beberman, University of Illinois.

Statement by John A. Barlow, Emory University

Socrates was so effective as a teacher that the notes kept by one of his pupils are read and discussed all over the world some twenty-three hundred years later.

Socrates taught primarily by asking questions, by presenting problems to his students.

Haylcon Skinner was so ineffective as an educational innovator that his fame in education rests entirely on the rediscovery of his long expired patent almost 100 years later. The Ohio inventor influenced the carpet industry with his power loom inventions but his machine to teach spelling was never widely used.

Socrates apparently did not pass on to his pupils the art of teaching by asking questions. Other teachers have used the method with varying degrees of effectiveness but it has remained an art. Haylcon Skinner's invention of a machine which taught by presenting problems did nothing to turn the art of teaching into a science of learning.

In the late 1920's, Sidney Pressey began to forecast a "coming 'industrial revolution' in education." Pressey published several papers describing devices which would immediately inform the student as to whether he were right or wrong when the student indicated his answers to multiple-choice questions. Pressey and his pupils also published several papers presenting evidence that students learned from taking such tests and that when such tests "are used systematically in college courses, gains are substantial and sufficiently generalized to improve understanding of the subject as a whole."

In the 1950's B. F. Skinner added the element which some of us feel shows real promise of resulting in the development of a new science of schoolroom learning. The teaching devices designed by the inventor, Haylcon Skinner, by the educator, Sidney Pressey, and in various forms by others, kept a record of the errors of the student so that the student could be graded. The new element added by Fred Skinner was the keeping of a *record of the success of each item* as a link of a sequence of problems and questions which leads the student to new usages, concepts, and insights.

There is a pertinent cartoon which originally appeared in the Columbia University *Jester*. It shows an anthropomorphized rat with a lever and a food-trough in front of him. The rat is saying, "Boy, have I got this psychologist well trained. Every time I press the lever he drops me some food." This portrayal of a rat in a Skinner Box seems to me to exemplify much that is *teaching* at its best. The rat gets a pellet of food in the trough whenever he presses the bar. *Who controls whom?* My students taught me what I know about writing auto-instructional lessons. My goal was to construct a series of questions which would lead the student along toward learning and understanding of certain material. Instead of pellets of food I wanted correct responses—especially correct responses to the last few items of a lesson. I had to try about 20 complete revisions of my first lesson before I began to be rewarded by about 95 per cent correct answers. The same pattern was followed by my colleagues at Earlham College, Howard Alexander and Rowland Smith, in the preparation of their statistics program.

Fred Skinner generalized from his succesful work with another teaching device, the Skinner box caricatured in the cartoon, to instruction in the classroom

and invented a new research tool, the teaching machine. Teaching machines have been greeted by gasps of horror on one hand and howls of glee on the other. Both of these emotional reactions result from an impression that teaching machines are going to replace teachers and that the schoolroom is to become an automatic Skinner box. Personally, I enjoy science fiction but I see no indication that this research tool is likely to replace teachers.

Furthermore, I do not regard it as likely that professional programming specialists who are not themselves content specialists in the area to be programmed will ever be able to prepare programs which make full use of the possibilities of this new method for lesson preparation.

At present most of the programs being prepared are what Douglas Porter calls "expediency programs." A currently established text (or a combination of portions of several such texts) is converted into sentences with blanks and called a Skinner-type program, or it is interspersed with diagnostic test items and called a Crowder-type program. When this conversion is skillfully done the result is apparently in many cases a sequence of material that some students can undertake to study independently and with considerable profit. The student can work at his own pace and at times of his own choosing. He is provided with regular checks on his progress and with reinforcement in the form of being immediately able to check each answer. The resulting lessons can be handled by various machines which add novelty to the process, and which may save student time in getting from item to item. Such machines force the student to follow the rule of answering first and then checking, and in some instances may be essential in order to most efficiently handle specific lesson material. All of this works, at least some times and to some extent; but it is far short of what can be done.

With expediency programming a programming specialist needs only guidance and advice from a content specialist. At this stage of the art of programming, however, it seems as easy and more efficient in many instances, for a content specialist to learn how to do expediency programming than for the programming specialist to attempt to gain any real mastery of academic areas with which he previously had little or no familiarity.

The preparation of lessons with blanks or diagnostic tests interspersed in texts, even if these both are presented to students by means of the most attractive toys, seems to me a current fad which may either have its day and die out or, and for this I hope, which may be a preliminary stage to a really new view of education and new method of preparing educational materials.

It is my experience with material in Psychology and it has been the experience of my colleagues at Earlham College (particularly in language and mathematics) that experience by a content specialist with the feedback and interaction with students that the auto-instructional method of preparing lessons can provide results in a complete re-orientation in view toward teaching as a communication process and to changes in the ordering of lessons and topics and even of course sequence. This is possible only when the programmer is also a

content specialist in the area being programmed. It is this total reorganization of the viewpoint and method of the teacher or writer which seems to me to promise a true educational revolution.

Statement by Norman A. Crowder, U. S. Industries, Inc.

It has been very gratifying to hear the emphasis placed by the preceding speaker on the role of the subject-matter expert in organizing material for automated instruction, and I think I also detect growing recognition of the skill of the programmer as being a branch of the language arts, rather than of experimental psychology as such. We used to hear the field of automated instruction discussed as if it were the unique creation and exclusive province of the psychological learning theorist. Whenever I have heard this point of view expressed, I have been reminded of a line from one of Mark Twain's lectures, as latterly interpreted by Mr. Holbrook. The line goes something like this: "There is an old Maxim," says Twain, "that assures us that man is the noblest work of God. Now, who found that out?"

The organization of a project producing automated training materials will depend heavily on how one views the tools with which he intends ultimately to work with the student. If one believes that he has in hand a new and powerful principle of learning basically different from the principles employed in teaching previously, then he will place primary emphasis on the use of this new principle. On the other hand, if one conceives of automated instruction as a new technique permitting the better utilization of familiar principles, the organization of the project will proceed along relatively conventional lines.

I will confess that in our work with intrinsically programmed materials I am not conscious of the use of any essentially new or revolutionary principles of learning or teaching. Rather it seems that we are working with a new technique that allows a number of old principles, some not very explicitly stated, to be implemented. If we look at the structure of intrinsically programmed materials we find what is really a quite conventional and familiar pedagogical situation. We present symbolic material to the student, i.e. we give him something to read. On the basis of considerable experience, we expect that when the student has read the material, he will be able to perform tasks, i.e. answer questions on the material he has read, that he could not perform previously. We test our expectation by confronting the student with such questions. If he can indeed answer the question correctly, we automatically pass on to the next material to be learned. If we find that the student cannot perform the criterion task, we automatically take appropriate remedial action. This remedial action will ordinarily involve re-teaching the material, perhaps in a somewhat more detailed and less sophisticated presentation. The student is then given another try at the criterion task.

I submit that no one in education is unfamiliar with this "teach-test-advance or re-teach" cycle. It is exactly what all good teachers have done for as long as we have records of teaching. In particular, it is the form of instruction

that has been used by the private tutor. What is new and revolutionary is that the recently developed techniques permit this "teach-test-advance or re-teach" cycle to take place once every minute or so, rather than once in twenty four hours or more; furthermore, the continuous intercession of the live tutor is not required to allow the cycle to take place. The cycle itself, however, and the relatively modest assumptions on which the desirability of such a cycle is postulated, are common coin in educational philosophy and have been for a long time.

It will perhaps clarify the point if I discuss our view of the role of the student's response in the automated instruction process. We are interested in getting the student's response primarily as a measure of how well the communication process is proceeding; i.e. we want to know if he is actually learning from the material we have given him to read. We want this information for the perfectly practical purpose of appropriately modifying the material presented, if and when it appears to be necessary. We want the student's response to control the material he sees. We are not unmindful of the auxiliary benefits resulting from the student's being required to respond frequently. These auxiliary benefits include such things as the fact that the student is required to be active, that the requirement to answer questions may clarify for the student what he is expected to learn, that the experience of succeeding in answering questions has a desirable motivational effect, and, as far as I know, the practice on specific items may do the student some good. The primary purpose of requiring the student's response, however, is to allow the material presented to the student to be modified according to that student's manifest needs, within the limits imposed by the remedial materials that have been built in to the program.

With this view of the philosophy underlying intrinsic programming, it will be clear that the roles we assign to the subject-matter specialist, the programming specialist, and the psychologist or educational theorist, will not be substantially different from the roles such specialists would play in the preparation of conventional text material. In particular, we are at present inclined to place great stress on the subject-matter qualifications and writing skills of a prospective programmer, and to be relatively unimpressed with theoretical knowledge of programming *per se*. The reasons for this are clear, I think, from the description of how we view this whole matter of automated instruction. Aside from a few technical details and a few tricks-of-the-trade, there is really not a great deal to the programming art that is not presently contained in the general body of knowledge about efficient teaching methods. The tasks required of the programmer seem to call primarily on his subject-matter knowledge and skills. The crucial decisions the programmer makes have not seemed to me to depend on principles of programming, but primarily on subject-matter considerations.

This is not to say that any subject-matter expert can write a good program; far from it. I have been careful to speak of the writing skill of the programmer as equal in importance to his subject-matter skill. The programmer who writes intrinsically programmed material does not have a set of rules to guide his choice of size of step, diction, etc. Rather, he is required to be able to visualize

and write a multi-way conversation between the instructor and a number of students. He must be able to put himself in the place of the student, to inspect a paragraph of material that he himself has just written to find the places where the student might possibly misinterpret what was said, to write questions that will reveal these misunderstandings (if he cannot remove the possible causes by rewriting), and to provide explanations that will correct the misunderstandings he foresees. The chief peculiar skills in programming are, to my mind: first, this ability to read one's own material with the eye of a student; and second, the ability to keep in mind the logical course that the development of the subject will take. Both of these skills depend very greatly on subject-matter knowledge for their efficient utilization.

Much of programming is merely peculiarly disciplined, careful writing. For example, suppose we are required to define the sine of an angle, A, with respect to a conventionally labeled right triangle diagram. It would be thought sufficient in most cases to define the sine as the ratio of side a to side c, supposing the hypotenuse of the defining diagram to be labeled c. The careful programmer would define the sine as the ratio of the *length* of side a to the *length* of side c. In doing so, he anticipates a difficulty that might bother perhaps thirty per cent of the pupils in an elementary class. This kind of attention to the details of meaning in explanations, and the ability to include the necessary detail without obscuring the main point, is the essence of good programming, just as it is the essence of good conventional expository writing. Needless to say, mere subject-matter qualifications do not guarantee that a candidate will possess the requisite writing skill.

As a practical matter, we feel that to select among qualified or qualifiable subject-matter experts those who possess the necessary writing skill and to teach these people the programming techniques is a much more promising approach than to attempt to teach the subject-matter to a skillful writer. At least we believe this to be true in any field requiring very much specialized knowledge.

To summarize what I have been saying, we view the preparation of intrinsically programmed materials as a specialized problem in communication, governed by certain structural features of the technique used, but depending heavily on judgments that are essentially artistic judgments, in that they are not reducible to rules that will decide each particular case. It follows that we do not distinguish operationally between the programming function and the subject-matter function, except as the latter is employed in guidance and review. We feel that to write a good program requires at least those skills and knowledges that would be required to write a good conventional text in the same subject, plus some rather specialized technical skills. The technical skills of programming as such are teachable, however. The requisite subject-matter knowledge is teachable also, of course, but ordinarily it is not practical to teach it to a prospective programmer unless the subject is very simple. The judgment and sensitivity to detail that are required to write good programs, and the perception and ability to plan the flow of exposition, are not likely to be im-

proved much by training, I fear; this is why I stressed *selection* among sufficiently literate subject-matter experts for these skills. I suspect that good programmers, like any other kind of good writers, are more born than made.

There is one final point that argues for the primary role of the combination subject-matter specialist and programmer whom I have been advocating here. This is that a good program, like a good book, must have something to say; that is, it must have a unifying purpose and point of view. Such a purpose and point of view is most likely to be possessed by the person having the information to be transmitted than by the ancillary specialists involved in the project.

Statement by Lewis D. Eigen

The Center for Programed Instruction, Inc.

If one were to examine the authorship of the many mathematics programs which exist today, one would find that the authors range from mathematicians who are expert in their field to college students who have been strong in mathematics; from excellent, highly experienced behavioral psychologists to individuals whose only experience with psychology was having avoided it in college. Some programmers have made careers of teaching, while others have never taught a day in their lives. Programming attempts have been made by individuals and teams which have represented various and sundry specialists ranging from technical writing experts to so-called experienced programmers. In trying to determine what combination of skills one should have represented in a program writing team, it is unfortunately the case that one cannot rely on any experimental data. Most of the writing teams, including those at the Center, which have produced mathematics programs, have been composed of people whose skills represent what the administrators of the particular project felt was ideal, and this was tempered by the realistic problem of the availability and willingness of personnel. The ideal programmer for a high school mathematics program probably would hold Ph.D.'s in both mathematics and experimental psychology as well as an Ed.D. degree in education. In addition to holding these degrees, he would know a great deal about these fields. He has probably taught in a high school classroom for many years and should certainly have participated with one or more of the groups which in recent years have done very exciting curriculum research in secondary school mathematics. He would be an experienced programmer, of course, having done many programs before, and would also be quite sophisticated in educational evaluative procedures. It goes without saying that he could express himself in the English language.

If such a person were even to exist, it is highly doubtful that he would be at all interested in programming mathematics, and he would undoubtedly be called to Washington to be a member of the New Frontier so that he could supervise and coordinate all the mathematics programming done in this country.

Since we at the Center have not been able to find such an individual, we have had to settle for other stock. In making up the writing team, we have felt that certain skills must be represented by at least one member of the team. Someone

should know the subject matter very well. Some member of the team should have had experience teaching this type of subject matter at the grade level for which it is intended. At least one of the programmers should be knowledgeable in the curriculum problems in mathematics and be acquainted with the most recent curriculum research in the field of secondary mathematics. Someone should be knowledgeable in the problems of programming and should understand the psychological background from which these techniques spring. There should be on the team at least one person who will be able to write clear and precise English and have at least a token respect for the traditions of grammar, punctuation and spelling. Someone should be aware of the methods of educational evaluation and be able to deal with the many problems of evaluation which always arise in programming.

Now it is entirely possible that two or more of these six skills can be housed in the same body. The writing team does not necessarily consist of specialists in each of these fields, but certainly each team should contain at least one member who has a high degree of skill in each of the areas. There is always of course a question as to how much expertise is required. For example, must the individuals with mathematical skills be leading figures in American mathematics? Must the person or persons with teaching skills have taught for twenty years? We do not think so. Certainly, if there were no curriculum work which had been done in mathematics in the last twenty years, the mathematics writing team would probably have a major curriculum job to do, and this in turn might require the specialties of the best mathematicians the country has to offer. On the other hand, if the recent work such as that done at Yale with the SMSG and at the University of Illinois were to be used, the skill of these outstanding mathematicians can be brought into the writing team without ever having their physical presence there. As a result, we conclude that we do not need the best mathematicians in the country, but we do need competent mathematicians. Likewise, we do not need the best teachers that our schools have to offer; what we need is a very competent teacher.

Our basic writing team is made up of five individuals. Four of them are competent mathematicians. Three have taught mathematics: two at the secondary level and one at the college level. Two have had previous experience in programming. Nonprogrammed instructional materials have been written previously by two of the team members. One team member has done extensive work in psychology though he is not primarily a psychologist. One thing that our experience at the Center has taught us is that it is much easier to make a programmer out of a mathematician than it is to make a mathematician out of a programmer. Since the most experienced programmers around have really only had three or four years of experience, it is doubtful that it would be efficient to cram 8 to 10 years of mathematics experience into the head of a programmer. We have found it far more efficient to teach the techniques of programming to mathematicians and mathematics teachers.

We have found our writing teams to be reasonably satisfactory. One prob-

lem which has developed is perennial in the development of mathematics instructional material. That is, the meta-mathematical problem of logical consistency. I would recommend the use of a mathematical logician in the creation of mathematics programs. This person should be *outside* the writing team. Whereas many of the writers may have the necessary abilities, their involvement in the creation of the program often impedes their examination of the content from an outside point of view. A "gadfly" is always a stimulating phenomenon, and in my view, an excellent contribution to the mathematics programming team.

Some of the very early mathematics programs disturbed me greatly. I was convinced that the content which I considered questionable was the result of a psychologist's programming mathematics. Later, I saw some examples of programs which had been done by mathematicians, and these not only lacked some of the obvious programming techniques, but were not much better in content. There is another important element I have discovered which is a necessary but not sufficient condition for the development of a mathematics program of high quality. That is in administration, which may take the form of a project director, a publisher, a foundation, and the like. I have seen one company go to a great deal of trouble to obtain fine mathematicians and then tell these same mathematicians that they could not program the type of mathematics which they would have wanted because the size of the potential market might be decreased. Very fine psychologists who have felt that they had ample behavioral evidence to justify an attempt at a slightly different type of programming technique have been instructed by their superiors to cease and desist on the grounds that the particular organization was advocating a "different" programming technique. Thus the administration of the project is a very important element of the writing team. In some cases, detrimental administrative interference has been caused by the obvious profit motive, in others, it was just plain ignorance. One major project engaged in the programming of mathematics materials has been set up so that half of the programmers are using the so-called "Skinner" style and the other half the "Crowder" style. The administrative insistence upon pitting one of these methods against the other will undoubtedly, in my view, lead to a great waste of time and effort. There are certain advantages to each of the programming methods, and many other techniques have been developed which would not fit into either category. The freedom of the programmers to use any and all methods to achieve their goals is essential. I have not yet seen a programmed sequence which would teach students how to prove original theorems. I suspect that we will probably see sequences of this type in the years to come, but one thing of which I am sure is that when these sequences are written they will neither be in the Skinner nor the Crowder styles. New techniques will have to be developed to handle these problems. At the Center for Programed Instruction we have maintained this freedom for the programmers to determine to the best of their ability what the content shall be and to utilize any programming

means, orthodox or not, to achieve their goals. I would strongly urge that any programming effort be characterized by the same type of intellectual freedom from the tyranny of monetary interest or an individual's professional status.

In summation, I should like to say that I believe that any moderate attempt at forming a program creation team has a reasonably good chance of being successful. The extremes, of course, are to be avoided. Psychologists writing mathematics programs without the aid of mathematicians and mathematics teachers run a high risk of ending up with a well programmed sequence of materials that should not be taught in schools. Mathematicians writing in a vacuum without any psychological or educational help may well end up with a product which has a fairly low degree of technical excellence from a programming point of view. Further, they might find that the form of the program makes it useless in American schools the way they are constituted.

Thus, where I cannot recommend exactly what proportion of subject matter specialists, psychologists, and educational skills should be brought to bear on the entire effort, I would certainly say that extremes are to be avoided. Given a sensible combination of individuals willing to put in the 95% perspiration and 5% inspiration which programming entails, we can expect a relatively high probability of producing a mathematics program of fine quality and also of pushing back the frontiers of knowledge in developing new programming techniques to handle the many types of mathematical behavior which we would hope mathematics students of the future would be able to accomplish.

Statement by R. C. Buck, University of Wisconsin

The professional mathematician must be deeply concerned both with the development of new mathematical knowledge, and with its impartment to others. Whether the latter be carried out by the printed word, or in a classroom, it equally deserves to be called teaching. It is therefore vital that mathematicians be concerned about the possible effects of recent innovations in technology and psychology on this total process.

Before making comments on the preceding papers, I want my own bias to be quite clear. As a research mathematician and classroom teacher, I am interested in communicating both content and enthusiasm for mathematics, and in perpetuating my own kind. I also represent the loyal opposition; I have written one college text, and will probably write others in the course of my lifetime.

I will start by saying that I feel there is much that is or will be valuable in the use of these new methods of instruction. In terms of technology alone, they offer a means to increase the efficiency and capabilities of those who are teaching; in this direction, Ramo painted an intriguing picture of the possibilities in a widely quoted article [1].

Anyone who has seen the results that Skinner has obtained with pigeons cannot help but be amazed at the plasticity of this living organism, and the range within which its behavior can be modified [2]. It is certainly worth in-

vestigating the relevance of these experiments for human learning, and the extent to which they can assist in the achievement of the usual educational objectives.

In some areas, success is generally admitted—e.g. the teaching of typing, of Morse code, of certain maintenance routines, of spelling, of bridge, etc. All of these have something in common—which, as I will point out later, they do not share with mathematics. In approaching a new area of knowledge as a candidate for programmed instruction, one must first ask: "What are the objectives of the learning process, and how does one decide if the learning has taken place?" Skinner has stated the basic problem very clearly: [3].

"We must decide upon the behavior we want to impart to the student. It does no good to say that we are interested in setting up mathematical *ability*, or artistic *appreciation*, or a language *skill*. We must identify the relevant behavior much more closely if we are to set the goals of the program in useful terms. If we seem to be neglecting some well-known goal such as the inculcation of *abilities* or *understanding* or *insight*, then those who defend these goals should be asked to specify them in terms of the behavior they would accept as indicating a successful realization."

This task is clearly one for the subject specialist. An outsider is apt to be misled by surface phenomena, mistaking the shadow for the reality and the instance for the concept. It is at this point that I feel the greatest concern for most of the current efforts in mathematics programming. First, let us ask why it is that mathematics is so often chosen as the first field in which to try one's wings as a programmer? Although there may be other reasons, it is certainly true that it is widely regarded as a field where there is no ambiguity, where every stimulus (question) has a unique correct response (answer), where the correct response is a number, symbol or formula, and where the essence of understanding of the subject is in the acquisition of an easily described collection of basic skills. Unfortunately, this belief only emphasizes the widespread lack of understanding of the nature of mathematics. To revert to the criterion suggested by Skinner which was quoted above, if mathematicians say that they want students to *prove theorems*, the programmer must reply: "Fine. Tell us which theorems, and which proofs, and we will see to it that they can do this!"

All the speakers except Norman Crowder have made it clear that, in the present state of the *art*, they are not attempting to instill the creative aspects of mathematics. In this, I feel that a grave mistake has been made; to put it bluntly, creativity is the heart and soul of mathematics at all levels. The collection of special skills and techniques is only the raw material out of which the subject itself grows. To look at mathematics without the creative side of it, is to look at a black-and-white photograph of a Cezanne; the outlines may be there, but everything that matters is missing.

Creativity takes place in the elementary school classroom of a master teacher such as David Page; it is present any time that mathematics is applied to other disciplines; it should be visible in every college mathematics course. This brings

me to the factor that I see in mathematics but find absent in many of the areas where programmed instruction has been most successful. We spend much of each classroom hour in mathematics in the effort to elicit "why's" from the students, training them to seek explanations for events, and to penetrate into the structure of the subject. At the college level, we are likely to pass to the student the responsibility for acquiring routine skills as a form of self-paced autoinstruction, using the valuable class time for the more important and more difficult creative and conceptual matters. Curiosity is certainly a keystone in mathematics as it is in all sciences, and we want to encourage students to seek problems as well as answers, to question as well as respond, and even to be ready to doubt or distrust what seems to be dogma. One prominent physicist attributes his success to his mother who would ask him each day, on his return from school: "Hans, did you ask some good questions today?" However, I do not feel that anyone has become a better radio operator by asking for the reasons that underlie the construction of the dot-dash patterns of the Morse code! Programmed instruction has been most successful in those fields where the *how* takes precedence over the *why*.

Until research in these aspects of education and programming has made it possible to present mathematics as a complete picture, in full color, we cannot afford to entrust the primary task of mathematical education to these devices without running the danger of doing grave and possibly permanent damage to the supply of mathematicians and to technology as a whole. In contrast, I do not oppose the use of programmed texts and other devices as useful supplements to able classroom instruction. (There may be other circumstances, perhaps abnormal from an administrative viewpoint, where a major portion of the teaching function can be left to the program even now; this may be the case, for example, with transfer students or with students who are confined to bed.) To answer those who see only virtues in the fact that a teaching machine allows self-paced instruction, thereby rewarding those who are most capable, let me remind you that *student-student* interactions are also important in learning, and that at the professional level, much mathematical research springs from discussions *between* mathematicians. Moreover, a test of understanding is often the ability to communicate it to others; and this act itself is often the final and most crucial step in the learning process.

There is one more remark which I regret that I must make. We are all outraged by those who have been preying upon gullible persons under the guise of the nation wide controversy over fall-out protection. I have heard of persons selling empty metal boxes labeled (correctly!) "radiation deflectors"! I hope that the present concern for educational improvement will not produce a similar attempt to stampede the public for commercial profit. If the benefits of programmed instruction are exaggerated and oversold, then not only will there be harm done to the total educational process, but the true value of this instrument will be prevented from developing.

Lest you think that I am seeing dragons where there are only angels, let

me quote an advertisement from my local paper:

"Teens learn atomic theory in 7 minutes. Young people are learning more at a faster rate using the new teaching procedure called programmed learning. In the new issue of Better Homes and Gardens, it is described for you with a sample program which teaches atomic theory to 13 and 14 year olds in about 7 minutes. Put it on your shopping list! Buy it today!"

Let us render unto Mammon the things that are Mammon's; but may I plead for restraint, understanding, and sober judgment when it comes to the use of teaching machines in mathematics education.

References

1. Simon Ramo, A new technique of education, Eng. & Sci. Monthly, 21 (1957) 17–22. Reprinted in Teaching Machines and Programmed Learning, Lumsdaine and Glaser, 1960 (Nat. Educ. Assoc.) pp. 367–381.

2. B. F. Skinner, Teaching machines, Sci. Amer., 205 (1961) 91–102.

3. ———, a quotation on pp. 683–684 of Lumsdaine & Glaser, cited above.

Statement by Robert M. Gagné, Princeton University

My reaction to these papers is that there is a large area of agreement about the functions involved in preparing learning programs in mathematics, and at the same time two positions which appear at first glance to be extremely divergent. Both these aspects of the papers are worth looking at more closely.

First of all, the agreement is that one needs a *content* expert, in this case, a mathematician. Then, one needs a person whose primary function is worrying about *communication to the learner*. Often, the speakers say, this is a skilled mathematics teacher. Thirdly, one needs an individual whose function is to *construct a learning program*, which has certain formal characteristics including number and variety of frames, and certain other characteristics which derive from its "evaluation." Most speakers agree that an ideal state of affairs would be achieved if one individual could do all of these things, and this point of view is not inconsistent even with Crowder's statement.

Still speaking of the area of agreement, I am struck with the parallel here between learning programming and a much older enterprise which also began in psychology—namely, the construction of mental tests. The same three functions occur in that area of effort. There is, first, the content expert, who must be concerned with the technical adequacy and accuracy of the material. As is well-known, university experts in subject-matter fields are usually employed for this purpose. Second, there is the function of communication, which in this case is embodied in the skilled item-writer. Many people who do this have been teachers, but full-time employment at it brings about a somewhat more specialized occupational title. And third, there is the test developer, who is concerned with designing tests to meet particular objectives, having certain formal characteristics, and also certain other characteristics defined in terms of behavioral data such as validity and reliability. This last kind of person is often trained as an educational psychologist.

I am inclined to suggest, therefore, that the enterprise of developing learning programs is very likely to end up with a high degree of similarity to that of developing mental tests. It seems to me that it will continue to require the separate functions of content specification, communication, and development-coupled-with-evaluation.

Having mentioned mental tests, I turn from the area of agreement to the area of divergency. This is most strikingly apparent in the remarks of Crowder on the one hand, who says there is no science to this business of programming which is not known already, and of Evans on the other, who suggests that maybe even the structure of knowledge can be investigated scientifically. Now, it seems to me that this divergency can be simply understood as a difference in emphasis between technology and science. The effort of constructing learning programs will surely be translated from an art to a technology. It will have its own techniques and its own standards, and I hope will be as successful in its own way as mental testing, another technology, has been. But the amount of science in mental testing, I guess most everyone would agree, is very small. There have been few theories about what is being measured by mental tests. Learning programming may well develop in the same way, and if it does, the psychologist's role in it will be no more than that of a "program developer and tester."

The proper role of the psychologist, as a scientist, is that of investigating *why* knowledge is imparted by learning programs, which of course involves the question of what knowledge is, anyhow, and what are the causal factors responsible for its acquisition by an individual. Answers to such questions, which are likely to have the most far-reaching educational implications, are not likely to come simply from experience in constructing learning programs. They require instead the development of theory and the testing of its implications, whether this is done in the context of a learning program and a teaching machine, or in the context of a laboratory.

In summary, when the matter is looked at from this point of view, I see almost no important disagreement among the speakers. Certain functions have to be performed, and many of them can be formulated and communicated as a technology. The knowledge of a mathematician cannot, because it is not a technology. The knowledge of a skilled teacher in communicating cannot, because it is an art; many people, however, can learn to do it. As for the rest, the detailed construction of programs, it will probably become a technical skill that is specialized, but not unusually difficult to acquire. What else is there? Well, there is science, which concerns itself not with constructing successful programs, but with understanding why they work.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED by HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1521. *Proposed by Frank Hawthorne, Education Department, State of New York.*

It has been claimed that if one "draws a circle" with a pair of compasses on a piece of paper rolled on a tin can, the result, when unrolled, will be an ellipse. Prove or disprove this.

E 1522. *Proposed by Azriel Rosenfeld, Budd Electronics, Long Island City, New York*

(1) Prove that a group which has only finitely many subgroups must be finite. (2) Prove the corresponding statement about commutative integral domains with identity. (3) Show by example that the corresponding statement about commutative fields is false.

E 1523. *Proposed by J. G. Hocking, Michigan State University*

The transformation of the euclidean plane (referred to complex coordinates) onto the open unit disk given by $z \to z/(1+|z|)$ is one-to-one and continuous. It provides a finite model of the plane. Describe the image of a straight line under this transformation.

E 1524. *Proposed by Albert Wilansky, Lehigh University*

Let $\beta(n)$ be a sequence of positive integers such that $\beta(1)=1$, and $\beta(n+1)-\beta(n)$ is either 0 or 1 for $n=1, 2, 3, \cdots$. Let I be the set of all $n/\beta(n)$ which are integers. Show that I is a segment of the integers.

E 1525. *Proposed by Sidney Heller, Brookhaven National Laboratory, Upton, New York*

Find the inverse of the lower triangular matrix of order $n+1$ whose triangle is Pascal's triangle.

SOLUTIONS

Equilateral Triangles Inscribed in a Parabola

E 1491 [1961, 1005]. *Proposed by J. W. Andrushkiw, Seton Hall University*

Show that there exists at least one and at most three equilateral triangles inscribed in a parabola and having a given point P on the parabola for a vertex.

I. *Solution by L. D. Goldstone, Watervliet, N. Y.* Let p be the parabola and let p' be the parabola obtained from p by a counterclockwise rotation of 60° about point P. Then the intersections other than P of p and p' are second vertices of sought equilateral triangles. There is at least one such point (because the axes of the parabolas are nonparallel) and at most three (because two parabolas can intersect in no more than four points).

II. *Solution by C. F. Pinzka, University of Cincinnati.* Since all parabolas are similar, it suffices to take $y=x^2$ for the parabola and (a, a^2) for P. Let Q: (t, t^2) be any other point on the parabola. Then R: (x, y), where

$$x = (a + t)/2 + (\sqrt{3}/2)(a^2 - t^2), \qquad y = (a^2 + t^2)/2 + (\sqrt{3}/2)(t - a),$$

is the third vertex (reading counterclockwise) of the equilateral triangle whose other vertices are P and Q. The condition $y=x^2$ leads, after rejection of the root $t=a$, to the cubic

$$a - t - 2\sqrt{3} = 2\sqrt{3}(a + t)^2 + 3(a + t)^2(a - t),$$

which has at least one but at most three real roots.

Also solved by J. Basile, D. I. A. Cohen, Michael Goldberg, R. D. H. Jones, D. C. B. Marsh, C. C. Oursler, and the proposer. Late solutions by D. C. Kay and Y. E. Yeager.

Positive Integers as Sums of Distinct Primes

E 1492 [1961, 1005]. *Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, Pennsylvania State University*

Considering unity as a prime, show that every positive integer can be written as a sum of distinct primes.

Solution by G. M. Benson, IBM, Endicott, N. Y. Let n be a positive integer. If n is a prime the result is trivial, hence assume n is not a prime (thus $n\neq 1, 2, 3$ in particular). Assume for every $m<n$ that m can be written as a sum of distinct primes. Now there is a prime p such that $n>p>n/2$ for $n\geqq 4$. Let $m=n-p$. By the assumption, $m=m_1+\cdots+m_q$, where $m_1, \cdots, m_q$ are distinct primes. Now $m<p$, whence $p\neq m_i$ for $i=1, \cdots, q$. Therefore $n=p+m_1+\cdots+m_q$ as required. The result follows by strong induction.

Also solved by W. A. Beyer, Robert Bowen, W. Brisley, Leonard Carlitz, P. L. Chessin and E. H. Weiss (jointly), D. I. A. Cohen, Samuel Cohen, D. T. Cottingham, Andrew Di Girolamo, Underwood Dudley, Thyrsa A. Frazier, N. V. Glick, Michael Goldberg, Jerry Goodman, J. D. Haggard, J. O. Herzog, J. M. Horváth, A. G. Konheim, Sidney Kravitz, D. L. Linfield, P. B. Manchester, D. C. B. Marsh, Robert Melter, William Moser, Sam Newman, F. D. Parker, C. F. Pinzka, D. A. Robinson, Peter Rosenthal, D. L. Silverman, Denis Sjerve, Michael Skalsky, E. L. Spitznagel, Jr., Rory Thompson, Charles Wexler, and the proposer. Late solutions by P. A. Cloutier, W. G. Darnell, Serge Dubuc, Erwin Just and Norman Schaumberger (jointly), J. B. Muskat, R. J. Oberg, Paul Stygar, W. C. Waterhouse, and Y. E. Yeager.

Goldberg pointed out that if $n>6$, then it is not necessary to consider unity as a prime. See H. E. Richert, "Über Zerfällungen in ungleiche Primzahlen," *Math. Zeit.* 52 (1949), 342–343, reviewed in *Mathematical Reviews* 11 (1960), 502.

Coefficients of a Polynomial

E 1493 [1961, 1005]. *Proposed by L. Flatto, N. C. Hsu, and A. G. Konheim, IBM Research Center, Yorktown Heights, New York*

Prove that a nonzero integral polynomial $h(x)$ (*i.e.*, one with integer coefficients) which vanishes at $x=1$ and $x=2$ must have a coefficient $\leqq -2$.

Solution by N. G. Gunderson, University of Rochester. Let $h(x)/(x-2) = b_0x^{n-1} + \cdots + b_{n-1}$. Since $h(1)=0$, not all of the b's have the same sign. If $b_{n-1} \geqq 1$, then $-2b_{n-1} \leqq -2$. Otherwise let $b_k \geqq 1$ with $b_j \leqq 0$ for $j>k$. Then $b_{k+1} - 2b_k \leqq -2$. In either case, $h(x)$ has a coefficient $\leqq -2$.

Also solved by Ronald Alter, Robert Bowen, Leonard Carlitz, D. I. A. Cohen, Gus Di Antonio, N. V. Glick, Erwin Just and Norman Schaumberger (jointly), D. C. B. Marsh, Michael Skalsky, R. P. Tapscott, Lincoln Teng, and the proposers. Late solution by E. L. Magnuson.

Inverse of a Matrix

E 1494 [1961, 1005]. *Proposed by George Sadowsky, Combustion Engineering, Inc., Windsor, Connecticut*

Invert the matrix $[a_{ij}] = [x^{|i-j|}]$, $x \neq 0, \pm 1$.

Solution by Leonard Carlitz, Duke University. The inverse matrix $[b_{ij}]$ is tridiagonal with $b_{11} = b_{nn} = 1/(1-x^2)$, $b_{ii} = (1+x^2)/(1-x^2)$ for $i = 2, \cdots, n-1$, $b_{i+1,i} = b_{i,i+1} = -x/(1-x^2)$ for $i = 1, \cdots, n-1$. Indeed, for $j \neq 1, n$ we have

$$-x \cdot x^{|i-j+1|} + (1 + x^2)x^{|i-j|} - x \cdot x^{|i-j-1|} = \begin{cases} 0 & (i \geqq j+1) \\ 1 - x^2 & (i = j). \\ 0 & (i \leqq j-1) \end{cases}$$

The excluded cases $j = 1, n$ are easily checked.

Also solved by W. J. Blundon, A. P. Boblétt, Norman Brenner, Maurice Brisebois, P. G. Carr, D. I. A. and S. P. Cohen (jointly), Romae J. Cormier, Richard Cottle, J. C. Eaves, S. H. Eisman, N. V. Glick, J. C. Hickman, P. G. Kirmser, D. C. B. Marsh, F. D. Parker, M. J. Pascual, Walter Penney, C. F. Pinzka, D. A. Robinson, E. M. Scheuer, B. T. Sims, Michael Skalsky, G. E. Smith, E. L. Spitznagel, Jr., R. P. Tapscott, Lincoln Teng, Dale Woods, P. W. Zehna, David Zeitlin, and the proposer. Late solutions by R. J. Oberg, L. J. Schneider and Y. E. Yeager.

Editorial Note. See Problem E 1389 [1960, 476].

The Vertex of a Parabola

E 1495 [1961, 1005]. *Proposed by Dunstan Hayden, The Priory School, Washington, D.C.*

Let $f(z) = z^2$, and let the domain of this function be any straight line in the complex plane. The graph of the range of f is then a parabola in the complex plane. What are the coordinates of its vertex?

Solution by John Brown, Lower Canada College, Montreal. Let OP be perpendicular to the given line, P: $z_0 = pe^{i\alpha}$. Let Q and R be points on the line symmetric about OP, say Q: $re^{i(\alpha+\theta)}$, R: $re^{i(\alpha-\theta)}$. Then the image points

$Q':r^2e^{i(2\alpha+2\theta)}$ and $R':r^2e^{i(2\alpha-2\theta)}$ are clearly symmetric about OP', since $P':p^2e^{2i\alpha}$. Hence OP' is the axis of the parabola, and its vertex is $P':z_0^2$.

Also solved by A. P. Boblétt, Robert Bowen, Norman Brenner, P. G. Carr, D. I. A. and S. P. Cohen (jointly), D. M. Danvers, Gus Di Antonio, N. V. Glick, Michael Goldberg, Emil Grosswald, Alvin Hausner, R. D. Hawkins, D. C. B. Marsh, F. D. Parker, Laurel Ann Phillips, C. F. Pinzka, Gertrude and Malcolm Pownall (jointly), J. A. Tierney, Charles Wexler, David Zeitlin, and the proposer. Late solutions by Serge Dubuc, Ragnar Dybvik, L. J. Schneider and Y. E. Yeager.

Editorial Note. The above solution holds for all lines not passing through the origin. A line through the origin does not map into a parabola, but into a repeated ray emanating from the origin. But since such a repeated ray may be regarded as a degenerate parabola with vertex at the origin, the answer given above can be said still to hold. If the line is $y=mx+b$, the sought vertex is $(b^2(m^2-1)/(m^2+1)^2, -2mb^2/(m^2+1)^2)$; if the line is $x=c$, the sought vertex is $(c^2, 0)$. It is interesting that all the parabolas under consideration have a common focus at the origin.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5031. *Proposed by Marvin Marcus, University of British Columbia*

Let A be an $n\times n$ matrix with n elements equal to 1 and the rest 0. Let S_r be the sum of all $\binom{n}{r}$ principal $r\times r$ subdeterminants of A. Show that $S_r=\binom{n}{r}$ for some r, $1\leqq r<n$ if and only if $A=\mathrm{I}$, the $n\times n$ identity matrix.

5032. *Proposed by Albert Wilansky, Lehigh University*

Let f be a real continuous function of a one real variable such that $\int_{-\infty}^{\infty} fg''\geqq 0$ whenever g is a twice differentiable function such that the integral exists and such that $g(t)\geqq 0$ for all real t. Show that f is convex.

5033. *Proposed by C. C. Faith, Pennsylvania State University*

E. Steinitz' well-known *Satz der Zwischenkörper* states: If K/L is a finite field extension, then K/L is simple if (and only if) the number of intermediate

fields of K/L is finite. His proof (*Algebraische Theorie der Körper*, Berlin, 1930, p. 74) falls into two categories: (1) L is infinite, (2) L is finite. In case (2) K is a Galois field and the result follows from the theory of finite fields. If (1) holds, the result can be restated in greater generality. Prove the following theorem:

Let A be a ring with identity 1, and let D be a division subring, $1 \in D$, and suppose that A is algebraic of bounded degree over D in the sense that there exists a natural number N such that for each $a \in A$ the subring $D[a]$ generated by D and a has left dimension $\leqq N$ over D. Furthermore, assume that the collection $\{R\}$ of all intermediate rings of A/D, $A \supseteq R \supseteq D$ for each $R \in \{R\}$, has cardinality less than the cardinality of D. Then A/D is simply generated in the sense that there is an $a \in A$ such that $A = D[a]$.

5034. *Proposed by Ronald Pyke, University of Washington*

We say that a random variable X satisfies the weak (strong) law of large numbers, denoted by W.L.L.N. (S.L.L.N.), if for a sequence of independent random variables, $\{X_n\}$, each with the same distribution function as X, $n^{-1}\sum_{i=1}^{n} X_i$ converges in probability (almost surely) to a finite constant. It is known that if one may write $X = Y + Z$ where Y and Z are independent, then X satisfies the S.L.L.N. if and only if both Y and Z do. Prove or disprove this proposition for the W.L.L.N.

5035. *Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan*

Let α be a fixed positive number. Prove that

$$\int_0^{\frac{1}{2}\pi} t^{\alpha} \cos^{2n} t\,dt = \frac{1}{2}\,\Gamma\left(\frac{\alpha+1}{2}\right) \Big/ n^{(\alpha+1)/2} - \frac{1}{12}\,\Gamma\left(\frac{\alpha+5}{2}\right) \Big/ n^{(\alpha+3)/2} + O(1/n^{(\alpha+5)/2}) \qquad (n \to \infty).$$

5036. *Proposed by Allan N. Wilson, Convair Astronautics, San Diego, California*

In playing the gambling system described below, what is the probability P_N that in a given sequence, the largest bet which the player will have to put up is exactly N units? Also what is the probability that he will be obliged to abandon the system, due to reaching a house limit of M units?

The system, applicable to games where the payoff is 1:1, is such that in each completed *sequence* of plays, the player gains one unit net profit, and starts over. A sequence obeys these rules:

(1) A sequence ends when a net profit of one unit is attained.

(2) The first bet is one unit. Subsequent bets may be one unit, or an integral number of units as dictated by rules (3) and (4).

(3) When a bet is lost, the next bet is the same as before.

(4) When a bet is won, the next bet is one unit larger—unless winning the

next bet would yield for the sequence a net profit exceeding one unit, in which case the bet is reduced to an amount just sufficient to yield a net profit of one unit.

Some illustrative sequences:

Bet outcome																											
	L	L	L	L	L	W	W	W	L	W	L	W	W	L	L	W	L	W	L	L	W	W	L	L	W	W	
Bet size																											
	1	1	1	1	1	1	2	3	1	1	1	1	1	1	1	1	2	2	2	2	2	3	1	1	1	2	
Net amount																											
	−1	−2	−3	−4	−5	−4	−2	1	−1	0	−1	0	1	−1	−2	−1	−3	−1	−3	−5	−3	0	−1	−2	−1	1	

The solution should be formulated for any game, in terms of the customary p and q. If an exact general expression for P_N is not attainable, give an approximation for $\sum_{N=1}^{M} P_N$.

SOLUTIONS

Jordan Curve with a Property of the Circle

4959 [1961, 383]. *Proposed by H. S. Shapiro and A. L. Shields, New York University*

(a) Exhibit an analytic Jordan curve Γ in the complex plane (not a circle about the origin) such that $\int_\Gamma z^n ds = 0$ for $n = 2, 3, \cdots$, where s is arc length.

(b) Exhibit an analytic arc Γ (not a circle) such that $\int_\Gamma z^n ds = 0$ for all $n \geqq 1$.

Solution by the proposers. (a) Let $f(z) = (1+\alpha z)^2$, where $|\alpha| < 1$, and $g(z) = \int_0^z f(t)dt = z + \alpha z^2 + \alpha^2 z^3/3$. Then, for small $|\alpha|$, $w = g(z)$ is univalent in $|z| \leqq 1$ and maps $|z| = 1$ onto an analytic Jordan curve Γ in the w-plane. For $n \geqq 2$,

$$\int_\Gamma w^n ds = \int_{|z|=1} [g(z)]^n \, |\, g'(z)\,| \cdot |\, dz\,|,$$

and since for $|z| = 1$, $|g'(z)| = |1+\alpha z|^2 = (1+\alpha z)(1+\bar{\alpha}/z)$, the last integral is

$$\int_{|z|=1} [g(z)]^n (1 + \alpha z)(1 + \bar{\alpha}/z) dz/iz = \int_{|z|=1} h(z)dz = 0,$$

since the total integrand, $h(z)$, is regular in $|z| \leqq 1$.

(b) Let Γ be the image of $|z| = 1$ under the mapping $w = g(z) = \int_0^z f(t)dt$, where f is a non-constant function, regular in $|z| < 1$ having modulus 1 on $|z| = 1$. [Example: $f(z) = \exp\{-(1+z)/(1-z)\}$]. Proof just as in case (a).

Special Cases of a Summation Due to Abel

4960 [1961, 383]. *Proposed by D. L. Shell, Cincinnati, Ohio*

Prove for every positive integer n,

$$\sum_{p=1}^{n} (-1)^p \binom{n}{p} (p+1)^{p-1} p^{n-p} = (-1)^n.$$

4984 [1961, 807]. *Proposed by Y. Matsuoka, Kagoshima-shi, Japan*

Prove

$$\sum_{k=0}^{n}\binom{n}{k}(k+1)^{k-1}(n-k+1)^{n-k}=(n+2)^n.$$

Note by H. W. Gould, West Virginia University. Both problems are special cases of the well-known formula found by Abel:

$$(1)\qquad x\sum_{j=0}^{n}\binom{n}{j}[y+(n-j)z]^{n-j}(x+jz)^{j-1}=(x+y+nz)^n.$$

Crelle's Journal, v. 1 (1826), p. 159. See also *Oeuvres Completes*, v. 1, p. 102. For 4960, set $y=-n$, $x=z=1$. For 4984, set $x=y=z=1$.

Special cases and extensions of (1) have appeared frequently. Two valuable papers in this connection are: H. Salie, *Über Abels Verallgemeinerung der binomischen Formel*, Ber. Verh. Sächs. Akad. Wiss., Leipzig, Math. Nat. Kl., v. 98, no. 4 (1951), 19–22; Jensen, *Sur une identité d'Abel et sur d'autres formules analogues*, Acta Mathematica, v. 26 (1902), pp. 307–318. See also papers by Gould, this MONTHLY, v. 63 (1956), 84–91 and v. 64 (1957), 409–415, in which are found further references and historical notes.

Also solved by M. T. L. Bizley, W. G. Brown, L. Carlitz, S. H. Greene, Andrzej Makowski, R. G. Nath, R. H. C. Newton, F. C. Smith, David Zeitlin; (4960 only) N. J. Fine, C. Ligtmans, and the proposer; (4984 only) A. A. Aitken, Marshall Freimer, R. Frucht, W. C. Harris, B. J. M. Morselt, F. D. Parker, G. N. Raney, R. C. Read, Michael Skalsky, W. F. Trench, and the proposer.

An Integral Transform Problem

4961 [1961, 383]. *Proposed by I. S. Gâl, University of Minnesota*

For σ, t real, $\alpha(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ and $a(u)=u^{-s}\sum_{n\le u}a_n$. Let

$$\hat{a}(t)=\int_0^{\infty}a(u)u^{-it}\,\frac{du}{u},$$

and define $\beta(s)$ and $b(v)$ similarly. Let

$$(a*b)(u)=\int_0^{\infty}a\left(\frac{u}{v}\right)b(v)\,\frac{dv}{v}.$$

Determine $\hat{a}$ and $a*b$ in terms of α and β.

Solution by A. G. Konheim, IBM Research, Yorktown Heights, N. Y.

In the following, σ is chosen sufficiently large to insure convergence of the integrals and series which involve σ. By a change of variable in the first integral, we obtain

$$\hat{a}(t)=\int_{-\infty}^{\infty}a(e^u)e^{-iut}du,\qquad a(e^u)=e^{-u\sigma}\sum_{\log n\le u}a_n.$$

Then substituting for $a(u)$ and inverting the order of integration and summation, we have

$$\hat{a}(t) = \int_{-\infty}^{\infty} e^{-u(\sigma+it)} \Big(\sum_{\log n \leqq u} a_n \Big) du$$

$$= \sum_{n=1}^{\infty} a_n \int_{\log n}^{\infty} e^{-u(\sigma+it)} du = \frac{\alpha(\sigma + it)}{\sigma + it}. \tag{1}$$

Similarly, we obtain

$$(a * b)(e^u) = \int_{-\infty}^{\infty} a(e^{u-v}) b(e^v) dv.$$

In order to express the latter integral in terms of α, β we note that the transform of $(a * b)(e^u) \equiv c(u)$ is

$$\hat{c}(t) = \int_0^{\infty} c(u) u^{-it} \frac{du}{u} = \hat{a}(t)\hat{b}(t) = \frac{\alpha(\sigma + it)\beta(\sigma + it)}{(\sigma + it)^2}.$$

See Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Th. 41, p. 59.

Using the inversion formula $c(u) = (1/2\pi)\int_{-\infty}^{\infty} e^{iut} c(t) dt$, we obtain

$$c(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha(\sigma + it)\beta(\sigma + it)}{(\sigma + it)^2} e^{iut} dt,$$

$$(a * b)(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha(\sigma + it)\beta(\sigma + it)}{(\sigma + it)^2} u^{it} dt. \tag{2}$$

By direct substitution of the series for α, β we obtain another form for $(a * b)(u)$ as follows:

$$c(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum a_n n^{-s} \sum b_n n^{-s}}{s^2} e^{iut} dt, \qquad (s = \sigma + it)$$

$$= \frac{e^{-u\sigma}}{2\pi} \int_{-\infty}^{\infty} \frac{\Big(\sum_1^{\infty} c_n n^{-s} \Big) e^{us}}{s^2} dt = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_n e^{-u\sigma} \int_{-\infty}^{\infty} \frac{(e^u/n)^s}{s^2} ds.$$

Now, by contour integration, the last integral is $(2\pi i) \log (e^u/n)$ if $e^u/n \geqq 1$, and 0 otherwise. Therefore

$$(a * b)(u) = u^{-\sigma} \sum_{n \leqq u} \log (u/n) \cdot c_n, \quad c_n = \sum_{j \cdot k = n} a_j b_k. \tag{3}$$

Also solved by R. G. Buschman and by the proposer. Buschman notes that the results (1) and (2) can be obtained directly by applying inversion formulae for Mellin's transformations. See Titchmarsh, *loc. cit.*, Th. 44, pp. 60.

Simultaneous Solution

4962 [1961, 383]. *Proposed by D. J. Newman, Yeshiva University*

Let $P(x, y)$ and $Q(x, y)$ be polynomials with real coefficients. Suppose $xP+yQ=1$ for all points on the unit circle. Prove that $P=0$, $Q=0$ are satisfiable simultaneously.

Solution by Robert Breusch, Amherst College. Clearly, $P(1, 0)=1>0$. Consider the set $\{(x, y) \mid x^2+y^2 \leqq 1\} \cap \{(x, y) \mid P(x, y)>0\}$, and let R be the component of this set which contains $(1, 0)$. Since $P(-1, 0)<0$, part of the boundary of R is a continuous arc Γ, connecting a point A on the lower half of the unit circle, with a point B on the upper half; along Γ, $P=0$. Thus at A and at B, $y \cdot Q(x, y)=1$. Therefore $Q(A)<0$, $Q(B)>0$, and consequently, $Q(S)=0$ for some S on Γ between A and B. Thus $P(S)=Q(S)=0$.

Nothing in this argument requires that P, Q, be polynomials. The result is true for any functions continuous on $x^2+y^2 \leqq 1$.

Also solved by Juan J. C. Aldemira, William G. Brown, Harley Flanders, John B. Kelley, Allan Trojan, J. H. van Lint and K. A. Post, and the proposer.

A Run of k Consecutive Objects

4963 [1961, 383]. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let k, r, n be three given integers such that $0<k \leqq r \leqq n$. From a set of n objects arranged in a line, r objects may be selected in $\binom{n}{r}$ ways. How many of these selections have the property that at least one set of k or more consecutive objects is included in the selection?

I. *An unsigned solution.* We shall count the number of selections having no sets of k or more consecutive objects. The required result will then follow immediately.

Let $A_1, \cdots, A_{n-r}$ be the $n-r$ objects that are not selected, and let the number of selected objects between A_{i-1} and A_i be α_i. We let α_1 be the number of selected objects which precede A_1, and α_{n-r+1} be the number which follow A_{n-r}. Any selection with the desired property is obtained by choosing the α_i so as to satisfy

$$0 \leqq \alpha_i \leqq k-1, \qquad \sum_{i=1}^{n-r+1} \alpha_i = r.$$

Clearly the number of ways of doing this is the coefficient of x^r in

$$(1+x+x^2+\cdots+x^{k-1})^{n-r+1} = \left(\frac{1-x^k}{1-x}\right)^{n-r+1}.$$

This coefficient can be expressed in terms of a single summation, so that the

answer to the proposed problem is

$$\binom{n}{r} - \sum_{\nu=0} (-1)^{\nu} \binom{n-r+1}{\nu} \binom{n-k\nu}{n-r} = \sum_{\nu=1} (-1)^{\nu+1} \binom{n-r+1}{\nu} \binom{n-k\nu}{n-r}.$$

II. *Note by A. J. Bosch, Technical University, Eindhoven, Netherlands.* The problem has already been solved. See: P. S. Olmstead, *Runs determined in a sample by an arbitrary cut*, The Bell System Techn. Journ., Jan. 1958, p. 72. For a different and somewhat more complicated solution see: F. Mosteller, *Annals of Math. Statist.*, v. 12 (1941), p. 231.

Also solved by M. T. L. Bizley, W. G. Brown, A. E. Currier, W. C. Waterhouse, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association

An Introduction to Ordinary Differential Equations. By Earl A. Coddington. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961. xi+292 pp. $9.00.

Professor Coddington has written a text which will be happily received by instructors who wish to introduce serious students to the thoretical aspects of differential equations. He begins with a carefully prepared summary of some basic results from the calculus and algebra of complex-valued functions, setting the tone for the succeeding chapters.

The central theme for most of the book is the existence and uniqueness of solutions of ordinary linear differential equations. The author carries the reader from first order linear equations to equations with analytic coefficients and then to a detailed treatment of second order equations with regular singular points. It is only after this thorough examination of linear equations that the existence and uniqueness of solutions of initial-value problems with a nonlinear equation are treated. The existence proof here uses the successive approximation method. The final chapter uses n-dimensional vectors to show that most of the results previously demonstrated for single equations remain valid for systems of equations.

The theoretical nature of the book is emphasized by the fact that the exercises, of which there are an abundance, nearly always fail to mention their obviously physical motivation. Indeed, the only illustration which admits to a physical background is a neat presentation of the problem of central forces related to planetary motions, which is presented in the last chapter.

A student who has a good command of the elementary calculus will find that he has all the tools necessary to follow the development throughout. This work, however, certainly probes more deeply into the analytic character of differential equations than do most elementary texts on the subject.

PHILLIP E. BEDIENT
Franklin and Marshall College

Handbook of Numerical Methods for the Solution of Algebraic and Trascendental Equations. By V. L. Zaguskin (translated from Russian by G. O. Harding). Pergamon Press, New York, 1961. 195 pp. $6.50.

Chapter I contains an adequate summary of information about polynomials. Domoryad's method and MacLaurin's theorem, introduced here, are too vaguely discussed.

Chapter II presents the author's very lucid ideas on the arithmetic of approximate numbers and concludes with excellent analysis of the accuracy of approximations to roots of equations. His brief discussion of loss of accuracy suffers from the absence of an illustration.

Chapters III through VI introduce and illustrate a variety of methods, some well known, for obtaining real and complex approximations to roots of equations. An excellent discussion of types of convergence is given in Chapter VI. Chapters III and IV conclude with useful summaries. There are a number of misprints which are clear from the context. Chapter VI on simultaneous equations is much too brief and might well be omitted. Since the behavior of almost every root approximation discussed must be judged "by eye," the needs of users of electronic computers are not completely met, and an automatic procedure for finding *all* roots of an equation should be described.

The many excellent illustrative examples, alone, make this a useful book for the practical computer.

ROBERT H. OWENS
University of New Hampshire

Statistical Analysis and Optimization of Systems. By E. L. Peterson. Wiley, New York, 1961. 190 pp. $9.75.

This book is presumably intended as an introduction for engineers. While the table of contents leads one to expect an attractively comprehensive development, the writing is somewhat careless. A reader who has had little previous contact with the mathematical background of the subject will be confused by some of the serious errors. Example: in equations (3.43)–(3.44) the author equates a random variable with its expectation (an omitted factor n^{-1} compounding the confusion).

There are some simple instructive examples of applications in the book, but not much is new theoretically. Thus, the cover mentions "the powerful *adjoint method* of statistical analysis," which turns out to mean substituting τ for $t-t_1$ in the expression

$$\int_0^t x(t_1)a(t_1)g(t - t_1)dt_1.$$

Contents: Chapter 1, *Linear System Theory*; Chapter 2, *Statistics of Random Variables*; Chapter 3, *Response to Distributed Inputs*; Chapter 4, *Systems Analysis and Design*; Chapter 5, *Optimum Systems*; Chapter 6, *Applications in Optimal Synthesis*; Chapter 7, *Optimization with Multiple Inputs and Outputs*; Chapter 8, *Optimizing Inputs for Specified Linear Systems*; Chapter 9, *Multi-Dimensional, Nonlinear, Boundary-Valued Variational Problems*; Appendix I, *General Stability Considerations*; Appendix II, *Minimum Mean-Squared Error for Linear Systems with Multiple Inputs and Outputs*; Appendix III, *Parametric expansion for Minimum Weighted Squared Error*; Appendix IV, *The Variational Problem of Mayer.*

J. KIEFER
Cornell University

Numerical Weather Analysis and Prediction. By P. D. Thompson. Macmillan Company, New York, 1961. xiv+170 pp. $6.50.

This is the first book-length treatment of the analysis and prediction of the behavior of planetary atmospheres by numerical mathematical techniques with an electronic computer. It is written by a recognized authority and an outstanding contributor to the field of numerical weather prediction (NWP).

Consisting of fifteen chapters, the book is addressed primarily to the nonspecialist, i.e., the geophysicist, mathematician, etc. "to appeal and cater to the interest of the outside contributor" and to the graduate student of dynamic meteorology. Thus, while a working knowledge of vector calculus and the numerical methods of solution of differential equations is required, no special knowledge of meteorology is assumed.

Significantly, the book is dedicated in part to the late Professor John von Neumann who "with an incomparable grasp of the theory of computing" developed the means for solving the equations expressing the essential dynamical processes of the atmosphere.

Chapter 1 provides an excellent introduction to the subject matter by tracing from an historical point of view the meteorological, mathematical and computational difficulties and advances leading up to the recent significant developments in, and operational practice of, NWP.

The hydrodynamic equations which represent an expression of the fundamental physical principles underlying NWP, i.e., conservation of momentum, mass and energy, are derived in Chapter 2 and their solution by finite-difference methods is described in Chapter 3.

Chapters 4 and 5 contain a particularly lucid discussion of pure types of wave motion, i.e., sound, gravity and Rossby waves, corresponding to solutions of the linearized hydrodynamic equations and the finite-difference methods of solution corresponding to each wave type.

The filtering problem i.e. the modification of the hydrodynamic equations to exclude solutions corresponding to sound and gravity waves without substantially modifying the solutions corresponding to Rossby waves, is discussed in Chapters 6 and 11.

Chapters 7, 8 and 9 describe the two- and three-dimensional models consisting of nonlinear differential equations which have provided the mathematical and physical basis for most of the NWPs prepared to date, and the numerical techniques utilized to solve these equations.

Chapters 10 and 12, respectively, discuss atmospheric energy transformations related to the three-dimensional model and methods of solution of the primitive, i.e., unmodified, hydrodynamic equations.

A description of the operational aspects of numerical weather analysis and prediction, a discussion of forecast accuracy and an analysis of forecast errors and a brief look into the future of NWP are contained in the remaining chapters.

The book is well written in a clear and vivid style and can be recommended without qualification to meteorologists and nonmeteorologists alike.

ALLAN H. MURPHY
The University of Michigan

Advanced Calculus, Second Edition. By David V. Widder. Prentice-Hall, Englewood Cliffs, N. J., 1961. xvi+520 pp. $9.00.

The new edition of Widder's Advanced Calculus continues in his clear, precise style. The examples bring out the exact content of the theorems, as they should. General theoretical concepts are frequently used before they are proved. For example, in the chapter on Stieltjes Integrals, the existence theorem is stated and used early, but the proof is left until the last section. More real variable theory is included than the chapter titles suggest. The coverage of Laplace transforms (in real variables) is by far the most complete which this reviewer has ever seen in an advanced calculus book. In addition to expected topics, this book presents the Heine-Borel theorem, Schwartz's example obtaining other than surface area from the limiting area of inscribed polyhedra, an analytic proof of the invariance of the inner and outer products of vectors, developable surfaces, Cesaro summability, Stirling's formula, infinite products, Fourier series, the bilateral Laplace transformation, and a treatment of the plucked string. The material is carefully selected and well treated. It would be difficult for most classes to cover all the material in this book in two semesters.

EARL LAFON
University of Oklahoma

CORRECTIONS

The price of *Probability, A First Course*, by Mosteller, Rourke and Thomas, was incorrectly given as $5.00 (69 (1962) 176). The correct price is $5.33.

The price of *Probability and Experimental Errors in Science*, by Lyman G. Parratt, was incorrectly given as $6.00 (69(1962)456). The correct price is $6.50.

BRIEF MENTION

Some of these books are classics which, by chance, have not been reviewed in the MONTHLY; they are now too well known to require more than brief mention when a new edition appears.

DOVER PUBLICATIONS, NEW YORK

(1) *Higher Geometry, An Introduction to Advanced Methods in Analytic Geometry.* By Frederick S. Woods. 1961. x+423 pp. $2.00 (paper).
An unabridged and unaltered republication of the work first published by Ginn and Company in 1922.

(2) *An Elementary Treatise on Elliptic Functions.* By Arthur Cayley. 1961. xii+386 pp. $2.00 (paper).
An unabridged and corrected republication of the second edition of the work published by George Bell and Sons in 1895.

(3) *An Introduction to the Theory of Canonical Matrices.* By H. W. Turnbull and A. C. Aitken. 1961. xiii+200 pp. $1.55 (paper).
An unabridged and corrected republication of the third (1952) edition of the work first published by Blackie and Son Limited in 1932.

(4) *Continuous Groups of Transformations.* By Luther Pfahler Eisenhart. 1961. ix+301 pp. $2.00 (paper).
An unabridged and corrected republication of the work first published by Princeton University Press in 1933.

(5) *Modern Theories of Integration* (Second Revised Edition). By H. Kestelman. 1960. x+309 pp. $2.00 (paper).
The revisions of the 1937 work consist in the addition of a chapter on Riemann-Stieltjes Integration and of a collection of exercises covering most of the topics dealt with in the text.

CHELSEA PUBLISHING COMPANY, NEW YORK

(1) *The Development of Mathematics in China and Japan.* By Yoshio Mikami. 1961. x+347 pp. $3.95.
A reprint of the 1913 edition.

(2) *Theory of Numbers* (Second Edition). By G. B. Matthews. 1961. xii+323 pp. $3.50.
A reprint of the first edition, with corrections of a number of errata, and with some improvements in notation.

(3) *Projective Methods in Plane Analytical Geometry* (Third Edition). By Charlotte Angas Scott. 1961. xii+290 pp. $3.50.
This edition incorporates into the text a number of corrections and additions indicated by the author, who completed the first edition in 1894.

(4) *Calculus of Variations* (Second Edition). By Oskar Bolza. 1960. ix+271 pp. $1.19 (paper); $3.25 (cloth).
In this reprint of the first edition, published by the University of Chicago Press, some of the addenda have been incorporated into the text, and a number of corrections and improvements in notation have been made.

(5) *Elements of Algebra* (Fourth Edition). By Howard Levi. 1961. 189 pp. $3.25.
Some exercises have been added to the third edition.

(6) *Calculus of Finite Differences* (Second Edition). By Charles Jordan. 1960. xxi+652 pp. $6.00.

CAMBRIDGE UNIVERSITY PRESS, NEW YORK

(1) *On Growth and Form.* By D'Arcy Wentworth Thompson. 1961. xiv+346 pp. $5.95.
An abridged edition, edited by John Tyler Bonner.

(2) *A First Course in Mathematical Statistics.* By C. E. Weatherburn. 1961. xv+277 pp. $2.75 (paper).

The first edition, published in 1946 and reviewed in this MONTHLY in 1948, has had some corrections in subsequent printings.

(3) *The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coordinates.* By E. A. Maxwell. 1960. xix+230 pp. $1.95 (paper).

A reprint, with minor corrections, of the first edition, published in 1946.

(4) *A Book of Curves.* By E. H. Lockwood. 1961. xi+199 pp. $4.95.

Part I, Special Curves: parabola, cardiod, deltoid, tractrix, etc.;

Part II, Ways of Finding New Curves: conchoids, roulettes, evolutes, caustic curves, etc.

Emphasis on drawing.

Science Awakening (Second Edition). By B. L. Van der Waerden. Oxford University Press, New York, 1961. 306 pp. $7.50.

This magnificent book, translated into English by Arnold Dresden, was originally published by P. Noordhoff, Ltd. in 1954. The new edition incorporates "an important discovery of P. Huber, which sheds new light upon the role of geometry in Babylonian algebra. . . . The section on Heron's Metrics . . . was written anew, following a suggestion of E. M. Bruins."

Mathematical Models (Second Edition). By H. Martyn Cundy and A. P. Rollett. Oxford, at the Clarendon Press, 1961. 286 pp. $2.75.

The first edition (1951) has been corrected and supplemented, notably by a chapter on models for logic and computing.

Cybernetics (Second Edition). By Norbert Wiener. The M.I.T. Press and John Wiley and Sons, Inc., New York, 1961. xvi+212 pp. $6.50.

Part I consists of a corrected version of the first edition (1948); Part II consists of new material: On Learning and Self-Reproducing Machines; Brain Waves and Self-Organizing Systems.

Guide to Tables in the Theory of Numbers. By Derrick Henry Lehmer. First published in 1941 and out of print since 1954, has been reprinted and is now available as Bulletin No. 105, 177 pages, for $3.00 a copy. Also a number of copies of *Linear Algebras*, a report of a conference on Linear Algebras held in June 1956, are still available as Publication No. 502, 59 pages, $1.50 a copy. Both of these bulletins can be ordered from the Printing and Publishing Office of the National Academy of Sciences—National Research Council, Washington 25, D. C.

Apollonius of Perga: Treatise on Conic Sections. By T. L. Heath. Barnes and Noble, New York, 1961. clxx+254 pp. $9.00.

Theory of Algorithms. By A. A. Markov. Published for the National Science Foundation and the Department of Commerce by the Israel Program for Scientific Translations, 1961. 444 pp. $4.50.

This important work (volume 42 of the publications of the Steklov Institute) treats of the possibility and impossibility of the existence of algorithms, and of the unsolvability of some problems. It is obtainable from the Department of Commerce, Washington 25, D. C. by referring to Office of Technical Services accession number 60-51085.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. P. Bailey, United States Naval Academy, represented the Association at the inauguration of John A. Logan, Jr. as Fifth President of Hollins College on April 14, 1962.

Mr. F. S. Agenbroad, Frankford Arsenal, Philadelphia, Pennsylvania, has accepted a position as Systems Engineering Trainee in the Data Processing Division of International Business Machines, Dayton, Ohio.

Mr. F. T. Baker, International Business Machines, Washington, D. C., has accepted a position as Development Engineer with International Business Machines, Omaha, Nebraska.

Mr. R. C. Balentine, Redstone Arsenal, Alabama, has accepted a position as Aero-Space Technologist with the NASA Manned Spacecraft Center, Langley Air Force Base, Virginia.

Dr. J. R. Blum, Sandia Corporation, Albuquerque, New Mexico, has been appointed Professor at the University of New Mexico.

Mr. Edward Brennan, Philadelphia, Pennsylvania, has accepted a position as Engineer's Assistant with H. R. B. Singer, State College, Pennsylvania.

Mr. E. A. Briss, Berea College, has accepted a position as Research Mathematician and Programmer with the Battelle Memorial Institute, Columbus, Ohio.

Professors J. W. Cell, North Carolina State College, and W. M. Whyburn, University of North Carolina, have been appointed to serve on the Scientific Advisory Committee of the State of North Carolina.

Mr. C. H. Cook, The Martin Company, Denver, Colorado, has been appointed Assistant Professor at the University of Oklahoma.

Mr. A. T. Ewald, Medical College of Virginia, has accepted a position as Analyst with General Electric, Santa Barbara, California.

Dr. D. C. Gerneth, Temco Aircraft Corporation, Dallas, Texas, has accepted a position as Assistant Technical Manager with Schlumberger Limited, Houston, Texas.

Mr. M. E. Grant, Boeing Airplane Company, Seattle, Washington, has accepted a position as Programmer with International Business Machines, San Jose, California.

Mr. J. K. Hancock, Boeing Airplane Company, Seattle, Washington, has accepted a position as Aerodynamic Engineer with Stanley Aviation, Denver, Colorado.

Mr. C. R. Hilland, Carswell Air Force Base, Texas, has accepted a position as Computer Programmer and Analyst in the Methods and Systems Department of Texas Eastern Transmission, Shreveport, Louisiana.

Mr. C. O. Hopperstead, Temco Aircraft Corporation, Dallas, Texas, has accepted a position as Senior Engineer with The Martin Company, Orlando, Florida.

Professor C. A. Hutchinson, University of Colorado, has been appointed Associate Dean of the College of Engineering.

Dr. J. H. Jordan, University of Colorado, has been appointed Assistant Professor at Washington State University.

Mr. Raymond Knodel, Bemidji State College, has been promoted to Assistant Professor.

Mr. W. E. Kopka, Service Bureau Corporation, New York, New York, has accepted a position as Advisory Programmer with International Business Machines, Poughkeepsie, New York.

Mr. D. E. Morrill, Thiokol Chemical Corporation, Denville, New Jersey, has accepted a position in the Data Systems Division, Remote Access Computing Systems, of International Business Machines, Poughkeepsie, New York.

Assistant Professor E. O. Nelson, University of Utah, has been appointed Associate Professor at the University of North Dakota.

Dr. R. W. Rector, Ramo-Woolridge Corporation, Los Angeles, California, has accepted a position as Administrative Manager of the Computation Center at Aerospace, Los Angeles, California.

Dr. W. C. Rheinboldt, Syracuse University, has been appointed Director of the Computer Science Center of the University of Maryland.

Mr. W. C. Rounds, Ford Apprentice Training School, Chicago, Illinois, has accepted a position as Mathematician-Programmer with the Statistical Tabulating Corporation, Chicago, Illinois.

Mr. R. A. Schwartz, Brooklyn, New York, has been awarded a Woodrow Wilson Fellowship for graduate study in the academic year 1962–63.

Dr. I. S. Tolins, Arma Division, American, Long Island City, New York, has accepted a position on the Research Technical Staff at the Institute for Defense Analyses, Washington, D. C.

Mr. A. W. Wallace, Massachusetts Institute of Technology, has been appointed Instructor at Northeastern University.

Professor H. R. Beveridge, Monmouth College, died on December 31, 1961. He was a member of the Association for 32 years.

Dr. H. T. Engstrom, Remington Rand Corporation, New York, New York, died on March 8, 1962. He was a member of the Association for 29 years.

Professor Emeritus M. G. Gaba, University of Nebraska, died on February 11, 1962. He was a charter member of the Association.

Assistant Professor Belle Kearney, Montclair State College, died on December 11, 1961.

Mr. W. D. Morgan, D. L. Piazza Company, Minneapolis, Minnesota, died on November 24, 1961. He was a member of the Association for 36 years.

Commander W. R. Wallis, United States Navy-Retired, died on October 21, 1961.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-ninth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Tulane University on February 16–17, 1962. The Friday afternoon meeting was held in two concurrent sessions. Professor W. B. Temple, Louisiana Vice-Chairman and Professor S. R. Knox, Mississippi Vice-Chairman presided. Professor B. O. Van Hook, Section Chairman, presided at the Friday evening and Saturday morning sessions. There were 216 persons registered including 99 members of the Association.

The following officers were elected for the coming year: Chairman, Professor W. B. Temple, Louisiana Polytechnic Institute; Vice-Chairman for Louisiana, Professor T. K. Maddox, Southeastern Louisiana College; Vice-Chairman for Mississippi, Professor Virginia I. Felder, Mississippi Southern College; Secretary-Treasurer, Professor Z. L. Loflin, University of Southwestern Louisiana.

At the business meeting, reports on the progress of the High School Contests for Mississippi and Louisiana were given by Professors N. A. Childress and H. T. Karnes. Professor Arthur Ollivier reported on the meeting of the Board of Governors.

The invited speaker of the meeting was Professor R. J. Wisner, Executive Director of the Committee on the Undergraduate Program in Mathematics. He lead a panel discussion on *Pre-graduate training* on Friday evening with Professors R. D. Anderson and A. D. Wallace. His Saturday morning address was entitled *C. U. P. M. progress and projects*.

The following papers were presented:

1. *A remark on Riemann-Stieltjes integral*, by Professor Ta-Sun Wu, Saint Mary's Dominican College.

Let Φ be a function defined on the real line and f a function defined on $[a, b]$. Then the following implications (1)$\Rightarrow$(2)$\Rightarrow$(3) are well known, where (1) $\int_a^b \Phi[f(x)]df(x)$ is $R-S$ integrable, (2) given $\epsilon>0$, there exists a $\delta>0$ such that $\left|\sum_{i=1}^{n}\{\Phi[f\eta_i)]-\Phi[f(\tilde{\eta}_i)]\}\{f(x_i)-f(x_{i-1})\}\right|<\epsilon$ for every finite partition by the points $a=x_0<x_1<\cdots<x_n=b$ with mesh less than δ and $\eta_i, \tilde{\eta}_i \epsilon [x_i, x_{i-1}]$, in particular

$$\left|\sum\{\Phi[f(x_i)]-\Phi[f(x_{i-1})]\}\{f(x_i)-f(x_{i-1})\}\right|<\epsilon. \tag{3}$$

In this note we prove that (2)$\Rightarrow$(1), and (3) is not equivalent to (1) in general. This answers a question in the paper Coll. Math. vol. V, Fasc 1 (1957), pp. 74–77.

2. *Existence and uniqueness of interpolating rational functions*, by Professor D. E. Dupree, Northeast Louisiana College.

The problem of existence and uniqueness of a rational interpolating function may be stated as follows: Given $n+1$ pairs of numbers (x_i, y_i), $i=0, 1, \cdots, n$, with the x_i distinct, and a nonnegative integer $k\leq n$, under what conditions does there exist an irreducible rational function $y(x_i)=y_i$, $i=0, 1, \cdots, n$, where $p(x)$ is of degree $r\leq k$ and $q(x)$ is a degree $s\leq n-k$? Necessary and sufficient conditions are derived for the existence of $y(x)=p(x)/q(x)$ by studying the system of homogeneous linear equations: $a(x_i)-b(x_i)y_i=0$, $i=0, 1, \cdots, n$, where $a(x)=\sum_{i=0}^{k} a_i x^i$, and $b(x)=\sum_{j=0}^{n-k} b_j x^j$.

3. *On the algebraic nature of Green's function*, by Professor R. D. Sheffield, University of Mississippi.

To give an algebraic structure for Green's function it is suggested that the linear differential operator L and the boundary operator B be composed into a nonsingular linear transformation $T=(L, B)$. Conditions for such a composition are given and T^{-1} is then shown to be of the form (M, C) where M and C are right inverses of L and B respectively.

4. *On the classification of subdirect products*, by Professor J. D. Gilbert, Louisiana Polytechnic Institute.

If B_i is a normal subgroup of A_i for $i=1, 2$, and if ϕ is an isomorphism from A_1/B_1 to A_2/B_2, then $A=[(a_1, a_2)|\phi(a_1B_1)=a_2B_2]$ is a subdirect product. We say that ϕ is the isomorphism which determines A. As a means of classifying the subdirect products which have kernels isomorphic to two given groups K_1 and K_2, an equivalence relation is introduced. Then, necessary and sufficient conditions are given for two subdirect products to be equivalent.

5. *Rings in which ideals having prime radicals are primary*, by Professor Robert Gilmer, Louisiana State University.

Let R be a commutative ring with unit. An ideal A of R will be called semi-primary if the radical of A is prime. We consider the following problem: Under what conditions will every semi-primary ideal in R be primary? The main result obtained is this: An integral domain R with unit has this

property if and only if nonzero prime ideals are maximal in R. Other results are obtained in rings in which non-maximal prime ideals are finitely generated.

6. *A result in the representation of linear functionals by an integral*, by Professor J. R. Webb, Louisiana State University.

A bounded linear functional F on the space of quasi-continuous functions over the unit interval (with uniform norm) may be represented by a mean-Stieltjes integral in the manner, $F(x)=\int_0^1 x[u]dg$, where substitution function u is a continuous nondecreasing function from the unit interval onto itself and g is of bounded variation and may be chosen so that $\|F\| = V_0^1 g$. However, an arbitrary functional $G(x)=\int_0^1 x[v]dh$ need not have a norm equal to the variation of h. A sufficient condition for equality is that h be linear on the intervals of constancy of v and, for each t in (0, 1), $h(t)$ be equal either to $h(t-)$ or $h(t+)$.

7. *Report on an experiment in the use of programmed materials in teaching elementary algebra*, by Professor N. F. Williamson, Louisiana State University New Orleans.

8. *A geometrical and physical explication of the Gibbs dyad*, by Professor Emeritus B. E. Mitchell, Millsaps College.

Professor Gibbs called the product of two vectors whose symbols stand in juxtaposition, the indeterminate product, and indeterminate it is. However, in its employ as an operator, together with certain geometrical and physical interpretations of the direct and skew products, some explication of its nature is perceived.

9. *Some properties of groupoids*, by Professor R. J. Warne, Louisiana State University New Orleans.

10. *On the number of twofold relations among three statements any two of which are unrelated*, by Professor G. J. Corley, Northwestern State College.

If a set of statements can have any combination of truth values, they are said to be unrelated. If one or more of the possible combinations of truth values on unrelated statements are excluded, the statements are related. These relations are called onefold, twofold, etc., depending upon the number of combinations excluded. On three statements, there are some relations that leave any two of the statements still unrelated. Apparently the number of twofold relations of this type is different from what some have thought.

11. *NASDTEC certification standards in mathematics*, by Professor Gail Young, Tulane University.

The Guidelines for Preparation Programs of Teachers of Secondary School Mathematics were distributed to each person attending, and the purpose and use of the Guidelines were discussed.

12. *A relationship between continuity, singularity, and the length of graphs under the hypothesis of monotonicity*, by Professor W. R. Alford, Tulane University.

The following theorem is known but the method of proof is new: If f: $[0, 1]\to[0, 1]$ is monotone, $f(0)=0$, $f(1)=1$, then f is continuous and $f'(x)=0$ almost everywhere if and only if the length of the graph of f is 2.

13. *Tangent elements to projections of a surface generated by an involution*, by Professor John Kenelly, University of Southwestern Louisiana.

14. *A two-point boundary condition for the n-th order linear differential equation*, by Professor Jackie Garner, Louisiana Polytechnic Institute.

15. *Approximate roots of polynomials*, by Professor W. M. Perel, Louisiana State University New Orleans.

Z. L. LOFLIN, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-second regular meeting of the Southern California Section of the Mathematical Association of America was held at Long Beach State College, Long Beach, California, on March 10, 1962. Professor Clifford Bell, Chairman of the Section, presided. The registered attendance was 201, including 146 members of the Association.

At the business meeting the report of Professor Morgan Ward, Chairman of the Nom-

inating Committee, was read and accepted. This report stated that—1) the following officers were elected by mail ballot to serve from July 1, 1962 to June 30, 1963: Chairman, Professor P. A. White, University of Southern California; Vice-Chairman, Professor Fred Marer, Los Angeles City College; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College—2) the following members were elected to the Program Committee for the March, 1963 Meeting to be held at University of California, Riverside: Professor E. F. Beckenbach, University of California, Los Angeles, Chairman; Professors Irving Drooyan, Charles Halberg, and Olga Taussky Todd, Dr. Robert Hayes and Dr. Harry Lass.

Professor F. A. Valentine, Chairman of the Committee to Study Activities of the Section, reported for the committee and presented two motions, which were approved—1) the Executive Board shall appoint a committee to draw up By-Laws for the Section, incorporating recommendations from the *Handbook for Officers of the Sections*, and including a provision for a three-year (renewable) term for the Secretary—2) the Executive Board shall appoint members to a standing committee on the High School Lectures and a standing committee on High School Contests.

The Secretary reported on the Program of Visiting Lecturers to Secondary Schools of the Section. During April and May of 1962, eleven lecturers from nine colleges in the area are presenting the program for a third year, visiting forty-seven schools in Southern California.

The following program was presented:

1. *Insights into the teaching of modern mathematics*, by Professor F. L. Wren, San Fernando Valley State College.

"Modern mathematics" at the elementary, high school, and junior college levels of instruction means primarily a new emphasis upon classical abstract developments rather than modern subject matter. This emphasis is on structure rather than techniques or applications. One danger of such emphasis is that mathematics will be presented as a self-justifying subject, detached from its vital uses. Consequent indifference of students can render such presentations useless. Effective instruction at all levels must be framed in the context of a consistent pattern of motivation, with a deliberate plan to challenge students, and a diligent effort to inspire them to excel.

2. *Latin squares and some related topics*, by Professor O. L. Laing, Pasadena City College.

The construction of $n-1$ pair-wise, mutually orthogonal latin squares of order n, n being a power of a prime is discussed. The solution of this problem in terms of the related ν, κ, λ problem of finite projective geometries and the corresponding difference sets is shown.

3. *Adaptive control processes*, by Dr. Richard Bellman, the RAND Corporation, Santa Monica, California.

The modern mathematical theory of adaptive control processes deals with multistage decision procedures in which information of the type required for classical treatments is either not present or present in huge quantities. Large classes of interesting and significant problems of this nature await recognition, formulation, analysis, and numerical solution. By using the theory of dynamic programming, many processes can be precisely formulated, and handled analytically and computationally. References are: R. Bellman, *Adaptive Control Processes: A Guided Tour*, Princeton University Press, 1961. R. Bellman and R. Kalaba, "On adaptive Control Processes," *IRE National Convention Record*, Part 4, 1959, pp. 3–11.

4. *Distribution-free statistics*, by Professor C. B. Bell, Jr., San Diego State College.

Simple examples in probability and statistics are given to illustrate the equivalence of certain families of probability distributions with respect to certain statistics. Known results related to statistics which are distribution-free with respect to the normals are extended to other classes of distributions. In particular, the classical Studentization Theorem is shown to be a corollary of the distribution-free independence theorem and of the distribution-free generation theorem. This last theorem is a result of the extension of the work of Lukacs, for the normals and the affine group of transformations, to the case of an arbitrary distribution and its transformation group.

5. *Some combinatorial theorems*, by Professor H. F. Bohnenblust, California Institute of Technology.

Finite sets of real numbers are analyzed in terms of the maximum of their partial sums in a given ordering, in terms of their nonnegative sums taken over the cycles of a given permutation. It is shown that the average of the former taken over all possible orderings is equal to the average of the latter taken over all possible permutations. Similar results are discussed for other functions of these numbers. These results form the basis for the proofs of important probability-theoretical theorems.

6. *Set theory in the primary grades*, by Professor Patrick Suppes, Stanford University.

An approach through set theorey to arithmetic in the primary grades is described. The central point is to describe numbers as properties of sets. Set theory provides both a good mathematical basis for arithmetic and a gradual transition from the concrete to the abstract. The notation for addition using Arabic numerals is the last of four stages of abstraction. Putting together sets of concrete objects is the first stage; the notation for union of disjoint sets is second; the pictorial N notation for describing numbers, prefacing a set with a capital N, is third; and addition in terms of Arabic numerals is the final stage.

R. B. HERRERA, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

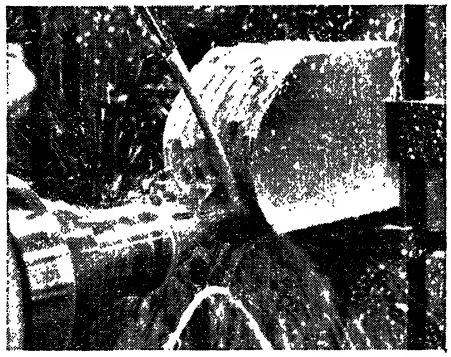
ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Evansville College, October 5, 1962.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 16–17, 1963.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Harvard University, Washington, D. C., December 1, 1962.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Bemidji State College, November 3, 1962.
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962
NORTHEASTERN, November 24, 1962.
NORTHERN CALIFORNIA, University of California, Berkeley, January 1963.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.
PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, University of California, Riverside, March 9, 1963.
SOUTHWESTERN
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IBM asks basic questions in computer software

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This textbook covers simply and concisely the Laplace transform method of solution of differential equations connected with electrical, mechanical, and electro-mechanical systems. Mechanical and electrical circuit analysis and necessary mathematics are included. Numerous photographs and figures help clarify the subject. Table of Laplace transform pairs included.

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By ATHANASIOS PAPOULIS, Polytechnic Institute of Brooklyn. *Electronic Sciences Series.* Available August.

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PROGRAMMING AND UTILIZING DIGITAL COMPUTERS

By ROBERT S. LEDLEY, National Biomedical Research Foundation, Inc. and The Johns Hopkins University. *McGraw-Hill Series in Information Processing and Computers.* Available August.

For advanced courses. Discussing programming on both scientific and business levels, this book covers machine language programming; the three-, two-, and one-address instruction systems; automatic programming languages, with chapters on ALGOL and COBOL; and methods for utilizing computers designed as a source for programming problems. Concrete examples illustrate all examples. Approximately 500 exercises follow each chapter.

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Introducing some ideas and techniques involved in communicating to a computer the solution of a problem. The basic approach is the gradual development of a computer language, accomplished by examining several typical problems. Each problem is examined in detail, showing the need for adding new features to the language. Language developed is MAD, an existing language, running on the IBM 704, 709, 7090 computers. An appendix shows how to convert from one computer language to another.

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THE SPIN MODEL OF EUCLIDEAN 3-SPACE*

W. F. EBERLEIN, University of Rochester

1. Introduction. "The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures. In some way Euclid's geometry must be deeply connected with the existence of the spin representation." (Hermann Weyl, [6])

We shall attempt to illuminate this connection for ordinary Euclidean 3-space, starting from a coordinate-free mathematical model. In the process the structure of 2-component spinor analysis becomes particularly simple and transparent. It turns out that, contrary to the usual statement in the physical literature, electron spin then appears as a nonrelativistic effect.

2. Euclidean space and its representations. Denote by $E_n(H_n)$ abstract Euclidean (unitary) n-space, defined as an n-dimensional linear vector space over the real (complex) field plus an inner product (Cf. [2], [5]). The standard Cartesian representation $R^n(C^n)$ of $E_n(H_n)$ consists of all real (complex) n-tuples $x=(x_1, \cdots, x_n)$ with component-wise addition and scalar multiplication and the inner product $(x, y)=\sum_1^n x_j\bar{y}_j$. When $n=3$ one obtains the traditional model R^3 of physical 3-space with imbedded coordinate axes. In an effort to eliminate the unphysical notion of distinguished coordinate system one is tempted to identify E_3 itself as the proper model of physical 3-space, but difficulties arise. In particular, the simple intuitive notions of left- and right-handedness of coordinate systems and the vector product become complex and ambiguous entities in E_3 (Cf. [5]). A coordinate-free representation of E_3 with a richer algebraic structure is required.

DEFINITION. $\mathfrak{E}_3$ *consists of all self-adjoint linear transformations of* H_2 *into itself of trace* 0.

$\mathfrak{E}_3$ is a real linear space under operator addition and scalar multiplication. To establish the dimension introduce an orthonormal basis (ϕ_1, ϕ_2) in H_2. Then any A in $\mathfrak{E}_3$ has the matrix representation

$$A \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

* Written with financial support from the Office of Naval Research and the Bureau of Ships.

Clearly $\sigma_1, \sigma_2, \sigma_3$ are linearly independent and form a basis for the matrix representation of $\mathfrak{E}_3$.

The matrices $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices; they satisfy the following identities:

(1) $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$, (Anticommutation relations);

(2) $\sigma_1\sigma_2\sigma_3 = iI$,

where I is the identity operator on H_2. (1) and (2) imply

(3) $\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2;$

(4) $\sigma_1\sigma_2 - \sigma_2\sigma_1 = 2i\sigma_3, \quad \sigma_2\sigma_3 - \sigma_3\sigma_2 = 2i\sigma_1,$

$\sigma_3\sigma_1 - \sigma_1\sigma_3 = 2i\sigma_2$ (Commutation relations).

We note in passing that $(\sigma_1, \sigma_2, \sigma_3) = -i(\mathbf{k}, \mathbf{j}, \mathbf{i})$, where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the canonical matrix representation of Hamilton's quaternion units (cf. MacDuffee, [4]).

Introduction of an inner product $A \cdot B$ into $\mathfrak{E}_3$ rests on the following observation:

LEMMA. *A, B in $\mathfrak{E}_3$ implies $AB+BA=kI$, where k is real.*

Proof. The self-adjointness of any A in $\mathfrak{E}_3$ implies the existence of an orthonormal basis (ϕ_1, ϕ_2) in H_2 with respect to which A has a diagonal matrix representation. Since A has trace 0,

$$A \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

Then

$$A^2 \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} = a^2 I,$$

and the lemma is proved in the special case $A=B$. But the general case then follows from the identity $AB+BA=(A+B)^2-A^2-B^2$.

We can now define the inner product as the normalized Jordan product:

DEFINITION. $\frac{1}{2}(AB+BA)=(A \cdot B)I, \quad (A, B \text{ in } \mathfrak{E}_3).$

One can then readily verify

COROLLARY. *$A \cdot B$ is a true inner product—i.e., the mapping $(A, B) \to A \cdot B$ of $\mathfrak{E}_3 \times \mathfrak{E}_3$ into the reals satisfies*

(a) $(aA) \cdot B = a(A \cdot B)$,

(b) $(A_1+A_2) \cdot B = A_1 \cdot B + A_2 \cdot B$,

(c) $A \cdot B = B \cdot A$,

(d) $A \cdot A \geqq 0$,

(e) $A \cdot A = 0$ if and only if $A = 0$.

To summarize,

THEOREM. *$\mathfrak{E}_3$ with the above inner product is a Euclidean 3-space.**

Given now the coupled spaces H_2, $\mathfrak{E}_3$ we term H_2 *spin space* and its elements *2-component spinors* and call $\mathfrak{E}_3$ the *spin model of Euclidean 3-space.*

3. The Pauli representation. The significance of the identities (1), (2) satisfied by the Pauli matrices may now be clarified. In terms of our inner product the condition $e_i \cdot e_j = \delta_{ij}$ that three vectors e_1, e_2, e_3 constitute an orthonormal basis for $\mathfrak{E}_3$ takes the form $e_ie_j + e_je_i = 2\delta_{ij}I$—the anticommutation relations (1). Hence

THEOREM. *The Pauli matrices represent an orthonormal basis for $\mathfrak{E}_3$.*

The converse is almost valid:

THEOREM. *Given an orthonormal basis (e_1, e_2, e_3) in $\mathfrak{E}_3$ there exists an orthonormal basis (ϕ_1, ϕ_2) in H_2 such that*

$$(e_1, e_2, e_3) \underset{(\phi)}{\longleftrightarrow} (\sigma_1, \pm\sigma_2, \sigma_3).$$

Proof. Since $e_3^2 = I$, one can choose an orthonormal basis (ψ_1, ψ_2) such that

$$e_3 \underset{(\psi)}{\longleftrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$e_1 \underset{(\psi)}{\longleftrightarrow} \begin{pmatrix} a & \alpha \\ \bar{\alpha} & -a \end{pmatrix}.$$

But

$$0 = e_1e_3 + e_3e_1 \underset{(\psi)}{\longleftrightarrow} \begin{pmatrix} 2a & 0 \\ 0 & +2a \end{pmatrix},$$

whence $a = 0$. Moreover,

$$I = e_1^2 \underset{(\psi)}{\longleftrightarrow} \begin{pmatrix} \alpha\bar{\alpha} & 0 \\ 0 & \alpha\bar{\alpha} \end{pmatrix},$$

whence $|\alpha| = 1$. In terms of the basis (ψ_1, ψ_2) these relations take the form

$$\begin{aligned} e_3\psi_1 &= \psi_1 & e_1\psi_1 &= \bar{\alpha}\psi_2 \\ e_3\psi_2 &= -\psi_2 & e_1\psi_2 &= \alpha\psi_1 \end{aligned} \qquad (|\alpha| = 1).$$

Now set $\phi_1 = \alpha\psi_1$, $\phi_2 = \psi_2$. Then (ϕ_1, ϕ_2) is an orthonormal basis such that

$$\begin{aligned} e_3\phi_1 &= \phi_1 & & e_1\phi_1 = \phi_2 \\ & \text{and} & & \\ e_3\phi_2 &= -\phi_2 & & e_1\phi_2 = \phi_1, \end{aligned}$$

* Note that $A \cdot A = -\det A$ and that $|A| \equiv (A \cdot A)^{1/2}$ is the norm of A as an operator on H_2.

or in matrix form

$$e_3 \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e_1 \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider now the representation of e_2. As before the identities $0 = e_2e_3 + e_3e_2$ and $e_2^2 = I$ imply that

$$e_2 \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} 0 & \gamma \\ \bar{\gamma} & 0 \end{pmatrix} \quad \text{where} \quad |\gamma| = 1.$$

But then

$$0 = e_1e_2 + e_2e_1 \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} \gamma + \bar{\gamma} & 0 \\ 0 & \gamma + \bar{\gamma} \end{pmatrix}$$

implies $\gamma = -\bar{\gamma}$, whence $\gamma = \pm i$ since $|\gamma| = 1$. Thus

$$e_2 \underset{(\phi)}{\longleftrightarrow} \pm \sigma_2.$$

That either sign is possible follows on replacing e_2 by $-e_2$.

Since σ_1, $-\sigma_2$, σ_3 offer an equally natural matrix basis for $\mathfrak{E}_3$, the ambiguity of sign is not surprising. Which sign occurs is implicit in the

COROLLARY. *Let* (e_1, e_2, e_3) *be an ordered orthonormal basis in* $\mathfrak{E}_3$. *Then* $e_1e_2e_3 = \pm iI$ (Hamilton-Clifford identity, [3], [1]).

A direct coordinate-free proof is of interest. Let $u = e_1e_2e_3$. The anticommutation relations (1) then imply that $u^2 = -I$ and that u commutes with e_1, e_2, e_3 Since e_1, e_2, e_3, I form a basis for $B(H_2)$, the algebra of linear transformations of H_2 into itself, u lies in the center of $B(H_2)$ and hence $u = kI$. But $k = \pm i$ since $u^2 = -I$.

4. Orientation and the vector product. Following established mathematical practice we turn ambiguity of sign into a virtue by means of a

DEFINITION. *The coordinate system defined by the ordered orthonormal basis* (e_1, e_2, e_3) *for* $\mathfrak{E}_3$ *is right/left-handed according as* $e_1e_2e_3 = \pm iI$.

It is convenient to equate "coordinate system" with "ordered orthonormal basis." The above results then become:

THEOREM. *The Pauli matrices represent a right-handed coordinate system for* $\mathfrak{E}_3$. *Conversely, any right-handed coordinate system has a Pauli matrix representation.*

To make contact with the usual notion of orientation in E_3 we must establish the

THEOREM. *Let T be an orthogonal transformation of $\mathfrak{E}_3$ into itself and let (e_1, e_2, e_3) be a right-handed coordinate system. Then* $(Te_1)(Te_2)(Te_3) = i(\det T)I = \pm iI$.

Proof. Let

$$T \underset{(e)}{\longleftrightarrow} (a_{ij}).$$

Then $Te_m = \sum_{i=1}^{3} a_{im}e_i$ and

$$(Te_1)(Te_2)(Te_3) = \sum_{i,j,k} a_{i1}a_{j2}a_{k3}e_ie_je_k.$$

The anticommutation relations (1) imply that if two of the indices i, j, k coincide $e_ie_je_k = \pm e_n$, where n is the remaining index, while if i, j, k are all distinct $e_ie_je_k = \pm e_1e_2e_3 = \epsilon_{ijk}iI$, where $\epsilon_{ijk} = \pm 1$ according as (i, j, k) is an even/odd permutation of (1, 2, 3). Since we know *a priori* that $(Te_1)(Te_2)(Te_3) = \pm iI$ and that e_1, e_2, e_3, I form a basis for $B(H_2)$, terms of the coincident type must cancel among themselves. Then

$$(Te_1)(Te_2)(Te_3) = \sum_{i\neq j\neq k} a_{i1}a_{j2}a_{k3}e_ie_je_k = \left[\sum_{i\neq j\neq k} \epsilon_{ijk}a_{i1}a_{j2}a_{k3}\right] iI = i(\det T)I.$$

Given now two elements A, B, in $\mathfrak{E}_3$ it is natural to consider their Lie product $[A, B] = AB - BA$. Since trace (AB) = trace (BA), trace $[A, B] = 0$, but $[A, B]$ is skew-Hermitian rather than Hermitian. To obtain an element of $\mathfrak{E}_3$ we need only divide by $2i$, the factor 2 arising by analogy with the scalar product.

Vector Product Definition. $A \times B = (AB - BA)/2i$.

To make contact with the ordinary definition of vector product choose a right-handed coordinate system (e_1, e_2, e_3). The commutation rules (4) then take the familiar form:

$$(4') \qquad e_1 \times e_2 = e_3, \qquad e_2 \times e_3 = e_1, \qquad e_3 \times e_1 = e_2.$$

(This formula is the real justification for equating right-handedness with the plus sign in the Hamilton-Clifford identity.) The (obvious) distributive law then implies the standard Cartesian formula:

$$(x_1e_1 + x_2e_2 + x_3e_3) \times (y_1e_1 + y_2e_2 + y_3e_3) = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} e_1 + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} e_2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} e_3.$$

To illustrate how vector algebra falls out of the above definition we derive the identity for the triple vector product, a favorite topic of contributors to this MONTHLY.

THEOREM. $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$.

Proof. Collect all the terms on the left and recall that factors of the form $(AB+BA)=2(A\cdot B)I$ commute with everything. Then

$$A\times(B\times C)-(A\cdot C)B+(A\cdot B)C$$
$$=\frac{1}{2i}[A(B\times C)-(B\times C)A]-\frac{B}{2}(AC+CA)+\tfrac{1}{2}(AB+BA)C$$
$$=\tfrac{1}{2}[-\tfrac{1}{2}A(BC-CB)+\tfrac{1}{2}(BC-CB)A-B(AC+CA)+(AB+BA)C]$$
$$=\tfrac{1}{4}[ABC-BCA+ACB-CBA]=\tfrac{1}{4}[A(BC+CB)-(BC+CB)A]=0.$$

5. The rotation group and its covering group. The orthogonal transformations of E_3 form a group which splits into 2 parts: the subgroup $\mathfrak{R}(3)$ of rotations or proper orthogonal transformations characterized by determinant 1 (orientation preserving) and the coset $-\mathfrak{R}(3)$ of improper orthogonal transformations of determinant -1 (orientation reversing). The structure of $\mathfrak{R}(3)$ as a topological group is particularly transparent in the spin model.

Denote the group of unitary transformations in H_2 by $\mathfrak{U}(2)$ and its subgroup of transformations of determinant 1 (unimodular) by $\mathfrak{SU}(2)$.

Given now any U in $\mathfrak{U}(2)$ set $R_UA=UAU^{-1}$ $(A\in\mathfrak{E}_3)$. Since the inner automorphism R_U preserves trace and adjoints, R_U is a linear transformation of $\mathfrak{E}_3$ into itself.

THEOREM. *The mapping $U\to R_U$ is a homomorphism of $\mathfrak{U}(2)$ onto $\mathfrak{R}(3)$—i.e.*
(a) *R_U is a rotation of $\mathfrak{E}_3$.*
(b) *$R_{U_1U_2}=R_{U_1}R_{U_2}$.*
(c) *Given a rotation T in $\mathfrak{E}_3$ there exists a U in $\mathfrak{U}(2)$ such that*

$$TA=UAU^{-1},\ (A\in\mathfrak{E}_3).$$

Proof. (a) R_U is orthogonal since an inner automorphism preserves the algebraically defined inner product:

$$\tfrac{1}{2}[(R_UA)(R_UB)+(R_UB)(R_UA)]=\tfrac{1}{2}[UAU^{-1}UBU^{-1}+UBU^{-1}UAU^{-1}]$$
$$=U[\tfrac{1}{2}(AB+BA)]U^{-1}=\tfrac{1}{2}(AB+BA).$$

Similarly, R_U preserves sign in the Hamilton-Clifford identity and hence is a rotation:

$$(R_Ue_1)(R_Ue_2)(R_Ue_3)=R_U(e_1e_2e_3)=U(e_1e_2e_3)U^{-1}=e_1e_2e_3.$$

(b) follows by straightforward calculation:

$$R_{U_1U_2}A=(U_1U_2)A(U_1U_2)^{-1}=U_1(U_2AU_2^{-1})U_1^{-1}=U_1(R_{U_2}A)U_1^{-1}$$
$$=R_{U_1}(R_{U_2}A)=(R_{U_1}R_{U_2})(A).$$

To establish (c) it is sufficient to pick a right-handed coordinate system (e_1, e_2, e_3) and produce a U in $\mathfrak{U}(2)$ such that $Te_j=Ue_jU^{-1}(j=1, 2, 3)$. Choose an orthonormal basis (ϕ_1, ϕ_2) in H_2 such that

$$(e_1, e_2, e_3) \underset{(\phi)}{\longleftrightarrow} (\sigma_1, \sigma_2, \sigma_3),$$

the Pauli matrices. Since T is a rotation, (Te_1, Te_2, Te_3) is also a right-handed coordinate system, hence admits a Pauli representation

$$(Te_1, Te_2, Te_3) \underset{(\psi)}{\longleftrightarrow} (\sigma_1, \sigma_2, \sigma_3)$$

with respect to some orthonormal basis (ψ_1, ψ_2). Let U be the unitary transformation in H_2 defined by $U\phi_j = \psi_j (j = 1, 2)$. Then it is clear that $Te_j = Ue_jU^{-1}$ $(j = 1, 2, 3)$.

In studying the inverse of the mapping $U \to R_U$ note that we may restrict the U to $\mathfrak{SU}(2)$: If $U \in \mathfrak{U}(2)$, then $|\det U| = 1$, kU is in $\mathfrak{U}(2)$ if $|k| = 1$, and $\det(kU) = k^2 \det U = 1$ on setting $k = \pm(\det U)^{-1/2}$. But $R_{kU} = R_U$, whence the image of $\mathfrak{SU}(2)$ is still all of $\mathfrak{R}(3)$.

THEOREM. *The kernel of the homomorphism $U \to R_U$ of $\mathfrak{SU}(2)$ onto $\mathfrak{R}(3)$—the set of elements mapping into the identity—consists of $\pm I$.*

Proof. $R_U = I$ is equivalent to $UAU^{-1} = A$ or $UA = AU$ for all A in $\mathfrak{E}_3$. As before it follows that U lies in the center of $B(H_2)$, whence $U = kI$ and $k = \pm 1$, since $\det(U) = 1$.

Thus $R_{U_1} = R_{U_2} (U_1, U_2 \text{ in } \mathfrak{SU}(2))$ is equivalent to $U_1 = \pm U_2$, and the mapping inverse to $U \to R_U$ is double-valued. Consider now $\mathfrak{R}(3)$ and $\mathfrak{SU}(2)$ as topological groups. In any coordinate system the defining equations $\det(U) = 1$ and $U^* = U^{-1}$ for an element U of $\mathfrak{SU}(2)$ amount to the representation

$$U \leftrightarrow \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} = a_0 I + i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3),$$

where a_0, a_1, a_2, a_3 are real and $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. The topology of $\mathfrak{SU}(2)$ is then that of S_3, the 3-sphere in R^4, while that of $\mathfrak{R}(3)$ is obtained by identifying antipodal points on S_3. Clearly $\mathfrak{R}(3)$ is connected but not simply connected. $\mathfrak{SU}(2)$ is termed the *covering group* of $\mathfrak{R}(3)$, while the double-valued inverse to the covering map $U \to R_U$ is called the *spin representation* of $\mathfrak{R}(3)$.

The covering map also yields the classical justification for calling a proper orthogonal transformation a "rotation".

EULER'S THEOREM. *Every proper orthogonal transformation T in $\mathfrak{E}_3$ is an ordinary rotation about some axis.*

Proof. Write T as $Tx = UxU^{-1} (x \in \mathfrak{E}_3)$, where U lies in $\mathfrak{SU}(2)$, and pick an orthonormal basis (ϕ_1, ϕ_2) in H_2 yielding a diagonal matrix representation for U. Then

$$U \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{+i\theta} \end{pmatrix}, \quad \text{where } \theta \text{ is real.}$$

Now introduce the orthonormal basis (e_1, e_2, e_3) represented by the Pauli matrices. Since diagonal matrices commute, $Te_3 = e_3$. Moreover, the equation $(x_1' e_1 + x_2' e_2 + x_3' e_3) = T(x_1 e_1 + x_2 e_2 + x_3 e_3)$ becomes

$$\begin{pmatrix} x_3' & x_1' - ix_2' \\ x_1' + ix_2' & -x_3' \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{+i\theta} \end{pmatrix} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \begin{pmatrix} e^{+i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Then

$$\begin{aligned} x_1' &= x_1 \cos 2\theta - x_2 \sin 2\theta \\ x_2' &= x_1 \sin 2\theta + x_2 \cos 2\theta \\ x_3' &= x_3 \end{aligned}$$

or

$$T \underset{(e)}{\longleftrightarrow} \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

a rotation about the e_3 axis through the angle 2θ.

Now let θ vary from 0 to π. Then $T = R_U$ describes a continuous rotation of 2π radians about e_3, but U runs continuously only from I to $-I$. We note in passing that

$$U(\theta) \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{+i\theta} \end{pmatrix} = \exp(-i\theta\sigma_3)(-\infty < \theta < \infty)$$

is a 1-parameter subgroup of $\mathfrak{SU}(2)$ but shall not enter into the Lie algebra theory of $\mathfrak{SU}(2)$ and $\mathfrak{R}(3)$ thus arising.

6. Electron spin and the neutrino equation. Let us now turn to the quantum mechanical origin of the term "spin." For simplicity we consider only 1-particle systems.

The conventional nonrelativistic treatment in Cartesian coordinates $r = (x_1, x_2, x_3)$ may be sketched as follows: Given a particle of mass m and momentum vector $P = (p_1, p_2, p_3)$ (where $p_i = mv_i$) moving in a scalar potential field $V(r, t)$ the energy equation is

$$\mathbf{H} \equiv \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V = \mathbf{E}, \tag{5}$$

where $\mathbf{H}$ is the Hamiltonian and $\mathbf{E}$ is the energy. The quantization process consists of the transition

$$\mathbf{E} \to i\hbar \frac{\partial}{\partial t},$$

$$P \to -i\hbar\nabla = -i\hbar\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

where $\hbar$ is Planck's constant divided by 2π. The differential operators act on all sufficiently smooth functions $\psi(r, t)$ from $R^3 \times R^1$ to the complex numbers such that

$$\int_{R^3} |\psi(r, t)|^2 d^3x < \infty.$$

The equation $\mathbf{H}=\mathbf{E}$ then transforms into the Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m}\Delta + V\right]\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{6}$$

For a particle of charge e in an electric field, V has the form $V=e\Phi$, where Φ is the electrostatic potential.

We note that the classical angular momentum $r \times P$ transforms into the operator $M=r\times(-i\hbar\nabla)=(M_1, M_2, M_3)$, whose components satisfy the commutation relations

$$[M_1, M_2] = i\hbar M_3, \qquad [M_2, M_3] = i\hbar M_1, \qquad [M_3, M_1] = i\hbar M_2.$$

But it is clear from (4) that these are just the commutation relations satisfied by the components of $\frac{1}{2}\hbar(\sigma_1, \sigma_2, \sigma_3)$.

Consider now a particle of charge e moving in an external electro-magnetic field described by a scalar potential Φ and a vector potential $\mathfrak{A}=(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3)$. The electric vector $\mathcal{E}$ and the magnetic vector $\mathfrak{H}$ are then given by the equations

$$\mathcal{E} = -\nabla\Phi - c^{-1}\frac{\partial\mathfrak{A}}{\partial t}, \qquad \mathfrak{H} = \operatorname{curl}\mathfrak{A},$$

c being the velocity of light, while the Hamiltonian takes the form

$$\mathbf{H} = \frac{1}{2m}\left[\left(p_1 - \frac{e\mathfrak{A}_1}{c}\right)^2 + \left(p_2 - \frac{e\mathfrak{A}_2}{c}\right)^2 + \left(p_3 - \frac{e\mathfrak{A}_3}{c}\right)^2\right] + e\Phi.$$

The above quantization rule then leads to the equation

$$-\frac{\hbar^2}{2m}\Delta\psi + \frac{ie\hbar}{2mc}(\mathfrak{A}\cdot\nabla + \nabla\cdot\mathfrak{A})\psi + \left[\frac{e^2(\mathfrak{A}\cdot\mathfrak{A})}{2mc^2} + e\Phi\right]\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{7}$$

The predictions of equation (7) for an electron in a magnetic field do not agree with experiment. To obtain agreement* it is necessary to double the components of ψ—that is, replace the scalar valued ψ by a mapping $\psi=(\psi_1, \psi_2)$ of $R^3\times R^1$ into C^2—and to add an extra term representing the energy of an in-

* We neglect higher order (relativistic) effects such as spin-orbit interaction and the electromagnetic self-field.

trinsic magnetic moment of magnitude $e\hbar/2\,mc$, ascribed to an internal "spin." This extra term is either postulated in the nonrelativistic case or obtained by a limiting analysis from a 4-component relativistic equation due to Dirac. We now show that both the doubling of components and the correct extra term occur automatically on the nonrelativistic level in the model $\mathfrak{E}_3$.

Let $\zeta_1, \zeta_2, \zeta_3$ denote an arbitrary orthonormal basis for $\mathfrak{E}_3$. Then the momentum vector has the form $P=p_1\zeta_1+p_2\zeta_2+p_3\zeta_3$ and $(P\cdot P)I=P^2=(p_1\zeta_1+p_2\zeta_2+p_3\zeta_3)^2$. Let the $\psi=\psi(x, t)$ be sufficiently smooth mappings of $\mathfrak{E}_3\times R^1$ into some nontrivial finite dimensional complex vector space W. The transition $p_i\rightarrow -i\hbar\partial_i$, where

$$\partial_i\psi = \frac{\partial\psi}{\partial x_i} = \lim_{h\to 0}\frac{\psi(x+h\zeta_i, t)-\psi(x, t)}{h},$$

carries P into $-i\hbar\nabla$, where now $\nabla=\zeta_1\partial_1+\zeta_2\partial_2+\zeta_3\partial_3$. But $\nabla\psi=\zeta_1(\partial_1\psi)+\zeta_2(\partial_2\psi)+\zeta_3(\partial_3\psi)$ makes sense only in the following 3 cases: a) $W=C^1=H_1$; b) $W=H_2$; c) $W=B(H_2)$. Then $\nabla\psi$ maps $\mathfrak{E}_3$ respectively into: a) the complexification of $\mathfrak{E}_3$=the linear transformations of H_2 into itself of trace 0; b) H_2; c) $B(H_2)$. Moreover, in all 3 cases we have the honest operator square $\nabla^2=(\partial_1^2+\partial_2^2+\partial_3^2)I=\Delta I$. In the first 2 cases there arises the natural Hilbert space $L^2(\mathfrak{E}_3, H_i)$ $(i=1, 2)$ of functions f from $\mathfrak{E}_3$ to H_i such that $\int_{\mathfrak{E}_3}(f(x), f(x))d^3x<\infty$. A space $L^2(\mathfrak{E}_3, B(H_2))$ arises in case c) on introducing the inner product $(A, B)=\frac{1}{2}$ trace AB^* into $B(H_2)$.

When ψ has range in $C^1=H_1$, $\nabla\psi=\text{grad }\psi$, which is independent of the basis for $\mathfrak{E}_3$. We hasten to provide an invariant definition of $\nabla\psi$ when ψ has range in H_2. Consider the following (complex-valued) inner product in H_2: (su, v) where s is in $\mathfrak{E}_3$ and u, v lie in H_2. Since the expression is linear in s it has the unique representation $(su, v)=s\cdot F_1(u, v)+is\cdot F_2(u, v)$, where F_1, F_2 are well defined mappings of $H_2\times H_2$ into $\mathfrak{E}_3$. Then

$$(u, \nabla\psi) = \left(u, \sum_1^3 \zeta_j\partial_j\psi\right) = \sum_1^3 (u, \zeta_j\partial_j\psi) = \sum_1^3 \partial_j(\zeta_j u, \psi) = \sum_1^3 \partial_j[\zeta_j\cdot F_1(u, \psi)]$$

$$+ i\sum_1^3 \partial_j[\zeta_j\cdot F_2(u, \psi)] = \text{div } F_1(u, \psi) + i\text{ div } F_2(u, \psi).$$

Equating the first and last terms then yields the desired invariant definition of $\nabla\psi$. The case where ψ has range in $B(H_2)$ is treated similarly.

Now let the ψ have range in H_1, H_2, or $B(H_2)$. The energy equation (5) may be written

$$\mathbf{E}I = \frac{1}{2m}P^2 + VI.$$

The transition $P\rightarrow -i\hbar\nabla$, $\mathbf{E}\rightarrow i\hbar\partial/\partial t$ then yields (6), multiplied by the cancellable factor I. Consider now an electron in the above electro-magnetic field. The

energy equation takes the form

$$\mathbf{E}I = \frac{1}{2m}\left(P - \frac{e\mathfrak{A}}{c}\right)^2 + e\Phi I,$$

where in terms of a basis ζ_1, ζ_2, ζ_3, which it is convenient to take right handed, $\mathfrak{A} = \mathfrak{A}_1\zeta_1 + \mathfrak{A}_2\zeta_2 + \mathfrak{A}_3\zeta_3$ and $\mathfrak{B} = \operatorname{curl} \mathfrak{A} = \mathfrak{B}_1\zeta_1 + \mathfrak{B}_2\zeta_2 + \mathfrak{B}_3\zeta_3$. Quantization yields

$$i\hbar I\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{e\mathfrak{A}}{c}\right)^2 + e\Phi I\right]\psi$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}(\nabla\cdot\mathfrak{A} + \mathfrak{A}\cdot\nabla) + \left(\frac{e^2\mathfrak{A}^2}{2mc^2} + e\Phi I\right)\right]\psi,$$

or in terms of the basis

$$-\frac{\hbar^2}{2m}I\Delta\psi + \frac{ie\hbar}{2mc}[(\zeta_1\partial_1 + \zeta_2\partial_2 + \zeta_3\partial_3)(\mathfrak{A}_1\zeta_1 + \mathfrak{A}_2\zeta_2 + \mathfrak{A}_3\zeta_3)$$

$$+ (\mathfrak{A}_1\zeta_1 + \mathfrak{A}_2\zeta_2 + \mathfrak{A}_3\zeta_3)(\zeta_1\partial_1 + \zeta_2\partial_2 + \zeta_3\partial_3)]\psi + \left(\frac{e^2\mathfrak{A}^2}{2mc^2} + e\Phi I\right)\psi = i\hbar I\frac{\partial\psi}{\partial t}.$$

Expansion by means of the identities 1) and 3) gives

$$-\frac{\hbar^2}{2m}I\Delta\psi + \frac{ie\hbar}{2mc}(\partial_1\mathfrak{A}_1 + \partial_2\mathfrak{A}_2 + \partial_3\mathfrak{A}_3)I\psi + \frac{ie\hbar}{2mc}(\mathfrak{A}_1\partial_1 + \mathfrak{A}_2\partial_2 + \mathfrak{A}_3\partial_3)I\psi$$

$$+ \left[\frac{e^2\mathfrak{A}^2}{2mc^2} + e\Phi I\right]\psi - \frac{\hbar e}{2mc}[\zeta_1\{(\partial_2\mathfrak{A}_3 - \mathfrak{A}_3\partial_2) - (\partial_3\mathfrak{A}_2 - \mathfrak{A}_2\partial_3)\} + \zeta_2\{(\partial_3\mathfrak{A}_1 - \mathfrak{A}_1\partial_3)$$

$$- (\partial_1\mathfrak{A}_3 - \mathfrak{A}_3\partial_1)\} + \zeta_3\{(\partial_1\mathfrak{A}_2 - \mathfrak{A}_2\partial_1) - (\partial_2\mathfrak{A}_1 - \mathfrak{A}_1\partial_2)\}]\psi = i\hbar I\frac{\partial\psi}{\partial t}.$$

Since $(\partial_i\mathfrak{A}_j - \mathfrak{A}_j\partial_i)\psi = \partial_i(\mathfrak{A}_j\psi) - \mathfrak{A}_j(\partial_i\psi) = \psi(\partial_i\mathfrak{A}_j)$, the equation reduces to

$$-\frac{\hbar^2}{2m}I\Delta\psi + \frac{ie\hbar}{2mc}(\mathfrak{A}\cdot\nabla + \nabla\cdot\mathfrak{A})I\psi + \left[\frac{e^2(\mathfrak{A}\cdot\mathfrak{A})}{2mc^2} + e\Phi\right]I\psi$$

$$- \frac{\hbar e}{2mc}(\mathfrak{B}_1\zeta_1 + \mathfrak{B}_2\zeta_2 + \mathfrak{B}_3\zeta_3)\psi = i\hbar I\frac{\partial\psi}{\partial t}. \tag{8}$$

Recall that ζ_1, ζ_2, ζ_3, I form a basis for $B(H_2)$. If $\psi \not\equiv 0$ is complex-valued it follows that $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = 0$ and the magnetic field must vanish. Hence in the presence of a magnetic field—no matter how weak—ψ must take values in H_2 or $B(H_2)$, and the factor I is absorbed. When ψ has values in H_2 we obtain equation (7) with doubled components plus the correct extra term $-(\hbar e/2mc)\mathfrak{B}\psi$. The case where ψ has values in $B(H_2)$ is clearly redundant: the resulting 4-component equation is just the 2-component equation written twice.

Now the operator $i\nabla$ is the original Hamilton operator (Cf. [3]). It is readily seen to be a symmetric operator in the Hilbert space $L^2(\mathfrak{E}_3, H_2)$ and is in fact essentially self-adjoint. Thus for arbitrary choice of positive constant c it generates the flow

$$\psi(x, t) = e^{-ict(i\nabla)}\psi_0(x) = e^{ct\nabla}\psi_0(x).$$

Differentiating, we obtain

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = \nabla\psi,$$

or in terms of a basis

$$\left[\zeta_1\frac{\partial}{\partial x_1} + \zeta_2\frac{\partial}{\partial x_2} + \zeta_3\frac{\partial}{\partial x_3} - \frac{1}{c}\frac{\partial}{\partial t}\right]\psi = 0. \tag{9}$$

When c equals the velocity of light this is just the Lorentz invariant 2-component neutrino equation of Hermann Weyl (1929), revived by T. D. Lee and C. N. Yang (1956–57) to predict the now well-known phenomenon of nonconservation of parity in weak interactions. Further analysis, however, must await extension of our considerations in a later paper to the model of space-time afforded by *all* self-adjoint linear transformations of H_2 into itself under the (now indefinite) metric $A \cdot A = -\det A$.

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PICARD'S GREAT THEOREM ON RIEMANN SURFACES*

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An analytic function in the plane is, by definition, meromorphic if it has no essential singularities. One of the most beautiful theorems of classical analysis is Picard's Great Theorem: Every meromorphic function in the plane takes on every value, finite or infinite, except at most two. A typical example is the exponential function e^z. It omits the values 0 and ∞, but assumes all other

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values. Picard's theorem has been the basis of the classical value distribution theory. It was the starting point of the work of Borel, Hadamard, Julia, and Valiron, and such theorems as the Picard-Borel theorem, Nevanlinna's second fundamental theorem, and the defect relation can all be considered to be far-reaching extensions of Picard's theorem.

Despite the elegance of the classical theory it suffers from the following drawback. A meromorphic function is, of course, a locally defined concept. It is, therefore, somewhat artificial to restrict the theory to a globally chosen special carrier such as the plane or the disk. General results can be expected only on the corresponding locally defined carrier, the most general one on which the concept of meromorphic function still makes sense. Such a carrier is the 2-manifold with conformal structure, that is a Riemann surface. Building up a general theory of meromorphic functions on Riemann surfaces thus appears to be of compelling importance.

The main questions here are: Do Nevanlinna's first and second fundamental theorems remain valid on open Riemann surfaces and can any counterpart of Picard's theorem be established on them? It is these questions I would like to discuss in this lecture. (The proofs in extenso will be contained in [19].)

On earlier work reference is here made to the classical method of R. Nevanlinna [13], its further development in the plane by F. Nevanlinna [12] and in particular by L. Ahlfors [1], and to generalizations in various directions given by L. Ahlfors [2], L. Carleson [4], S. Chern [5, 6], G. af Hällström [7], K. Kunugui [9], L. Myrberg [11], J. Tamura [20], M. Tsuji [21], and Y. Tumura [22].

The first fundamental theorem. A Riemann surface is, by definition, a 2-manifold which we can imagine as a surface W in 3-space. It is covered by open sets, each mapped homeomorphically by some function, say φ, onto a disk in the z-plane. Whenever two open sets overlap, producing several parametric images of the intersection, the mapping from z_1 to z_2 is required to be directly conformal, that is, the function $z_2=\varphi_2(\varphi_1^{-1}(z_1))$ is analytic. This conformal structure permits us to speak of harmonic, analytic, and meromorphic functions on W. In fact, because of conformality, if a function is meromorphic in z_1, then it is meromorphic in z_2 as well.

In order to be able to speak of the rate of growth of functions on W, we need a metric on the surface. To this end we choose a point ζ on W and a relatively compact subregion Ω with $\zeta\in\Omega$ whose boundary β_Ω consists of a finite number of analytic Jordan curves. On Ω we construct a harmonic function p_Ω with a negative logarithmic pole at ζ, such that

$$p_\Omega(z)-\frac{1}{2\pi}\log|z-\zeta|\to 0 \text{ as } z\to\zeta.$$

Moreover, we require that $p_\Omega(z)=k_\Omega(\text{const.})$ on β_Ω. These conditions obviously determine the function p_Ω uniquely. As Ω exhausts W, the functions p_Ω tend to a

function $p(z)=\lim_{\Omega\to W} p_\Omega(z)$, harmonic on all of W except for the same logarithmic singularity at ζ [3]. The limiting function $p(z)$ is called the capacity function. The constants k_Ω increase as Ω increases and tend to a limit which we assume in this lecture to be infinite. (Only minor modifications in our reasoning will be needed if this condition is not satisfied.) We consider surfaces on which the capacity function tends to infinity for every approach of z to the ideal boundary of W.

We shall make use of the level lines of p. For any $k\geqq 0$, denote by β_k the set of points with $p(z)=k$. For the smallest value $k=0$ we have the curve β_0 and denote the region between β_0 and β_k by W_k. Along β_k we choose the metric $ds=dp^*$, where p^* is the harmonic conjugate of p. In this metric the length of β_k is $\int_{\beta_k} ds=1$. The function $P=p+ip^*$ maps W_k, suitably cut, into a rectangle of the P-plane with width k and height 1.

We are now ready to consider an arbitrary meromorphic function w on W. It maps W_k onto a covering surface of the w-plane. The various sheets hang together at certain branch points.

The first function the value distribution theory is interested in is the following. Given a point a of the w-plane, how many points of the image of W_k are there above a, that is, how many inverse images of a are there in W_k? We set $n(k, a)=$number of a-points in W_k, counted with their multiplicities, and $N(k, a)=\int_0^k n(k, a)dk$. (We make the convention for this entire lecture that capital letters will stand for the integrals from 0 to k of the corresponding small letters.) This $N(k, a)$ we take as the counterpart of Nevanlinna's *counting function*. It gives us information on the growth of the number of a-points as $k\to\infty$. For future reference we also introduce for the special point $a=\infty$ the notation $n(k, a)=n(k, w)$ to indicate the function w we are dealing with. Thus in the sequel $n(k, w)$ is the number of poles of w in W_k.

The other function with which the value distribution theory deals concerns the distance of the image of the boundary of W_k from a. It is convenient to measure the distances on the Riemann sphere A with unit diameter above the w-plane. The image of $\beta_k-\beta_0$ is a set of curves on the w-plane. It has a stereographic image on the sphere. The point a has its image that we continue denoting by a, and a point w on our curve has its image w on the sphere. We are interested in the chordal distance $[w, a]$ between the stereographic images of w and a. We know that

$$[w, a]=\frac{|w-a|}{\sqrt{(1+|w|^2)}\sqrt{(1+|a|^2)}}.$$

The proximity of our image curve to a is indicated by the function

$$m(k, a)=\frac{1}{2\pi}\int_{\beta_k-\beta_0}\log\frac{1}{[w, a]}\,ds.$$

We take this to be the counterpart of Nevanlinna's *proximity function*. It is, in

a sense, the logarithmic potential of the image curve at a or, if you prefer, the mean logarithmic distance of the curve from a.

The first question is the following: How do the counting function $N(k, a)$ and the proximity function $m(k, a)$ vary as a ranges over the w-plane? In particular, is there any relation between N and m? The answer will be given by the first fundamental theorem which we shall now prove.

Take any two points a, b in the w-plane. We have

$$\frac{dm(k, b)}{dk} - \frac{dm(k, a)}{dk} = \frac{1}{2\pi}\int_{\beta_k} \frac{d}{dk} \log \left|\frac{w - a}{w - b}\right| ds.$$

Indeed, in $[w, a]$ the a-factor goes out in the differentiation process, the w-factor is the same for b and a and goes out when we take the difference, and all that remains is $w-a$. Now we make use of the Cauchy-Riemann relations which give us

$$\frac{1}{2\pi}\int_{\beta_k} d \arg \frac{w - a}{w - b}.$$

This integral is, by the argument principle, equal to the number of zeros less the number of poles. Since $n(k, a)$ only gives the zeros in W_k, we must add the number $n_0(a)$ of zeros in the compact region bounded by β_0. We obtain

$$n(k, a) + n_0(a) - n(k, b) - n_0(b).$$

On combining the a-terms and on integrating from zero to k we get

$$m(k, a) + N(k, a) + n_0(a)k,$$

which is equal to the same expression for b. But a and b are arbitrary points. It follows that our expression is independent of a and can be denoted by $T(k)$. This we take as the counterpart of Nevanlinna's characteristic function and we have reached the *first fundamental theorem.* It is easy to see that $T(k)$ is a convex function of k and that the third term is a negligible remainder term. The essence of the theory is that m and N add up to T. This expresses a beautiful balance. If a point a is only lightly covered, then N is small, but m must be correspondingly large, that is, the image of β_k comes close to the point a. Conversely, if a point a is strongly covered, then the image of β_k stays at a great distance from a. In short, what a point a loses in its coverage, it gains in the proximity of the image of β_k.

We observe that we made no use of the properties of w on W beyond β_k. Consequently, the first fundamental theorem expresses a universal property. It is valid for all meromorphic functions on all subregions of all Riemann surfaces.

The second fundamental theorem. The next question is: What are the comparative contributions of m and N to T? In particular, for how large a set of points a can m make any substantial contribution? The answer will be given by the second fundamental theorem which we shall now prove. It will give us an

upper bound for the sum of the proximity functions for any finite number of points a.

We consider our rectangle in the P-plane, our region W_k, the w-plane, and the sphere above the w-plane. On the sphere we distribute a mass

$$d\mu(a) = \rho(a)d\omega(a),$$

where $\rho(a)$ is the density, and $d\omega(a)$ is the area element of the sphere. We require that the total mass is unity. The area element is the image of some area element at the point z of W_k by way of the function w and the subsequent stereographic projection. The area element at z also has an image in our P-rectangle, and we consider the induced density $\delta(z)$ at this image. Clearly it is equal to the corresponding density $\rho(w(z))$ multiplied by the change of scale of the area element:

$$\delta(z) = \frac{|w_P|^2}{(1 + |w|^2)^2}\rho(w(z)).$$

Here w_P is the P-derivative of w, defined by $w_P = w_z/P_z$. We consider the logarithm of $\delta(z)$ and integrate over β_k. The theorem on the geometric and arithmetic mean gives

$$\int_{\beta_k} \log \delta ds \leqq \log \int_{\beta_k} \delta ds,$$

or, equivalently,

$$\int_{\beta_k} \log \frac{\delta}{\rho} ds + \int_{\beta_k} \log \rho ds \leqq \log \int_{\beta_k} \delta ds.$$

This is the preliminary form of the second fundamental theorem. The right-hand side can be shown to be an insignificant remainder term. The second term on the left depends, of course, only on the mass distribution. If the mass is suitably concentrated at q given points a_ν, the term will give the sum of the proximity functions for these points. But that is precisely what we want to evaluate. The first term depends only on w and P. It will be evaluated in terms of the characteristic function, of the counting function of multiple points of w, and the counting function of the zeros of P_z.

We start on the right-hand side. Since δ is the density at a point, we obtain the total mass $M(k)$ distributed on the image surface of W_k by first integrating δ along β_k and then integrating from 0 to k:

$$M(k) = \int_0^k \int_{\beta_k} \delta dk ds.$$

On the other hand, the total mass is obtained by multiplying the mass $d\mu(a)$ at a by the number $n(k, a)$ of times a is covered and by integrating over the entire A-sphere:

$$M(k) = \int_A \int n(k, a) d\mu(a).$$

We integrate this from zero to k, change the order of integration to obtain the integrand N, and estimate N upward by T. But T does not depend on a; it can be taken in front of the integral sign, and the integral reduces to the total mass which equals one:

$$\int_0^k M(k)dk = \int_A \int N(k, a)d\mu(a) \leqq T(k).$$

Thus the integral of the integral of the quantity $\int_{\beta_k} \delta ds$ is bounded above by $T(k)$, and easy computation shows that, except for negligibly small intervals,

$$\log \int_{\beta_k} \delta ds = O(k + \log T(k)).$$

We proceed to the evaluation of $\int_{\beta_k} \log \rho ds$. Consider the density

$$\frac{1}{2} \log \rho = \sum_1^q \log \frac{1}{[w, a_\nu]} - \log \sum_1^q \log \frac{1}{[w, a_\nu]} + C,$$

where the second term on the right is added to make the total mass finite, and the constant is so chosen that this total mass is unity. The essence is the first term on the right indicating a heavy concentration of the mass at the q chosen points a_ν. We use the notation

$$m(k) = \sum_1^q m(k, a_\nu)$$

and obtain on dividing by 2π, on integrating along β_k, and on applying the theorem on the geometric and arithmetic mean,

$$\frac{1}{4\pi} \int_{\beta_k} \log \rho ds \geqq m(k) - \log T(k) + O(k),$$

where we have also made use of the inequality $T(k) \geqq m(k)$.

It remains to evaluate the term

$$L(k) = \frac{1}{4\pi} \int_{\beta_k} \log \frac{\delta}{\rho} ds = \frac{1}{2\pi} \int_{\beta_k} \log \frac{|w_P|}{1 + |w|^2} ds.$$

We form the derivative

$$L'(k) = \frac{d}{dk} \frac{1}{2\pi} \int_{\beta_k} \log \frac{1}{1 + |w|^2} ds + \frac{1}{2\pi} \int_{\beta_k} d \arg w_P ds,$$

where we have again used the Cauchy-Riemann relations in the last term.

Denote the first and second terms on the right by I_1 and I_2. For I_1 one obtains immediately

$$I_1 = -2\frac{d}{dk}\frac{1}{2\pi}\int_{\beta_k}\log\sqrt{(1+|w|^2)}ds = -2\frac{dm(k,\infty)}{dk}.$$

For I_2 we first have by the argument principle

$$I_2 = n\left(k,\frac{1}{w_z}\right) - n(k,w_z) - n\left(k,\frac{1}{P_z}\right) + C,$$

where the constant comes from the integral along β_0. Consider the number $n_1(k)$ of multiple points in W_k, each point of multiplicity i being counted $i-1$ times. It is known that

$$n_1(k) = n\left(k,\frac{1}{w_z}\right) - n(k,w_z) + 2n(k,w).$$

Indeed, a finite value of w with multiplicity i is a zero of w_z with multiplicity $i-1$. As to a pole of w of multiplicity i, it is a pole of w_z of multiplicity $i+1$, and the second term gives us $-i-1$. But the third term contributes $2i$, and the total of the last two terms is again $i-1$ as desired. With this notation I_2 becomes

$$I_2 = n_1(k) - 2n(k,w) - n\left(k,\frac{1}{P_z}\right) + C.$$

It remains to add I_1 and I_2 and to integrate to obtain

$$\frac{1}{4\pi}\int_{\beta_k}\log\frac{\delta}{\rho}ds = N_1(k) - 2T(k) - N\left(k,\frac{1}{P_z}\right) + O(k).$$

Here, according to our convention, the capital letters N_1 and N stand for the integrals of the corresponding small letters, and T resulted from an application of the first fundamental theorem.

The term $N(k,P_z^{-1})$ has an interesting geometric meaning. We are to evaluate the number $n(k,P_z^{-1})$ of the zeros of P_z in W_k. If W_k is doubly connected, the level lines of p^* cover it smoothly without any branchings and the above $n=0$. If, however, we have two contours constituting β_k, then some level line of p^* must branch off to reach both contours and we have a zero of P_z. In general, if the connectivity of W_k is c, the number of zeros of P_z is $n=c-2$. Suppose then we have a "handle" in W_k, that is, a torus-shaped part between β_0 and β_k. Then some level line of p^* must branch off before entering the tubes of the torus, and again combine after completing its passage through the tubes. Thus we have two zeros of P_z, and if the genus is g we obtain $n=2g$. In the case of connectivity c and genus g we conclude that

$$n\left(k,\frac{1}{P_z}\right) = 2g + c - 2.$$

But this is known to be the Euler characteristic $e(k)$ of W_k,

$$e(k) = -n_0 + n_1 - n_2,$$

where the n's give the numbers of vertices, edges, and faces in any triangulation of W_k. We denote the integral of $e(k)$ from 0 to k by $E(k)$ and have

$$N\left(k, \frac{1}{P_z}\right) = E(k).$$

We combine the results and arrive at the *second fundamental theorem on W*:

$$\sum_1^q m(k, a_\nu) < 2T(k) - N_1(k) + E(k) + O(k + \log T(k)).$$

The meaning of this inequality is that there are only relatively few points a_ν that contribute anything substantial to the proximity functions m. No matter how many points q we choose, the sum of the corresponding proximity functions can never exceed, in essence, $2T(k)+E(k)$.

Picard's Great Theorem. The meaning of the second fundamental theorem becomes clearest if we give some applications. Let the first be to Picard values. To do this we first substitute for m from the first fundamental theorem and obtain

$$(q-2)T(k) < \sum_1^q N(k, a_\nu) - N_1(k) + E(k) + O(k + \log T(k)).$$

We introduce the notation

$$\eta(W, w) = \liminf_{k\to\infty} \frac{E(k)}{T(k)},$$

divide the above inequality by $T(k)$, and let $k\to\infty$. The last term contributes nothing because $T(k)$ grows more rapidly than k and, of course, more rapidly than $\log T(k)$. If we consider Picard points a_ν, that is, points not covered at all, then $N(k, a_\nu)=0$. We denote by P the number of Picard values and state the following *Picard-type theorem on W*: $P \leqq 2+\eta$. Schematically, we could write

$$\text{Picard} \leqq \frac{\text{Euler}}{\text{Nevanlinna}} + 2.$$

The number of Picard values exceeds by at most two the lower limit of the integrated Euler characteristic divided by the Nevanlinna characteristic.

The following natural question now arises: What about values a, which, although not completely uncovered, are so sparsely covered that $N(k, a)/T(k)\to 0$. The answer is given by the following *Picard-Borel-type theorem on W*: *There are at most* $2+\eta$ *values a with this property.*

The proof is immediate: If we substitute $q=2+\eta+\epsilon$ with $\epsilon>0$ in the main inequality, we arrive at the contradiction $\eta+\epsilon\leqq\eta$.

Suppose then that a point a is even more strongly covered, however not with the complete power that makes $N(k, a)/T(k)$ tend to 1. The counterpart of Nevanlinna's defect is

$$\delta(a) = 1 - \limsup \frac{N(k, a)}{T(k)} = \liminf \frac{m(k, a)}{T(k)} .$$

The question here is: What can be said about the sum of all defects? The answer is given by the following *defect relation on W*:

$$\sum \delta(a_\nu) = 2 + \eta.$$

The defective values a need not be located exclusively at the $2+\eta$ Picard points or the $2+\eta$ Picard-Borel points, but may be spread to a countable point set provided the sum of the defects does not exceed $2+\eta$. If $\eta<\infty$, it is easy to see that the set of defective values must indeed be countable and the defect relation is a direct consequence of the second fundamental theorem. If $\eta=\infty$, no such conclusion can be made on countability.

The above applications concern defective coverage of a point a. We now ask about multiple coverage, that is coverage by algebraic branch points. What can be said about the set of points that are strongly covered by multiple points? The counterpart of Nevanlinna's ramification index is

$$\vartheta(a) = \liminf \frac{N_1(k, a)}{T(k)},$$

where $N_1(k, a)$ is the counting function of multiple a-points. In the second fundamental theorem we transpose the counting function $N_1(k)$ of all multiple points to the left and observe that it equals the sum of the counting functions for all a-points. On taking lower limits on the left we obtain the *ramification relation on W*:

$$\sum \vartheta(a_\nu) \leqq 2 + \eta.$$

There may be a countable number of "ramified" points a_ν, if $\eta<\infty$, provided that the sum of the ramification indices does not exceed $2+\eta$. An easy corollary states that the number of "totally ramified" points, that is, points not covered by any smooth sheets, cannot exceed $4+2\eta$.

We note in passing that the above theorems lead in a natural manner to the corresponding classical theorems in the plane. In fact, the Euler characteristic for an exhausting disk is constantly -1 and consequently $\eta=0$. For this value of η we have the classical second fundamental theorem of Nevanlinna, Picard's theorem, the Picard-Borel theorem, the defect relation, and the ramification relation. It is of some interest to note that in this original form the theorems remain valid more generally on all open Riemann surfaces with finite Euler char-

acteristics, that is, with finite genus and a finite number of boundary components.

In reaching the universal bound $2+\eta$ in the above theorems we were rather generous in suppressing various terms and one might wonder whether the bound is at all sharp. We conclude by showing that the bound is the best possible for even integers. For odd integers we refer to the proof given by B. Rodin in his doctoral dissertation [15]. Take the Riemann surface above the z-plane consisting of any number $q>0$ of sheets with branch points of multiplicity q at $i(\frac{1}{2}\pi+j\pi)$, $j=0, \pm 1, \pm 2, \cdots$. On the surface consider the meromorphic function

$$w = \sqrt[q]{\left(\frac{e^z+i}{e^z-i}\right)}.$$

It gives an elementary conformal mapping, namely, first an exponential mapping, then a linear mapping, and finally a root transformation. It is easy to see that the image surface above the w-plane consists of infinitely many identical sheets, each with q slits on the unit circle, the $2q$ end points of the slits equally spaced along the circle. The inner edge (of radius $1-0$) of each slit is identified with the outer edge (of radius $1+0$) of the slit on the sheet above it. We exhaust the surface above the z-plane by q-tuple disks $|z|<2\pi m$, $m=1, 2, \cdots$. By Hurwitz' formula the Euler characteristic is, apart from insignificant bounded terms, the sum of the orders of the branch points, $e\sim 4m(q-1)$. The poles of the function w are at $z=i(\frac{1}{2}\pi+j\cdot 2\pi)$, $j=0, \pm 1, \pm 2, \cdots$, and the number of the poles is $n=2m$. The ratio we are interested in is $E/T<E/N$, but here E and N are the integrals of the above e and n. We conclude that $\eta \leqq e/n = 2(q-1)$. Our Picard-type theorem states that the number of Picard values cannot exceed $\eta+2=2q$. But the end points of the slits on the image surface are all logarithmic branch points, consequently Picard points. The number is precisely $2q$, and we have proved that our bound $2+\eta$ is sharp.

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BOOLEAN ALGEBRA, MAP COLORING, AND INTERCONNECTIONS

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The problem of coloring a mathematical map with the minimum number of distinct colors may be described abstractly as follows. A finite set X (of regions of the map) has a symmetric relation R (xRy means x and y may be colored with the same color). Denote by "$\overline{R}$" the complementary relation of R; $x\overline{R}y$ if and only if xRy is false (x and y have a boundary in common). Let a nonnull subset of X be called an *R-compatible* set, or *$\overline{R}$-compatible* set, if and only if xRy, or $x\overline{R}y$, whenever x and y are distinct members of the subset. The problem is to choose a smallest set s of nonoverlapping R-compatible sets whose union is X. Call such a set s a *solution*. In this guise a variation of the famous 4-color problem is as follows. The statement—if the largest $\overline{R}$-compatible set has four or less elements (a necessary condition for the map to be on the plane or the sphere), then each solution contains four or less sets—is false (and is false no matter which integer replaces four) by example and no such example found can be drawn on the plane or sphere.

The abstract description can be generalized by considering a given partition

p of X and requiring that a solution be a smallest set of nonoverlapping R-compatible sets whose union contains exactly one element of each $P \in p$ (for the map coloring application p is the trivial partition where each $P \in p$ contains exactly one member of X). This generalization is an abstraction of the following physical problem which can be of some importance in the design of electronic equipment.

Imagine pins sticking out of a plane surface of a box of electronic components. A list of some pairs of these pins is available. Each pair of pins in this list is to be connected electrically by etching a conducting path on a board through which all the pins protrude. Many such boards may be used, stacked one upon the other with the pins protruding through all. Each etched path must follow one of some predescribed routes between the pair of pins to be connected. Two such path-routes for different pairs of pins are said to *interfere* if and only if, when paths following these routes are etched on the same board, the paths touch and establish an unwanted electrical connection. The problem is to choose one path-route for each pair of pins to be connected and to assign each path-route to a board so that on any one board no two path-routes interfere and so that the number of boards used is the minimum. Think of the set of all path-routes for all pairs of pins to be connected as X. Each set of predescribed routes for a pair of pins to be connected is a set of the partition p. xRy means path-routes x and y do not interfere. Then a solution to this physical problem is a solution to the generalization of the abstraction of the map coloring problem.

A procedure was sought by which a large scale computer could compute a solution of this so-called etched interconnection problem. However, the number of R-compatible sets and the number of combinations of R-compatible sets to be examined in the quest for a solution to a problem of worthwhile proportions can easily exceed the capacities of all existing computers. The argument below will show that attention may be confined to only the *maximal* R-compatible sets—those that are not proper subsets of other R-compatible sets.

Let $t = \{T_1, \cdots, T_n\}$ be a smallest set of maximal R-compatible sets whose union contains at least one element from each $P \in p$. Form the set C by choosing one element of each $P \in p$ that is also in some T_i. Let $t' = \{T_1', \cdots, T_n'\}$ where $T_i' = T_i \cap C$. Let $t'' = \{T_1'', \cdots, T_n''\}$ where $T_i'' = T_i' - \bigcup_{j=1}^{i-1} T_j'$. The set of nonnull sets in t'' (actually no T_i'' is empty, as this argument will prove) is a set of nonoverlapping R-compatible sets whose union contains exactly one element of each $P \in p$. If m is the number of R-compatible sets in a solution, then $m \leqq n$. Now let $s = \{S_1, \cdots, S_m\}$ be a solution. Choose the maximal R-compatible sets $M_1, M_2, \cdots, M_m$ so that each $S_i \subset M_i$. Then the set of all distinct M_i (actually it is easy to prove that all are distinct) is a set of maximal R-compatible sets whose union contains at least one element from each $P \in p$. Since t is a smallest set with this property, $n \leqq m$. Hence, any smallest set t of maximal R-compatible sets whose union contains at least one element from each $P \in p$ has the same number of sets as a solution and from t a solution may be constructed.

Despite the significant percentage decrease from the number of sets of R-compatible sets to the number of sets of maximal R-compatible sets that the preceding argument affords and despite an impressive statistic that we will quote below, there is still some doubt whether a present-day computer could handle the immensities of a worthwhile etched interconnection problem by means of the ideas stated here. A procedure that has been programmed and is under study will be illustrated here by means of the sample map in Figure 1.

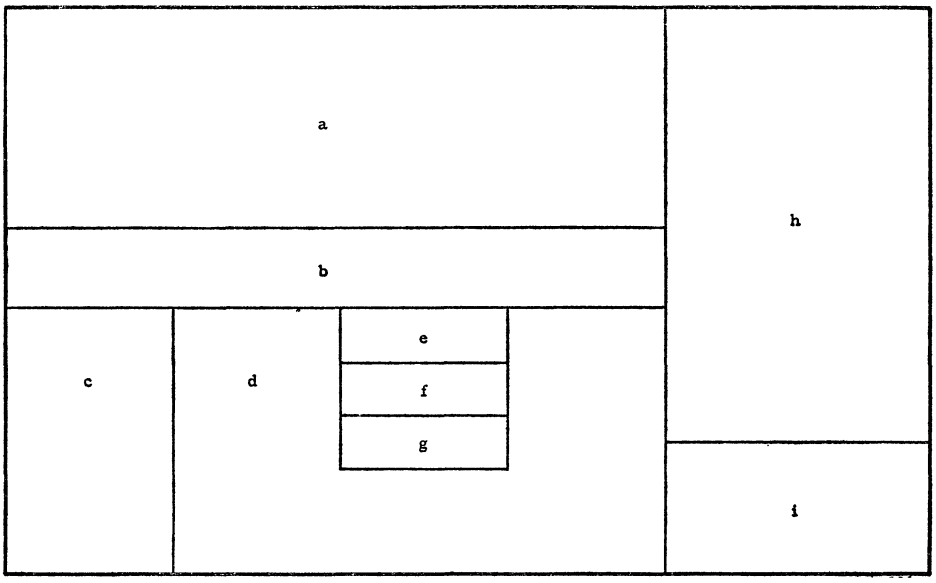


FIG. 1

In this procedure a Boolean conversion determines the complements of all maximal R-compatible sets and the same type of Boolean conversion determines a proper set of maximal R-compatible sets from which a solution may be constructed.

The Boolean expression

$$(1) \qquad (b + a)(c + b)(d + bc)(e + bd)(f + de)(g + df)(h + abd)(i + dh)$$

results from the observations that in the map of Figure 1, a precedes b in alphabetical ordering and a and b have a common boundary ($a\overline{R}b$), b precedes and has a common boundary with c, b and c precede and have a common boundary with d, etc. Expression (1) is equal to the following standard nonredundant form of a sum of products

$$(2) \qquad bdfh + abdfi + bdegh + abdegi + bcefghi + acdegh + acdefh.$$

For each product in (2) (like $abdegi$), the set of all regions not in that product

$(\{c, f, h\})$ is a maximal R-compatible set. Furthermore, these are the only maximal R-compatible sets.

To prove this in general, prescribe some nonreflexive linear (in the example, alphabetical) ordering for X. Let

$$\overline{R} = \{(x_1, y_1^1), \cdots, (x_1, y_1^{f(1)}), \cdots, (x_n, y_n^1), \cdots, (x_n, y_n^{f(n)})\},$$

where the subscripts and superscripts are assigned so that x_i precedes x_j in the prescribed ordering if and only if $i<j$, each y_i^j precedes x_i, and y_k^i precedes y_k^j if and only if $i<j$. Let C be an arbitrary R-compatible set. Then the following formal logical statement (where " $\vee$ " is "or" and " $\cdot$ " is "and") is obviously true.

$$(x_1 \notin C \vee y_1^1 \notin C). \cdots .(x_1 \notin C \vee y_1^{f(1)} \notin C). \cdots .(x_n \notin C \vee y_n^1 \notin C) \cdots .(x_n \notin C \vee y_n^{f(n)} \notin C). \tag{3}$$

By the distributive and associative rules, (3) is logically equivalent to the following which is analogous to (1).

$$(x_1 \notin C \vee (y_1^1 \notin C. \cdots .y_1^{f(1)} \notin C)). \cdots .(x_n \notin C \vee (y_n^1 \notin C \cdots .y_n^{f(n)} \notin C)). \tag{4}$$

Now by elementary considerations, (4) is logically equivalent to a statement of form (5) below, the standard nonredundant OR of AND's and the analog of (2).

$$(z_1^1 \notin C. \cdots .z_1^{g(1)} \notin C) \vee \cdots \vee (z_p^1 \notin C. \cdots .z_p^{g(p)} \notin C). \tag{5}$$

Statement (5) has the following properties as a result of the elementary considerations by which it is formed. First, for each "$(x_j \notin C \vee y_j^k \notin C)$" in (3) and for each "$(z_m^1 \notin C. \cdots .z_m^{g(m)} \notin C)$" in (5), either x_j is some z_m^i or y_j^k is some z_m^i. Second (and this is nonredundancy), no set $\{z_i^1, \cdots, z_i^{g(i)}\}$ is a subset of any other set $\{z_j^1, \cdots, z_j^{g(j)}\}$. By elementary set theory, (5) is equivalent to

$$C \subset (X - \{z_1^1, \cdots, z_1^{g(1)}\}) \vee \cdots \vee C \subset (X - \{z_p^1, \cdots, z_p^{g(p)}\}). \tag{6}$$

Since C is an arbitrary R-compatible set, every maximal R-compatible set is a subset of some $X - \{z_i^1, \cdots, z_i^{g(i)}\}$. Each $X - \{z_i^1, \cdots, z_i^{g(i)}\}$ is R-compatible; for if aRb and both a and b are in $X - \{z_i, \cdots, z_i^{g(i)}\}$, then neither a nor b is in $\{z_i^1, \cdots, z_i^{g(i)}\}$, then by the first property of statement (5) above "$(a \notin C \vee b \notin C)$" does not appear in (3) contrary to its definition. If each $X - \{z_i^1, \cdots, z_i^{g(i)}\}$ is not a maximal R-compatible set as well as an R-compatible set, then it is a subset of some other $X - \{z_j^1, \cdots, z_j^{g(j)}\}$ contrary to the second property of statement (5). Hence, the set of all $X - \{z_i^1, \cdots, z_i^{g(i)}\}$ is the set of all maximal R-compatible sets.

Returning to the sample map of Figure 1, the complete list of its maximal R-compatible set of regions as computed from (2) appears in the right column below.

$$
\begin{aligned}
bdfh &\longrightarrow A = \{a, c, e, g, i\} \\
abdfi &\longrightarrow B = \{c, e, g, h\} \\
bdegh &\longrightarrow C = \{a, c, f, i\} \\
abdegi &\longrightarrow D = \{c, f, h\} \\
bcefghi &\longrightarrow E = \{a, d\} \\
acdegh &\longrightarrow F = \{b, f, i\} \\
acdefh &\longrightarrow G = \{b, g, i\}.
\end{aligned}
$$

From the fact that region a is in sets A, C, and E, region b is in sets F and G, region c is in sets A, B, C, and D, etc., we set up

$$
(7) \qquad \begin{aligned}(A + C + E)(F + G)(A + B + C + D)E(A + B)(C + D + F) \\ \cdot(A + B + G)(B + D)(A + C + F + G).\end{aligned}
$$

In a general problem (remember this example has a trivial partition) each factor of (7) corresponds to a set of the partition p and is the sum of all maximal R-compatible sets containing at least one member of that set. Expression (7) is converted to the same standard form

$$
(8) \qquad ADEF + BEF + ADEG + BCEG + BDEG.
$$

It is obvious that the sets $\{A, D, E, F\}$, $\{B, E, F\}$, $\{A, D, E, G\}$, $\{B, C, E, G\}$, and $\{B, D, E, G\}$ are the minimal sets of maximal R-compatible sets whose unions contain at least one element from each $P \in p$. Therefore, from $\{B, E, F\}$ a solution in three colors can be constructed but in this example that solution is $\{B, E, F\}$ itself, as shown in Figure 2. Incidentally, from $\{A, D, E, F\}$, for example, the sixteen 4-color schemes of Figure 3 can be constructed.

Unless the reader is unusually interested and enjoys excessive leisure time, he is advised not to attempt this procedure on any problem where X contains more than 10 or 11 elements. I once attempted to compute, by hand and by the above Boolean conversion, all maximal R-compatible sets of a problem where X contained 32 elements. I gave up after a few hours with the task half finished. The IBM 7090, well programmed by our Autonetics staff, did the job and found the 186 maximal R-compatible sets in less than 3 **seconds**. An unsolved problem that would aid both human and computer computation is how to prescribe either aRb or $a\bar{R}b$ for any two elements a and b in the same $P \in p$ so that the number of maximal R-compatible sets is the minimum. Notice that the structure of R within each $P \in p$ (indeed, whether R is reflexive or not) has no effect on a solution. Notice also that if computation becomes oppressive for either human or

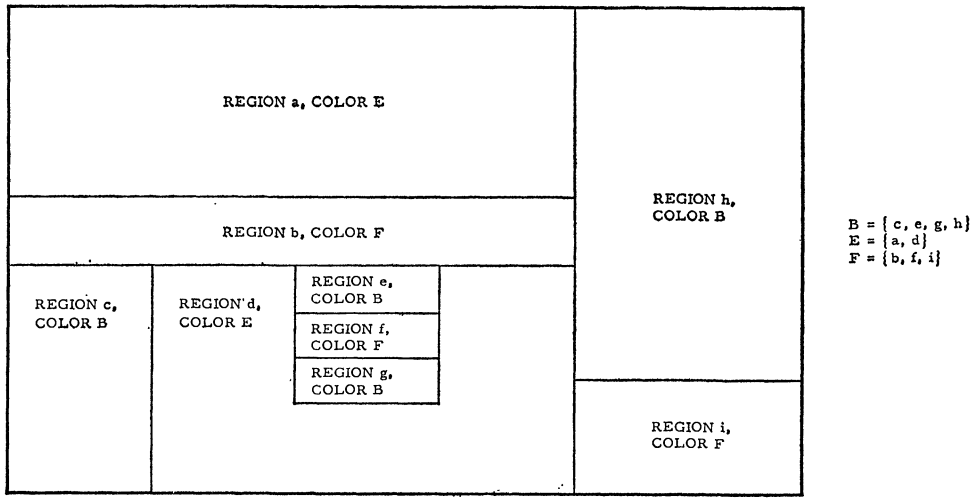


FIG. 2.

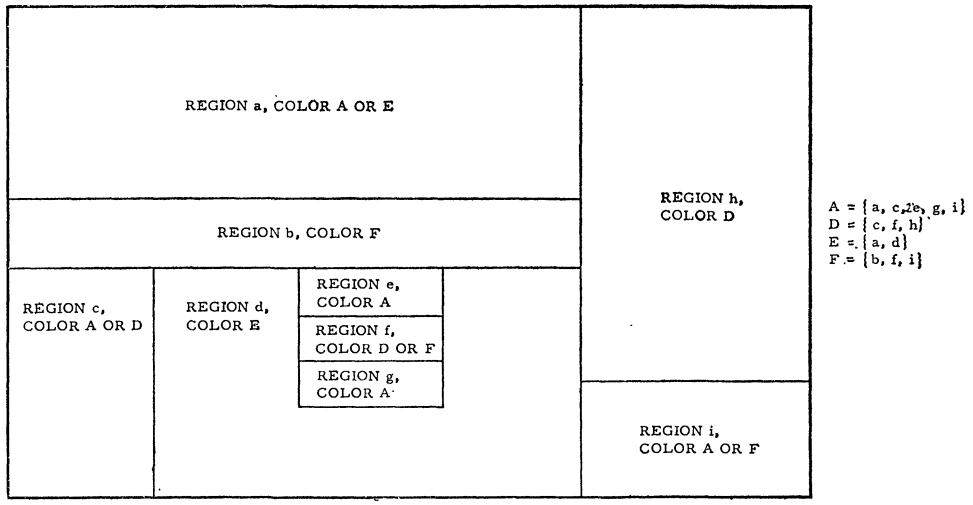


FIG. 3

computer during the first Boolean conversion, he, she, or it may stop at a point (say after finding in the example above that the product of the first six factors of (1) equals $bdf+bdeg+bcefg+acdeg+acdef$) and then get a solution for a subproblem (the submap consisting of only regions a through g) for whatever it is worth.

FIVE POINTS IN THREE-SPACE

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1. Notations. Given a tetrahedron $(T)=DABC$ and a point M, the lines DM, AM, BM, CM meet the respectively opposite faces of (T) in the vertices of the cevian tetrahedron $(T')=D'A'B'C'$ of M for (T).

Three lines MUX, MVY, MWZ drawn through M to meet the three pairs of opposite edges DA, BC; DB, CA; DC, AB determine on those edges the pairs of points U, X; V, Y; W, Z.

The vertices D', A', B', C' of (T') have XYZ, XVW, YWU, ZUV for their cevian triangles with respect to the corresponding faces of (T).

The tetrahedral polar plane, or, briefly, the harmonic plane $\mu=U'V'W'X'Y'Z'$ of M for (T) determines on the edges DA, DB, DC; BC, CA, AB of (T) the points U', V', W'; X', Y', Z' harmonically separated from the points $U, \cdots, Z$ by the respective pairs of vertices of (T) [1; pp. 230–232].

The lines $s=X'Y'Z'$, $p=X'V'W'$, $q=Y'W'U'$, $r=Z'U'V'$ are the harmonic lines of the vertices D', A', B', C' of (T') for the respective faces of (T).

2. The plane μ (§1) is the plane of perspectivity of the two tetrahedrons (T), (T') and passes through the lines s, p, q, r [1; p. 235, §720].

The point D' is a vertex of the cevian tetrahedron of the point D for the tetrahedron $MABC$, hence the harmonic plane δ of D for $MABC$ passes through the line s.

For analogous reasons the harmonic planes α, β, γ of the points A, B, C for the tetrahedrons $MBCD$, $MCDA$, $MDAB$, pass through the lines p, q, r, respectively.

Thus the four pairs of planes δ, ABC; α, DBC; β, DCA; γ, DAB intersect along the four lines s, p, q, r which lie in the plane μ, hence: The two tetrahedrons $\delta\alpha\beta\gamma$, (T) are perspective.

The plane of perspectivity is the harmonic plane of M for (T).

3a. The line $DA'X$ passes through D and meets the pair of opposite edges MA, BC of the tetrahedron $MABC$ (§2) in the points A', X, respectively, hence the harmonic plane δ of D for $MABC$ passes through the harmonic conjugate D_a of D with respect to A', X [1; p. 230]. Moreover, the face $A'B'C'$ of (T') passes through the line s [1; p. 233, §717].

Thus the four planes of the pencil $s(D, \delta, ABC, A'B'C')$ cut the line DX in the four harmonic points D, D_a, X, A', hence the four points D, S, D', D'' which that pencil of planes determines on the line DM are also harmonic, that is, the point S which the plane $\delta=s-D_a$ determines on the line DM is separated harmonically from D by the traces D', D'' which DM determines in the faces ABC, $A'B'C'$ of (T), (T'), or, in other words, the harmonic plane δ of D for the tetrahedron $MABC$ is the harmonic conjugate of the plane $s-D$ with respect to the faces ABC, $A'B'C'$ of (T), (T').

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b. A similar argument may show that the plane α is the harmonic conjugate of the plane pA with respect to the faces DBC, $D'B'C'$ of (T), (T'). Now the latter two planes obviously meet the line DMD' in the points D, D', hence if D_0 is the trace of pA on DM, the harmonic conjugate D^* of D_0 for D, D' is the trace of α on DM.

The plane pA is a face of the anticevian tetrahedron (T_0) of M for (T) [2; p. 89], hence the trace D_0 of pA on the line DM is a vertex of (T_0). Consequently the planes qB, rC will also pass through D_0 and therefore the planes β, γ will pass through D^*, and D^* is therefore the vertex of the tetrahedron $\alpha\beta\gamma\delta$ opposite the face δ.

The vertex D of the tetrahedron (T) and the corresponding vertex D^* of the tetrahedron $(T^*) = \delta\alpha\beta\gamma = D^*A^*B^*C^*$ are thus aligned with the point M, and the same holds for the other pairs of corresponding vertices of those two tetrahedrons, for analogous reasons. Hence: The center of perspectivity of the two perspective tetrahedrons (T), $\delta\alpha\beta\gamma$ (§2) is the point M.

4. LEMMA. *If five collinear points P, Q, R, S, Z are connected by the two biratios (that is, anharmonic ratios) $(PQRZ) = r$, $(PQSZ) = s$, then*

$$(PQRS) = r:s. \tag{1}$$

Indeed, expanding the two given biratios, we have

$$(PR:RQ):(PZ:ZQ) = r, \qquad (PS:SQ):(PZ:ZQ) = s.$$

Dividing the first equality by the second we obtain on the left hand side the expansion of (1).

5. The two planes ABC, δ are homologous faces of the two tetrahedrons (T), (T^*) which correspond to each other in the perspectivity (M, μ) (§§2, 3). Now the line MD meets those two planes in the points D', S, hence if L is the trace of DM in the plane μ, the value of the biratio $(MLD'S)$ is the constant of the homology of (M, μ).

To evaluate that constant we observe that we have (1) $MLDD' = -3$, (2) $MLD'D'' = -3$ [2; p. 89], (3) $(DSD'D'') = -1$ (§3a). Replacing -3 by n, for convenience, and using the lemma (§4), we may eliminate L between (1) and (2), as follows [3; pp. 59–61]:

$$\begin{aligned} MLDD' &= n & MLD'D'' &= n \\ MLD'D &= 1/n & MD'LD'' &= 1 - n \\ MD'LD &= (n-1)/n & MD'D''L &= -1/(n-1) \\ MD'DL &= n/(n-1) \end{aligned}$$

hence

$$MD'DD'' = -n. \tag{4}$$

Eliminating D'' between (4) and (3) we obtain

$$(DD'MS) = n/2(n+1). \tag{5}$$

Finally, eliminating D between (5) and (1) we obtain

$$MLD'S = (2n+1)/(n+2), \tag{6}$$

which for $n=-3$ is equal to 5.

Considering the pair of homologous vertices D, D^*, the same result may be obtained by making use of the three relations: (1) $(MLD_0D) = -3$, (2) $(MLDD') = -3$, (3) $(D_0D^*DD') = -1$.

6a. Summarizing the results obtained in §§2, 3, 5 we may formulate the

THEOREM. *Given a tetrahedron (T) and a point M, if the harmonic plane is constructed of each vertex of (T) for the tetrahedron determined by the three remaining vertices of (T) and the point M, the four planes thus obtained form a tetrahedron perspective to (T). The center and plane of this perspectivity are the point M and its harmonic plane for (T). The constant of the perspectivity is equal to* 5.

b. The proposition remains valid if, instead of a point M, a plane μ is given. Indeed, the relation between the point M and its harmonic plane μ for (T) is a one-to-one reciprocal correspondence.

7. Some interesting results may be obtained by taking for the point M a "remarkable" point associated with the given tetrahedron (T). Thus, if M coincides with the centroid G of $(T) = DABC$, the harmonic plane of G for (T) is the plane at infinity, so that the faces of the tetrahedron (T^*) are parallel to those of (T), that is, (T) and (T^*) are homothetic.

The anticevian tetrahedron of G for (T) coincides with the anticomplementary tetrahedron $(T_0) = D_0A_0B_0C_0$ of (T), and the cevian tetrahedron of G for (T) has for its vertices the centroids of the faces of (T). Thus the vertex D^* of (T^*) situated on the line D_0DG_d is the harmonic conjugate of D_0 with respect to the points D, G_d, and so on. Thus we may state the following

THEOREM. *Given a tetrahedron (T), if for each of its vertices the harmonic plane is constructed with respect to the tetrahedron formed by the three remaining vertices of (T) and the centroid of (T), the four planes thus obtained form a tetrahedron homothetic to (T), and the two tetrahedrons have the same centroid. Their homothetic ratio is equal to* 5.

In the present case the interested reader may readily derive the value of this constant directly.

8. In the proposition of §6a, the point M plays a different role from the role played by the vertices of (T). But M is any point in space, independent of the vertices of (T). Our proposition is therefore about five points in space to one of which we assigned a special role. This role may equally well be assigned to any one of the five points. We are thus led to the following conclusion:

THEOREM. *Given five points in space, if for each of them we construct the harmonic plane with respect to the tetrahedron determined by the remaining four, the five planes thus obtained are such that any four of them form a tetrahedron perspective to the tetrahedron determined by their four corresponding points. The center and plane of perspectivity are in each case the remaining point and the remaining plane. The constants of the five perspectivities considered are equal, their common numerical value being* 5.

9a. The constructions which determine the harmonic plane μ of a point M for a tetrahedron (T) are projective, and the relation of M and μ with respect to (T) is one-to-one and is reciprocal [1; pp. 230–232]. Hence if the figure M, μ, (T) is subjected to a projective transformation, the transforms of M, μ will be harmonically associated with respect to the transform of (T), that is, *the harmonic relation of a point and a plane for a tetrahedron is projective* [4].

b. On the other hand the constructions and the arguments used in arriving at the proposition of §8 are projective, hence this proposition is itself projective and therefore may be dualized. We have thus the following

THEOREM. *Given five planes, if for each we construct the harmonic pole with respect to the tetrahedron determined by the remaining four planes, the five points thus obtained are such that any four of them and their four corresponding planes form two tetrahedrons which are perspective. The center and plane of perspectivity are the remaining fifth point and the remaining fifth plane, the constant of perspectivity in each case being equal to* 5.

10a. As an application of §8 consider five points forming an orthocentric group [1; p. 272, §826]. Each point of the group has for its harmonic plane for the tetrahedron determined by the remaining four points, the orthic plane of that tetrahedron [1; p. 265, §811 b.]. That same proposition may be worded differently, using the property that the point and its orthic plane are pole and polar with respect to the first twelve-point sphere of the orthocentric tetrahedron [1; §808].

b. By starting with the orthic planes the above propositions may be worded as illustrations of the proposition §9. The reader may find it interesting to formulate those propositions explicitly.

Conjecture. The proposition which corresponds, in the plane, to §8, is valid [5]. The proposition may, presumably, hold for $n+2$ points in an n-dimensional Euclidean space.

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TOTAL DIFFERENTIAL EQUATIONS IN THE LIGHT OF DIMENSIONAL ANALYSIS

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1. Introduction. We shall write the total differential equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0, \tag{1}$$

in vector form

$$\mathbf{f}\cdot d\mathbf{r} = 0, \tag{1}$$

where $\mathbf{r} = [x, y, z]$ and the vector $\mathbf{f} = [P, Q, R]$ has continuous first partial derivatives in a rectangular region $\mathfrak{R}$. Equation (1) is exact if and only if $\mathbf{f} = \nabla\varphi$, a gradient vector; and its solution is then $\varphi(x, y, z) = \text{const}$. The condition $\mathbf{f} = \nabla\varphi$ implies rot $\mathbf{f} = \mathbf{0}$; and conversely if rot $\mathbf{f} = \mathbf{0}$, the scalar φ is given by

$$\varphi(x, y, z) = \int_{r_0}^{r} \mathbf{f}\cdot d\mathbf{r}, \tag{2}$$

where the path of integration is an arbitrary curve of $\mathfrak{R}$ joining $\mathbf{r}_0$ to $\mathbf{r}$. It is usually advantageous to integrate over one of the six step paths joining $\mathbf{r}_0$ and $\mathbf{r}$. Thus for the path

$$(x_0, y_0, z_0) - (x_0, y_0, z) - (x_0, y, z) - (x, y, z)$$

we have

$$\varphi(x, y, z) = \int_{x_0}^{x} P(t, y, z)dt + \int_{y_0}^{y} Q(x_0, t, z)dt + \int_{z_0}^{z} R(x_0, y_0, t)dt,$$

where the integrals are written in reverse order. The limits x_0, y_0 are chosen to simplify the following integrands as much as possible.

If rot $\mathbf{f} \neq \mathbf{0}$, equation (1) is integrable if and only if

$$\mathbf{f}\cdot\text{rot}\,\mathbf{f} = 0. \tag{3}$$

It then admits multipliers (integrating factors) λ such that $\lambda\mathbf{f}\cdot d\mathbf{r} = 0$ is exact or

$$\text{rot}\,(\lambda\mathbf{f}) = \nabla\lambda \times \mathbf{f} + \lambda\,\text{rot}\,\mathbf{f} = \mathbf{0}. \tag{4}$$

If μ is a second multiplier,

$$\nabla\mu \times \mathbf{f} + \mu\,\text{rot}\,\mathbf{f} = \mathbf{0}. \tag{5}$$

Multiply (4) by μ, (5) by λ and subtract the equations: then

$$(\mu\nabla\lambda - \lambda\nabla\mu) \times \mathbf{f} = \mathbf{0}$$

or, on division by μ^2,

$$\nabla\left(\frac{\lambda}{\mu}\right) \times \mathbf{f} = \mathbf{0}. \tag{6}$$

Thus $\mathbf{f}$ is parallel to $\nabla(\lambda/\mu)$ and (1) is equivalent to $d(\lambda/\mu)=0$ and admits λ/μ $=$const. as an integral.

2. Isobaric Equations. We now regard the variables x, y, z as having the dimensions U^a, U^b, U^c relative to some unit U. Then equation (1) is said to be *isobaric* (Greek *isos* equal, *baros* heavy) if each term has the same dimensions (or "weight"). Thus the equation (see [1])

$$2xzdx + 2yz^2dy + (x^2 + 2y^2z - 1)dz = 0$$

is isobaric when

$$a = \text{wt}\, x = 0, \qquad b = \text{wt}\, y = -1, \qquad c = \text{wt}\, z = 2;$$

for the last term implies $a=0$, $2b+c=0$; and the weights of the terms will be equal if

$$c = 2b + 2c \quad \text{or} \quad 2b + c = 0.$$

If an equation is isobaric when $a=b=c=1$, it is said to be *homogeneous* (Greek *homos*, same, *genos* kind).

3. Multiplier for Homogeneous Equation.

THEOREM 1. *If the integrable equation* $\mathbf{f}\cdot d\mathbf{r}=0$ *is homogeneous it admits the multiplier*

$$\lambda = 1/(\mathbf{r}\cdot\mathbf{f}) = 1/(xP + yQ + zR) \qquad \text{if } \mathbf{r}\cdot\mathbf{f} \neq 0.* \tag{7}$$

Proof. Since P, Q, R are homogeneous functions of the same degree (say n),

$$\mathbf{r}\cdot\nabla P = nP, \qquad \mathbf{r}\cdot\nabla Q = nQ, \qquad \mathbf{r}\cdot\nabla R = nR$$

from Euler's theorem. The vectorial analogue of this theorem is

$$\mathbf{r}\cdot\nabla f = \mathbf{r}\cdot(\nabla P\mathbf{i} + \nabla Q\mathbf{j} + \nabla R\mathbf{k}) = n\mathbf{f}.$$

We now show that $\lambda=1/(\mathbf{r}\cdot\mathbf{f})$ satisfies the multiplier equation (4). We have (see [2])

$$\nabla(\lambda^{-1}) = \nabla(\mathbf{r}\cdot\mathbf{f}) = \mathbf{r}\cdot\nabla\mathbf{f} + \mathbf{f}\cdot\nabla\mathbf{r} + \mathbf{f}\times\text{rot}\,\mathbf{r} + \mathbf{r}\times\text{rot}\,\mathbf{f}\cdot$$

or since $\nabla\mathbf{r}=\mathrm{I}$ (the idemfactor) and rot $\mathbf{r}=\mathbf{0}$,

$$-\lambda^{-2}\nabla\lambda = (n+1)\mathbf{f} + \mathbf{r}\times\text{rot}\,\mathbf{f},$$

$$-\lambda^{-2}\nabla\lambda\times\mathbf{f} = (\mathbf{r}\times\text{rot}\,\mathbf{f})\times\mathbf{f} = \lambda^{-1}\,\text{rot}\,\mathbf{f}$$

since $\mathbf{r}\cdot\mathbf{f}=\lambda^{-1}$ and $\mathbf{f}\cdot\text{rot}\,\mathbf{f}=0$, the integrability condition. But the last equation is equivalent to (4).

* When $\mathbf{r}\cdot\mathbf{f}=0$, the homogeneous equation $\mathbf{f}\cdot d\mathbf{r}=0$ is always integrable. For $\mathbf{r}\cdot\mathbf{f}=0$ implies $\nabla(\mathbf{r}\cdot\mathbf{f})=\mathbf{f}\cdot\nabla\mathbf{r}+\mathbf{r}\times\text{rot}\,\mathbf{f}+\mathbf{r}\cdot\nabla\mathbf{f}+\mathbf{f}\times\text{rot}\,\mathbf{r}=\mathbf{0}$; or since $\nabla\mathbf{r}=\mathbf{I}$, the idemfactor, rot $\mathbf{r}=\mathbf{0}$, and $\mathbf{r}\cdot\nabla\mathbf{f}=n\mathbf{f}$ for a homogeneous equation of degree n, $\nabla(\mathbf{r}\cdot\mathbf{f})=(n+1)\mathbf{f}+\mathbf{r}\times\text{rot}\,\mathbf{f}=\mathbf{0}$. If $n+1\neq 0$, we have on dot multiplying by rot $\mathbf{f}$, $\mathbf{f}\cdot\text{rot}\,\mathbf{f}=0$; if $n+1=0$, we have $\mathbf{r}\times\text{rot}\,\mathbf{f}=\mathbf{0}$ or rot $\mathbf{f}=\lambda\mathbf{r}$ and again $\mathbf{f}\cdot\text{rot}\,\mathbf{f}=\lambda\mathbf{f}\cdot\mathbf{r}$ $=0$. Thus the integrability condition (3) is satisfied in either case. See [2].

4. Multiplier for Isobaric Equation. Let equation (1) be isobaric when

$$\text{wt } x = a, \qquad \text{wt } y = b, \qquad \text{wt } z = c.$$

Case 1. $a, b, c \neq 0$. Change to the variables X, Y, Z given by

$$x = X^a, \qquad y = Y^b, \qquad z = Z^c;$$

then wt $X = $ wt $Y = $ wt $Z = 1$ and the transformed equation

$$aPX^{a-1}dX + bQY^{b-1}dY + cRZ^{c-1}dZ = 0$$

is homogeneous. By Theorem 1 it admits a multiplier λ whose reciprocal is

$$aX^aP + bY^bQ + cZ^cR,$$

that is

$$\lambda = 1/(axP + byQ + czR). \tag{8}$$

Case 2. $a, b \neq 0, c = 0$. Change to the variables X, Y, Z given by

$$x = X^a, \qquad y = Y^b, \qquad z = Z/X;$$

then wt $X = $ wt $Y = $ wt $Z = 1$ and the transformed equation

$$aPX^{a-1}dX + bQY^{b-1}dY + R\left(\frac{dZ}{X} - \frac{ZdX}{X^2}\right) = 0$$

is again homogeneous. By Theorem 1 it admits the multiplier λ whose reciprocal is

$$\left(aX^aP - \frac{ZR}{X}\right) + bY^bQ + \frac{ZR}{X};$$

that is

$$\lambda = 1/(axP + by\,Q). \tag{9}$$

Case 3. $a \neq 0, b = c = 0$. Change to the variables X, Y, Z given by

$$x = X^a, \qquad y = Y/X, \qquad z = Z/X;$$

then wt $X = $ wt $Y = $ wt $Z = 1$ and the transformed equation

$$aPX^{a-1}dX + Q\left(\frac{dY}{X} - \frac{YdX}{X^2}\right) + R\left(\frac{dZ}{X} - \frac{ZdX}{X^2}\right) = 0$$

is homogeneous and admits a multiplier λ whose reciprocal is

$$\left(aX^aP - \frac{YQ}{X} - \frac{ZR}{X}\right) + \frac{YQ}{X} + \frac{ZR}{X};$$

that is

$$\lambda = 1/axP. \tag{10}$$

All cases may be combined in

THEOREM 2. *If the integrable equation* $\mathbf{f}\cdot d\mathbf{r}=0$ *is isobaric when* x, y, z *have the respective weights* a, b, c, *not all zero, it admits the multiplier*

$$\lambda = 1/(axP + byQ + czR),$$

provided the denominator is not zero.

5. Examples. Nearly all total differential equations that appear in text books have multipliers given by Theorem 2.

Example 1 (see [3]). The equation

$$2xyzdx + z(1 - yz^2)dy + y(3 - 2yz^2)dz = 0$$

is isobaric when wt $x=0$, wt $y=2$, wt $z=-1$; hence a multiplier λ is given by

$$\lambda^{-1} = 2yz(1 - yz^2) - yz(3 - 2yz^2) = -yz.$$

Multiply by $1/yz$ and integrate the exact equation

$$2xdx + (1 - yz^2)\frac{dy}{y} + (3 - 2yz^2)\frac{dz}{z} = 0;$$

$$x^2 + \int_1^y \left(\frac{1}{y} - z^2\right)dy + \int_1^z \left(\frac{3}{z} - 2z\right)dz = \text{const.},$$

whence

$$\log(yz^3) + x^2 - yz^2 = C.$$

Example 2 (see [4]). The equation

$$(1 + yz)dx + x(z - x)dy - (1 + xy)dz = 0$$

is isobaric when wt $x=1$, wt $y=-1$, wt $z=1$; and

$$\lambda^{-1} = x(1 + yz) - xy(z - x) - z(1 + xy) = (x - z)(1 + xy).$$

The integral is $(1+xy)/(x-z)=C$.

Example 3 (see [5]). The equation

$$(2xz^2 + y)dx + xdy + (2x^2z + 3z^2)dz = 0$$

is exact. It is also isobaric when wt $x=1$, wt $y=5$, wt $z=2$; then each term has the weight 6. Hence $\mu=1$,

$$\begin{aligned}\lambda^{-1} &= x(2xz^2 + y) + 5xy + 2z(2x^2z + 3z^2)\\ &= 6(x^2z^2 + xy + z^3).\end{aligned}$$

Its integral is therefore $\mu/\lambda = \text{const.}$ or $x^2z^2+xy+z^3 = C$.

Example 4. The isobaric equation in §2 has the multiplier λ where

$$\lambda^{-1} = -2y^2z^2 + 2z(x^2 + 2y^2z - 1) = 2z(x^2 + y^2z - 1).$$

Example 5. A method of integrating certain exact equations was given by Pennisi [8] in this Monthly and applied to the equation

$$z^2(y^2 - 1)xdx + z^2(x^2 + 1)ydy + (x^2y^2 - x^2 - y^2 - 1)zdz = 0.$$

This equation is isobaric when wt $x = $ wt $y = 0$, wt $z = 1$, and its integral is

$$\lambda^{-1} = (x^2y^2 - x^2 + y^2 - 1)z^2 = \text{const.}$$

Example 6. The theorem also applies to integrable equations with more than three variables. For example Forsyth [9] integrates the equation

$$(2x + y^2 + 2xw - z)dx - 2xydy - xdz + x^2dw = 0$$

by a method applicable to integrable equations. This equation is exact and also isobaric when wt $x = $ wt $z = 2$, wt $y = 1$, wt $w = 0$. Its integral is therefore immediate:

$$\lambda^{-1} = 2x(2x + y^2 + 2xw - z) + 2xy^2 - 2xz = 4(x^2 + xy^2 + x^2w - xz) = \text{const.}$$

6. Ordinary Equations. Theorem 2 also gives a multiplier λ for ordinary isobaric equations

$$P(x, y)dx + Q(x, y)dy = 0; \tag{11}$$

since for these the integrability condition (3) is always fulfilled. Such equations may have other known multipliers μ; and in this event the integral is given by $\mu/\lambda = \text{const.}$ For example the equation (see [6])

$$x^p(Aydx + Bxdy) + y^q(Cydx + Dxdy) = 0$$

always has a multiplier of the form $\mu = x^my^n$ if $AD - BC \neq 0$. But this equation is obviously isobaric when wt $x = q$, wt $y = p$, and hence admits a second multiplier λ. Its integral may therefore be written down at once.

Finally certain multipliers deduced by the aid of Lie's theory of continuous transformation groups follow readily from Theorem 2 (see [7]):

Equation	wts x, y	λ^{-1}
$xy' = yF(xy)$	1, −1	$xy\{F(xy) + 1\}$
$yy' = F(y^2/x)$	2, 1	$2xF(y^2/x) - y^2$
$y' = xF(y/x^2)$	1, 2	$x^2F(y/x^2) - 2y$

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A THEOREM ON n-COLORING THE POINTS OF A LINEAR GRAPH

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For the definitions, see [1].

We say that the points of an (unoriented) linear graph are *n-colored* if they are partitioned into n sets (some of which may be empty) so that, for each edge of the graph, the end-points of this edge are in different sets. In an *oriented* graph, consider a cycle (simple closed curve). I shall use the term *flow-ratio* of the cycle to mean the ratio m/n, where m and n are the numbers of edges of the cycle directed one way and the other around the cycle and $m \geqq n$. (The flow-ratio may be $+\infty$.)

THEOREM. *The points of a graph can be n-colored if and only if there exists an orientation of the graph such that the flow-ratio of each cycle does not exceed* $n-1$.

Proof. First I show the necessity of the condition. Assume that the points are n-colored, using the integers $0, 1, \cdots, (n-1)$ as colors (labels). Orient each edge in the direction from the smaller label to the greater. I now leave it to the reader to show that each cycle has flow-ratio $\leqq (n-1)$.

The sufficiency-proof is more entertaining. Let us first consider the case where the graph is *finite* and *connected.* Assume an orientation satisfying the flow-ratio condition; distinguish one point, called *the starting point*, and label it with the integer 0. Label some other point with the letter p.

I assign to a traveller the problem of finding a path from the starting point to p which maximizes his gain, subject to the rules: he is paid \$1 for each road (edge) he traverses *with* its orientation, and is fined $\$(n-1)$ for each road he traverses *opposite* to its orientation. I now claim that this maximum gain *exists*, for: any path P which is not a *simple* path contains a *cycle.* Because of the flow-ratio condition, this cycle cannot contribute to the "gain" $G(P)$ of P, and can be deleted from P, yielding a path P_1 of equal or greater gain. Again, if P_1 is not simple, a cycle can be deleted, etc., until a *simple* path P_n is found whose gain is not less than that of P. Thus, in determining maximum gain, nonsimple paths can be omitted from consideration. But there are only finitely many simple paths

from the distinguished point to p, so that the maximum gain *exists* (and is obviously an integer).

Now let *each* point p be labelled with the maximum gain $g(p)$; the distinguished point is still labelled 0. It is easy to show (by contradiction) that if p_1 and p_2 are two points joined by an edge, then

$$0 < |\, g(p_1) - g(p_2) \,| < n,$$

so that the function $g(p)$, reduced mod n, yields an n-coloring of the points. (The reader may discover for himself why it was necessary to permit nonsimple paths in the definition of $g(p)$.)

If the graph is *infinite*, a different argument must be used to establish, for each fixed p, an upper bound on the numbers $G(P)$. By the above argument, the gain of any *closed* path is nonpositive, and hence, if P_1 is *any* path from the starting-point to p, $G(P)$ is bounded above by the negative of the gain of the *reverse-path* of P_1.

If the graph is *not connected*, we choose a starting-point in each component (by the Axiom of Choice) and proceed as above.

Remark. It is well known that the 4-color problem can be re-stated as a problem of 4-coloring the *points* of a planar graph. Thus we have found an equivalent of the 4-color conjecture: *The edges of a planar graph can be oriented so that the flow-ratio of each cycle is* $\leqq 3$.

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HOW NEWTON DISCOVERED THE LAW OF GRAVITATION*

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A tremendous increase in the number of vigorous young workers in the scientific vineyard has been one of the happiest results of the recent expansion, encouraged and nourished by our Federal Government, of scientific research in this country. These neophytes, left to their own devices by harassed research directors, have often found themselves without adequate guidance through the intricacies of governmental sponsorship; but, fortunately, they can find inspiration in the story of Sir Isaac Newton, his development of the law of gravitation, and his experience as director of the Subproject for Apples of the Fruit-Improvement Project, sponsored by His Majesty's Government of Great Britain in co-operation with a syndicate of British fruit growers.

Few are familiar with the details of Newton's twenty-year search for a proof

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of his hypothesis: the frustrations and failures, the need for accurate measurements of the earth's radius and for a mathematical tool that Newton himself was forced to invent, and the integration of his scattered efforts by the splendid organization of the Fruit-Improvement Project. These details have been collected from his *Principia*, personal letters, notebooks and other papers, and a series of personal interviews arranged by a medium of the author's acquaintance.

In 1665 the young Newton became a professor of mathematics in the University of Cambridge, his alma mater. His devotion to his work and his capabilities as a teacher and friend of the student may be assumed without question. It is well to point out also that he was no dreamy, impractical inhabitant of an ivory tower. His services to his college went far beyond the mere act of classroom teaching. He was an able and active member of the college's curriculum committee, the board of the college branch of the Young Noblemen's Christian Association, the dean's advisory committee on scholarships and awards, the committee for discipline, the grounds committee, the publications committee, the *ad hoc* committee, and numerous other committees essential to the proper administration of a college in the seventeenth century. An exhaustive compilation of Newton's work along these lines reveals that, during a five-year period, he served on 379 committees, which investigated an aggregate of 7924 problems of campus life and solved 31 of them.

Newton the genius was yet a human being; and though, in energy and ability, he far surpassed the great majority of his fellow men, he found himself ultimately limited in his powers. His unselfish devotion to the important work of his committees absorbed so much time that he was constrained to turn more and more of his teaching duties over to one of his students. He reasoned, quite correctly, that the substitution of a student as teacher in his place would benefit both the student and the student's students: the former because, in teaching, his own knowledge would be enhanced; and the latter because, in being taught by one near to them in age and interests, they would more eagerly grasp at the scraps of knowledge that came their way. Newton, whose stipend was small, did not spoil this idyllic arrangement by offering pay to his student substitute: a prime example of his sense of values and his restraint. Eventually, when his substitute had proved his ability as a teacher, Newton turned all the classroom work over to him and was thus able to channel all his tremendous powers into the administrative work of the college.

At about this time Newton, whose mind was too active ever to let scientific problems recede from his attention, occasionally mulled over the great discoveries of Kepler on planetary motions and the hypothesis advanced by a number of astronomers, that these motions were governed by an attraction that varied inversely as the square of the distance between planets. One evening of a crowded day in the year 1680, a committee that was scheduled to meet at eleven o'clock, no earlier time being available, was unable to muster a quorum because of the sudden death from exhaustion of one of the older committee members. Every waking moment of Newton's time was so carefully budgeted that he

found himself with nothing to do until the next committee meeting at midnight. So he took a walk—a brief stroll that altered the history of the world.

It was on this excursion into the night air of Cambridge that Newton was struck by a flash of insight which set off a chain of events culminating in his announcement of the law of gravitation to the world in 1686. The season was autumn. Many of the good citizens in the neighborhood of the modest Newton home had apple trees growing in their gardens, and the trees were laden with ripe fruit ready for the picking. Newton chanced to see a particularly succulent apple fall to the ground. His immediate reaction was typical of the human side of this great genius. He climbed over the garden wall, slipped the apple into his pocket, and climbed out again. As soon as he had passed well beyond that particular garden, he removed the apple from his pocket and began munching it. Then came inspiration. Without prelude of conscious thought or logical process of reasoning, there was suddenly formed in his brain the idea that the falling of an apple and the motions of planets in their orbits may be governed by the same universal law. Before he had finished eating the apple and discarded the core, Newton had formulated his hypothesis of the universal law of gravitation. By then it was three minutes before midnight, so he hurried off to the meeting of the Committee to Combat Opium Eating Among Students Without Nobility.

In the following weeks Newton's thought turned again and again to his hypothesis. Rare moments snatched between the adjournment of one committee and the call to order of another were filled with the formulation of plans for testing the hypothesis. Eventually, after several years during which, according to evidence revealed by diligent research, he was able to spend 63 minutes and 28 seconds on his plans, Newton realized that the proof of his hypothesis would take more spare time than might become available during the rest of his life. He had to find accurate measurements of a degree of latitude on the earth's surface, and he had to invent the calculus.

Finally he concluded that he must find some relief from his collegiate administrative burdens. He knew that it was possible to get the King's support for a worthy research project of definite aims, provided a guarantee could be made that the project would be concluded in a definite time at a cost exactly equal to the amount stipulated when the project was undertaken. Lacking experience in these matters, he adopted a commendably simple approach and wrote a short letter of 22 words to King Charles, outlining his hypothesis and pointing out its far-reaching implications if it should prove to be correct. It is not known whether the King ever saw this letter, and he may not have, being overwhelmed with problems of state and plans for pending wars. There is no doubt that the letter was forwarded, through channels, to all heads of departments, their assistants, and their assistants' assistants, who might have reason to make comments or recommendations.

Eventually, Newton's letter and the bulky file of comments it had gathered on its travels reached the office of the secretary of HMPBRD/CINI/SSNBI—His Majesty's Planning Board for Research and Development, Committee for

Investigation of New Ideas, Subcommittee for Suppression of Non-British Ideas. The secretary immediately recognized its importance and brought it before the subcommittee, which voted to ask Newton to testify before the Committee for Investigation of New Ideas. Some discussion of Newton's idea—whether it could really be called British in intent—preceded this decision, but the transcript of the discussion, filling several quarto volumes, clearly shows that no real suspicion ever fell upon him.

Newton's testimony before HMPBRD/CINI is recommended to all young scientists who may wonder how they will comport themselves when their time comes. His college considerately granted him two months' leave without pay while he was before the committee, and the Dean of Research sent him off with a joking admonition not to come back without a fat contract. The committee hearing was open to the public and was well attended, though it has been suggested that many of the audience had mistaken the hearing room of HMPBRD/CINI for that of HMCEVAUC—His Majesty's Committee for the Exposure of Vice Among the Upper Classes.

After Newton was sworn to tell the truth and had denied that he was a member of His Majesty's Loyal Opposition, had ever written any lewd books, had traveled in Russia, or had seduced any milkmaids, he was asked to outline his proposal. In a beautifully simple and crystal-clear ten-minute speech, delivered extemporaneously, Newton explained Kepler's laws and his own hypothesis, suggested by the chance sight of an apple's fall. At this point one of the committee members, an imposing fellow, a dynamic man of action, demanded to know if Newton had a means of improving the breed of apples grown in England. Newton began to explain that the apple was not an essential part of his hypothesis, but he was interrupted by a number of committee members, all speaking at once in favor of a project to improve apples. This discussion continued for several weeks, while Newton sat in characteristic dignity waiting until the committee wished to consult him. One day he arrived a few minutes late and found the door locked. He knocked circumspectly, not wishing to disturb the committee's deliberations. The door was opened by a guard who told him there was no more room and sent him away. Newton, with his logical way of reasoning, deduced that the committee did not wish to consult him further, and forthwith he returned to his college and his important committee work.

Several months later Newton was surprised to receive a bulky package from HMPBRD/CINI. He opened the package and found it contained a variety of governmental forms, each in quintuplicate. His natural curiosity, the main attribute of the true scientist, provoked him into a careful study of the forms. After some time he concluded that he was being invited to submit a bid for a contract for a research project on the relationship between breed, quality, and rate of fall of apples. The ultimate purpose of the project, he read, was to develop an apple that not only tasted good but also fell so gently that it was not bruised by striking the ground. Now, of course, this was not what Newton had had in mind when he had written his letter to the King. But he was a practical

man and he realized that, in carrying out the proposed project, he could very well test his hypothesis as a sort of sideline or by-product. Thus, he could promote the interests of the King and do his little bit for science in the bargain.

Having made his decision, Newton began filling out the forms without further hesitation. One of the questionnaires asked how the funds allotted for the project were to be spent. Newton was somewhat taken aback to read that £12,750 6*s.* 3*d.*, the surplus remaining in the horticultural development fund for the current fiscal year, had been estimated as the total cost of his project. Methodically, he put down his own stipend first, and after a moment's thought he added the item: "Other salaries, travel, supplies, and overhead: £12,750 0*s.* 0*d.*"

A true believer in correct administrative procedures, Newton sent the completed forms by special messenger to the Dean of Research, for transmittal through proper channels to HMPBRD/CINI. His adherence to established procedure was rewarded a few days later when the Dean of Research summoned him and outlined a new plan, broader in scope and more sweeping in its conception. The Dean pointed out that not only apples but also cherries, oranges, lemons, and limes fell to the earth, and while they were about it they might as well obtain a real, man-sized government contract to cover all the varieties of fruit that grow above the ground. Newton started to explain the misunderstanding about the apples but he stopped rather than interrupt the Dean, who was outlining a series of conferences he proposed to organize among fruit growers and representatives of various departments of His Majesty's Government. The Dean's eyes began to glaze as he talked, and he became unaware that anybody else was in the room. Newton had an important committee meeting at that time, so he quietly went out the door, leaving the Dean of Research in an ecstasy of planning.

The seasons passed, while Newton led a busy, useful life as a member of many committees and chairman of some. One dark winter's day he was called again to the office of the Dean of Research. The Dean was beaming; he proudly explained to Newton all about the new contract he had obtained to study the relationship between breed, quality, and rate of fall of all the varieties of fruit that grow above the ground. The project was to be supported by no less than five different branches of His Majesty's Government plus a syndicate of seven large fruit growers. Newton's part in the project was to be small but important: he was to direct the Subproject for Apples.

The following weeks were busy ones for Newton. Though relieved from his committee work (a young instructor of Greek, Latin, history, and manual training took his place on the committees), he found himself cast into a morass of administrative problems: forms to be filled out for the governmental departments, for the fruit growers, for the Dean of Research, for the Assistant Dean of Research, and for the financial office of the college; prospective research assistants to be interviewed and hired; office and laboratory space to be wangled from other projects on the campus. The wide abilities of our great genius are

fully demonstrated by the way he piloted his subproject during its first formative weeks. He personally filled out 7852 forms, often in quintuplicate and sextuplicate; he interviewed 306 milkmaids and hired 110 of them as technical assistants. With his own hands he cleaned out an abandoned dungeon in a nearby castle for use as subproject headquarters; and, turning carpenter in typically versatile fashion, he erected twelve temporary buildings to house his staff. These buildings, used today as classrooms, stand as a monument to Newton's career.

Soon the subproject was fully implemented, documented, and regimented. Newton was not quite sure what his reconverted milkmaids could do for his hypothesis (he was a lifelong bachelor and hence not well acquainted with the ways of women), but he abhorred the thought of idleness in his staff. So he divided them into six teams, each of which was to measure and tabulate the rate of fall of one variety of apple, using sufficient apples to establish a statistically significant result. All went well except with the winesap team, who discovered a new way of making applejack, and consequently ran short of apples. Newton made a note of their recipe, wisely comprehending long before his fellow scientists the advantages of serendipity, or finding good things while looking for other things.

This period of his life was a happy and profitable one for Newton. From the time he arose in the morning until, exhausted with honest labor, he dropped late at night back into his humble bed of straw, he spent each day filling out payroll forms for his milkmaids, ordering pens and paper, answering the questions of the financial office, and showing distinguished visitors and the Dean of Research around his subproject. Often he discussed the past, present, and future work of his project with representatives of the five governmental departments and seven fruit growers who had been sent to check on his progress. He was frequently invited to give progress reports in person at the central offices of these twelve sponsors. Each week he wrote out a full progress report which was duplicated and sent by special messenger to 3388 other projects sponsored by His Majesty's Government throughout the British Isles.

One of these remarkable documents, in an excellent state of preservation, can be found in the Museum of the Horticultural Society of Western Wales, in the village of Merthyr Tydfil. In typically logical style, the report, bound in a dark red stiff cover bearing the project number, HM2wr3801-g-(293), stamped in gold leaf, opens with a succinct table of contents:

1. Administration
2. Conferences
3. Correspondence
4. Supplies
5. Results of research

The last section, "Results of research," may have been lost during the intervening years, or it may not have been specifically required under the terms of

His Majesty's contracts of that era. At any rate it is not there. But the other sections remain to gladden the hearts of those permitted to read them. Is it too much to hope that this report can be published and distributed among our young scientists in America? Such a precept should accomplish miracles for the morale and spirit of our neo-geniuses.

One day in 1685 Newton's precise schedule was interrupted through no fault of his own. He had set aside a Tuesday afternoon to receive a committee of vice presidents of the fruit growing syndicate when, much to his horror and Britain's deep sorrow, the news spread that the whole committee had been destroyed in a three-stagecoach smashup. As once before, Newton found himself with a hiatus. He took a leisurely walk through the luscious vineyards of the Subproject on Grapes, but not, of course, until he had obtained security clearance at the gate. While on this walk there came to him, he knew not how ("Ye thought just burst upon me," he later wrote), a new and revolutionary mathematical approach which, in less time than it takes to tell about it, could be used to solve the problems of attraction in the neighborhood of a large sphere. Newton realized that the solution to this problem provided one of the most exacting tests of his hypothesis; and furthermore, he knew, without need of pen and paper to demonstrate the fact to himself, that the solution fully supported his hypothesis. We can well imagine his elation at this brilliant discovery; but we must not overlook his essential humility, which led him forthwith to kneel and offer thanks to the King for having made the discovery possible.

On his return from this walk, Newton stopped a moment to browse in a bookstore, where he accidentally knocked a book to the floor. With apologies to the proprietor, who seemed in a mood to toss him out upon his ear, Newton retrieved the book and dusted it off. It was Norwood's *Sea-Mans Practice*, dated 1636. Opening the book at random, Newton found it contained the exact information on the length of a latitude degree that he required for the complete test of his hypothesis. Almost instantaneously, one part of his brain performed several lightning calculations and presented the result for the other part to examine; and there it was: the proof complete and irrefutable. Newton glanced at the hourglass in the shopkeeper's window and with a start remembered that he was due back at the dungeon to sign the milkmaids time slips as they checked out for the day. He hurried out of the bookshop with the book under his arm, forgetting in his zeal that he had not paid for it.

Thus it was that His Majesty's Government supported and encouraged Newton during the trying years in which he was putting his hypothesis to the test. Let us not dally with the story of Newton's efforts to publish his proof, the misunderstanding with the editor of the *Horticultural Journal*, the rejections by the editors of *The Backyard Astronomer* and *Physics for the Housewife*. Suffice it to say that Newton founded his own journal in order to make sure that his proof would be published without invalidating alterations. Regrettably, he named his journal *Star and Planet*, with the result that he was branded a subversive since Star could mean Red Star and Planet could mean Plan-It. Newton's subsequent

testimony before the Subcommittee for Suppression of Non-British Ideas remains as a convincing demonstration of the great qualities that combined to make him a genius. Eventually he was exonerated, and after enjoying many years of the fame that was due him, reigning one day each year as King of the Apple Festival, Newton died happily.

A CORRECTION

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There is an error in my paper CHARACTERIZATION OF THE SINE AND COSINE [this MONTHLY, 62 (1955) 707–713]. Fortunately, it is not difficult to correct. The error occurs on page 712, near the end of Section 4, and consists in assuming that the limit which occurs in the last displayed formula is independent of x. This is the case but, since $h=x/n$, it requires proof. To supply this proof and to make certain of (28b) as well as of (28a), lines 10b through 7b of page 712 should be replaced by the following more extended argument.

* * *

Since g and f are continuous and $f(0)=0$, it follows [for $f\neq 0$] that

$$(*) \qquad \int_0^x g(t)\,dt = f(x) \lim_{n\to\infty} \frac{x/2n}{f(x/2n)}$$

for all $x>0$ for which $f(x)\neq 0$. If b has been chosen so that g is positive in $(0, b)$ [as well as so that f has no zeros in $(0, b)$], then the limit in question exists and is nonzero for x in $(0, b)$. Using (21b) and the fact that g is continuous at 0 and $g(0)=1$, we see that the limit exists and is nonzero for all $x>0$. Since f is odd, the same holds for all $x\neq 0$. Consequently, the function L for which

$$L(x) = \lim_{n\to\infty} \frac{f(x/2n)}{1/2n}$$

is defined for all x and has 0 as its only zero. Moreover, from (*) and the continuity of f and of indefinite integrals, it follows that L is continuous in $(0, b)$. Using (19) and the fact that g is continuous at 0 and $g(0)=1$, it follows that, for all x and y,

$$L(x+y) = L(x) + L(y).$$

Consequently, there is an $a\neq 0$ such that, for all x, $L(x)=ax$. So, for all $x\neq 0$, the limit on the right side of (*) is $x/L(x)$—that is, is $1/a$.

Now, since g and f are continuous, (*) holds for all $x>0$ for which the limit on the right exists; and, since g is even and f is odd, it follows that (*) holds for all $x\neq 0$ for which the limit exists. Hence [since $f(0)=0$], for all x,

$$f(x) = a\int_0^x g(t)dt.$$

Since g is continuous, it follows [in the case $f\not\equiv 0$] that

$$f'(x) = ag(x), \qquad \text{for all } x,$$

and, since $g(0)=1$, $a=f'(0)$. This completes the proof of (28a).

The proof of (28b) is, up to a point, similar to that of (28a). The case $f=0$ is trivial. In the case $f\not\equiv 0$, an argument like that above shows that

$$(**) \qquad \int_0^x f(t)dt = [1 - g(x)] \lim_{n\to\infty} \frac{x/2n}{f(x/2n)}$$

for all $x\neq 0$ for which the limit on the right exists. Now, since f is continuous, so is g and, by (28a), f is differentiable at 0. If $f'(0)=0$ then, by (28a), f is constant—but, since $f(0)=0$ and $f\not\equiv 0$, this is not the case. So, $f'(0)\neq 0$ and, for all $x\neq 0$, the limit in question is $1/f'(0)$. The conclusion of (28b) now follows at once from this and (**).

MATHEMATICAL NOTES

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Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida.

ON A PARTICULAR ONE-DIMENSIONAL RANDOM WALK

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Consider the familiar one-dimensional random walk in which a particle, initially at the origin, moves either one unit to the right or one unit to the left at each step. Let X_i be a random variable that assumes the values 1 and -1 depending on whether the ith step is to the right or to the left, and let $S_m = \sum_{i=1}^{m} X_i$. Then $S_m=0$ is the event of a return to the origin at the mth step (which can occur, of course, only if m is even), and $\sum_{m=1}^{2n} P\{S_m=0\}$, which we will denote by E_n, is the expected number of returns to the origin in a $2n$-step walk. The following theorem is well known ([1], p. 298):

THEOREM. *If $P\{X_i=1\}=\frac{1}{2}$, for $i=1, 2, \cdots$, then $E_n\sim C\sqrt{n}$, where C is independent of n.*

As Feller mentions in [1], this result seems at first glance to be paradoxical. We are led to ask: what requirements must the distributions $P\{X_i=1\}$ satisfy in order that $E_n=O(n)$? In its generality, this question appears to present many difficulties; in this note we will merely present one case in which $E_n=O(n)$.

THEOREM. *Suppose that* $P\{X_i=1\}$ *is equal to:* $p>\frac{1}{2}$ *whenever* $S_{i-1}<0$, $q=1-p$ *whenever* $S_{i-1}>0$, $\frac{1}{2}$ *whenever* $S_{i-1}=0$; *then* $E_n\sim(2-1/p)n$.

Proof. Let $P\{S_{2m}=2k\}$ be abbreviated by $P_m(k)$, and consider the expected value of $|S_{2m}|$.

$$E|S_{2m}| = \sum_{k=-m}^{m} |2k| P_m(k) = 4\sum_{k=1}^{m} kP_m(k).$$

The following recurrence relations are easily established:

$$\begin{aligned}
P_{m+1}(0) &= 2p^2P_m(1) + pP_m(0),\\
P_{m+1}(1) &= p^2P_m(2) + 2pqP_m(1) + \frac{q}{2}P_m(0),\\
P_{m+1}(k) &= p^2P_m(k+1) + 2pqP_m(k) + q^2P_m(k-1) \qquad \text{for } 1<k<m \qquad (1)\\
P_{m+1}(m) &= 2pqP_m(m) + q^2P_m(m-1),\\
P_{m+1}(m+1) &= q^2P_m(m).
\end{aligned}$$

Consequently,

$$\begin{aligned}
E|S_{2m+2}| &= 2\sum_{k=1}^{m+1} 2kP_{m+1}(k)\\
&= 4p^2\sum_{k=1}^{m-1} kP_m(k+1) + 8pq\sum_{k=1}^{m} kP_m(k) + 4q^2\sum_{k=2}^{m+1} kP_m(k-1) + 2qP_m(0)\\
&= 4p^2\sum_{k=1}^{m} (k-1)P_m(k) + 8pq\sum_{k=1}^{m} kP_m(k) + 4q^2\sum_{k=1}^{m} (k+1)P_m(k) + 2qP_m(0)\\
&= E|S_{2m}| + 4(q^2-p^2)\sum_{k=1}^{m} P_m(k) + 2qP_m(0)\\
&= E|S_{2m}| + 2[1-2p+pP_m(0)].
\end{aligned}$$

Since $E|S_2|=2q$, summation over $m=1, 2, \cdots, n$ results in

$$E_n = \sum_{m=1}^{n} P_m(0) = \frac{2n(2p-1)+E|S_{2n+2}|-2q}{2p}. \qquad (2)$$

Since $E|S_{2n+2}|>0$, we have a lower bound for E_n. To find an upper bound, we show that

$$P_m(k+1) < \frac{q}{p}P_m(k), \qquad k=0, 1, \cdots, m-1,$$

for all $m>0$. The proof, which is omitted, is by induction on m and depends on the relations (1). It follows that

$$E|S_{2n+2}| < 4\sum_{k=1}^{n+1} k\left(\frac{q}{p}\right)^k P_{n+1}(0)$$

$$< \frac{4q}{p}\sum_{k=1}^{\infty} k\left(\frac{q}{p}\right)^{k-1} = \frac{4q}{p}\cdot\frac{1}{(1-q/p)^2}.$$

Therefore, from (2),

$$\frac{n(2p-1)}{p} + C_2 > E_n > \frac{n(2p-1)}{p} + C_1$$

and this implies the theorem.

Reference

1. William Feller, An Introduction to Probability Theory and its Applications, vol. 1, 2nd ed., Wiley, New York, 1957.

THE NUMBER OF POSSIBLE AUCTIONS AT BRIDGE

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The game of bridge has given rise to a wide variety of interesting problems in combinatorial analysis, most of which have been collected by Borel and Cheron.* Many of these problems are intimately related to the strategy of bidding and play. In fact, a player cannot proceed beyond the novice stage without acquiring at least an intuitive knowledge of suit-split probabilities, etc. On the other hand, some combinatorial problems of bridge have nothing to do with the strategy of the game: It is of no value at the bridge table to know there are 635,013,559,600 possible hands and 53,644,737,765,488,792,839,237,440,000 ($\approx 5.4\times 10^{28}$) possible deals.

One interesting nonstrategic problem which, as far as I can tell, has never appeared in the literature of either bridge or mathematics, concerns the number of possible bidding auctions. At a first glance, it appears that this number could be calculated only by a laborious tallying of the possibilities, perhaps using a few obvious shortcuts. As we shall see, however, counting the auctions in a slightly unnatural way leads quite directly to the answer. We shall assume that the reader is familiar with bridge terminology, but not necessarily with bridge strategy.

We exclude from consideration the infinity of possible auctions in which improper calls are made and condoned. An auction without such irregularities consist of at most thirty-five bids, at least three *pass's*, possibly some *double's* and *redouble's*. There is one null auction, consisting entirely of four *pass's*; we omit this case from the calculation and add it in at the end.

Consider an auction in which k bids are made, where $1 \leqq k \leqq 35$. As each bid must outrank its predecessor, there are $\binom{35}{k}$ possible sequences of bids leading to such an auction.

* Borel, É. and Cheron, A., La Théorie Mathématique du Bridge, Gauthier-Villars, Paris, 1955.

The auction can begin four ways: any one of the players can open the bidding. After the final bid is made, any one of seven sequences of calls can end the auction: denoting *pass*, *double*, and *redouble* by their initials, these are

ppp	*drppp*	*ppdrppp*
dppp	*dpprppp*	*ppdpprppp*.
	ppdppp	

Between each pair of successive bids, several sequences of calls can take place. After a player has bid, his left hand opponent, if he chooses not to bid, can double or pass. Each of these choices then presents certain options to the next player. We find that, between any two consecutive bids, there are 21 possible sequences of calls:

no intervening call	*ppdp*	*ppdrpp*
p	*ppdpp*	*dppr*
pp	*dr*	*dpprp*
d	*drp*	*dpprpp*
dp	*drpp*	*ppdppr*
dpp	*ppdr*	*ppdpprp*
ppd	*ppdrp*	*ppdpprpp*.

Since there are $(k-1)$ consecutive pairs of bids in an auction consisting of k bids, there are $4\cdot 7\cdot \binom{35}{k}\cdot 21^{(k-1)}$ such auctions. In order to determine the total number N of possible auctions without improper calls, we sum over the permitted range of k, then add 1 to account for the null auction. Thus,

$$N = 1 + 28\sum_{k=1}^{35}\binom{35}{k}21^{(k-1)}.$$

Rearranging slightly, we find

$$N = 1 + \frac{28}{21}\left[\sum_{k=0}^{35}\binom{35}{k}21^{k} - 1\right].$$

Since the summation is now in the form of a binomial expansion, we have immediately,

$$N = 1 + \frac{4}{3}(22^{35} - 1).$$

Carrying out the arithmetic yields

$$N = 128,745,650,347,030,683,120,231,926,111,609,371,363,122,697,557$$
$$\approx 1.3 \times 10^{47}.$$

Thus, although the number of possible deals is huge, the number of possible auctions greatly exceeds it, the ratio being about 2.4×10^{18} to 1.

π IS NOT ALGEBRAIC OF DEGREE ONE OR TWO

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In [1] I. Niven gives a simple proof that π is irrational. The following is a modification of his proof to show that π does not satisfy an equation

$$P(x) = ax^2 + bx + c = 0, \tag{1}$$

where a, b and c are rational integers, not all zero.

The following identity is given in [1]. Let $f(x)$ be a polynomial with successive derivatives $f^{(1)}(x), f^{(2)}(x), \cdots$ and let

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots.$$

Then $(d/dx)\{F'(x) \sin x - F(x) \cos x\} = f(x) \sin x$ and so

$$\int_0^\pi f(x) \sin x dx = F(\pi) + F(0). \tag{2}$$

Now suppose π satisfied equation (1). To prove the theorem we obtain a contradiction by putting

$$f(x) = \frac{1}{n!}[P(x)\{P(x) - P(0)\}]^{2n}$$

for some integer n and showing that the right side of (2) is then an integer while the left side lies strictly between 0 and 1 for sufficiently large n.
Since for each integer $r \geqq 0$

$$\begin{aligned}\{P(x)^r\}'' &= r(r-1)P(x)^{r-2}P'(x)^2 + rP(x)^{r-1}P''(x) \\ &= r(r-1)P(x)^{r-2}\{4aP(x) + (b^2 - 4ac)\} + 2arP(x)^{r-1},\end{aligned}$$

the successive even order derivatives $f(x), f^{(2)}(x), \cdots$ must all be polynomials in $P(x)$ with rational coefficients. Since the factors $P(x)$ and $P(x) - P(0)$ each occur $2n$ times in $f(x)$, those expressions in $f^{(2k)}(x)$ which are nonzero for $x = 0$ or $x = \pi$ must have a factor of $(2n)!$ in the numerator. This cancels the $n!$ in the denominator and $f^{(2k)}(0)$ and $f^{(2k)}(\pi)$ are integers for $k = 0, 1, 2, \cdots$. Hence $F(\pi) + F(0)$ is an integer.

On the other hand if C is an upper bound of $[P(x)\{P(x) - P(0)\}]^2$ for $0 \leqq x \leqq \pi$ then $|f(x)| < C^n/n! \to 0$ for $n \to \infty$. Therefore $\int_0^\pi f(x) \sin x dx < 1$ for sufficiently large n. At the same time $f(x) \sin x \geqq 0$ for $0 \leqq x \leqq \pi$, and since $f(x) \sin x$ is continuous and not zero everywhere in the interval we have $\int_0^\pi f(x) \sin x dx > 0$. Thus the integral lies strictly between 0 and 1 and we have the required contradiction in (2).

Reference

1. I. Niven, A simple proof that π is irrational, Bull. Amer. Math. Soc. **53** (1947) 509.

A NOTE ON PYRAMIDAL NUMBERS

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The (triangular) pyramidal numbers may be defined as the binomial coefficients $\binom{m}{3}$, m an integer $\geqq 3$, or, alternatively, as the successive partial sums of the series of triangular numbers. It is to this latter representation that they owe the name pyramidal. If one writes out the first few pyramidal numbers: 1, 4, 10, 20, 35, 56, $\cdots$, one notices that $10+10=20$, and this raises the question of which pyramidals are the sums of two other pyramidals. It is easy to see that since $t^3-t=(t-1)t(t+1)$, an equivalent formulation of this problem is: what are the nontrivial solutions in positive integers of the Diophantine equation:

$$x^3 + y^3 - z^3 = x + y - z? \tag{1}$$

Recently M. Wunderlich and V. Keiser programmed the CDC 1604 computer at the National Bureau of Standards Laboratory in Boulder, Colorado, in an attempt to find pyramidals which are the sums of two pyramidals. They found the first 81 such numbers, unaware that some of these had already been found in 1919. (See Dickson [1] p. 599. Dickson's reference was unfortunately unavailable.) Of all these, the only one in which there was a repeated summand was $10+10=20$. This note will prove the ensuing conjecture that 10 and 20 are the only pyramidals such that twice a pyramidal is pyramidal. We shall prove the obviously equivalent

THEOREM. *The only nontrivial solution in positive integers of*

$$2(m^3 - m) = n^3 - n \tag{2}$$

is $m=4$, $n=5$.

Proof. Let m, n be positive integral solutions of (2). Write $(m, n)=d$, $m=ad$, $n=bd$, $(a, b)=1$, where as usual (m, n) denotes the g.c.d. of m and n. Substituting in (2) and dividing by d we get

$$2a^3d^2 - 2a = b^3d^2 - b. \tag{3}$$

Hence, $d^2 \mid 2a-b$. Define c by

$$2a - cd^2 = b. \tag{4}$$

Then, since $(a, b)=1$, $(a, c)=1$, and $(b, c)=1$ or 2. Substituting (4) for b in (3) and dividing by d^2 gives $2a^3=(2a-cd^2)^3+c$. Cubing and simplifying we get

$$6a^3 = 12a^2cd^2 - 6ac^2d^4 + c^3d^6 - c. \tag{5}$$

Now, since $ad=m$, $bd=n$, (4) gives $2m-n=cd^3$, and so from (2), $2m^3-n^3=cd^3$, or, dividing by d^3

$$2a^3 - b^3 = c. \tag{6}$$

Hence, $6a^3-3b^3=3c$. Substituting into (5) gives

$$3b^3 = 12a^2cd^2 - 6ac^2d^4 + c^3d^6 - 4c. \tag{7}$$

So, $c \mid 3b^3$. If $(c, b) = 1$, then $c = 1$ or 3, since c is clearly positive by (4), since n must be less than $2m$. If $(c, b) = 2$, then since $(c, a) = 1$, from (5) we see that $c = 2$ or 6. In this case we can put $b = 2k$ in (6) and divide through by 2. Thus all solutions of (2) in positive integers correspond by (6) to solutions of the equations

$$2a^3 - b^3 = 1, \tag{8}$$

$$2a^3 - b^3 = 3, \tag{9}$$

$$a^3 - 4k^3 = 1, \tag{10}$$

$$a^3 - 4k^3 = 3, \tag{11}$$

where a, b, k are positive integers.

But (8)–(11) are just very special cases of a theorem of Nagell [2] on solutions of $Ax^3 + By^3 = C$, $C = 1$ or 3, (for $B = C = 1$, see [3]). Nagell's results show that: (8) has no integral solutions other than $a = 1$, $b = 1$; (9) no positive integral solutions other than $a = 4$, $b = 5$; (10) no integral solutions; and (11) no positive integral solutions. Since the conditions of our problem require a, $b > 1$ ($a = b = 1$ corresponds to the trivial solution $m = n = 1$ of (2)), we have $a = 4$, $b = 5$.

Hence $m = 4d$, $n = 5d$, d an integer ≥ 1. Substitution in (2) gives $128d^3 - 4d = 125d^3 - d$, whence $d = 1$. Hence the only nontrivial solution of (2) is $m = 4$, $n = 5$, proving the theorem.

S. Chowla has recently shown that there are infinitely many pyramidals which are the sums of two pyramidals. The frequency of these numbers is an interesting unsolved question.

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NOTE ON A THEOREM OF DAY

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Let p be a real number for which $0 < p < 1$ and consider Lebesgue measure on the unit interval $[0, 1]$. Define L_p to be the set of all real-valued Lebesgue measurable functions f on $[0, 1]$ for which $\int_{[0,1]} |f|^p < \infty$ (two functions being identified if they are equal almost everywhere). Endow L_p with the metric topology where the metric ρ is given by $\rho(f, g) = \int_{[0,1]} |f - g|^p$; it follows that L_p is a real topological vector space under the usual operations of scalar multiplication and vector addition (see [1]). In [1] M. M. Day showed that there exists no nonzero continuous linear functional on L_p. The Radon-Nikodym theorem played a prominent role in his arguments. In the present paper we will give a simpler proof of

On the other hand

$$\sum_{i=1}^{n} n^{-1}f_i = n^{-1}[n^{(1+p)/2p}\chi_E] = n^{(1-p)/2p}\chi_E,$$

and

$$1 > \left|T\left(\sum_{i=1}^{n} n^{-1}f_i\right)\right| = |T[n^{(1-p)/2p}\chi_E]| = n^{(1-p)/2p}|T\chi_E| > 1,$$

which is impossible. This concludes the proof.

Now consider the set V of all real-valued Lebesgue measurable functions on $[0, 1]$ (two functions being identified if they are equal almost everywhere). Endow V with the metric topology where the metric ρ is given by $\rho(f, g) = \int_{[0,1]} |f-g|(1+|f-g|)^{-1}$; it follows that V is a real topological vector space under the usual operations of scalar multiplication and vector addition (see [2]). By methods similar to the proof of our theorem we now show that there exists no nonzero continuous linear functional on V (see also [2]).

Our argument is indirect. Suppose that T is a nonzero continuous linear functional on V. Because T is continuous and linear there exists a measurable subset E of $[0, 1]$ for which $T\chi_E \neq 0$. Of course $m(E) > 0$. There is a $\delta > 0$ such that if f is any function in V for which $\rho(f, 0) < \delta$, then $|Tf| < 1$. Select an integer $n > 1$ for which $n^{-1}m(E) < \delta$ and $n|T\chi_E| > 1$. Let $E_1, \cdots, E_n$ be mutually disjoint measurable subsets of E for which $E = \bigcup_{i=1}^{n} E_i$ and $m(E_i) = n^{-1}m(E)$ for all i. Define $f_i = n^2\chi_{E_i}$, $i = 1, \cdots, n$; then $\rho(f_i, 0) < m(E_i) = n^{-1}m(E) < \delta$ for all i. Consequently

$$|Tf_i| < 1 \quad \text{and} \quad \left|T\left(\sum_{i=1}^{n} n^{-1}f_i\right)\right| \leqq n^{-1}\sum_{i=1}^{n} |Tf_i| < 1.$$

But on the other hand $\sum_{i=1}^{n} n^{-1}f_i = n^{-1}\sum_{i=1}^{n} f_i = n\chi_E$ and

$$1 > \left|T\left(\sum_{i=1}^{n} n^{-1}f_i\right)\right| = |T(n\chi_E)| = n|T\chi_E| > 1,$$

which is impossible.

References

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ON SOME MATHEMATICAL RECREATIONS

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1. Introduction. There are a number of arithmetic puzzles which may be put in the following general form: "Given a sequence of positive integers, $[a_1, a_2, \cdots, a_N]$, and a set of permissible operations upon these integers, such as

addition, subtraction, multiplication, division, the taking of a square root, a logarithm, or of a factorial, it is required to obtain representation of all integers in another set, $[b_1, b_2, \cdots, b_M]$, in the form

$$b_i = R_1(a_1R_2(a_2 \cdots R_Na_N) \cdots),$$

where the R_i belong to the set of admissible operations. A well-known problem of this type is that involving the four 4's, and one which we shall discuss below involves the use of the integers $1, 2, \cdots, 9$ in consecutive order so as to arrive at 100. In this note we wish to show how recurrence techniques, and dynamic programming techniques in particular [1], enable us to obtain the number of representations of different types and minimal representations.

If only addition and multiplication are allowed, a representation due to S. Markowitz, as quoted by Willy Ley [2], is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100.$$

It is natural to ask whether there are other representations, how they might be found, and to look for a minimal representation, say one involving the smallest total number of $+$ signs. We shall consider only this simple problem involving the operations of addition and multiplication. It is easy to see how our methods can be extended to treat more general problems.

2. Number of representations. Let $\{a_i\}$, $i=1, 2, \cdots$, be a given infinite sequence of positive integers and let $f_N(n)$ denote the number of ways in which a positive integer n can be represented in the form

$$n = a_1 * a_2 * \cdots * a_N, \tag{1}$$

where the a_i must be used in the designated order, but where there is a choice of a $+$ or $\times$ for each asterisk. The function $f_N(n)$ is defined for $N=1, 2, \cdots$, $n=1, 2, \cdots$.

Since $6=1+2+3=1\times2\times3$, it is clear that multiple representations exist. Since the last asterisk can be a $+$, or the last r asterisks all $\times$'s, $r=1, 2, \cdots$, $N-2$, we see that

$$\begin{aligned} f_N(n) = {} & f_{N-1}(n - a_N) + f_{N-2}(n - a_Na_{N-1}) \\ & + f_{N-3}(n - a_Na_{N-1}a_{N-2}) + \cdots \\ & + f_2(n - a_Na_{N-1} \cdots a_4a_3) + f_1(n - a_Na_{N-1} \cdots a_3a_2) + g(n), \end{aligned}$$

where

$$\begin{aligned} g(n) &= 1 \quad \text{if } n = a_1a_2 \cdots a_N, \\ &= 0 \quad \text{if } n \neq a_1a_2 \cdots a_N. \end{aligned}$$

3. Minimal representation. Let us now seek to minimize the number of $+$ signs. Define $F_N(n)$ = minimum total number of $+$ signs in a representation of the form of (1), when at least one representation exists. If the representation is

not possible, we define $F_N(n) = \infty$. Arguing as above, we see that we have a choice of using a $+$ sign before a_N, in which case

$$F_N(n) = 1 + F_{N-1}(n - a_N),$$

or we can use r $\times$ signs between a_N, a_{N-1}, $\cdots$, and a_{N-r}, in which case

$$F_N(n) = F_{N-r-1}(n - a_N a_{N-1} \cdots a_{N-r}).$$

Since we wish to minimize $F_N(n)$, we have

$$F_N(n) = \min\left[1 + F_{N-1}(n - a_N), \min_{1 \leqq r \leqq N-2} F_{N-r-1}(n - a_N a_{N-1} \cdots a_{N-r})\right].$$

This is a particular application of the principle of optimality [1].

4. An example.* Let us compute $F_9(100)$. We have

$$\begin{aligned}
F_9(100) &= \min[1 + F_8(91), 1 + F_7(28)], \\
F_8(91) &= \min[1 + F_7(83), 1 + F_6(35)], \\
F_7(83) &= \min[1 + F_6(76), 1 + F_5(41)], \\
F_6(76) &= \min[1 + F_5(70), 1 + F_4(46)] = 1 + F_5(70), \\
F_5(70) &= \min[1 + F_4(65), 1 + F_3(50), 1 + F_2(10)] = \infty, \\
F_4(46) &= \infty,
\end{aligned} \tag{2}$$

Hence

$$F_7(83) = 1 + F_5(41) = 1 + \min[1 + F_4(36), 1 + F_3(21)] = \infty.$$

Thus

$$\begin{aligned}
F_8(91) &= 1 + F_6(35) = 1 + \min[1 + F_5(29), 1 + F_4(5)] \\
&= 2 + F_5(29) = 2 + \min[1 + F_4(24), F_3(9)] \\
&= 3 + F_4(24) = 3,
\end{aligned}$$

since $24 = 1 \cdot 2 \cdot 3 \cdot 4$.

Consequently, returning to the first equation of (2),

$$F_9(100) = \min[4, 1 + F_7(28)].$$

Since

$$\begin{aligned}
F_7(28) &= 1 + F_6(21) = 2 + F_5(15) = 3 + F_4(10) \\
&= 4 + F_3(6) = 4,
\end{aligned}$$

(since $6 = 1 \cdot 2 \cdot 3$) we see that $F_9(100) = 4$, and, indeed, retracing our steps, we have

$$100 = 1 \cdot 2 \cdot 3 \cdot 4 + 5 + 6 + 7 \cdot 8 + 9.$$

* I wish to thank L. Shapley for correcting my original calculations.

5. Discussion. The method used here can be used to study the design of complex systems, and, particularly, questions of the reliability of multistage systems, (see [3]).

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EXTENSIONS OF THE BROUWER FIXED POINT THEOREM

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THEOREM 1. *Let S be a closed bounded subset of E^n, C a closed n-cell or a single point in E^n, with $C \subset S$. Let f be a continuous map from S into E^n which carries the boundary B of S into C. Then f has a fixed point.*

Proof. Let T be a closed n-ball containing S and $f(S)$, and $T_1 = T$ minus all inner points of S. As f maps B, which is a closed subset of T_1, into C, it follows by Tietze's extension theorem [1] that f can be extended to a continuous map of T_1 into C. Now, with f applied to all of T, Brouwer's theorem shows that there is a fixed point. But any point of T not in S is mapped into $C \subset S$, hence cannot be fixed. Thus the original map has a fixed point.

The following is an extension of Theorem 1.

THEOREM 2. *Same as Theorem* 1, *but with the hypothesis "C a closed n-cell" replaced by "C a homeomorphic image of a retract C' of a unit k-cube I^k, with $k \geqq 1$."*

Proof. The only step that is different is the proof that f can be extended to a continuous map of T_1 into C. Let h map C into C' homeomorphically. Then hf maps B into $C' \subset I^k$. Let g be an extension of hf from B to all of T_1, so that g maps T_1 into I^k. Let m be a retraction from I^k onto C', so that mg maps T_1 into C'. Then $h^{-1}mg$ is the required extension.

Reference

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* a) The writer and Dr. Smbat Abian have been working on extensions of the Brouwer theorem, and have previously announced Theorem 1 for the case $S = C$, with proof not depending on the Brouwer theorem or the Tietze extension theorem. b) The proof of Theorem 1 is based largely on ideas and information obtained from Drs. P. S. Doyle, A. M. Gleason and Leo Zippin. c) The writer learned subsequently that Dr. Robert Ellis had earlier had the same, or substantially the same, proof of Theorem 1, at least for the case $S = C = a$ closed n-cell and that Dr. C. T. Yang had a proof of Theorem 1 for the same case, based on the Lefschetz fixed-point formula.

AN INTRODUCTION OF SPHERICAL COORDINATES

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The following is a description of a method by which spherical coordinates may be introduced into the n-dimensional Euclidean space. Some generalizations of the method will be given later together with a sketch of a proof that the method is valid.

For our purposes it will be useful to introduce the variant of the gamma-function defined for $\mathrm{Re}(x) > -1$ by

$$k(x) = \int_0^\infty t^x e^{-\frac{1}{2}t^2} dt \tag{1}$$

in terms of which the Euler beta-integral takes the form

$$\int_0^{\frac{1}{2}\pi} \cos^a\theta \sin^b\theta d\theta = k(a)k(b)/k(a+b+1), \quad (\mathrm{Re}(a), \mathrm{Re}(b) > -1). \tag{2}$$

To illustrate our method let us introduce the usual spherical coordinates in E_4. Let a, b, c and d be arbitrary numbers subject to the same restriction as are the numbers which appear above. We have the obvious identity

$$k(a)k(b)k(c)k(d) = k(a+b+c+d+3)\cdot\frac{k(a)k(b+c+d+2)}{k(a+b+c+d+3)}\,\frac{k(b)k(c+d+1)}{k(b+c+d+2)}\,\frac{k(c)k(d)}{k(c+d+1)}$$

which is our starting point. We write the product on the left-hand side using the expression (1) as the quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u^a x^b y^c z^d e^{-\frac{1}{2}(u^2+x^2+y^2+z^2)} du\,dx\,dy\,dz.$$

Our interchange of the orders of integration is easily justified by the absolute convergence of these integrals. We now do the same thing with the four factors which appear on the right-hand side of our identity. The factor $k(a+b+c+d+3)$ is written as an integral of the form (1) using the integration variable R. The next three factors we write as trigonometric integrals using (2). Thus for the first factor we have

$$\int_0^{\frac{1}{2}\pi} \cos^a w\cdot\sin^{b+c+d+2} w\,dw = k(a)k(b+c+d+2)/k(a+b+c+d+3).$$

If we use for the second two factors the integration variables ϕ and θ and write the resultant product as a quadruple integral we have

$$\int_0^\infty\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} (R\cos w)^a(R\sin w\cos\phi)^b(R\sin w\sin\phi\cos\theta)^c(R\sin w\sin\phi\sin\theta)^d \cdot e^{-\frac{1}{2}R^2}(R^3\sin^2 w\sin\phi)\,dw\,d\phi\,d\theta\,dR.$$

If we compare the integrands of the two quadruple integrals at which we have arrived we see that the transformation equations

$$(3)\qquad \begin{aligned} u &= R\cos w \\ x &= R\sin w\cos\phi \\ y &= R\sin w\sin\phi\cos\theta \\ z &= R\sin w\sin\phi\sin\theta \end{aligned} \qquad \begin{aligned} dV = dudxdydz &= (R^3\sin^2 w\sin\phi dwd\phi d\theta dR) \\ R^2 &= u^2+x^2+y^2+z^2 \end{aligned}$$

suggest themselves. With the equations on the left before us it may be verified that the limits of integration of the two integrals actually agree and that the value of the Jacobian of the transformation is that given on the right-hand side of (3), namely $J=R^3\sin^2 w\sin\phi$. If we write our identity for the product of n factors,

$$k(a_1)k(a_2)\cdots k(a_n) = k(a+n-1)\cdot\frac{k(a_1)k(a-a_1+n-2)}{k(a+n-1)}\cdots\frac{k(a_n)k(a_{n-1})}{k(a_n+a_{n-1}+1)},$$

and use it as above, we get the usual spherical coordinates for E_n, the pyramid of equations which generalizes (3): here we have written $a=a_1+a_2+\cdots+a_n$, the a_i being subject only to the usual condition $\mathrm{Re}(a_i)>-1$.

Various other spherical coordinates may be introduced besides those considered above. Thus the identity

$$k(a)k(b)k(c)k(d) = k(a+b+c+d+3)\cdot\frac{k(a+b+1)k(c+d+1)}{k(a+b+c+d+3)}\cdot\frac{k(a)k(b)}{k(a+b+1)}\cdot\frac{k(c)k(d)}{k(c+d+1)}$$

gives us the spherical coordinates

$$(4)\qquad \begin{aligned} u &= R\cos w\cos\phi \\ x &= R\cos w\sin\phi \\ y &= R\sin w\cos\theta \\ z &= R\sin w\sin\theta \end{aligned} \qquad \begin{aligned} dV = dudxdydz &= R^3\cos w\sin w\cdot dwd\phi d\theta dR \\ \text{and } R^2 &= u^2+x^2+y^2+z^2 \end{aligned}$$

We may verify these results by the usual methods, but we do have before us the complete results of those methods. We may summarize our results by saying that we get the spherical coordinates which correspond to the decomposition of the unit sphere in E_n into lower-dimensional spheres in various ways. Thus (3) corresponds to the decomposition of the sphere in E_4 into a diameter (a sphere in E_1) and a sphere in E_3, and (4) corresponds to the decomposition of this same sphere into two unit circles (spheres in E_2) in the $w-x$ and $y-z$-planes; we learn this by setting w equal to 0 or π and then setting $w=\frac{1}{2}\pi$. If we write s_n for the number of distinct coordinate systems which this method gives us for E_n then

the relation $s_n = s_{n-1}s_1 + s_{n-2}s_2 + \cdots + s_{n-[\frac{1}{2}n]}s_{[\frac{1}{2}n]}$ (where $[x]$ is the greatest integer less than x) is easily proved; it merely accounts for the different ways in which the unit sphere may be decomposed in E_n. We find that $s_1 = s_2 = s_3 = 1$, $s_4 = 2$ (the equations (3) and (4) given here), $s_5 = 3$, $s_6 = 6$, $s_7 = 11$ and so on.

This discussion may be carried over without change if we use the (in general) hyperelliptic functions discussed in the paper [1] instead of the trigonometric functions. There the relation which corresponds to the Euler beta-integral is given, and what corresponds to the equations (3) for an n-dimensional space in which we are interested in the surfaces $R^s = x_1^s + x_2^s + \cdots x_n^s$ is arrived at. In fact the spherical coordinates given in that paper were first derived by the author using the present method. This technique also works when certain spherical coordinates in which are involved the Jacobi elliptic functions are considered. This is developed as above on the relation

$$\int_0^K sn^a\theta cn^b\theta dn\theta d\theta = k(a)k(b)/k(a+b+1), \quad (\mathrm{Re}(a), \mathrm{Re}(b) > -1),$$

which is easily proved in the usual way.

That this method actually gives the correct values of the Jacobians of the transformations may be proved by a method which we sketch here.

Beginning with the integral identity from which the various parts of the transformation are taken, we make the same transformation on the cartesian integral and find ourselves with an integral identity involving two trigonometric integrals over the $n-1$-angle variable $\theta = (\theta_1, \theta_2 \cdots \theta_{n-1})$: to do this we make use of the fact that the Jacobian which we seek is of the form $J = r^{n-1}j(\theta)$, where $j(\theta)$ is a trigonometric polynomial, to eliminate r-integrations. It is now possible to show that $j(\theta)$ is equal to the coefficient of r^{n-1} in the expression which our method gives for the Jacobian. This is done by reducing the identity which we have to the following statement: Let S be the region on the unit sphere in E_n for which $x_i \geqq 0$ $(i = 1, 2, \cdots, n)$ and let f be a continuous function on S such that

$$\int_S f \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} dA = 0, \qquad (\mathrm{Re}(a_i) > -1 \text{ for each } i);$$

then $f = 0$. In our calculation f is

$$j(\theta) - r^{1-n} \frac{\partial(x_1, x_2 \cdots x_n)}{\partial(r, \theta_1 \cdots \theta_{n-1})},$$

a trigonometric polynomial. By the use of this fairly familiar completeness statement, which we may assume is well known, we prove our assertion.

Reference

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CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

A CONTINUOUS FUNCTION

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Teachers of freshman calculus may find it troublesome, when introducing students to the concept of continuity, to concoct a function that isn't either easily proved continuous (usually with the aid of the limit theorems) or only with difficulty proved continuous (usually the difficulty lies with the definition of the function). I suggest that the logarithm function is suitable for a student exercise in continuity. Specifically, ask the student to prove the following proposition (giving the necessary hints).

PROPOSITION. *Let f be a monotone function whose domain is the positive real numbers. Suppose that f satisfies*

$$f(xy) = f(x) + f(y). \tag{*}$$

Prove that f is continuous everywhere.

Proof. Observe first that all the usual rules for logarithms follow from (*). In particular we have

$$f(1) = 0, \tag{1}$$

$$f\left(\frac{x}{y}\right) = f(x) - f(y), \tag{2}$$

$$f(x^q) = qf(x) \text{ for } q \text{ rational}. \tag{3}$$

It follows from (2) that continuity at x is a consequence of continuity at 1. (If this leap seems too long for the class, one can merely ask it to prove continuity at 1.) Thus, in view of (1) we must prove that $\lim_{x\to 1} f(x) = 0$. To this end choose positive numbers a and b such that $a<1<b$. Now, from (3) it follows that $f(a^{1/n})$ and $f(b^{1/n})$ both approach zero as n approaches infinity. Hence, given $\epsilon>0$ there is an n such that $|f(a^{1/n})| <\epsilon$ and $|f(b^{1/n})| <\epsilon$. Let δ be the minimum of $|a^{1/n}-1|$ and $|b^{1/n}-1|$. Then $|x-1| <\delta$ implies $|f(x)| <\epsilon$ by the monotonicity and we have the assertion.

Observe that we do not use the fact that for a positive $\lim_{n\to\infty} a^{1/n}=1$: in fact, this is a consequence of our Proposition provided there exists a strictly monotone function satisfying (*). One can mention this corollary to the class and hope to convince it thereby that proving functions continuous may be worthwhile.

AN INTERESTING RESULT ON ALMOST SQUARE MATRICES AND THE CAUCHY-BINET THEOREM

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Unfortunately a number of traditional theorems about matrices are given scant attention in present day courses. A case in point is the Cauchy-Binet theorem on the determinant of the product of two rectangular matrices. In this note we propose to give a new application of this classic result to illustrate our contention that this theorem is an important tool that should not be relegated to its present status of relative neglect. We state the new theorem first after a preliminary definition.

DEFINITION. *If the dimensions of A are $(n-1)\times n$, we say A is almost square.*

THEOREM. *If A is an almost square matrix with integral values such that the row sums are all zero, then*

$$|AA^T| = nS,$$

where n is the number of columns of A, and S is a perfect square.

By analogy with the square case, the unwary could easily be led into a false step in the following way: add all the columns to the last column producing a final column of zeros since the row sums vanish. Then this "extra" column can be neglected in forming AA^T so that we have the square case with $|AA^T| = |D_{n-1,n-1}|^2$ where $D_{n-1,n-1}$ is the minor consisting of the first $n-1$ rows and columns of A. The reasoning is completely false as a simple counter-example like the matrix $[1, -1]$ shows. Here $|AA^T| = 2$ whereas the fallacious procedure outlined above would yield the new matrix $B = [1, 0]$ with $|BB^T| = 1$. There is thus no necessary analogue of operations which are legal for square matrices for the nonsquare case.

Since the usual reduction methods are not applicable in the rectangular case, our theorem appears to be nonelementary. Actually it can be established with ease using the Cauchy-Binet theorem, which we now state.

CAUCHY-BINET THEOREM. *If A is $m\times n$ and B is $n\times m$ with $m \leqq n$, then*

$$|AB| = \sum |M_k| \cdot |N_k|,$$

where M_k is an $m\times m$ minor of A and N_k the corresponding minor of B using the same rows of B as the columns employed in forming M_k with the summation extending over the $\binom{n}{m}$ possible choices of $m\times m$ minors.

A proof of this theorem can be found on page 85 of the reference. For our case, we choose $m = n-1$ and $B = A^T$. Then we have:

$$|AA^T| = \sum |M_k|^2,$$

where there are n terms in the sum.

Since the rows have sum zero, we can write A in the form:

$$\begin{pmatrix} -\sum a_{1k} & a_{12} & \cdots & a_{1n} \\ -\sum a_{2k} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum a_{n-1,k} & a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix},$$

where the summations on k are from 2 to n. Using the specialization of the Cauchy-Binet theorem just developed, observe that one choice for M_k is secured by eliminating, say, the last column. Call this minor M_n. In obtaining $|M_n|$, using the theorem on the determinant of a sum, we have

$$|M_n| = |L_2| + \cdots + |L_n|,$$

where each L_j has the same columns except for the initial column while the first column depends on j and consists of the elements $-a_{1j}, -a_{2j}, \cdots -a_{n-1,j}$. Evidently only $|L_n|$ fails to vanish since in all the other cases the first column is the negative of some other column. Aside from sign,

$$|L_n| = \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix}.$$

Similarly, if the jth column is omitted and we call this minor M_j, then again

$$|M_j| = \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix}$$

to within sign. Hence $|AA^T| = nd^2$ where

$$d = \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix}.$$

An immediate consequence is that all of the minors of A of dimension $n-1$ have the same determinant to within sign. This follows since we could have replaced the elements in the jth column by their equivalents $-\sum_{k\neq j} a_{1k}$, $-\sum_{k\neq j} a_{2k}, \cdots, -\sum_{k\neq j} a_{nk}$.

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A GENERALIZATION OF THE MEDIAN THEOREM FOR TRIANGLES

ROGER BURR KIRCHNER, Harvard University

Let $p_1, p_2, \cdots, p_n$ be points in three-space, and let $P_n=[p_1, p_2, \cdots, p_n]$ be the n-gon obtained by connecting p_1 to p_2, p_2 to p_3, $\cdots$, and p_n to p_1. Regarding the points as triples of numbers (their coordinates with respect to some basis) we can operate with them as vectors.

DEFINITION 1. *$p=(p_1+p_2+\cdots+p_n)/n$ is defined to be the center of $P_n=[p_1, p_2, \cdots, p_n]$.*

The center of a 1-gon (a point) is the point itself, and the center of a 2-gon (line segment) is its midpoint. It is the purpose of this note to give a geometrical characterization of the center of an n-gon and thereby provide examples of applications of vector algebra. We first make the following definition.

DEFINITION 2. *If $Q_k=[q_1, \cdots, q_k]$ and $Q'_{n-k}=[q'_1, \cdots, q'_{n-k}]$ are such that $P_n=[q_1, \cdots, q_k, q'_1, \cdots, q'_{n-k}]$, then Q_k and Q'_{n-k} are called opposite (k and $n-k$) subgons of P_n* for $0<k<n$.

Our main result is the following theorem.

THEOREM. *The line segments joining the centers of opposite subgons of an n-gon P_n meet in a point, the center of P_n. If Q_k and Q'_{n-k} are opposite subgons of P_n, this point is $(n-k)/n$ of the way from the center of Q_k to the center of Q'_{n-k}.*

Proof. Let $q=(q_1+\cdots+q_k)/k$ and $q'=(q'_1+\cdots+q'_{n-k})/(n-k)$ be the centers of Q_k and Q'_{n-k} respectively. We observe that $p=kq/n+(n-k)q'/n$ is the center of P_n. Since $k/n+(n-k)/n=1$, it follows that p is on the line segment joining q and q'. The first statement of the theorem follows from the symmetry of p in the vertices of P_n. The second statement is obvious if we write $p=q+(n-k)(q'-q)/n$.

The case $n=3$ is a familiar application of vector algebra, and motivated the present note. We state it as

COROLLARY 1. *The medians of a triangle meet in a point two-thirds of the way from a vertex to the midpoint of its opposite side.*

Not so well known is

COROLLARY 2. *The segments joining the centers of opposite n-gons of a $2n$-gon bisect each other.*

This has the special case,

COROLLARY 3. *The lines joining the midpoints of opposite sides of a quadrilateral bisect each other.*

These theorems are especially interesting as we do not need to assume that the vertices lie on a plane.

SOME CURIOUS LIMITS INVOLVING DEFINITE INTEGRALS

H. W. GOULD AND T. A. CHAPMAN, West Virginia University

One of the problems posed in the twenty-second annual William Lowell Putnam Mathematical Competition (held Dec. 2, 1961) was to evaluate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n^2} \frac{n}{n^2+k^2} \tag{1}$$

which is of a type not commonly found in the average textbook.

In fact perusal of quite a number of textbooks turns up little more than the obvious limit

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}, \tag{2}$$

which follows at once from the definition of the integral.

The two limits are related of course. For any fixed n we see that the finite series in (1) has more terms than the series in (2) and hence $\frac{1}{4}\pi$ is easily seen to be a lower bound for the series (1). However this is not a very good lower bound. A little experimentation with inequalities suggests that the limit (1) exists and has the value $\frac{1}{2}\pi$.

In this note we wish to give a generalization and also point out some of the history behind the limit (1).

The limit (1) is a special case of a problem posed by R. D. Carmichael [5]. A published solution was given by G. B. M. Zerr, who was among the most active problemists to contribute to the MONTHLY.

Carmichael's problem was to investigate the limit

$$L = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{k=1}^{m} \frac{n}{n^2+k^2}, \tag{3}$$

where the manner in which m and n are allowed to increase is to be carefully considered. The heuristic, nonrigorous solution given by Zerr was that (and here we quote practically verbatim)

$$L = \int_0^{mh} \frac{dx}{1+x^2} = \arctan mh, \qquad (h = 1/n)$$

so that three cases occur. Zerr concluded that "if m is *small* compared to n so that $mh=0$ then $L=\arctan 0=0$; if m is *equal* to n so that $mh=1$ then $L=\arctan 1=\frac{1}{4}\pi$; if m is *large* compared to n so that $mh=\infty$ then $L=\arctan \infty=\frac{1}{2}\pi$." The result is correct but the method of handling the limits leaves much to be desired if we wish a rigorous proof.

We shall first need the

LEMMA. *Let $f(x, z)$ be continuous and nonincreasing in x for $a-1 \leqq x < \infty$ and for any fixed real z. Also let $f(x, z) \geqq 0$. Then*

$$\int_a^{m+1} f(x, z)dx \leqq \sum_{k=a}^{m} f(k, z) \leqq \int_{a-1}^{m} f(x, z)dx. \tag{4}$$

Aside from the appearance of the parameter z this is essentially a well-known inequality, found in most advanced calculus texts and used to establish the Cauchy-Maclaurin integral test. As for the proof, we have $f(k+1, z) \leqq f(x, z) \leqq f(k, z)$ for $k \leqq x \leqq k+1$. Integrating each member from $x=k$ to $x=k+1$ and then summing from $k=a$ to $k=m$, we obtain the inequality

$$\sum_{k=a+1}^{m+1} f(k, z) \leqq \int_a^{m+1} f(x, z)dx \leqq \sum_{k=a}^{m} f(k, z), \tag{5}$$

from which (4) follows by an iteration of this result.

In case we take $a=1$, $z=n$ and $f(x, z)=z/(x^2+z^2)$ we get from (4)

$$\text{Arctan}(m/n + 1/n) - \text{Arctan}(1/n) \leqq \sum_{k=1}^{m} \frac{n}{n^2 + k^2} \leqq \text{Arctan}(m/n). \tag{6}$$

This gives a confirmation of the value of the limit L in (3). For if we suppose that m is a function of n and m/n tends to c as $n \to \infty$, then $L = \text{Arctan } c$. If $m=n$ we have $L=\frac{1}{4}\pi$. If m increases more rapidly than n, say e.g. when $m=n^p$ then $L=\frac{1}{2}\pi$.

It is easy to find other examples where (4) allows us to evaluate such limits. Thus with $f(x, z)=x/(x^2+z^2)^2$ we find

$$\lim_{n\to\infty} n^2 \sum_{k=1}^{n} \frac{k}{(k^2 + n^2)^2} = 1/4, \tag{7}$$

$$\lim_{n\to\infty} n^2 \sum_{k=1}^{n^p} \frac{k}{(k^2 + n^2)^2} = 1/2, \qquad (p \geqq 2). \tag{8}$$

If we do not introduce the factor n^2 we get the inequality

$$\frac{1}{z^2 + 1} - \frac{1}{z^2 + (m + 1)^2} \leqq \sum_{k=1}^{m} \frac{2k}{(k^2 + z^2)^2} \leqq \frac{1}{z^2} - \frac{1}{z^2 + m^2},$$

so that, e.g.,

$$\sum_{k=1}^{\infty} \frac{2k}{(k^2 + z^2)^2} \leqq 1/z^2, \qquad \text{for all real } z \neq 0. \tag{9}$$

This inequality was conjectured by Mathieu [2] for $z=n$ and strengthened forms have been given by van der Corput and others. Makai [3] has given a particularly simple proof and extension.

For a final example let us take $f(x, z)=1/\sqrt{(x^2+z^2)}$, and we find, e.g.,

$$\lim_{n\to\infty}\sum_{k=1}^{pn}\frac{1}{\sqrt{(k^2+n^2)}}=\log\left[p+\sqrt{(p^2+1)}\right],\qquad (p\geqq 1). \tag{10}$$

The three sums

$$\sum_{k=1}^{n}\frac{1}{\sqrt{(k^2+n^2)}},\qquad \sum_{k=1}^{n}\frac{1}{\sqrt{(k+n^2)}},\qquad \sum_{k=1}^{n^2}\frac{1}{\sqrt{(k+n^2)}}$$

whose behavior could be discussed by way of (4) were the subject of Problem 9 in the fourth annual Putnam Competition (held March 1, 1941).

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3. E. Makai, On the inequality of Mathieu, Publ. Math. Debrecen, **5** (1957) 204–205. See Math. Reviews, **19** (1958) 955.
4. Fourth William Lowell Putnam Mathematical Competition, this MONTHLY, **48** (1941) 346–349.
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AN AXIOMATIC APPROACH TO THE G.C.D.

ROY DUBISCH, University of Washington

Many properties of the g.c.d., (a, b), of two natural numbers are known: $(a, b)=(b, a)$, $(a, a)=a$, etc. The question as to whether or not the g.c.d. can be characterized by some of its properties is answered in the

THEOREM: *Let N be the set of natural numbers and $*$ be an operation on $N\times N$ to N. If for all $a, b\in N$,*

$$(1)\quad a*b=b*a;\qquad (2)\quad a*a=a;\qquad (3)\quad a*(a+b)=a*b,$$

then it follows that $a*b=(a, b)$.

Proof. We first let $a+b=c$, $b=c-a$ in (3) to obtain, for $c>a$,

$$a*c=a*(c-a). \tag{3'}$$

Then, to calculate $a*c$, we note that we can assume $a\leqq c$ by virtue of (1). If $a=c$ we have $a*c=a$ by (2). Then for $a<c$ we use (3') and continue [using (1) if necessary] until eventually we arrive at $a'*a'=a'$. (For example, $9*12=9*3=3*9=3*6=3*3=3$.)

Since, however, properties (1)–(3) hold for $a*b=(a, b)$ we see that (a, b) can be computed by the same process and hence that $a*b=(a, b)$.

Note that the hypotheses of the theorem cannot be weakened. For $a*b=\min(a, b)$ satisfies (1) and (2) but not (3). Next, $a*b\equiv 1$ satisfies (1) and (3) but not (2). Finally, $a*b=a$ satisfies (2) and (3) but not (1).

REMOVABLE SINGULARITIES OF ANALYTIC FUNCTIONS

HARLEY FLANDERS, Purdue University

Here is another proof of the standard result:

If $f(z)$ is regular in $0<|z|<R$ and bounded, then there is a unique definition of $f(0)$ which makes f regular on $|z|<R$.

Consider $g(z)=z^2f(z)$. This is continuous on $|z|<R$ with $g(0)=0$, and differentiable at each point of the region. In particular

$$g'(0) = \lim_{z\to 0} g(z)/z = \lim_{z\to 0} zf(z) = 0.$$

Hence g is regular on $|z|<R$ and

$$g(z) = b_2z^2 + b_3z^3 + \cdots$$

is its Taylor expansion, since $g(0)=g'(0)=0$.
Thus $f(z)=b_2+b_3z+\cdots$, which makes the definability of f at 0 crystal clear.

GEOMETRIC SOLUTIONS OF ALGEBRAIC EQUATIONS

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A geometric method for finding the *real* roots of algebraic equations is developed by means of a simple graphical construction made up only of straight lines perpendicular to each other.

The process is derived for the general case of the nth-order algebraic equation which is written in the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0 \qquad (a_n > 0).$$

The set of coefficients $\{a_n, a_{n-1}, \cdots, a_2, a_1, a_0\}$ can be represented geometrically by a corresponding set of directed line segments of length proportional to the magnitude of these coefficients, so juxtaposed that each line segment is drawn at a right angle through the terminus of the preceding segment. The magnitude and direction of a particular line segment (a_k) is given by the vector $a_ke^{j(n-k)\pi/2}$, where $e^{j\pi/2}$ denotes a single 90° turn in the positive or anticlockwise sense taken with respect to the first vector (a_n) used as reference. The resulting geometric figure forms a continuous "nth-order path" with a start O at the origin of segment (a_n) and a finish T at the terminus of segment (a_0). A solution to the equation is determined by finding a new $(n-1)$-order path having the same start O and finish T. The new path progresses as follows: a first line segment drawn through the origin O is terminated at the point where it intersects the infinite line on which the segment (a_{n-1}) is located: at that point, a 90° turn is made defining a second segment which terminates at the intersection with the

line determined by segment (a_{n-2}): a further 90° turn is effected. The process is continued until the last intersection occurs at the finish T. One solution of the equation is given by $x = -\tan\theta$, where $\theta < \pm 90°$ is the constant angle between the two paths, counted positive for an anticlockwise angular shift and negative for a clockwise shift. The process is illustrated in Fig. 1 for the case of a 5th-order equation in which all coefficients are positive.

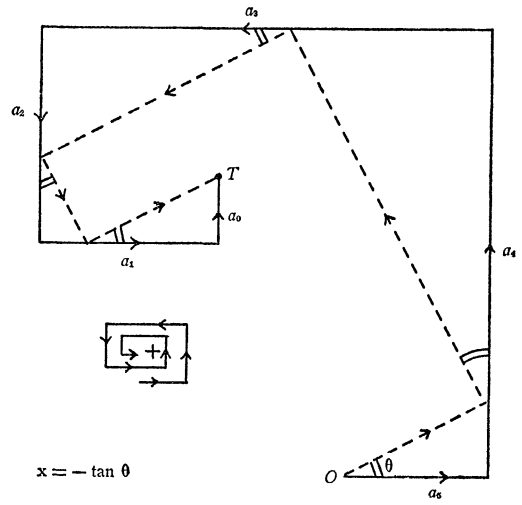


FIG. 1. Geometric solution of the equation

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

(all a's > 0)

That $x = -\tan\theta$ is indeed one of the roots of the algebraic equation may readily be verified by calculating the sequence of lengths corresponding to those sides of the similar rectangular triangles which are opposite to the angle θ:

$$xa_n;\quad x[a_{n-1} + xa_n];\quad x[a_{n-2} + x[a_{n-1} + xa_n]];\cdots;$$
$$x[a_1 + x[a_2 + \cdots + x[a_{n-2} + x[a_{n-1} + xa_n]]\cdots]] = -a_0,$$

and by observing that the last relation is identical to the original equation, though cast in a different form.

The $(n-1)$-order solution path represents the geometric equivalent of the reduced $(n-1)$-order algebraic equation obtained after extracting the first real root from the original nth-order equation. An $(n-2)$-order solution path taken with respect to the $(n-1)$-order path will in turn produce the second real root and permit a further reduction in the order. The continued reduction process can be followed until all the real roots are determined.

As an example, the cubic equation $x^3-7x-6=0$ is solved geometrically in Fig. 2 through finding a succession of reduction paths which turn out to correspond to the algebraic sequence $x^2-2x-3=0$ (after extracting $x_1=-2$) followed by $x-3=0$ (after extracting $x_2=-1$). Note that even though the coefficient of the x^2 term in the cubic is zero, the first reduction path (a quadratic path) has its first 90° turn located on the line corresponding to the x^2 coefficient of the cubic path. The particular choice of reduction sequence is a matter of convenience.

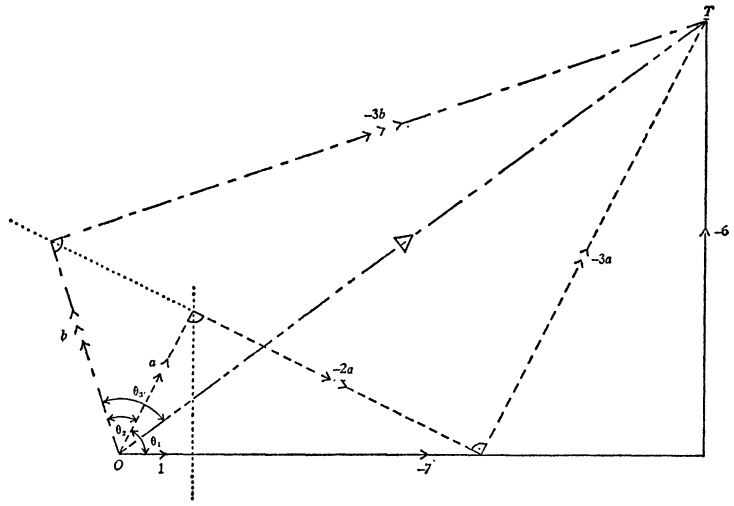


FIG. 2. Solutions of the cubic equation

$$x^3 - 7x - 6 = 0$$

$$x_1 = -\tan\theta_1 = -2$$

$$x_2 = -\tan\theta_2 = -1$$

$$x_3 = -\tan(-\theta_3) = +3$$

It must be pointed out that a reduction procedure as described is not necessary to find the real roots of an algebraic equation. These roots can be obtained directly by determining all the $(n-1)$-order paths that fit the original nth-order path.

The construction of solution paths inherently involves a cut-and-try procedure. However, the simple technique of using a transparent sheet of square millimeter paper can greatly facilitate the search for the solution paths. By turning around the transparent sheet over the drawing of the kth order path, one can readily observe the manner by which the angular shift between the two sheets of paper must be adjusted to meet the constraints of the graphical con-

struction yielding the $(k-1)$-order path. In fact, with a little practice, the adjustment may be effected by scanning visually appropriate lines of the transparency without recourse to actual drawing of construction lines.

The geometrical approach developed here for the solution of algebraic equation may be extended to cover particular situations: for instance, the calculation of the value of an algebraic polynomial for a given value of the variable x, or the calculation of the nth root of a real number. Figures 3 and 4 show the graphical determination of the square and cube roots, and illustrate the degenerate case of a polynomial in which all but the first and last coefficients are zero.

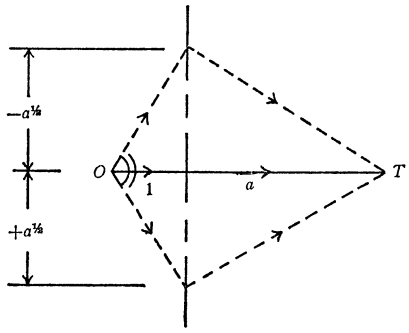


FIG. 3. Square root

$x^2 - a = 0$

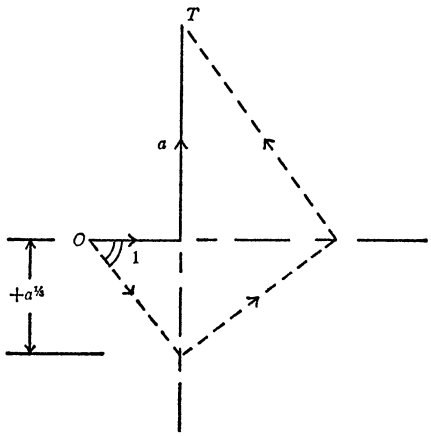


FIG. 4. Cube root

$x^3 - a = 0$

A basic advantage to this geometrical type of solution stems from the fact that it provides a ready visual "feel" for the solutions. The effects of changing the coefficients or the order of the original equation on its solution may be estimated rapidly. Certain symmetries in the coefficients of the algebraic form have their geometrical counterparts placed in clear evidence. Being a graphical method, it can only yield the real roots of an algebraic equation with an accuracy that is inherently limited. However, these roots may be calculated to any desired degree of accuracy by following up the graphical method with a numerical method (such as Newton's). By extracting the real roots from the original equation, the order is reduced and the depressed equation may then be solved for its complex roots through applying suitable algebraic or numerical methods.

To emphasize, in conclusion, the main features, this geometric formulation of the solutions of algebraic equations provides:

(a) a conceptually simple method of solution based on purely geometrical considerations,

(b) a simple graphical construction requiring for equipment two triangles, a linear scale and transparent square millimeter paper,

(c) an "analog" technique of solution in contradistinction to the usual "numerical" techniques of algebraic or digital computation,

(d) an isomorphism between algebraic and geometric methods in mathematics.

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MATHEMATICAL EDUCATION NOTES

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A DESK—SOME CHAIRS—AND A BLACKBOARD!

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Science, and two modes of thinking. History indicates that many of the basic laws of mathematics came from "laboratory-type" thinking. Early men of mathematics were primarily concerned with the laws of nature, laws of the universe, and the measurement of physical objects. For example, the need for some method of land measurement, by the Egyptians and Chinese, led to basic concepts in geometry. The stars, the moon, the sun, and the earth itself became laboratory objects for the derivation of many other mathematical concepts. Even earlier than this, one can assume that man's possession of five fingers on each hand influenced the basic concepts of arithmetic.

Although one might argue that early mathematical thinking was concerned with that which was *practical*, there is also evidence that an "introspective" or *theoretical* mode of thinking also existed. This is shown by the fact that man puzzled over the characteristics of numbers to the extent that secret cults were sometimes organized around these "discoveries."

History seems to indicate that for many hundreds of years, the two types of mathematical thinking, the "applied" and the "pure," walked hand in hand—each complementing the other. Nevertheless, as scientific knowledge increased, strong forces came into focus which tended to separate the two.

One such factor was the usurpation of many of the applied mathematical techniques by the other sciences. A second factor was the inertia of mathematics itself. For a long period of years, the number of accepted branches of mathematical thought remained constant. Man appeared to be content to study, evaluate, and analyze the logic of the past masters.

A third factor, one related to the static condition within the field, was the willingness of leaders in mathematics to relinquish many of the applied aspects of the science. It became a mark of respect (pseudo or otherwise) to indicate that one was a "pure mathematician" rather than an "applied mathematician." As a result, many of the departments of mathematics at colleges and universities cheerfully handed over "applied responsibilities" to schools of engineering, schools of education, etc.

Two world conflicts stopped the trend toward pure mathematical emphasis. The pendulum of emphasis was reversed. The theoretical approach became the vehicle by which the practical could be achieved. Urged on by government pressure and financial backing, men began to seek "new" applications and techniques for "old" theories, and "new" theories for "old" applications.

War provided the spark, and the advent of the modern electronic computer made a conflagration of conservatism toward practical mathematical philosophy. The computer gave the practical mathematician a priceless capability hitherto denied him—almost *instantaneous* computational power. So many operations once theoretical because of *time* considerations now became practical. From the mathematics of tactical weapons involving minute time intervals it was but a step to considerations applicable to satellites and space travel.

Mathematical thought, then, seems to have followed a circular route through the pages of history. It was influenced and developed by the laboratory made up of the world and its objects. It had its beginnings in the bonds that existed between the practical and the theoretical aspects of thinking. Today, it is looking toward the same laboratory but with larger and more complex problems.

At the present time mathematics finds itself in a position where it is important that a correct balance be maintained between that which is theoretical and that which is practical. To maintain this balance, it is necessary that opportunities for exploitation of both modes of thinking be made available to students and teachers of mathematics.

America is presently engaged in a school building campaign that is tremendous in scope. Basic to this program is the desire to advance. One means of doing this is to install adequate laboratory facilities for the science courses. Now, however, plans must also include provisions for modern mathematical laboratories!

Present status of school mathematics facilities. Many interested people have formed groups whose purpose is to improve facets of the American school system. Many of these have concentrated their efforts on school facilities. One such group is the Minnesota School Facilities Council. This is an incorporated, non-

commercial, nonprofit organization, which brings together people from education, architecture, engineering, and business to help the Minnesota schools by offering group suggestions and consulting services for improving school building design, school facilities, and school equipment.

The members of this Council believe that good school facilities are "educational tools" and that "efficient educational tools will enable good teachers to do better teaching." On May 15, 1961, the Minnesota School Facilities Council issued a report [1] designed to assist all those concerned in the planning and operation of educational facilities. The report contains a detailed description of the types of facilities that should be provided for efficient teaching in the three areas of biology, chemistry, and physics. These details can be summarized by stating that proposals in all three areas were organized around *combination lecture-laboratory* rooms. This lecture-laboratory room connected to other auxiliary rooms, such as the advanced projects room, plant room, animal room, storage room, conference room, and shop.

Two factors are significant about the Council report relative to "the mathematical community":

(1) The importance of the laboratory to the subjects of biology, chemistry, and physics was recognized.
(2) Mathematics was *not* included in the report.

These factors are closely related. Both are the result of the historical development mentioned previously. Stated briefly, the public, along with school administrators, have come to believe that efficient mathematical instruction merely requires a traditional classroom containing student chairs, a teacher's desk, and a little blackboard space.

The writer of this report recently had the opportunity to visit a new secondary school in the suburbs of Minneapolis. In this particular visit, the physics teacher acted as a guide. This devoted teacher took great pride in pointing out the physics rooms with interconnecting lecture rooms and laboratories. He was proud of the fact that he had helped design plans for these rooms. Near the end of the tour I asked him where the mathematics rooms were in the new school. His reply is classic.

"The math room is downstairs somewhere. I understand the mathematics teacher is a little unhappy with it—it doesn't have enough blackboard space."

Although there is evidence that many mathematics teachers have not yet awakened to the changing demands for better facilities in mathematics, there is also evidence that some leaders in the field are aware of the problem. One such group is the Conference Board of the Mathematical Sciences.

In January 1961, the Conference Board announced as its first project the study of the design of buildings and facilities for mathematics. Some excerpts from the announcement [2] of this proposed study are worth noting here.

"There are many reasons why it is desirable to make a study of the design of facilities for the mathematical sciences at the present time. In the first place,

mathematics has been poorly housed in the past. In the second place, enrollments are now expanding rapidly. Many colleges and universities have five times as many majors in mathematics as they had only four or five years ago, and the great increases in enrollments for the nation as a whole are still to come.

"In the third place, a study of the design of mathematics facilities is appropriate because of the many changes that have taken place in the mathematical sciences. The project will undertake a study of the design of facilities to support the total activities of the mathematical sciences. These activities include research and instruction in pure mathematics, applied mathematics, and statistics; preparation of the manuscripts of research papers; preparation of the manuscripts of textbooks and expository manuscripts for instructional purposes; teacher training; instruction in the operation of desk calculators and *electronic digital computers*; and the operation of summer and academic year institutes. Modern facilities for the mathematical sciences must provide headquarters space; classrooms; seminar rooms; offices for the staff; library space; a statistics laboratory with desk calculators; a *computation center for the electronic digital computer*; facilities for the use of films, television, and other teaching aids; and a commons room."

Reliable evidence seems to indicate that the present status of school mathematics facilities is poor. Steeped in the traditional classroom approach, mathematics teachers have been slow to recognize that the new world of mathematics requires new facilities.

Proposed school mathematics facilities. The facilities for the modern mathematics program cannot be standardized for all financial structures, varying student populations, and different educational levels. Nevertheless, there are unique characteristics of the modern approach to mathematics that should have a place in every program. First of all, it is important to recognize two aspects of the mathematical approach—the theoretical, or pure mathematics, and the practical, or applied mathematics. Once recognized, it is also important that plans contain provisions for each of these two.

For the theoretical aspect, it is important to provide for a *Lecture Area* containing some or most of the following: students grouped about a common lecture area; ample board and display space; the availability of lecture equipment, such as projectors, television, and special aids; a mathematics library.

For the practical aspect, it is important to provide for a *Laboratory Area* containing some or most of the following: an electronic digital computer; teaching machines; a conference area; special equipment for mathematical experiments and research projects; student experimental study tables; storage space for records, etc.

If the financial status of the local school is adequate, the mathematics facilities program can be patterned to a plan similar to that suggested by the Minnesota School Facilities Council for biology, chemistry, and physics.

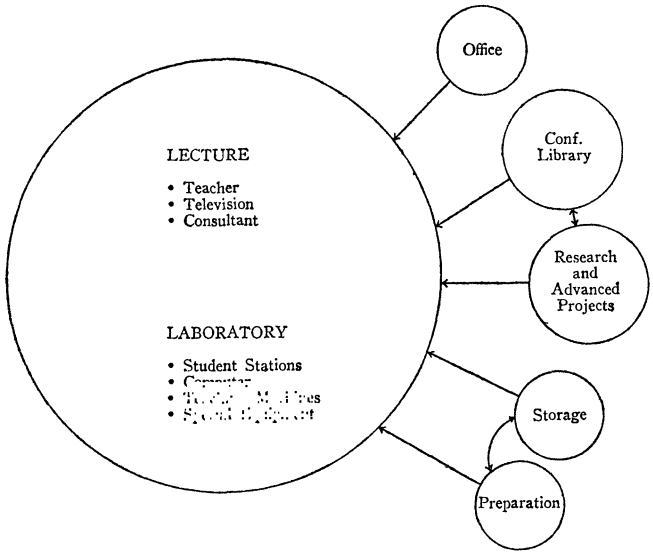


FIG. 1. A suggested facilities program.

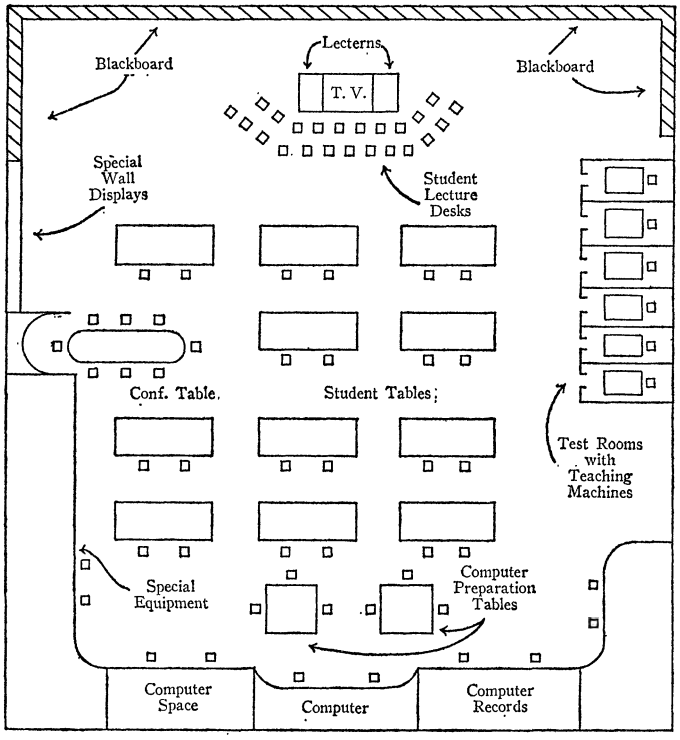


FIG. 2. Combination lecture-laboratory room.

Under this plan, separate room facilities connect into the main Lecture-Laboratory Room. The diagram in Figure 1 indicates this type of planning. There is much to recommend this organizational pattern for long-range planning.

Another possibility is to combine the Lecture-Laboratory into a single room unit. This single room would then contain opportunities for lectures as well as for experimental projects. Although this plan is not quite as flexible as the former, it is possibly more economical. This plan offers almost equivalent opportunities if adjustments are made, for example, using the school library for the mathematics library, etc.

Figure 2 is a suggestive diagram of the combination Lecture-Laboratory Room.

Conclusions

There are increasing indications of demands for new mathematics facilities in schools.

Modern mathematical thinking recognizes the importance of both the pure and applied approaches, wherein each complements the other.

There is a great need for the mathematical laboratory.

The mathematical laboratory will not only provide opportunities for applied mathematical research for students and teachers; it will also directly supply training in critical areas such as cybernetics and systems engineering.

School facilities in the mathematics laboratory will probably center around the electronic digital computer. However, the need for a mathematics laboratory does not hinge directly on the availability of a computer. There are more than enough other devices, equipments, applications, and techniques, unique to mathematics, that require laboratory facilities.

There is evidence to indicate that laboratory facilities for mathematics—including the electronic computer—are within the financial amounts now set aside for biology, chemistry, and physics laboratories.

Although establishment of mathematics laboratories is gaining momentum in American colleges and universities, there is evidence that the movement will rapidly extend to the better secondary schools as well.

References

1. Science Facilities in High School, Report One of Minnesota School Facilities Council, May, 1961, Burton Hall, Minneapolis 14, Minnesota.

2. G. Baley Price, A Study of the Design of Facilities for Mathematics, this MONTHLY **68** (1961) 85.

THE MATHEMATICS CURRICULUM IN THE SMALL COLLEGE

J. WILLIAM DREW, Virginia Union University

Questionnaires were sent to 35 small colleges making inquiry about courses given in mathematics, the number of hours required for graduation with a major in mathematics, and the number of mathematics teachers and their earned degrees. Replies were returned from 27 colleges and universities, none of which had graduate departments. Denominations represented were: Advent Christian, American Lutheran, Baptist, Church of Brethren, Evangelical and Reform, Free Methodist, Methodist, Presbyterian, Roman Catholic, Seventh Day Adventist, Southern Baptist, United Church of Christ, and United Lutheran.

Each was a college with full accreditation. Twenty-two per cent of the teachers in these colleges hold the Ph.D. degree; 70%, the master's degree; and the remainder, a bachelor's degree. The colleges are in Alabama, Arkansas, California, Georgia, Illinois, Kansas, Kentucky, Mississippi, New York, Ohio, Pennsylvania, Tennessee, and Virginia. The average number of semester hours required for graduation with a major in mathematics is 30.6.

Courses Offered	Number of Colleges Reporting	Average Semester Hours' Credit
Differential Equations	24	3
College Algebra	23	3.2
Calculus	22	9.3
Advanced Calculus	22	3
Modern Algebra	22	3
Trigonometry	20	3
Theory of Equations	20	3
Analytic Geometry	16	3.4
Statistics	15	3
Mathematics of Finance	14	3
College Geometry	14	3
Vector Analysis	13	3
Analytic Geometry and Calculus	12	7.2
General Mathematics—Basic Skills, Permutations, etc.	11	4
Basic Concepts, Structural Logic, Sets, etc.	10	4.4
Mathematics for Teachers	10	3.2
Mathematical Analysis, Trigonometry, etc.	9	9
Mathematics for Elementary Teachers	9	2.7
Programming for Digital Computers	9	5.2
Solid Geometry	7	3
Complex Variables	7	3.4
Solid Analytic Geometry	7	3
Theory of Numbers	6	3
Probability	5	3
History of Mathematics	5	2.6
Matrix Theory	3	
Synthetic Projective Geometry	3	3.3
Analytic Projective Geometry	3	3

FIRST INTER-AMERICAN CONFERENCE ON MATHEMATICS EDUCATION

Reprinted from *El Commercio*, Lima, Peru. December 17, 1961.

Bogotá. In an aura of extraordinary solemnity, Colombia's Minister of Education, Dr. Jaime Pasada opened the First Inter-American Conference on Mathematics Education, at the Capitol Building in Bogotá, on December 4th.

The conference was held under the auspices of the Organization of American States, the UNESCO, the Colombian Association of Universities, the Ford Foundation, as well as the Rockefeller Foundation; and attending it were representatives from every country in North, Central, and South America.

Designated as President of the conference was the illustrious professor and researcher of world fame, Marshall H. Stone, of the University of Chicago, who was chairman of the Organizing Committee and who will soon visit our country.

As a special guest, there was present, among others, the celebrated creator of the theory of distributions, and professor at the University of Paris, Laurent Schwartz; also Professor Gustave Choquet of the Institut Henri Poincaré, who took part in discussions as an advisor.

The First Inter-American Conference on Mathematics Education is a consequence of the urgent need, recognized by mathematicians throughout the world, to alter substantially the teaching of mathematics in schools and universities in order to render it in keeping with the advancement made by science and with the broadening of its applications, which in the last thirty years have made such extraordinary progress that their structure has been entirely modified.

Accordingly, the meetings held during the week from December 4th through the 9th dealt with subjects relative to production and perfection of teachers and researchers, programs of study in schools and universities, and widening the scope of knowledge of mathematics.

The conference agreed to establish an Inter-American Commission on Mathematics Education of a permanent character, designed to lend continuity to the projects and ideas discussed at the conference and to promote initiative which would elevate the level and efficiency of mathematics teaching in high school and colleges.

Those chosen to constitute the commission were the following professors: 1. Marshall H. Stone, of the United States; 2. Bernardo Alfaro Sagot, of Costa Rica; 3. Jose Tola Pasquel, of Peru; 4. Alfredo Pereira Gomez, of Brasil; and 5. Alberto Gonzalez Dominguez, of Argentina.

Translated from Spanish by
RICHARD BRIAN,
University of Maryland.

THE GREATER CLEVELAND MATHEMATICS PROGRAM

Science Research Associates, Inc., announced in late February plans for a major modern mathematics program for elementary school students; a program developed as a result of an unusual alliance between business, civic, and educational leaders in the Greater Cleveland Area and already in use, in experimental

form, by 100,000 children in three states. The new program was developed over a period of three years by the Educational Research Council of Greater Cleveland, as part of its over-all effort to create better educational materials for its member schools, and represents a "major departure" from traditional methods of teaching mathematics.

Lyle M. Spencer, president of Science Research Associates, Inc., states that the elementary program is a new, but thoroughly proven, method of bringing modern mathematics into the grades in such a way that even five- and six-year-olds can "discover for themselves" the basic principles upon which all mathematics is based. According to Spencer, children who study the Cleveland materials are brought to see the mathematical relationships which exist in the problems, create the symbolism needed to express these relationships in a meaningful way and, in a very real sense, play a part in "making" this or that piece of mathematics; also, careful testing has shown that students taught in this way learn more and learn it better than those taught in the traditional manner.

The Greater Cleveland Mathematics Program was built, revised, and refined in more than 3,000 classrooms in 27 public, private, and parochial school systems in suburban Cleveland; Owatonna, Minnesota; and Hanover, New Hampshire. One hundred thousand kindergarten through sixth grade children in these systems are presently working with it.

The new materials for grades K-3 became available nationally in the spring. Fourth grade materials will be out by summer 1963, and those for the fifth and sixth grades, by 1964. They will include films and other training and explanatory materials for teachers and school administrators, guide booklets for parents, and boxed sets of brightly colored padded worksheets for the students. Supplementary materials, including specially prepared standardized tests and such manipulative aids as plastic numerals, Countingman (a device for demonstrating the base-ten numerical system) and individual cardboard devices for helping students "discover" arithmetic facts and principles will also be available.

Consultants for the Cleveland program have included Max Beberman, Phillip S. Jones, and John G. Kemeny.

The Greater Cleveland Mathematics Program was developed in the belief that American schools have traditionally underestimated the mathematical capabilities of their children, and that advanced mathematical ideas and concepts can be successfully introduced much earlier than had hitherto been thought possible. Test results show that even though students using GCMP materials are learning the new and more difficult modern mathematics, they are also learning the content of traditional mathematics courses from two to six months earlier than those who are not using these materials.

Abstract from a news release by
Science Research Associates

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1526. *Proposed by Ned Harrell, Melo-Atherton High School, Atherton, California*

Construct an isosceles triangle given the base and the bisector of one of the base angles.

E 1527. *Proposed by R. G. Buschman, Oregon State University*

Let $N(a)$ be the number of solutions to the equation $[x]=ax$, where a and x are real and $[x]$ denotes the greatest integer not exceeding x. Find a simple formula for $N(a)$.

E 1528. *Proposed by L. A. Zalcman, Southwest High School, Kansas City, Missouri*

The 1958 Putnam Competition put forth the following problem: "Given an infinite number of points in a plane, prove that if all the distances determined between them are integers, then the points are all in a straight line." What is the greatest number of points that can be arranged such that the distance between each pair is integral, and yet the points do not all lie on the same straight line?

E 1529. *Proposed by C. R. MacCluer and G. A. Wilson, Ohio State University*

What is the order of the smallest noncommutative ring?

E 1530. *Proposed by Sidney Penner, Illinois Institute of Technology*

What is the smallest positive integer that can be increased by 50% by moving the digit on the extreme left to the extreme right?

E 1531. *Proposed by C. W. Trigg, Los Angeles City College*

There are two distinct hexahedra with equal edges. Each has congruent regular polygons as faces. Show that one can be placed inside the other.

E 1532. *Proposed by J. B. Love, Eastern Baptist College*

Let $P(x)$ be a polynomial of degree n with only real roots and real coefficients. Show that

$$(n-1)[P'(x)]^2 - nP(x)P''(x) \geqq 0.$$

SOLUTIONS

An Application of Euler's Theorem

E 1496 [1962, 57]. *Proposed by T. J. Kearns, University of Santa Clara*

Let a and b be relatively prime integers. Show that there exist integers m and n such that $a^m+b^n \equiv 1 \bmod ab$.

Solution by Erwin Just and Norman Schaumberger, Bronx Community College. We give two solutions:

I. Let $m=\phi(b)$ and $n=\phi(a)$. Then, by Euler's theorem, $a \mid (b^n-1)$ and $b \mid (a^m-1)$, from which it follows that $ab \mid (a^m+b^n-1)$ as required.

II. Since $(a, b)=1$, $(a+b, ab)=1$. Hence, by Euler's theorem, $(a+b)^{\phi(ab)} \equiv 1 \bmod ab$. But $(a+b)^{\phi(ab)} \equiv a^{\phi(ab)}+b^{\phi(ab)} \bmod ab$, from which we obtain $a^{\phi(ab)}+b^{\phi(ab)} \equiv 1 \bmod ab$ as required. This solution shows that the conclusion is true even under the restriction that $m=n$.

Also solved by R. H. Anglin, Kandiah Balachandran, Mark Bernard, R. L. Bernhardt III, Marjorie Bicknell, Walter Bluger, Robert Bowen, Anne E. Britton, Brother U. Alfred, Brother Joseph Heisler, Brother T. C. Wesselkamper, P. J. Brown, Bill Bynum, Leonard Carlitz, D. I. A. Cohen, D. E. Crabtree, Gus Di Antonio, Serge Dubuc, Underwood Dudley, Sarah Evangelista, Jane Evans, Harley Flanders, David Friedman, Michael Friedman, W. C. Gish, N. V. Glick, Michael Goldberg, L. D. Goldstone, Cornelius Groenewoud, Emil Grosswald, J. V. Herod, Sylvia M. Hubbard, W. O. Hulser, J. A. H. Hunter, Betty Levine, Jiang Luh, C. R. MacCluer, Andrzej Makowski, E. L. Magnuson, T. L. Markham, D. C. B. Marsh, E. V. Martin, Jean B. Mobley, Otto Mond, J. B. Muskat, R. J. Oberg, L. D. Olson, Walter Penney, W. A. Peterson, T. B. Pinkerton, C. F. Pinzka, R. M. Raphael, D. A. Robinson, F. A. Roescher, Peter Rosenthal, Alexandre Rossolimo, Donna J. Seaman, M. C. Seibel, R. Sibson, Jr., D. L. Silverman, Denis Sjerve, H. P. Smith, Dmitri Thoro, Dennis Travis, Wilfred Turner, Stephen Ullom, Emanuel Vegh, John Vinson, J. H. Wahab, L. J. Warren, W. C. Waterhouse, J. B. Weems IV, Charles Wexler, C. T. Whyburn, S. Irene Williams, J. A. Wolf, Dale Woods, J. E. Yeager, Kenneth Yocom, Art Young, David Zeitlin, and the proposer.

The following generalizations were stated by Oberg and Warren respectively:

(1) If $a, \cdots, a_n$ are relatively prime in pairs, there exist integers $x_1, \cdots, x_n$ such that

$$a_1^{x_1} + \cdots + a_n^{x_n} \equiv n-1, \bmod a_1 \cdots a_n.$$

(2) If $a_1, \cdots, a_n$ are relatively prime in pairs, $c=a \cdots a_n$, $b_i=c/a_i$, and b_i belongs to the exponent s_i modulo a_i, then

$$b_1^{s_1} + \cdots + b_n^{s_n} \equiv 1, \bmod c.$$

Moreover, if

$$b_1^{t_1} + \cdots + b_n^{t_n} \equiv 1 \bmod c$$

then $s_i \mid t_i$.

Re Fermat's Last Theorem

E 1497 [1962, 57]. *Proposed by A. W. Knapp and Albert Whitcomb, Dartmouth College*

Suppose x, y, z, n are integers such that $x^n+y^n=z^n$. (1) Show that if $2n+1$ is prime and n is sufficiently large, then $2n+1$ divides at least one of x, y, or z. (2) Show that if $4n+1$ is prime and n is sufficiently large, then $4n+1$ divides at least one of x, y, or z. (3) Can (1) and (2) be generalized to all primes of the form 2^qn+1?

Solution by Leonard Carlitz, Duke University. (1) Let $2n+1=p$, a prime. Assume $(p, xyz)=1$. Then $x^n=x^{(p-1)/2}\equiv \pm 1 \pmod p$. Similarly, $y^n\equiv \pm 1$ and $z^n\equiv \pm 1 \pmod p$. Thus the relation $x^n+y^n=z^n$ implies $(\pm 1)+(\pm 1)\equiv \pm 1 \pmod p$, which is impossible for $p>3$.

(2) Let $4n+1=p$ and assume $(p, xyz)=1$. In the Gaussian field $R(i)$ we have $p=\pi\pi'$, where π, π' are conjugate primes. Then $x^n=x^{(p-1)/4}\equiv i^a \pmod \pi$, where $a=0$, 1, 2, or 3. Similarly $y^n\equiv i^b$ and $z^n\equiv i^c \pmod \pi$, where b, $c=0$, 1, 2, or 3. Thus the relation $x^n+y^n=z^n$ implies $i^a+i^b\equiv i^c \pmod \pi$. The possibilities $i^a=i^c$ or $i^a+i^b=0$ may be ruled out immediately. Thus the only cases that remain reduce to $3\equiv 0$ or $2i\equiv 1 \pmod \pi$. The first is obviously impossible; the second is impossible for $p>5$.

(3) Let $rn+1=p$ and assume $(p, xyz)=1$. Set $\epsilon=e^{2\pi i/r}$. Then in the cyclotomic field $R(\epsilon)$ the prime p has the factorization $p=\mathfrak{P}_1\cdots\mathfrak{P}_k$, $k=\phi(r)$, where the $\mathfrak{P}_j$ are prime ideals of the first degree. Now, exactly as in case (2), we have $x^n=x^{(p-1)/r}\equiv\epsilon^a \pmod{\mathfrak{P}_1}$, and similarly $y^n\equiv\epsilon^b$ and $z^n\equiv\epsilon^c \pmod{\mathfrak{P}_1}$. We therefore get

$$\epsilon^a+\epsilon^b\equiv\epsilon^c \pmod{\mathfrak{P}_1}. \tag{*}$$

Set

$$A=\prod_{s,t=0}^{r-1}(1-\epsilon^s-\epsilon^t),$$

so that (*) implies $A\equiv 0 \pmod{\mathfrak{P}_1}$. Since A is a rational integer, we have

$$A\equiv 0 \pmod p. \tag{**}$$

Clearly A depends only on r. Hence (**) is impossible for p sufficiently large provided $A\neq 0$.

Assume $\epsilon^s+\epsilon^t=1$. Set $\epsilon^s=u+iv$, $\epsilon^t=u'+iv'$, where u, v, u', v' are real. Then $u+u'=1$ and $v+v'=0$. Since $u^2+v^2=u'^2+v'^2=1$, we get $u^2=u'^2$, so that $u=u'=\frac{1}{2}$, $v=\pm\sqrt{3/2}$. Hence ϵ^s, ϵ^t are cube roots of unity, and r is divisible by 3. But in this case the relation $x^r+y^r=z^r$ leads at once to $xyz\equiv 0 \pmod 3$.

For $r=2^q$ we get the desired generalization.

Also solved by D. I. A. Cohen, D. C. B. Marsh, Denis Sjerve, and the proposers. Some of these solutions did not answer part (3).

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5037. *Proposed by A. Mąkowski, Warsaw, Poland*

Let $J_k(n)$ denote Jordan's generalization of Euler's totient function, and let $(x_1, \cdots, x_p)$ be the greatest common divisor of $x_1, \cdots, x_p$. Prove that

$$J_k(n) = n^k + \sum (-1)^l (a_1, \cdots, a_l, n)^k,$$

summed for all systems of positive integers $a_1, \cdots, a_l$ such that $a_1 < a_2 < \cdots < a_l < n$. ($J_k(n)$ is the number of incongruent sets $\{a_1, \cdots, a_k\}$ mod n such that $(a_1, \cdots, a_k, n) = 1$.)

5038. *Proposed by G. H. Meisters, University of Nebraska*

Let Δ denote a division ring. Show that $a+b \neq 0$ is a necessary and sufficient condition that $ax+xb=c$ has a unique solution x in Δ for each c in Δ, if and only if Δ is a field.

5039. *Proposed by Seth Warner, Duke University*

Let G be a semigroup. If H is a subgroup of G, define xR_Hy (respectively, xL_Hy) to mean x and y belong to the same right (left) coset of H. A subgroup H is called normal if the identity element of H is an identity element of G, and if $xH = Hx$ for all elements x of G. An equivalence relation R on E and a composition on E are called compatible if xRy and uRv imply $xuRyv$. Prove that R_H and L_H are equivalence relations on G compatible with the composition of G if and only if H is a normal subgroup of G.

An equivalence relation R on G is called trivial either if xRy means $x=y$ or if xRy for all elements x, y of G. Prove that if G has an identity element and satisfies the left and right cancellation laws, then every nontrivial equivalence relation on G compatible with the composition of G is defined by a normal subgroup if and only if G is a group.

5040. *Proposed by D. J. Newman, Yeshiva University*

Suppose that a polynomial $a_0+a_1x+\cdots+a_nx^n$, $a_n \neq 0$, has all its zeros in $|x| < 1$. Prove that

$$\sum_{k=0}^{n} k|a_k|^2 \Big/ \sum_{k=0}^{n} |a_k|^2 > \tfrac{1}{2}n.$$

5041. *Proposed by Peter Ungar, New York University*

A lawn sprinkler with a rotating head delivers $f(r)$ units of water per unit area per unit time at distance r from the head. If the sprinkler travels at a low constant speed along a straight line a strip of width $2R$ is watered uniformly. The outside of the strip remains dry. Find $f(r)/f(0)$.

5042. *Proposed by P. Barrucand, Paris, France*

Let $(n|q)$ be the symbol of Legendre-Jacobi and let $q \equiv 7 \pmod 8$. Show that $\sum_{n=1}^{\lfloor \frac{1}{2}q \rfloor} n(n|q) = 0$.

5043. *Proposed by W. W. Funkenbusch, Michigan College of Mining and Technology*

The three dimensional deck consists of 64 cards, uniquely determined by 4 suits, 4 denominations, and 4 colors. In three dimensional poker, two cards are of the same kind if they have two common coordinates. Any one particular card can be used to form only one unit. Find the numbers of five card hands of the following types:

(a) Four of a kind, (b) Full house, (c) Three of a kind,
(d) Two pair, (e) One pair, (f) No pair.

SOLUTIONS

Union of Commutative Fields

4933 [1961, 299; 1961, 1014]. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set-theoretic union of a finite number of commutative fields having the same unit element. Prove that R must then be a commutative field.

III. *Solution by E. R. Gentile, University of Bahia Blanca, Argentina, and R. M. Cohn, Rutgers, The State University.* We prove, more generally, *A ring which is a union of a finite number of fields is either a field or a direct sum of (a finite number of) copies of* Z_2. (Refer to I, solution by Toskey, [1961, 1014].)

Let $R = \cup_{i=1}^{k} K_i$, where the K_i are fields whose identities are denoted by e_i. For any left ideal I of R and any K_i, $0 \neq I \cap K_i$ implies $K_i \subset I$. Consequently, every nonzero ideal of R is a set-theoretic union of some of the K_i's. This implies that R has only a finite number of left ideals, and, *a fortiori*, satisfies the descending chain condition for left ideals. Furthermore, since R has no nontrivial nilpotent element, R is Jacobson semisimple, therefore a direct sum of a finite number of left ideals which are rings of matrices over division rings $F_1, \cdots, F_m$. Since these rings of matrices contain no nontrivial nilpotent elements they must coincide with the division rings F_i. So R can be written as a direct sum $R = F_1 \oplus \cdots \oplus F_m$ of left ideals F_i which are division rings.

Let f_i be the identity of F_i. If all the F_i's are copies of Z_2 we are done. So, let us suppose that, say, F_1 is not isomorphic to Z_2. Observe that the identity of R is $1 = f_1 \oplus \cdots \oplus f_m$. Let $j \in F_1$, $j \neq 0$, f_1; and form the element $(f_1 - j) \oplus f_2$

$\oplus \cdots \oplus f_m$. This element has all the components different from zero, so must be contained in some field K_h whose identity is $f_1 \oplus \cdots \oplus f_m$. Then $j \in K_h$, and so $K_h \cap F_1 \neq 0$, which implies $K_h \subset F_1$, and so $f_1 \oplus \cdots \oplus f_m \in F_1$ which implies $m=1$. Therefore $R=F_1$ and is a division ring. As in the solution by Toskey it follows that R is a field.

Editorial Note. Several readers pointed out a gap in proof II [1961, 1014]. In the third line from the bottom the statement $abb^{-1}=a \in B$ assumes that B is closed under multiplication, a fact which has not been proved. The more general solution by Gentile and Cohn, given above, was requested by several readers. Because of solution I it was possible to shorten it considerably.

Partition of a Triangle—Perimeters

4964 [1961, 384]. *Proposed by Ernst Trost, Zürich, Switzerland, and Anders Bager, Hjørring, Denmark*

Consider a triangle abc divided into four smaller triangles, a central one def inscribed in abc and three others on the three sides of def. Show that def cannot have the smallest perimeter of the four unless all four perimeters are equal with d, e, and f the midpoints of the sides of abc.

Solution by Robert Breusch, Amherst College. If x represents an angle, set $X=\tan \frac{1}{2}x$. We need two lemmas.

Lemma 1. *If* $x+y+z=\pi$, *then* $XY+YZ+ZX=1$.

Proof. $\tan(\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z)=(X+Y+Z-XYZ)/(1-XY-YZ-ZX)$.

Lemma 2. *If* ABC_i $(i=1, 2)$ *are two triangles with common side* AB, *and if* x_i *and* y_i *are the angles at* A *and* B *in the triangle* ABC_i, *then the perimeter of* ABC_1 *is* $>(=, <)$ *the perimeter of* ABC_2 *if and only if* $X_1Y_1>(=, <)X_2Y_2$.

Proof.

$$\frac{AB+AC+BC}{AB}=1+\frac{\sin x+\sin y}{\sin(x+y)}=1+\frac{\cos(\frac{1}{2}x-\frac{1}{2}y)}{\cos(\frac{1}{2}x+\frac{1}{2}y)}$$

$$=1+\frac{1+XY}{1-XY}=\frac{2}{1-XY}.$$

Thus the larger XY corresponds to the larger perimeter. (Notice that certainly $XY<1$.)

Now let the notation be such that d, e, f are opposite a, b, c, respectively, and assume that the perimeters of afe and fbd are at least equal to that of def. Otherwise there is nothing to prove. If necessary, keeping def fixed, push a and b toward f until these three triangles have equal perimeters; this will increase the perimeter of edc. We wish to show that the latter is not greater than the perimeter of def.

Denote the nine angles at d, e, f as indicated in the figure. Then we know, by the lemmas, that

$$XX_1+XX_2+X_1X_2=1, \tag{1}$$

(2) $$YY_1 + YY_2 + Y_1Y_2 = 1,$$

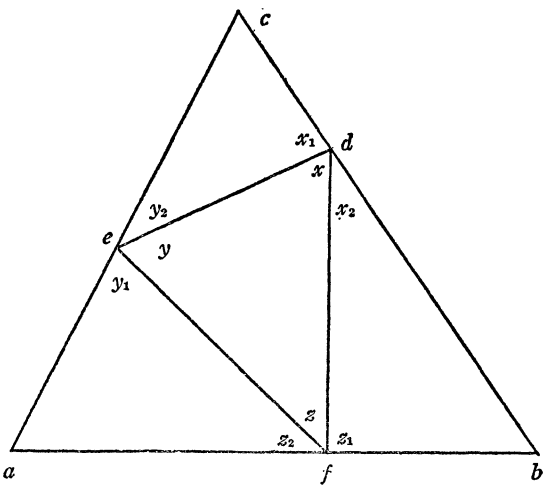
(3) $$ZZ_1 + ZZ_2 + Z_1Z_2 = 1,$$

(4) $$X_2Z_1 = XZ,$$

(5) $$Z_2Y_1 = ZY,$$

(6) $$Y_2X_1 = k \cdot YX \text{ (to show: } k \leqq 1),$$

(7) $$XY + YZ + ZX = 1.$$



Substituting for X_1, Y_1, Z_1, we obtain

(1′) $$k \cdot X^2Y + XX_2Y_2 + k \cdot XX_2Y = Y_2,$$

(2′) $$Y^2Z + YY_2Z_2 + YY_2Z = Z_2,$$

(3′) $$Z^2X + ZZ_2X_2 + ZZ_2X = X_2.$$

Expressing X_2 and Y_2 in terms of Z_2, we can (after some computation using (7)) write (1′) in the form

$$Z(1 + X^2)(Z_2 - Y)^2 + (k - 1)X^2Y^2(1 + Z^2)(Z + Z_2) = 0.$$

Since Z and Z_2 are positive, it follows that $k \leqq 1$, as desired. If $k = 1$, then $Z_2 = Y$; thus by (5), $Y_1 = Z$, and similarly, $X_2 = Z$, $Z_1 = X$, $Y_2 = X$, $X_1 = Y$. Therefore, if the four triangles have equal perimeters, then they are congruent, and d, e, and f are midpoints of their respective sides of abc.

Also solved by Juan J. C. Aldemira, Bernhard Andersen, Dragomir Djoković, L. A. G. Dresel, and Melvin Bloom and G. Lamar.

Editorial Note. The problem was proposed by A. Oppenheim in *Nabla* (Bulletin of the Malayan Mathematical Society), 7 (1960), p. 175, and Dresel's solution was published, *ibid* 8 (1961), p. 97. In the same issue Dresel also proved an analogous inequality for the radii of the incircles of the four triangles.

See the analogous theorem concerning areas of the four triangles, problem 4908 [1961, 386]. It has come to our attention that the area problem appears in H. S. M. Coxeter, *Introduction to Geometry*, ex. 7, p. 212, and that a brief and elementary solution is given in the *Answer Booklet*, p. 18.

Polynomial Solutions of a Partial Differential Equation

4965 [1961, 510]. *Proposed by H. S. Shapiro, New York University*

Let $P(t_1, \cdots, t_n)$ be a homogeneous polynomial with real coefficients and D the partial differential operator

$$D = P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right).$$

Prove that no nonzero polynomial solution u of the equation $Du=0$ can be divisible by $P(x_1, \cdots, x_n)$.

Solution by R. M. Thrall and J. G. Wendel, University of Michigan. More generally, let $P(x)$ be a nonzero polynomial in variables $x=(x_1, \cdots, x_n)$; P need not be homogeneous. Let $P(D)$ be the operator which results when each x_j is replaced by $\partial/\partial x_j$. Let $A(x)$ be an arbitrary nonzero polynomial. We prove: $P(D)\{A(x)P(x)\}$ *does not vanish identically.*

Suppose the contrary. Then also $A(D)P(D)\{A(x)P(x)\}$ vanishes identically, and in particular at $x=0$. But for any nonzero polynomial

$$Q(x) = \sum q_{r_1,\cdots,r_n} x_1^{r_1} \cdots x_n^{r_n},$$

we have

$$Q(D)Q(x)\,|_{x=0} = \sum r_1! \cdots r_n! q_{r_1,\cdots,r_n}^2 > 0.$$

Setting $Q=AP$ shows that we have a contradiction; thus the result is proved.

Also solved by W. G. Brown, L. Carlitz, J. B. Kelly, W. C. Waterhouse, and the proposer.

Surfaces of Revolution

4966 [1961, 510]. *Proposed by G. Di Antonio, Duquesne University, Pittsburgh, Pa.*

Find the equation of all surfaces all of whose normals intersect a given line, say $x=y=z$.

Solution by Joseph Hammer, University of Sydney, Australia. The equations of the normal at the point (x, y, z) of the surface $F(x, y, z)=0$ are

$$\frac{X-x}{\dfrac{\partial F}{\partial x}} = \frac{Y-y}{\dfrac{\partial F}{\partial y}} = \frac{Z-z}{\dfrac{\partial F}{\partial z}}.$$

The condition that the normal may intersect the line $x=y=z$ is $(y-z)\partial F/\partial x+(z-x)\partial F/\partial y+(x-y)\partial F/\partial z=0$. This is Lagrange's differential equation of which the solution is $x^2+y^2+z^2=f(x+y+z)$, where f is arbitrary. These are surfaces of revolution with the given line as axis.

The problem is not new. See Piaggio, *An Elementary Treatise on Differential Equations and Their Applications*, (1943) p. 241.

Note by W. C. Waterhouse, Harvard University. This is essentially Problem 1, part 1, in the 17th Putnam Competition. For solution see this MONTHLY, 64(1957) 649, 651.

Also solved by M. S. Demos, Michael Goldberg, C. B. Grosch, Emil Grosswald, A. J. Kokar, R. C. Read, P. D. Thomas, K. R. Unni, and the proposer.

A Definite Integral

4967 [1961, 510]. *Proposed by D. S. Mitrinovitch, Yugoslavia.* Evaluate the following definite integral

$$J(a, n, k) = \int_{-\pi}^{\pi} \frac{\cos (n - k)x dx}{(1 - 2a \cos x + a^2)^n}.$$

Here n and k are positive integers, and a is a real constant for which the integral has meaning.

Solution by D. C. B. Marsh, Colorado School of Mines. Put $p=|n-k|$, and set $z=e^{ix}$. We have

$$J(a, n, k) = -\frac{i}{2}\oint_C \frac{z^{n+p-1}+z^{n-p-1}}{(z-a)^n(1-az)^n}dz = -i\oint_C \frac{z^{n+p-1}dz}{(z-a)^n(1-az)^n} = 2\pi S,$$

where C is the unit circle $|z|=1$ and S is the sum of the residues at the poles of the integrand which are interior to C.

Take $|a|<1$, and let $w=z-a$. We compute $S_{w=0}$ for

$$(a + w)^{n+p-1}/w^n(1 - a^2 - aw)^n.$$

$S_{w=0}$ is also the coefficient of w^{n-1} in the Taylor expansion for

$$(a + w)^{n+p-1}\left(1 - \frac{aw}{1 - a^2}\right)^{-n}(1 - a^2)^{-n}.$$

Upon expanding the first two factors and collecting like terms, we find:

$$J(a, n, k) = \frac{2\pi a^{|n-k|}}{(1 - a^2)^n}\sum_{j=0}^{n-1}\binom{n - 1 + |n - k|}{n - 1 - j}\binom{n - 1 + j}{j}\frac{a^{2j}}{(1 - a^2)^j}.$$

Because of the identity $J(1/a, n, k)=a^{2n}J(a, n, k)$, the case $|a|>1$ is solved by an easy extension of the same formula.

Also solved by G. DiAntonio, A. J. Kokar, John Raleigh, M. R. Spiegel, and David Zeitlin.

Editorial Note. Zeitlin notes that the integral is given in Table 66, formula (5), p. 101 in Bierens de Haan, *Tables d'Intégrales Définies*, 1867, Stechert. Raleigh refers to W. Gröbner & N. Hoftreiter, *Integraltafel*, 2er Teil (Springer 1950), integral no. 332.31, p. 113. The proposer's solution appears in his *Zbornik Matematičkih Problema*, III, 69–70.

Characteristic Polynomials

4968 [1961, 510]. *Proposed by Basil Gordon, University of California, Los Angeles*

If M is any square matrix, let $\phi_M(\lambda)$ denote its characteristic polynomial, and M_i the matrix obtained by deleting the ith row and ith column of M. Show that if A and B are $n \times n$ matrices over a field of characteristic 0 or $p \geqq n$ with $\phi_{A_i}(\lambda) = \phi_{B_i}(\lambda)$, $(i = 1, \cdots, n)$, then $\phi_A(\lambda)$ and $\phi_B(\lambda)$ differ by a constant.

Solution by W. C. Waterhouse, Harvard University. We can use the algebraically defined derivative and differentiate the determinant row by row: $D_\lambda \phi_M(\lambda) = -\sum_{i=1}^{n} \phi_{M_i}(\lambda)$. Hence $D_\lambda(\phi_A - \phi_B) = 0$. This implies that ϕ_A and ϕ_B differ by a constant and possibly, when $n = p$, in the coefficient of λ^n. But this is $(-1)^n$ in both ϕ_A and ϕ_B.

Also solved by William G. Brown, Robert Breusch, R. A. Cunninghame-Green, D. C. B. Marsh, D. W. Robinson, and the proposer.

Contour Integral

4969 [1961, 511]. *Proposed by David Gale, Brown University and D. J. Newman, Yeshiva University*

Let C be a closed convex curve, p a measurable function defined on C such that $0 \leqq p \leqq 1$. Prove that there is a chord ζ of C which satisfies $\int_C p\,dz = \zeta$.

Solution by the proposers. First note that $\int_C p\,dz = \int_C q\,dz$ where $q = p - \frac{1}{2}$ lies between $-\frac{1}{2}$ and $\frac{1}{2}$. Now suppose that $\int_C q\,dz$ has direction θ, and note that all that need be proved is that $\left|\int_C q\,dz\right| \leqq L$, where L is the length of the longest chord of C which has direction θ. Draw this longest chord and observe that C must have parallel lines of support at its endpoints (otherwise a small perturbation would produce a longer such chord). Now introduce the oblique coordinate system ξ, η with directions given by those of the chord and the support lines. In this coordinate system $\int_C q\,dz$ has components $\int_C q\,d\xi$, $\int_C q\,d\eta$; but the latter is zero since $\int_C q\,dz$ lies completely in the ξ direction. Therefore

$$\left|\int_C q\,dz\right| = \int_C q\,d\xi \leqq \operatorname{Max} |q| \cdot V(\xi),$$

where $V(\xi)$ is the total variation ξ along C. Since $\operatorname{Max} |q| \leqq \frac{1}{2}$ and $V(\xi) = 2L$, the proposed result follows.

Also solved by Robert Breusch, and by D. Ž. Djoković.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Introduction to Hilbert Space. By Sterling K. Berberian. Oxford University Press, New York, 1961. xi+206 pp. $6.50.

The prerequisites for reading this charming little book are just the elements of what is usually called "real variables" (or possibly "advanced calculus"). Anyone who is not frightened by infinite series and can integrate a continuous function on an interval (even if it has complex values) can start to read. As he reads he will be gently but efficiently introduced to vector spaces (including a somewhat pedantic treatment of the general associative law), to metric spaces (including a simple example of an incomplete "pre-Hilbert" space—an unfortunate but common term), to set theory and baby topology (including points "adherent" to sets, i.e., cluster points), and to such more sophisticated words and concepts as injective mapping and Banach algebra.

The pace is leisurely; the book is more than half over before the Riesz representation theorem appears (the one about bounded linear functionals via inner products). Only non-Lebesgue aspects of the theory are treated; this accounts for the absence of even the Riesz-Fischer theorem and certainly the spectral theorem. Proper values and approximate proper values are defined (in English, not in that hybrid language in which "properWerte" is a word); the general concept of spectrum is mentioned only incidentally in an exercise. The climax is the "spectral theorem" (diagonal form) for completely continuous operators, briskly and clearly presented.

There are many examples described in detail, and there are many exercises. The last section is of an uncommon sort in a relatively advanced book; it consists of about a dozen pages of hints and outlines of solutions for the exercises, occasionally in the form of nothing more than a reference. The style is pleasantly light with some touches of humor, generally unobtrusive.

The book is highly recommended as an apéritif.

P. R. HALMOS
University of Michigan

A Book of Curves. By E. H. Lockwood. Cambridge University Press, New York, 1961. xi+199 pp. $4.95.

Men and affairs make changing fashions in Mathematics, and the demands of the day have relegated the study of curves to limbo. This is deplorable, since much good can come of their study, particularly in the modern world of machinery. Moreover, the instructor of mathematics can make profitable and interesting reference to properties of curves and their applications.

This book by an English author is a fruitful source of material for the mathe-

matician, engineer, and draftsman. Conics, Cycloids, Strophoids, Spirals, Ovals of Cassini, Catenary, Tractrix, and other less well-known curves are surveyed quite thoroughly. The book also includes means of finding or generating curves, including discussions of Roulettes, Pedals, Glissettes, Evolutes and Involutes, Inversion, and Caustics. The book is well illustrated, and should be in the libraries of all high schools and colleges, along with those of Loria, Wieleitner, Salmon, Frost, and even Yates. Although the author evidently intended it for classroom use, I do not know where in this country it might now serve as a text.

Certain details must be mentioned. Almost without exception each curve is strangely enough defined initially as an envelope instead of as a point locus thus often creating complexities in the derivation of other properties. Many concepts such as length, area, intrinsic equation, and gradient are apparently assumed to be known by the reader. The steady marshalling of facts in paragraphs marked *Properties* and *Exercises* leaves an awesome burden, some items forming short mathematical "careers" and calling for special ingenuity.

The demonstration of the double generation theorem for each curve of the family of Cycloids and the account of Roberval's clever derivation of the area under an arch of an ordinary Cycloid are highlights of special interest.

ROBERT C. YATES
University of South Florida

Topology. By John G. Hocking and Gail S. Young. Addison-Wesley, Reading, Mass., 1961. 374 pp. $9.75.

The authors state in their preface that this book is designed as a text for a one-year first course in topology, but that the student should have had the elements of set theory and should have had, or be taking, courses in group theory and the theory of functions. The reviewer feels that for most students an introductory course in topology should be added to the prerequisites. The authors further state that many important topics are mentioned briefly to stimulate interest and provide a broad framework for further study. This objective is certainly achieved.

The first three chapters (148 pages) are devoted to point set topology, including topological characterizations of an arc and of a simple closed curve, and considerable material on continuous functions. The remaining five chapters treat algebraic topology including the Eilenberg-Steenrod axioms and the theories of Cech and Vietoris. The chapter headings are as follows: 1. Topological Spaces and Functions; 2. The Elements of Point-Set Topology; 3. Further Topics in Point-Set Topology; 4. The Elements of Homotopy Theory; 5. Polytopes and Triangulated Spaces; 6. Simplicial Homology Theory; 7. Further Developments in Algebraic Topology; 8. General Homology Theories.

An extensive bibliography is included and, where an important topic is given brief mention, references are supplied for further study. The text contains a wealth of material; many illustrative examples are provided, as well as almost 200 exercises with a wide range of difficulty. Care has been taken to include

background material which would aid in the understanding of new concepts. The art work in the figures has been carefully conceived and well executed.

In the reviewer's opinion, the book would be improved, especially for reference purposes, by the addition of an Index of Notation. Quite a bit of notation is carried through the text and there might be some difficulty in reading a particular section in the middle of the book. More entries are also needed in the subject matter index. For example, the only page reference given for "arc" is 115, whereas a topological characterization of an arc appears on p. 54.

Only a few misprints were noted, but there are too many misstatements. Necessary hypotheses are omitted from some theorems. Examples: Theorems 1-24, 2-24 and 2-51 are false as stated; the "metric" ρ on p. 81 doesn't satisfy the triangle inequality. The proofs are concise—sometimes almost sketchy. As a first course in topology this book will offer considerable difficulty to the usual beginning graduate student.

B. H. ARNOLD
Oregon State University

Advanced Calculus. By John M. H. Olmsted. Appleton-Century-Crofts, New York, 1961. xviii+706 pp. $9.50.

This book is the third member of a set of texts by this author addressed to the student who has had a standard first course in the calculus. A review of the first, *Intermediate Analysis*, appeared in the MONTHLY, Vol. 66, pp. 158–9, and a review of the second, *Real Variables*, appeared in Vol. 67, pp. 932–3. The present work differs from *Real Variables* (the first nine chapters of which are almost identical with the entire *Intermediate Analysis*) principally in having chapters on Solid Analytic Geometry and Vectors, Complex Variables, Vector Analysis, Line and Surface Integrals, and Differential Geometry. Some of these topics are treated briefly in *Real Variables*. On the other hand, a chapter entitled Some Theoretical Considerations appears in *Real Variables* but not in *Advanced Calculus*. Most of the topics discussed in this chapter, however, are also discussed somewhere in the latter book.

There are many exercises of varying difficulty. Some of them invite the student to prove or to extend theorems. Both the average and above average student will find much to stimulate them.

As with the author's previous works, the present one is most carefully written and should bring excellent results from a class of ordinary or extraordinary students at the advanced undergraduate or first year graduate level. A choice of either book will give the instructor a text which will keep his class engrossed in the subject matter. The choice between the two may be made solely on the basis of the topics which the instructor wishes to include in his course. Either provides ample material for two semesters and more. Either could be used in a course with a title similar to the title of either book.

LLOYD L. LOWENSTEIN
Arizona State University

A Course of Mathematical Analysis. By A. Khinchin. Translated from the third Russian edition (1957). Gordon and Breach, New York, 1960. 668 pp. $15.00.

This book is a translation of a generally well-written Russian text designed for a basic analysis course in Russian universities for students in applied mathematics (and to some extent, prospective teachers). It is perhaps most closely comparable in spirit and content to Volume I of R. Courant's *Differential and Integral Calculus*, Nordemann (1940), and like Courant's book it is partly designed to bridge the gap between (European) school and university mathematics.

The first two chapters are concerned with functions and limits from the intuitive "18th century" point of view. Then follow chapters on the reals, limits, and continuity, shifting gradually to the more rigorous "ϵ, δ" formulations. The usual elementary calculus topics are treated in an orthodox but careful manner. Over 100 pages are devoted to a rather thorough elementary account of infinite series. Here topics not ordinarily encountered in American elementary calculus texts include the Raabe ratio test, an introduction to infinite products, the Weierstrass approximation theorem and its proof via the Bernstein polynomials, and a chapter on Fourier series.

There follow treatments of partial differentiation, directional derivatives, etc., through to the implicit functions theorem. In this connection, Jacobians are referred to as "Ostrogradskij's determinants." A footnote (p. 472) mentions Jacobi but asserts that "Ostrogradskij himself obtained most of the important results several years before Jacobi," though he (p. 661) "unfortunately failed to publish them." Incidentally, Ostrogradskij is also credited, in the account of multiple, line, and surface integrals, with the discovery of Gauss's formula. Improper integrals (called "generalized integrals"), including the gamma and beta functions, are also studied. (In connection with these functions the Swiss mathematician Euler is referred to (p. 655) as "the St. Petersburg scientist.")

Though the book is bound (quite cheaply) and marketed (quite expensively) by Gordon and Breach of New York, it is actually published in Delhi, India, by the Hindustan Publishing Corporation. The typography is reasonably clear, but the paper is of poor and rather translucent quality. There are a number of typographical errors. That the (unidentified) translators are probably Russian and probably several in number is suggested by occasional Russian constructions (like "infinitesimalness") and by the fact that terminology is not entirely consistent throughout. For instance, while intervals are sometimes called "intervals," they are also sometimes referred to as "sections" (p. 70, e.g.) or "lines" (p. 87, e.g.). There are no exercises, but frequent reference is made to exercises in B. P. Demidovich's *Problem Book on Mathematical Analysis*, 2nd Edition (1954), which is apparently not available in English.

TRUMAN BOTTS,
University of Virginia

BRIEF MENTION

Tables of Integrals and Other Mathematical Data (Fourth Edition). By H. B. Dwight. Macmillan, New York, 1961. x+336 pp. $3.50.

Incorporates minor changes and corrections. Section on definite integrals enlarged, including the inclusion of a group involving elliptic integrals.

Mathematical Recreations and Essays (Eleventh Edition). By W. W. R. Ball; revised by H. S. M. Coxeter. Macmillan, New York, 1962. xvi+418 pp. $1.95 (paper).

The Tides. By G. H. Darwin. Freeman, San Francisco, 1962. xx+378 pp. $2.75 (paper). A classic exposition (1898), still fresh and interesting.

Russian Journals of Mathematics: A Survey and Checklist. By H. A. Steeves. The New York Public Library, New York, 1961. 19 pp.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor W. T. Guy, University of Texas, represented the Association at the inauguration of Dr. L. D. Fleming as President of Southwestern University on May 4, 1962.

Professor A. E. Livingston, Montana State University, represented the Association at the inauguration of Sister Rita of the Sacred Heart as President of the College of Great Falls on May 3, 1962.

Professor G. R. MacLane, Rice University, represented the Association at the inauguration of Dr. P. G. Hoffman as President of the University of Houston on April 27, 1962.

Professor C. B. Morrey, Jr., University of California, Berkeley, has been elected a member of the National Academy of Sciences.

Mr. Frederick Renvyle, Arlington Senior High School, Arlington, Massachusetts, represented the Association at the Fiftieth Anniversary Convocation of Nasson College on May 5, 1962.

Connecticut College: Associate Professor Aileen Hostinsky, Pennsylvania State University, has been appointed Professor; Assistant Professor E. C. Schlesinger, Wesleyan University, has been appointed Assistant Professor; Mr. W. R. R. Transue, University of North Carolina, has been appointed Instructor; Professor Julia W. Bower is a Visiting Scholar at the University of Miami during the academic year 1962–63.

Rensselaer Polytechnic Institute: Associate Professors Kurt Bing, J. D. Campbell and R. C. DiPrima have been promoted to Professors; Assistant Professors L. A. Segel and E. E. Nash have been promoted to Associate Professors.

State University of Iowa: Associate Professor R. V. Hogg has been promoted to Professor; Mr. George Burke has been promoted to Assistant Professor; Professor Minoru Tomita, Okayama University, Japan, has been appointed a Visiting Lecturer; Dr. D. V. Meyer has been appointed Lecturer.

Tri-State College: Associate Professor C. L. Davis has been promoted to Professor; Assistant Professor R. A. Butler has been promoted to Associate Professor; Mr. J. M. Ott has been promoted to Assistant Professor; Professor C. L. Davis has been awarded a National Science Foundation Faculty Fellowship for graduate studies at Michigan State University.

University of Colorado: Professor J. R. Britton has been appointed Chairman of the Department of Applied Mathematics; Dr. G. E. Lewis, New York University, has been appointed Assistant Professor.

University of South Carolina: Associate Professor Johann Sonner has been promoted to Professor; Assistant Professor J. L. Boal has been promoted to Associate Professor.

Western Washington State College: Associate Professor Joseph Hashisaki, Montana State University, has been appointed Professor and Chairman of the Department of Mathematics; Assistant Professor W. R. Abel, University of Nebraska, and Associate Professor P. T. Rygg, University of South Dakota, have been appointed Associate Professors; Assistant Professor George Witter has been promoted to Associate Professor; Assistant Professor J. E. McFarland has been named Director of the recently established Computer Center.

Dr. D. W. Allan, University of British Columbia, has accepted a position in the Theoretical Division of Culham Laboratory, Near Abingdon, Berkshire, England.

Mr. G. E. Anderson, U. S. Army Chemical Research and Development Laboratories, Army Chemical Center, Maryland, has accepted a position as Mathematician with United Electrodynamics, Pasadena, California.

Professor B. H. Arnold, Oregon State University, will be on leave during the academic year 1962–63 as Lecturer at the National University of Taiwan, Taipei, Taiwan, Republic of China, under a Fulbright grant.

Mr. Dov Avishalom, University of Minnesota, has been appointed Assistant Professor at St. Thomas College.

Mr. R. J. Benice, Sylvania Electronic Systems, Buffalo, New York, has accepted a position as Senior Associate Mathematician with International Business Machines Communication Center, Rockville, Maryland.

Mr. Harold Bennett, Bell and Howell Company, Chicago, Illinois, has accepted a position as Optical Design Engineer with the John R. Miles Corporation, Skokie, Illinois.

Mr. D. R. Beuerman, Texaco Research Center, Beacon, New York, has been appointed Assistant Professor at Erie County Technical Institute.

Mr. D. G. Beverage, University of Florida, has accepted a position in the Logical Design Division of Lockheed Missiles and Space Division, Palo Alto, California.

Mr. J. R. Bienas, Philco Corporation, Philadelphia, Pennsylvania, has accepted a position as Engineer in the Missile and Space Vehicle Department of General Electric, Valley Forge, Pennsylvania.

Dr. H. J. Biesterfeldt, Jr., Pennsylvania State University, has been appointed Assistant Professor at Lebanon Valley College.

Mr. L. M. Bostick, Convair, Fort Worth, Texas, has accepted a position as Senior Mathematician with General Electric, Syracuse, New York.

Mr. H. H. Brown, Corporation for Economics and Industrial Research, Arlington, Virginia, has accepted a position as Mathematician-AST with the Goddard Space Flight Center, Greenbelt, Maryland.

Miss Jean S. Campbell, Office of Naval Research, Washington, D. C., has accepted a position in the Editorial Branch of the Technical Information Division of the National

Aeronautics and Space Administration, Greenbelt, Maryland.

Dr. George Cherlin, Mutual Benefit Life Insurance Company, Newark, New Jersey, has accepted a position as Actuary with the National Health and Welfare Retirement Association, New York, New York.

Dr. P. L. Chessin, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, has accepted a position as Permanent Staff Member with the Systems Research Institute of International Business Machines, New York, New York.

Mr. E. N. Clark, Pioneer Service and Engineering Company, Chicago, Illinois, has accepted a position with the Harza Engineering Company, Chicago, Illinois.

Professor V. F. Cowling, University in Kentucky, has been appointed Professor at Rutgers, The State University.

Mr. H. H. Crapo, Massachusetts Institute of Technology, has been appointed instructor at Northeastern University.

Mr. F. W. Dalleska, University of Arizona, has accepted a position as Mathematician in the Aeronautics Division of Minneapolis-Honeywell, Minneapolis, Minnesota.

Dr. R. S. Dinsmore, Stanford University, has accepted a position as Systems Engineer-Scientific with International Business Machines, Sacramento, California.

Professor E. K. Dorff, University of Minnesota, Duluth, has accepted a position with the Control Data Corporation, Minneapolis, Minnesota.

Mr. F. E. Dristy, Florida State University, has been appointed Assistant Professor at Florida Presbyterian College.

Dr. S. I. Drobnies, Rice University, has accepted a position as Senior Operations Analyst with the General Dynamics Corporation, Fort Worth, Texas.

Mr. G. R. Faulhaber, Haverford College, has accepted a position as a Member of the Technical Staff with Bell Telephone Laboratories, Holmdell, New Jersey.

Mr. R. A. Fellows, Fairchild Semi-Conductor Company, Mountain View, California, has accepted a position as Q. A. Reliability Engineer with Westinghouse, Newbury Park, California.

Mr. C. K. Fendall, Ford Motor Company, Newport Beach, California, has accepted a position in the Scientific Computation Laboratory of the General Products Division of International Business Machines, San Jose, California.

Mr. R. J. Fey, University of Oklahoma, has accepted a position as Oceanographer with the U. S. Navy Hydrographic Office, Washington, D. C.

Mr. S. M. Fisher, University of Illinois, has accepted a position as Research Engineer with the Autonetics Division of North American Aviation, Downey, California.

Dr. C. H. Franke, Rutgers, The State University, has accepted a position as Mathematician with Bell Telephone Laboratories, Whippany, New Jersey.

Mrs. Marjorie L. French, Supervisor of the Mathematics Departments for Topeka, Kansas, high schools and junior high schools was named National Teacher of the Year for 1962.

Mr. R. A. Gammill, Jr., Philco Corporation, Willow Grove, Pennsylvania, has accepted a position as a Member of the Machine Organization Section of Collins Radio, Newport Beach, California.

Mr. K. W. Garrison, Control Data Corporation, Beverly Hills, California, has accepted a position as Principal Programmer with the Lawrence Radiation Laboratory, University of California, Livermore.

Mr. Gilbert Goering, Colorado State University, has accepted a position as Mathematician with International Business Machines, Poughkeepsie, New York.

Associate Professor Donald Greenspan, Purdue University, has accepted a staff position with the Mathematics Research Center, U. S. Army, University of Wisconsin.

Dr. Felix Haas, Wayne State University, has been appointed Head of the Division of Mathematical Sciences, Purdue University.

Dr. B. F. Hadnot, International Business Machines, New York, New York, has accepted a position as Manager of Missile and Tracking Systems of International Business Machines, Washington, D. C.

Mr. Brindell Horelick, Villanova University, has been appointed Assistant Professor at Lafayette College.

Mr. G. L. Hatch, Central Washington State College, has accepted a position as Programmer Trainee with the Systems Development Corporation, Santa Monica, California.

Assistant Professor L. O. Hatch, Northern Michigan College, has been appointed Assistant Professor at the U. S. Coast Guard Academy.

Associate Professor H. A. Heckart, Simpson College, has been appointed Professor and Head of the Department of Mathematics at Illinois College.

Mr. J. J. Henninghan, Boeing Airplane Company, Seattle, Washington, has accepted a position as Technical Writer with Volt Technical Services, New York, New York.

Mr. J. J. Henrick, Convair, Fort Worth, Texas, has accepted a position as Research Engineer with the Boeing Airplane Company, Seattle, Washington.

Mr. T. L. Hillburn, Phillips Petroleum Company, Phillips, Texas, has accepted a position as Systems Engineer Physicist with the Phillips Petroleum Company, Bartlesville, Oklahoma.

Dr. A. A. J. Hoffman, University of Texas, has been appointed Assistant Professor and Director of the Computer Center at Texas Christian University.

Assistant Professor G. W. Horton, Jr., Upper Iowa University, has been appointed Associate Professor at Illinois College.

Assistant Professor L. C. Huffman, Midwestern University, has been promoted to Associate Professor.

Assistant Professor R. E. Hughs, Lehigh University, has accepted a position as Staff Mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Associate Professor H. G. Jacob, Louisiana State University, has been appointed Professor at the University of Massachusetts.

Mr. T. D. Jones, Michigan State University, has accepted a position as Mathematician with the Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota.

Mr. E. J. Keegan, Research Triangle Institute, Durham, North Carolina, has accepted a position as Operations Analyst with the Strategic Air Command Headquarters, Offutt Air Force Base, Nebraska.

Dr. R. B. Kelman, Mathematical Staff Consultant with the Remington Rand Univac Division of Sperry Rand, has been appointed Lecturer at Howard University.

Dr. H. P. Kerfoot, Jr., University of Southern California, has accepted a position as Director of the Mechanical and Mathematical Sciences Laboratory of Lockheed Missiles and Space Company, Palo Alto, California.

Mr. B. M. Kiernan, Jr., St. Peter's College, has been promoted to Assistant Professor.

Assistant Professor J. E. Kist, Pennsylvania State University, has been appointed Visiting Associate Professor at Purdue University.

Mr. G. D. Knott, American University, has accepted a position as Programmer with the National Institutes of Health, Bethesda, Maryland.

Mr. Nickolas Konopliv, General Mills, Minneapolis, Minnesota, has been promoted to Associate Engineer.

Dr. C. F. Kossack, International Business Machines, Yorktown Heights, New York, has been appointed Industry Operational Analyst with International Business Machines, Houston, Texas.

Dr. H. F. Kreimer, Jr., Yale University, has been appointed Assistant Professor at Florida State University.

Mr. F. W. Kroeck, McGraw-Hill Book Company, New York, New York, has accepted a position as Staff Member in the Director's Office of the National Center for Atmospheric Research, Boulder, Colorado.

Mr. J. B. Kuplic, Beloit College, has been appointed Mathematics Teacher at the Madison East High School, Madison, Wisconsin.

Dr. H. E. Kyburg, Rockefeller Institute, New York, New York, has been appointed Associate Professor in the Center for Humanistic Studies at the University of Denver.

Assistant Professor Richard Laatsch, Oklahoma State University, has been appointed Assistant Professor at Miami University.

Dr. H. N. Laden, Chief, New Systems Development, Chesapeake and Ohio Railway Company, Cleveland, Ohio, has been elected a member of the Board of Directors of National Computer Analysts.

Professor C. H. Lehmann, Cooper Union, retired June 1, 1962 with the title Professor Emeritus.

Mr. J. R. LeRoy, International Business Machines, Los Angeles, California, has accepted a position as Applied Systems Representative with Advance Data Systems, Los Angeles, California.

Assistant Professor A. E. Livingston, Montana State University, has been appointed Associate Professor at the University of Alberta.

Mr. A. H. Lumpkin, University of Texas, has been appointed Instructor at East Texas State College.

Mr. Thomas McFee, David Taylor Model Basin, Washington, D. C., has accepted a position with the Weapons Systems Evaluation Group, The Pentagon, Washington, D. C.

Mr. C. L. McKelvey, George V. Stennes and Associates, Minneapolis, Minnesota, has accepted a position as Associate Actuary with the Minnesota Mutual Life Insurance Company, St. Paul, Minnesota.

Associate Professor J. S. McNair, State University College, Plattsburgh, New York, has been promoted to Professor.

Mr. S. W. Malinowski, University of Pennsylvania, has accepted a position as Research Engineer with the Sylvania Electronic Laboratories, Waltham, Massachusetts.

Dr. J. J. Malone, Jr., Rockhurst College, has been appointed Assistant Professor at the University of Houston.

Associate Professor Marvin Marcus, University of British Columbia, has been appointed Professor at the University of California, Santa Barbara.

Assistant Professor G. H. Meisters, University of Nebraska, has been promoted to Associate Professor.

Mrs. Glenda D. Merhoff, University of Kentucky, has been appointed Instructor at Vanderbilt University.

Dr. W. L. Miser, Lincoln, Massachusetts, has been appointed Professor at Rose Polytechnic Institute.

Dr. D. A. Moran, University of Illinois, has been appointed Research Instructor at the University of Chicago.

Dr. D. D. Morrison, Ramo Woolridge Corporation, Los Angeles, California, has been appointed Assistant Professor at the University of Texas.

Dr. C. V. Newsom, Prentice-Hall, Englewood Cliffs, New Jersey, has been elected President of Educational Services, Watertown, Massachusetts.

Mr. T. P. Norris, Arizona State University, has accepted a position as Physicist with the Navy Electronics Laboratory, San Diego, California.

Associate Professor J. B. O'Toole, U. S. Naval Postgraduate School, has been appointed Associate Professor at Orange County State College.

Mr. V. E. Palmour, Office of Chief of Naval Operations, Washington, D. C., has accepted a position as OEG Representative with the Air Development Squadron Five at the U. S. Naval Air Facility, China Lake, California.

Professor R. S. Pierce, University of Washington, has been appointed Chairman of the Department of Mathematics to succeed Professor C. B. Allendoerfer. Professor

Allendoerfer will remain at the University of Washington as Professor of Mathematics.

Mr. B. W. Pollack, Massachusetts Institute of Technology, has accepted a position as Coder at the Computing Facility of the University of California, Los Angeles.

Dr. S. C. Port, Northwestern University, has accepted a position with the Rand Corporation, Santa Monica, California.

Mrs. Charlie J. Powers, Bureau of Census, Washington, D. C., accepted a position as Junior Scientist—Mathematics with the Avco Research and Advanced Development Corporation, Wilmington, Massachusetts.

Mr. L. H. Rhodes, II, Remington Rand Univac, Dallas, Texas, has accepted a position as AST-Data Systems with the National Aeronautic and Space Administration, Greenbelt, Maryland.

Professor R. F. Rinehart, Case Institute of Technology, has been appointed Director of Research of the Weapons Systems Evaluation Group of the Department of Defense.

Professor C. C. Robusto, St. John's University, has been appointed Director of the Junior College of St. John's University.

Mr. D. J. Roek, Silver Spring, Maryland, has accepted a position as Research Engineer with the Jet Propulsion Laboratory of the California Institute of Technology.

Mr. H. L. Roth, Bell Aircraft, Buffalo, New York, has accepted a position as Astro-Dynamicist with Aerospace, El Segundo, California.

Associate Professor D. C. Russell, Mount Allison University, has been appointed Professor and Chairman of the Department of Mathematics at York University.

Miss Elizabeth Ryan, Hartford, Connecticut, has accepted a position as Mathematician with Nasa-Lewis Research, Cleveland, Ohio.

Mr. H. C. Saar, Albion College, has been appointed Chairman of the Department of Mathematics and Physics at North Central Michigan College.

Mr. D. R. Saathoff, Arizona State University, has accepted a position as Associate Mathematician with the Southwest Research Institute, San Antonio, Texas.

Professor Charles Saltzer, University of Cincinnati, has been appointed Professor at Ohio State University.

Professor Alice T. Schafer, Connecticut College, has been appointed Professor at Wellesley College.

Assistant Professor John Schue, Oberlin College, has been appointed Associate Professor at Macalester College.

Professor R. F. Shortt, Bates College, has been appointed Chairman of the Department of Mathematics at Mansfield State College.

Assistant Professor G. F. Simmons, Williams College, has been appointed Associate Professor at Colorado College.

Associate Professor Paul Slepian, University of Arizona, has been appointed Associate Professor at Rensselaer Polytechnic Institute.

Professor C. D. Smith, University of Alabama, retired June 30, 1962 with the title Professor Emeritus.

Dr. C. V. L. Smith, National Aeronautics and Space Administration, Greenbelt, Maryland, has accepted the position as Head of the Mathematics and Computer Programs in the Research Division of the Atomic Energy Commission, Germantown, Maryland.

Mr. M. H. Smith, John Brown University, has been appointed Instructor at the College of the Ozarks.

Mr. P. F. Smith, Nicholls State College, has been appointed Assistant Professor at Adams State College.

Dr. T. J. Smith, University of Wisconsin, has been appointed Assistant Professor at Kalamazoo College.

Visiting Professor D. E. South, Florida Presbyterian College, has been appointed Professor.

Mr. T. M. Spencer, Sperry Utah Engineering Laboratory, Salt Lake City, Utah, has accepted a position as Specialist-Research with the North American Aviation, Downey, California.

Dr. Maria W. Steinberg, Institute for Advanced Study, has been appointed Assistant Professor at San Fernando State College.

Mr. J. P. Storto, Cook Research Laboratories, Morton Grove, Illinois, has accepted a position as Assistant Mathematician with the Armour Research Foundation, Chicago, Illinois.

Mr. L. R. Tanner, Westminster College, has been appointed Assistant Professor at Jamestown College.

Mr. D. B. Taylor, University of California, Los Angeles, has accepted a position as Manager of Computer Center Operations with Litton Systems, Woodland Hills, California.

Professor W. J. Thron, University of Colorado, has received a Fulbright Grant to lecture at Punjab University, India, during the 1962–63 academic year.

Assistant Professor W. E. Timon, Jr., Northwestern State College of Louisiana, has been promoted to Associate Professor and appointed head of the Department of Mathematics.

Professor J. W. Tukey, Princeton University, has been awarded an honorary degree of Doctor of Science from the Case Institute of Technology.

Mr. W. H. Turner, University of North Carolina, has accepted a position as Mathematician at the U. S. Navy Electronics Laboratory San Diego, California.

Mr. L. J. Watson, Jr., System Development Corporation, Santa Monica, California, has accepted a position as Senior Research Engineer in the Jet Propulsion Laboratory of California Institute of Technology.

Associate Professor M. S. Webster, Purdue University, has been promoted to Professor.

Dr. P. M. Young, Kansas State University, has been appointed Vice President for Academic affairs at the University of Arkansas.

Dr. O. S. Adams, Mount Vernon, Ohio, died March 4, 1962. He was a charter member of the Association.

Associate Professor Madeleine P. Grenard, West Virginia Institute of Technology, died November 29, 1961. She was a member of the Association for five years.

Dean L. R. Harper, Sr., Paine College, died February 20, 1961. He was a member of the Association for five years.

Professor Emeritus A. J. Lewis, University of Denver, died September 26, 1961. He was a member of the Association for thirty-five years.

Associate Professor F. L. Lynch, Jr., Seton Hall University, died March 12, 1962.

Professor E. B. Miller, Illinois College, died May 23, 1962. He was a charter member of the Association.

Professor Emeritus C. A. Noble, University of California, Berkeley, died May 7, 1962. He was a charter member of the Association.

Professor A. W. Recht, University of Denver, died January 8, 1962. He was a member of the Association for thirty-four years.

Charles Jean de La Vallée Poussin, noted Belgian Mathematician, died on March 2, 1962, at the age of 96.

Associate Professor L. E. Wear, California Institute of Technology, died April 2, 1962. He was a member of the Association for forty-six years.

Correction: In a news item concerning Miss Jean E. Sammet in this MONTHLY, *69* (1962) 247, the former and present positions were interchanged. The item should read:

Miss Jean E. Sammet, Sylvania Electric Products, has accepted a position as Boston Advanced Program Manager with International Business Machines, Cambridge, Massachusetts.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OBITUARY—WALTER BUCKINGHAM CARVER

With the death of Walter Buckingham Carver, on July 4, 1961, American mathematics lost one of its most long-serving contributors.

Born in Town Hill, Pennsylvania, on January 11, 1879, he began his career as a student at Gettysburg College and continued at Dickinson College, graduating in 1899. He taught for one year at the Troy Conference Academy in Vermont, then went to Johns Hopkins for graduate study and obtained his Ph.D. in 1904. After two years as Professor of Mathematics and Physics at Ursinus College, he left to join the Mathematics Department of Cornell University. Here he remained for the rest of his life. He was appointed Professor in 1924 and was Chairman of the Department from 1938 to 1940. In 1948 he officially retired, but nevertheless continued to contribute to Cornell mathematics by occasional teaching, student counselling, and the like.

Professor Carver was mathematically active to the very end of his long life. In 1957 he organized the mathematics portion of the Shell Merit Fellowship Program, one of the first of the summer institutes for high school teachers. He conducted this program for the last four years of his life, and was confidently planning for his fifth summer's work when he was struck down by his last illness.

His primary interest lay in the teaching of mathematics to undergraduates. It is not surprising, then, that he was a charter member of the Association and was continually active in its affairs. His first important office in the Association was that of Editor-in-chief of the MONTHLY from 1932 to 1936. In 1937 to 1939 he was an ordinary member of the Editorial Committee, and in 1940 (when the three-man Editorial Committee was abolished) he became Associate Editor until 1942. In the meantime he had served as President of the Association during 1939 and 1940. He succeeded his good friend W. D. Cairns as Secretary-Treasurer in 1943, continuing in this position until retirement. He then became a member of the Finance Committee, serving in this capacity until his final illness forced his resignation in 1960.

Throughout his long career, Carver was a frequent contributor to the MONTHLY. As was noted in the Otto Dunkel Memorial Problem Book, his problems span a period of fifty-six years, longer than that of any other contributor. His last article, "Cyclic Polygons," appeared in the June–July, 1961, issue of the MONTHLY, less than a week before his death, making a total span of sixty years for his contributions to this journal. His most prolific periods were his early years and the time of his retirement. In the intervening time he was busy with his students, with University and Association affairs, and with such matters as grading papers for the College Entrance Examination Board and refereeing books for publishers. (In the twelve year period from 1936 to 1948 he refereed nearly two hundred manuscripts.) His most interesting work is the Slaught Memorial Paper No. 5, "The Conjugate Coordinate System for Plane Euclidean Geometry," published in 1956. This article presents an exposition and elaboration of a method, developed by Frank Morley, of making extensive use of complex algebra to develop the geometric properties of figures in the complex plane.

Walter Carver made no pretense of being a research mathematician. At the dinner honoring his retirement, he remarked that he had decided early in his career that his most useful role would be that of teacher and counsellor. He felt that he had worked hard along these lines and that he need have no regrets at the part he had played in his university and in the world of mathematics. With these sentiments, all who knew him will wholeheartedly agree.

R. J. WALKER

THE 1962 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-third annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 1, 1962. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October first to the regular mailing list. If an application blank is not received by October 15, one may be secured by writing the director, Professor L. E. Bush, 308 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 5, 1962. For further details of the examination and the list of prizes (including the $3,000 scholarship to Harvard), see the announcement which will accompany the application blank.

Reports of the previous competitions and the examinations may be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August-September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958; August-September 1959; January (questions and solutions for eighteenth, nineteenth and twentieth competitions) 1961; and August–September 1961.

L. E. BUSH, *Director*

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1962 by a mail vote of the membership of the Association in the Sections indicated:

Illinois	M. M. Day, University of Illinois
Iowa	J. J. L. Hinrichson, Iowa State University
Louisiana-Mississippi	A. D. Wallace, Tulane University
Maryland-District of Columbia-Virginia	C. F. Stephens, Morgan State College
Michigan	L. E. Mehlenbacher, University of Detroit
Minnesota	J. E. Hafstrom, University of Minnesota
Philadelphia	D. W. Western, Franklin and Marshall College
Southern California	P. J. Kelly, University of California, Goleta
Texas	H. C. Parrish, North Texas State University

The Sections in which more than 50% of the members voted were Louisiana-Mississippi (58%) and Iowa (55%).

L. J. MONTZINGO, *Associate Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The sixth annual meeting of the New Jersey Section of the Mathematical Association of America was held at Saint Peter's College, Jersey City, New Jersey, on Saturday, November 4, 1961. Professor E. P. Starke, Chairman of the Section, presided at the morning session, and Dr. G. Y. Cherlin presided at the afternoon session. There were 126 persons present, including 75 members of the Association.

The following officers were elected: Dr. G. Y. Cherlin, Actuary for the National Health and Welfare Retirement Association, Chairman; Professor R. M. Walter, Douglass College of Rutgers, the State University, Member at Large of the Executive Committee, '63; Professor M. A. Sobel, Montclair State College, Member at Large of the Executive Committee, '64.

The following papers were presented at the morning session:

1. *A modern intuitive approach to calculus*, by Professor John Reckzeh, Jersey City State College. (By invitation).

Function was defined as a set of ordered pairs with no two first elements alike. The advantages of this definition were discussed from a pedagogical point of view rather than one of foundations. Intuitive graphical differentiation and integration were then discussed. These methods could enable a student to discover the fundamental theorem of the calculus without any formal differentiation or integration.

2. *The mathematical theory of mental testing*, by Mr. F. M. Lord, Educational Testing Service and Princeton University. (By invitation).

Standard problems in the mathematical theory of mental testing were outlined: how to describe test items quantitatively, how to relate the properties of a test to those of its items, how to select the items, how to evaluate the test. Inadequacies in the metric provided by the test score were discussed. The true-score-plus-error model was outlined, together with some characteristics that distinguish test theory from the classical theory of errors. Some peculiar effects arising from errors of measurement were described. Typical elementary formulas deduced from the model were presented illustratively.

At the afternoon session—a symposium on *The emerging curriculum in geometry*—the following papers were presented:

3. *Revision of the high school geometry curriculum*, by Professor Walter Prenowitz, Brooklyn College. (By invitation).

Deductive geometry is important in the high school curriculum, both for the mathematics and the liberal arts student. It should be taught not as an abstract deductive science nor merely as a body of facts, but as the simultaneous analysis of a body of empirical knowledge and its deductive organization. The conventional treatment of high school geometry was analyzed critically; suggestions for modification were given. Attention was called to the revised treatments of the Ball State, UICSM, and SMSG programs. Suggestions were made for a teacher training program to prepare for the teaching of a revised type of course.

4. *The UICSM geometry program*, by Mr. Arnold Petersen, Pascack Valley Regional High School. (By invitation).

The speaker described the purposes of a secondary geometry program as the presenting of geometry as a mathematical theory abstracted from physical experience and organized deductively; and the developing of insight enabling one to guess consequences of assumptions, followed by an understanding of logic to show the validity of these conclusions. The topics covered in the UICSM text (Unit 6) were outlined, along with comparisons and contrasts with the traditional presentation. The speaker also cited evidence of the success of the program in the secondary field.

5. *The emerging curriculum in geometry*, by Professor B. E. Meserve, Montclair State College. (By invitation).

The emerging curriculum in high school geometry is primarily concerned with the thinking required of the students, a relatively precise terminology, the recognition of abstractions, and the careful use of deductions. Specific examples were given showing that any teacher with almost any textbook can make substantial progress in implementing contemporary recommendations.

Our best hope lies in the continued evolution of modifications and improvements in traditional courses. The role of experimental programs was visualized as developing materials for the better students and testing conjectured topics.

The work in geometry of the Teacher Training Panel of CUPM was described briefly.

I. L. BATTIN, *Secretary*

THE DECEMBER MEETING OF THE TEXAS SECTION

The Texas Section of the Mathematical Association of America has decided to have two meetings each year, their regular spring meeting and the other in conjunction with the Texas Academy of Science, which is held sometime during the first two weeks of December. A special section was formed for mathematics. The meeting was held at Galveston, Texas, December 8 and 9. Professor W. T. Guy, University of Texas, sectional Vice-President, presided. Professor L. A. Colquitt, Texas Christian University, was elected Vice-President for the next year. The Secretary of the Texas Section absorbed the regular duties of that office for this meeting. About 100 members were present.

These papers were presented:

1. *The geometrical and physical interpretation of the Weyl conformal curvature tensor*, by Professor Alfred Schild, University of Texas, and Dr. F. A. E. Pirani, King's College, London, England, presented by Professor Schild.

A normal, hyperbolic conformal space C_N of N ($\geqq 4$) dimensions is a (sufficiently) differentiable manifold, endowed at each of its points with a real infinitesimal null cone, but with no other metrical properties. Components of the Weyl conformal curvature tensor can be determined by a purely conformal construction which uses four neighboring null geodesics. In the space-time of general relativity, null cones describe the propagation of light; the geometrical construction above corresponds to an idealized measurement of the Weyl tensor by light rays alone, and without the use of clocks or rigid rods; if Einstein's vacuum field equations are satisfied ($R_{mr}=0$), then the Riemann and Weyl curvature tensors coincide, and the Riemann curvature tensor can also be measured by light alone.

2. *An endomorphism theorem and summability*, by Professor E. P. Kelly, Stephen F. Austin State College.

The endomorphism theorem "if G is a group, H a normal subgroup of G and α an endomorphism on G such that $\alpha(h)=h$, for every $h\in H$, then $\alpha^{-1}(H)=HK$, where K is the kernel of α," was interpreted in terms of summability. The condition on α is the property enjoyed by regular summability methods. If G is the difference group of bounded real sequences modulo the null sequences, M/Z, and H is the normal subgroup of convergent sequences modulo the null sequences, C/Z, then the domain of summability of α is $C/Z+K$. In this setting the following theorem can be proved: "if T_1, T_2 are regular summability methods and the domain of summability of T_1 includes the domain of summability of T_2, then T_1 and T_2 are consistent on M/Z."

3. *A "random" number generating subroutine for the 1620*, by Mr. M. H. Burns and Professor G. H. Dubay, Texas Institute for Rehabilitation and Research and University of St. Thomas.

The purpose of the paper was to (1) characterize and (2) apply a set of pseudo-random numbers. The generator is available from T. I. R. R. as a relocatable subroutine supplement to the 1620 Fortran subroutine tape. The power residue method was used to generate 1890 normalized random numbers, X, and find $\overline{X}$ and s_x. Analysis was made (1) of the distribution using the Kolmogorov-Smirnov test, (2) of runs, both up and down and those above and below .5. Monte Carlo applications were presented and other number sources compared.

4. *Matrices over the real field with specified characteristic and minimum equations*, by Professor Louis Brand, University of Houston.

5. *The role of the solution set in mathematics*, by Professor H. J. Ettlinger, University of Texas.

Many suggestions are available for unifying various aspects of mathematical analysis. The basic concept of the solution set for a given number relation is an outstanding example of a fundamental idea which will accomplish this purpose. The objective is to find and gather together the collection of all elements which will satisfy the given linear or nonlinear algebraic equation or inequality, in number answers, or sets of functions with prescribed properties for given differential or integral equations. Illustrations were given from many important branches of mathematics.

6. *On quasi-inverses*, by Mr. Elbert Pirtle, Stephen F. Austin State College. (Introduced by Professor E. P. Kelly)

This problem is a direct analogue of a problem by Kaplansky and requires the Kaplansky problem for its proof. Kaplansky's problem reads as follows: If an element of a ring with identity has more than one right inverse, then it has infinitely many. The analogous problem reads thus: If an element of a ring with identity has more than one right quasi-inverse, then it has infinitely many. As an application, the following can be shown: If a solution to the matrix equation $AX-A-X=0$ exists, then it is unique.

7. *Linear mappings on two-dimensional Hansen spaces*, by Professor D. E. Edmondson and Mr. D. J. Hansen, University of Texas.

$\langle H, +, \cdot, = \rangle$ is a two-dimensional Hansen space if $H=B\times B$ where $\langle B, +, \cdot, = \rangle$ is a Boolean algebra, $+$ on H is $(x, y)+(u, v)=(x+u, y+v)$, $\cdot$ on H is $a\cdot(x, y)=(a\cdot x, a\cdot y)$ for $a\in B$, and $=$ on H is $(x, y)=(u, v)$ if $x=u$ and $y=v$. T, a mapping of H into H, is a linear mapping if $T(ap+bq)=aTp+bTq$ for $a, b\in B$ and $p, q\in H$. *Theorem*: If T is a linear mapping on a two-dimensional Hansen space, then the following are equivalent: (1) T is one-to-one, (2) T is onto, (3) T is invertible, (4) the Hansen determinant of the associated Boolean matrix of T is I and (5) the associated Boolean matrix is orthogonal.

8. *A Laplace integral and a related moment problem*, by Professor J. T. White, University of Texas.

In finding the necessary and sufficient condition that the function $f(s)$ be the Laplace-Stieltjes transform of a function $B(t)$, which is nondecreasing and attains its maximum value, the following moment theorem is needed. A necessary and sufficient condition that the sequence $C_0, C_1, \cdots$ should have the form $C_n=\int_a^1 x^n d\alpha(x)$, $n=0, 1, \cdots$, where $0<a<1$ and $\alpha(x)$ is of bounded variation, is that there exists a number B such that for each positive integer k,

$$\sum_{m=0}^{k} |A_{k,m}| \leqq B, \quad \text{where} \quad A_{k,m} = [(-1)^{k-m}/(1-a)^k]\binom{k}{m}\sum_{r=0}^{m}\binom{m}{r}(-a)^r\Delta^{k-m}C_{m-r},$$

and

$$\Delta^k C_n = \sum_{r=0}^{k}(-1)^r\binom{k}{r}C_{n+k-r}.$$

9. *Finding primitive roots of primes using synthetic division*, by Professor Roger Osborn, University of Texas.

The $\phi(p-1)$ primitive roots of an odd prime p are the ϕ simple roots of a ϕ-degree polynomial congruence. This paper discusses the determination of this ϕ-degree congruence and the subsequent determination of all of its roots (which are the primitive roots of the prime p) by synthetic division modulo p.

10. *On certain relations between Szàsz summability and Cesàro summability*, by Professor Louis Silverman, University of Houston.

In 1944, in a joint paper with the author, L. L. Silverman, Otto Szàsz considered definitions of summability simpler and less general than the arithmetic mean M. These definitions Z_k will be called Szàsz summability of rank k. The corresponding transformations Z_k are permutable among themselves, but not with M. It is proved that MZ_2 and Z_2M, while not identical, are nevertheless equivalent. A second result arises from the fact that if a sequence is summed by Z_2 and Z_3 it is necessarily convergent. It is proved that if a sequence is summed by both MZ_2 and MZ_3 it is necessarily summable by M. Generalizations of these results are also considered.

11. *A distortion theorem for certain conformal maps*, by Mr. G. S. Innis, Jr., University of Texas.

Let a, b and c be positive and let $b>c$. If D is the z-sphere cut along the real axis from c to ∞ and along the line $y=b$ from $-a$ and if f is a univalent analytic function defined on D such that

$f(0)=0$ and $f'(0)=1$, a positive number r is found such that the disk of radius r about the origin is contained in $f(D)$. This result is shown to be sharp and an approximation to r is given.

12. *Some practical aspects of power spectral analysis*, by Mr. A. A. J. Hoffman, University of Texas.

Power spectra of stochastic time series computed using the Tukey-Blackman method have confidence intervals which depend on the number of sampled values, the length of sampling interval, and the number of lags. It is shown how a low-pass filter represented by a numerical linear operator (followed by decimation) may be used to improve the frequency resolution of the spectral estimates without a corresponding increase in the size of the confidence intervals. Recordings of the magnetotelluric field of the earth are subjected to spectral analysis both with and without low-pass filtering to illustrate the improvement in frequency resolution. (This work was supported through a contract with Bureau of Naval Ordnance.)

13. *Confidence intervals for sums and products of binomial probabilities*, by Professor P. D. Minton, Southern Methodist University.

Since the distribution of a binomial variable p is approximately normal for large n, a linear function of k binomial variables is approximately normal if all n involved are large enough. A confidence interval on the linear function of binomial probabilities may be constructed on this basis. The present status of the problem of placing confidence limits on products of binomial probabilities is discussed in the paper.

14. *A topological space of analytic functions*, by Mr. P. J. Knopp, University of Texas.

Let A be the set of functions analytic in the unit disk. A function θ from A to the extended non-negative reals is defined having all the properties of a norm except that possibly $\theta(f)=\infty$. *Lemma*: If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is in A and if $g(z)=\sum_{n=0}^{\infty} b_n z^n$ is a formal power series, and if $\sup |a_n-b_n| = r < \infty$, then g is in A. *Theorem*: A is complete with respect to the function θ.

15. *A perturbation method for detecting ill-conditioned matrices*, by Mr. J. M. Nash, Agricultural and Mechanical College of Texas.

Let $AX=B$ denote a linear system of n equations in n unknowns. Let E be such that $C-EAE^{-1}$ is nearly A. Then for A ill-conditioned, in general $CX=B$ has a solution which differs widely from that of $AX=B$. For example let $A=\|a_{ij}\|$ where $a_{ii}=a$; $a_{ij}=b$, $i\neq j$. It is known that $|a_{ij}| = (a-b)^{n-1}[a+(n-1)b]$. Also let $E=\exp[(-1)^{i+1}\epsilon]$. If a is nearly b (or $[n-1]b$), let ϵ be of the order of magnitude of $a-b$ (or $a+[n-1]b$). More generally a random choice of ϵ_i (ϵ_i negligible in computation), with $E=[\exp(\epsilon_1), \cdots \exp(\epsilon_n)]$, seems indicated.

16. *On the boundary behavior of holomorphic functions in the unit disk*, by Sister Claude Marie Faust, C. C. V. I., Incarnate Word College.

The speaker proves the following theorem as a generalization of a theorem by W. Seidel: Let $f(z)$ be holomorphic in the unit disk. Let τ be a point on the circumference of the unit disk, and let $z_n=r_n\tau$, $0<r_n<1$, $\lim_{n\to\infty} r_n=1$, be a sequence of points for which (1) the non-Euclidean distance $\rho(z_n, z_{n+1})<M$, a positive constant, $n=1, 2, \cdots$, and (2) $\lim_{n\to\infty} f(z_n)=\infty$. Then, there exists a real number α_τ, with $0\leqq\alpha_\tau\leqq\frac{1}{2}\pi$, such that (1) $f(z)$ tends to infinity in every Stolz angle $\Delta_{\tau,\beta}$, where $\beta<\alpha_\tau$, and (2) the complement of the range of f (2) in $\Delta_{\tau,\beta}\, \rho R(f, \Delta_{\tau,\beta})$, consists of at most one point for every Stolz angle $\Delta_{\tau,\beta}$, where $\beta<\alpha_\tau$.

C. R. SHERER, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 24, 1962 at the University of Michigan in conjunction with the annual meeting of the Michigan Academy of Science, Arts and Letters. Professor F. L. Celauro of Central Michigan University, Chairman of the Section, presided at all sessions of the meeting. About 80 persons were in attendance.

At the business meeting, the following officers were elected: Professor J. H. McKay, Michigan State University Oakland, Chairman; Professor N. D. Kazarinoff, University of Michigan, Vice Chairman; Professor J. H. Powell, Western Michigan University, Secretary-Treasurer. Professor R. M. Thrall of the University of Michigan gave the report of the activities of the Board of Governors.

The following papers were presented:

1. *An application of finite symmetric linear operators*, by Professor G. H. Andrews, University of Michigan.

A symmetric linear operator is defined as a real linear combination of the operator G_m (on real valued functions of the reals) defined for m, an integer, by

$$G_m u_x = \begin{cases} u_{x+m/2} + u_{x-m/2} & m \neq 0 \\ u_x & m = 0. \end{cases}$$

For any natural number n, the set of symmetric linear operators of length $\leqq n$ is an n-dimensional vector space. Several bases are defined for this space and it is shown that changes in basis may be used to establish a number of trigonometric identities.

2. *Mathematical applications in operations research*, by Professor R. M. Thrall, University of Michigan.

The purpose of this paper is to discuss something of the range of mathematical applications in Operations Research with indications of some important unsolved problems and also some consideration of what kinds of mathematics belong in an Operations Research curriculum.

3. *Some results on the asymptotic completion of an ideal*, by Professor J. W. Petro, Western Michigan University.

The asymptotic completion A_s of an ideal A in a commutative ring with identity [introduced by Rees, Proc. London Math. Soc., 5 (1955) 105-128] defines a mapping $A \to A_s$ which satisfies the following conditions: i) $A \subseteq A_s$, ii) $A \subseteq B$ implies that $A_s \subseteq B_s$, iii) $A_{ss} = A_s$, iv) $(A_s B_s)_s = (AB)_s$, v) $(A_s + B_s)_s = (A+B)_s$, vi) $(A_s \cap B_s)_s = A_s \cap B_s$, vii) $(AC)_s \subseteq (BC)_s$ and $A \subseteq \text{Rad } C$ together imply that $A_s \subseteq B_s$.

4. *An n-stage delivery-lag inventory model with emergency*, by Professor K. H. Daniel, Michigan State University.

5. *On the structure of organizations*, by Professor Frank Harary, University of Michigan.

We define an *organization* in general as a set of objects on which there is a relation. The structure of the relation may be represented by the points and lines of a directed graph. Since research in each substantive area is amenable to a graph theoretic analysis, the theoretical findings about graphs developed in any one area is often usable in others.

6. *Computing, mathematics, and education*, by Professor P. C. Hammer, University of Wisconsin.

A computing machine may be interpreted in various ways in mathematical terminology—as a transformation mapping input into output or as an implication: input implies output. A computer has a possibility of any one of a number of finite states and the capability of altering a state dependent on that state. Since computing provides a means of becoming more appreciative of mathematics, pure or applied, and since it becomes a virtual necessity in complex situations, computing should be taught certainly in colleges and possibly earlier. It should best be taught by those with active interest in abstract models and interrelations among them—mathematicians

L. E. MEHLENBACHER, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The forty-first annual meeting of the Southeastern Section of the Mathematical Association of America was held March 30–31 at the Woman's College of the University

of North Carolina, Greensboro, North Carolina. Professors E. B. Shanks, Anne Lewis, W. M. Whyburn, and J. E. Maxfield presided over the General and Divisional Sections. Professor M. R. Hestenes, University of California, Los Angeles, was the featured lecturer at the meeting. There were 198 members and guests of the Association in attendance.

The following officers were elected for the coming year: Chairman, Professor Anne Lewis, Woman's College, University of North Carolina; Vice Chairman, Professor D. C. Sheldon, Clemson Agricultural College. The following motion was passed: That a committee on invitations be appointed, each member to serve three years, one member retiring and one new member being appointed each year. The duty of the Committee will be to consider invitations for places to hold the annual meetings of the Section and to make recommendations to the Section at each annual business meeting for final action.

The following papers were presented:

1. *An algorithm for systems of representatives*, by Professor R. R. Korfhage, North Carolina State College.

Let $S_1, \cdots, S_n$ be a collection of subsets of a finite set A, and let n_k be the number of occurrences of an element a_k of A among the sets S_j. Let $K=\{a_k \mid n_k=N\}$, where N is the least positive value of n_k. For S_j such that $S_j \cap K \neq \phi$, let $h_j=\sum_{a_k \in S_j} n_k$; and let H be the minimum value of h_j. If S_{j_0} is the first S_j such that $h_j=H$, let the first element, say a_{k_0}, in $S_{j_0} \cap K$ represent S_{j_0}. Delete the set S_{j_0} and all occurrences of a_{k_0}. Iteration of this process will yield a system of distinct representatives for $S_1, \cdots, S_n$ whenever such a system exists.

2. *On the continuity of the roots of algebraic equations* (1), by Professor A. C. Mewborn, University of North Carolina.

3. *On the continuity of the roots of algebraic equations* (2), by Professor A. T. Brauer, University of North Carolina.

4. *Sums of normal semi-endomorphisms*, by Professor D. A. Robinson, Georgia Institute of Technology.

Let $\{G, \cdot\}$ be a multiplicatively written group and let e denote its identity. A single-valued mapping ϕ of G into G is a *normal semi-endomorphism* of G if (1) $\phi(e)=e$, (2) $\phi(xyz)=\phi(x)\phi(y)\phi(x)$ for all $x, y \in G$, and (3) $\phi(x^{-1}yx)=x^{-1}\phi(y)x$ for all $x, y \in G$. Group $\{G, \cdot\}$ has *Property R* if its normal semi-endomorphisms generate a ring. A sufficient condition is established for $\{G, \cdot\}$ to have Property R and, in particular, it is shown that all finite groups of odd order and all non-Abelian simple groups have Property R.

5. *The quadratic equation of real quaternions*, by Professor H. S. Thurston, University of Alabama.

This paper is a discussion of the equation $X^2+PX+Q=0$ where P and Q are quaternions over the real field. The treatment is analogous to that of the same equation in binary matrices, discussed fully in this MONTHLY, Vol. 61, 1954. Unlike the corresponding matric equation, the so-called *irregular* case occurs if and *only* if P and Q are linear functions of a quaternion A.

6. *On the existence of an improperly primitive form*, by Professor E. H. Hadlock, University of Florida.

The purpose of this paper is to show the existence of a ternary quadratic form $f=ax^2+by^2+cz^2+2ryz+2sxz$, having given integral values of the invariants Ω, Δ and d associated with f, with an almost arbitrary integral value of a. Congruent relations are imposed on b modulo 4, and prime factors of a, Ω and Δ, so that f will be improperly primitive and have the given determinant d associated with it. Then s and r must satisfy certain congruences and finally an integral value of c is obtained in this order.

7. *"Quo Vadis" in mathematics*, by Professor W. L. Williams, University of South Carolina.

This paper was concerned with the present day trends in mathematics on the secondary school level. The author emphasized that colleges and universities should give serious thought to incorporating in their curricula suitable mathematics courses to prepare teachers to teach the new courses with which they will be confronted in the very near future.

8. *Let's be honest*, by Professor C. G. Phipps, Tennessee Polytechnic Institute.

In a deductive system, the definitions must be formed according to strict rules of composition. However, in their axiomatic approach to some topics, mathematicians often violate the rules that have been so carefully laid down. A case in point is furnished by the two so-called definitions of addition and multiplication in the development of the number system. Inspection shows these statements to be assumptions. This note is a plea that we strive to be logically honest at the same time we strive to be logically rigorous.

9. *An extension theorem for monotone functions*, by Professor M. K. Fort, Jr., University of Georgia.

The plane is partially ordered by defining $(x, y) \leqq (u, v)$ if and only if $x \leqq u$ and $y \leqq v$. The notions of *increasing* and *strictly increasing* for real valued functions on a subset of the plane are defined in the obvious way. Now let S be a closed 2-cell in the plane, and let A be a closed subset of S. If f is a continuous real valued function on A which is increasing (strictly increasing), then f can be extended to a continuous real valued function F on S which is also increasing (strictly increasing).

10. *Mathematics in Latin American universities*, by Professor W. L. Furman, Spring Hill College.

A study was made of courses offered in mathematics in Latin American universities and of mathematical publications in Latin America. One year was spent in Colombia in a detailed study of mathematical faculties, library facilities and mathematical research in the universities.

11. *A comparison of the effect of two grading systems on student learning*, by Professors J. E. Maxfield and S. G. Sadler (presented by Professor Sadler), University of Florida.

An experiment was run in an elementary mathematics course having uniform intermediate and final examinations on students learning of the inclusion of a classroom grade, in the final grade determination. Two divisions were made. In one the students were told that their grade would be determined exclusively by the uniform examination. In the other, that a classroom score determined by the instructor would also be considered. A comparison between the two groups of students on the uniform examinations was made. There was no significant difference between the scores of the two groups.

12. *On compact quasi-open mappings in the plane*, by Professor R. H. Kasriel, Georgia Institute of Technology.

Let $f: X \to P$ be a compact mapping from a simply connected plane region X into a plane P. Let sup [number of components in $f^{-1}(y)/y \in f(X)$] be designated as the component degree, $d(f)$, of f. (i) If f is quasi-open, $d(f)$ is finite and for each mapping $g: X \to P$, sufficiently close to f, $d(g) \geqq d(f)$. Furthermore, for each compact quasi-open mapping $g: X \to P$ sufficiently close to f, $d(f) = d(g)$. (ii) f is quasi-open if and only if for each $y \in f(X)$, $f^{-1}(y)$ does not separate X and f is minimal. [The definition of minimal is given by M. K. Fort on page 175 of *Proc. Amer. Math. Soc.* vol. 2 (1951)].

13. *Contraction mappings in uniform spaces*, by Professor W. J. Kammerer, Georgia Institute of Technology.

Let (X, ρ) be a compact metric space and let f map X into X. If there exists an $\epsilon > 0$ such that whenever $0 < \rho(x, y) \leqq \epsilon$, one has $\rho(fx, fy) < \rho(x, y)$, then the set of periodic points of f is a finite nonempty set $A = \{x_1, x_2, \cdots, x_n\}$. If p is a common multiple of the periods, then for each x in X there exists an x_i in A such that $\lim_{n \to \infty} f^{p \cdot n}(x) = x_i$. Furthermore, if any two points in X can be

joined by an ϵ^* chain ($\epsilon^*<\epsilon$) then f possess a unique fixed point x_1 such that $\lim f^n(x)=x_1$. With the appropriate definitions this theorem may be extended to uniform spaces.

14. *Interval integrals*, by Professor Wayman Strother, University of Florida.

15. *Isomorphisms of loop-isotopes*, by Professor B. F. Bryant, Vanderbilt University and Professor Hans Schneider, University of Wisconsin (presented by Professor Bryant).

Let $(L, ')$ be a loop, and denote by $L(a, b)$ the loop defined by $(x\cdot b)^0(a\cdot y)=x\cdot y$. It was shown that

(1) if $L(a, b)$ is isomorphic to $L(c, d)$ under θ, then $L(e, f)$ is isomorphic to $L((e\cdot b)\theta/d, c/(a\cdot f)\theta)$ under θ.
(2) $G=\{\theta \mid L(a, b)$ is isomorphic to $L(c, d)$ under θ for some $a, b, c, d\in L\}$ is a group.
(3) $L(a, b)$ is isomorphic to $L(c, d)$ under the identity mapping if and only if $c\cdot b$ and $a\cdot d$ are in the middle nucleus of $L(a, b)$ and $a\cdot b=c\cdot d$.

16. *Theorems on n-spheres*, by Professor A. R. Amir-Moez, University of Florida.

A theorem concerning the intersection of an orthogonal set of vectors and a real sphere was given in the Mathematische Annalen, 135 (1958) 388–390, where the modification of it to the complex case was left as an open question. Here we study the complex case. Let $(a_1, \cdots, a_n)$ be an orthonormal set in a unitary space E. Let $\beta\in E$ and r be a real positive number such that $|\beta|\leqq r$. Let $\xi\in E$ with $|\xi-\beta|=r$. Then for all h_i, satisfying $|h_ia_i-\beta|=r$ we have

$$\sum[(\max|h_i|)^2+(\min|h_i|)^2]=2nr^2-(2n-4)|\beta|^2.$$

Then a few other equalities follow as corollaries of this theorem.

17. *Signed elements in double nygroups*, by Professor R. G. Blake, University of Florida.

A set which is a semigroup for addition and for multiplication with multiplication distributive over addition and in which order is preserved by addition, multiplication by a positive element, and cancellation of addition, and reversed by multiplication by a negative element is called a double nygroup. The usual laws of signs for ordered rings are valid for double nygroups.

18. *Corrections for plane stress*, by Professor C. B. Smith, University of Florida.

In the problem of plane stress, it is assumed that certain stress components are functions of two variables only and that the remaining stress components vanish. In this paper these assumptions are not made, and the resulting three-dimensional problem is solved approximately by energy considerations. The method is used to discuss the stress distribution in a plate weakened by a small circular hole. The stress distribution arrived at reduces to the usual plane stress solution if the plate is taken to be very thin.

19. *Energy transfer in a beam-pendulum system*, by Professors R. A. Struble and J. H. Heinbockel (presented by Professor Struble), North Carolina State College.

The resonant oscillations of a beam-pendulum system consisting of a beam simply supported at its ends by pendulums is examined. Using a perturbational technique introduced in *Nonlinear Differential Equations*, McGraw-Hill (1962) by R. A. Struble, the pronounced transfer of energy between beam and pendulum oscillations, which occurs when the beam frequency is twice the pendulum frequency, is depicted in closed form as long-period beats modulating the principal harmonics of the oscillations.

20. *A stability theorem obtained with Laplace transforms*, by Professor J. W. Jayne, Georgia Institute of Technology.

Let $F(t)$, $G(t)$ have Laplace transforms. Using an inversion formula, it is shown that under certain assumptions on the transforms $F(t)\leqq G(t)$, $0\leqq t<\infty$.

Consider the nonlinear system $x'=Ax+f(t, x)$, $'=d/dt$, with solution $\phi(t)$ satisfying $\phi=\phi(0)$ at $t=0$. If $f(t, x)$ is continuous for $|x|<\infty$, $t\geqq 0$, and there exists a constant $M>0$ such that $|f(t, x)|\leqq M|x|$ for $|x|<\infty$, $t\geqq 0$, then $|\phi(t)|$ possesses a Laplace transform. [Here $|x|=\sum|x_i|$]. If in

addition the eigenvalues of A all have negative real parts and M is sufficiently small then there exist positive constants K, ϵ such that the transforms of $|\phi(t)|$ and $K|\phi(0)|e^{-\epsilon t}$ satisfy the hypotheses referred to in the first paragraph. Therefore $|\phi(t)| \leqq K|\phi(0)|e^{-\epsilon t}$, $0 \leqq t < \infty$.

21. *On a generalized mean value theorem*, by Professor Karl Matthies, University of South Carolina.

1) Let f be a continuous mapping from the compact interval $[a, b]$ in the reals into a Banach space with norm $\| \|$. Let g be a continuous mapping from $[a, b]$ into the reals. Let S be a countable subset of $[a, b]$. If for every t in the complement of S with respect to $[a, b]$, $\|f'(t)\| \leqq D_+g(t) < \infty$, then $\|f(b)-f(a)\| \leqq g(b)-g(a)$. (Generalization of a result of J. Dieudonné)

2) Let f and g be defined as in 1). Let N be a set of Lebesgue measure zero. In order that "$\|f'(t)\| < D_+g(t) < \infty$ for every t in the complement of N with respect to $[a, b]$" implies $\|f(b)-f(a)\| \leqq g(b)-g(a)$, it is necessary and sufficient that f be absolutely continuous.

22. *Some formulas involving Ramanujan sums*, by Professor C. A. Nicol, University of South Carolina.

23. *Remarks on conjugate points of linear differential equations of 3rd and 4th order*, by Professor J. H. Barrett, University of Tennessee.

Ordinary linear differential equations of second, third and fourth orders are discussed from the standpoint of zeros of certain equations of second and fourth orders forcing 3 zeros of a corresponding third order equation. Some recent results of Maurice Hanan (*Pacific Journal of Math.* 11 (1961) 919–944) are noted for a slightly more general form of third order equations and are obtained by a different approach using Polya's factored form of these equations.

C. L. SEEBECK, JR., *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 49th regular meeting of the Iowa Section of the Mathematical Association of America was held at Wartburg College, Waverly, on April 13 and the morning of April 14, 1962. Chairman Hazel M. Rothlisberger presided. Total attendance was 91, including 22 members of the Association.

The following officers were elected: Chairman, Professor L. E. Pursell, Grinnell College; Vice-Chairman, Professor C. H. Lindahl, Iowa State University; Secretary-Treasurer, Professor E. L. Canfield, Drake University. Discussion was held regarding the Iowa high school mathematics contest arrangement. It was agreed the present plan in conjunction with the Actuaries' Club of Des Moines was a satisfactory one.

The following papers completed the program:

1. *Newtonian gravitational potential for an oblate spheroid*, by Mr. L. Huehn, Iowa State University, introduced by the Chairman.

An expression for the Newtonian gravitational potential function for a homogeneous oblate spheroid is found by considering a boundary value problem for Laplace's differential equation. Of the four terms retained of the potential function, the first term is due to a sphere and the other three terms have coefficients which are functions of the oblateness.

2. *American vs. Russian mathematics education*, by Professor G. H. Miller, Parsons College.

3. *A connected topology for* $[0, 1]$, by Professor S. K. Hildebrand, Iowa State University.

A topology, τ, on $I=[0, 1]$ is given which has the following properties: τ is a connected topology finer than the usual topology. If τ_R is the topology generated by the open sets of τ and right-closed intervals and τ_L is the topology generated by the open sets of τ and by left-closed intervals, then there exist subsets of I, call them L and R, such that $L \cup R = I$, $0 \in L$, $1 \in R$, L open in τ_L, R open in τ_R and $L \cap R = \phi$. {This answers a question posed by J. Stallings in *Fundamenta Mathematicae* XLVII (1959).}

4. *Opaque subsets of a square*, by Professor R. E. D. Jones, Iowa State University.

An opaque set of degree α is defined for any cardinal number α. It is shown that if $2 \leqq \alpha \leqq c$, there exists an opaque set of degree α. From the definition it is easily seen that there is no opaque set of degree one or of degree greater than c.

5. *Application of the Mellin transform to boundary value problems*, by Mr. David Lomen, Iowa State University, introduced by the Chairman.

The Mellin transform is investigated with special emphasis on its applications to the solution of boundary value problems. A technique is given for solution of Laplace's equation in plane polar and spherical polar coordinates.

6. *On the construction of the measurable sets*, by Professor D. E. Sanderson, Iowa State University, introduced by the Chairman.

Given a measure function a, it is possible to determine a binary relation Q on the set of a-measurable sets according to the definition: If A and B are a-measurable sets, then AQB if and only if $a(A) \leqq a(B)$. On the basis of certain axioms for such a Q, we may construct, among other things, the Lebesgue measurable sets without using the completeness property of the real number system.

7. *Periodic integral surfaces for periodic systems of differential equations*, by Professor D. D. James, Iowa State University.

8. *Vector functions with linear norms*, by Professor George Seifert, Iowa State University.

A subset S_α of a Banach space of continuous vector functions on the real interval $[0, 1]$ is shown to be conditionally compact, convex, and to consist of functions all of which have the same linear norm bu, u in $[0, 1]$. If the vector space of function values is n-dimensional, real, and Euclidean, then S_α reduces to a single function $u(\alpha)$, α a vector. Examples are given to show that this is not true in general.

9. *The least-squares approximate solution of a linear ordinary differential equation*, by Mr. Derald Walling, Iowa State University, introduced by the Chairman.

10. *Stability of generalized predictor-corrector methods*, by Mr. R. L. Crane, Iowa State University introduced by the Chairman.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Annual Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Weapons Laboratory, Dahlgren, Virginia, on Saturday, April 28, 1962. Professor W. K. Morrill, Chairman of the Section, presided. Sixty-six persons were present, including 52 members of the Association.

The following officers were elected to serve during the year 1962–63: Chairman, Professor Herta T. Freitag, Hollins College; Vice-Chairmen, Professor H. C. Carter, Mary Washington College, and Dr. C. R. Phelps, National Science Foundation, Washington, D. C.; Secretary, Professor S. S. Saslaw, U. S. Naval Academy; Treasurer, Professor S. B. Jackson, University of Maryland.

The following papers were presented:

1. *Vector lattice analogues of two well known theorems*, by Mr. J. C. MacPherson, U. S. Naval Academy.

If we consider vector spaces on which there is defined an order relation, we are frequently able to replace topological properties with order concepts. In the classical version of the Hahn Banach theorem the existence of an extension for certain linear functionals defined on a subspace of a vector space is asserted. The first result of this paper demonstrates that this result depends only on the vector lattice properties of the real numbers. Observing next that the burden of the proof of the

Stone-Weierstrass theorem rests with the vector lattice properties of $C^R(\overline{X})$, a vector lattice analogue of this theorem is developed.

2. *Monte Carlo synthesis of computer subroutines*, by Dr. Carl Hammer, Manager, and Mr. L. G. Green, Representative, Scientific Applications, Electronic Data Processing, Radio Corporation of America, Washington, D. C.

Computer subroutines in the context of this paper shall imply mathematical subroutines. A routine is designed to operate on a *finite* set of numbers extending over a stated range. Designate this operation by $\phi(x)$ where $x_L \leqq x \leqq x_U$. The operation $\phi(x)$ is designed to simulate the function $f(x)$. The residual errors $r(x)=f(x)-\phi(x)$ determine the quality of the subroutine operator $\phi(x)$. If $\mathrm{MAX}\{|r(x)|\} \leqq L$, we have a subroutine with bounded (maximum) residuals. The probability distribution of $r(x)$ cannot always be determined with ease; statistical (sampling) techniques may then be used to assess $r(x)$ and to perform "quality control" on the subroutine operator.

3. *Maxima and minima without calculus*, by Professor Emeritus L. R. Ford, Illinois Institute of Technology.

It was shown that many standard problems in Calculus can be solved by using only algebra, trigonometry, or plane geometry. The problem of finding the largest polygon of a given number of sides which can be inscribed in an ellipse—which appears in Calculus only for the very simplest cases—was solved with generality by the use of an elastic band.

4. *Rational approximations of several functions in the complex plane*, by Dr. A. V. Hershey, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

Many functions have an ascending power series which is absolutely convergent for all values of their argument and they have an asymptotic series whose truncation error diminishes with increase in their argument. In the numerical evaluation of such functions, the ascending power series can be used for small arguments and the asymptotic series can be used for large arguments. However, there is an intermediate range of arguments in which rounding error interferes with the evaluation of the ascending power series and truncation error is too large to permit use of the asymptotic series. This range can be spanned by a rational polynomial approximation in the complex plane, valid over half of that part of the complex plane which lies outside of the unit circle.

5. *A simple proof of A. Grothendieck and L. Schwartz's theorem on kernels*, by Dr. Witold Bogdanowicz, University of Maryland.

In this paper was given a proof of a theorem of Grothendieck on kernels for the case of infinitely differentiable functions on a compact Lie group. From that theorem the author received Schwartz's theorem that every linear continuous operator L from the space $D(\Omega_2)$ into $D'(\Omega_1)$ has a kernel $T \in D(\Omega_1 \times \Omega_2)$.

6. *General linear approximations and general integration rules*, by Dr. E. W. Schwiderski, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

The advantage of general linear approximations of prescribed functions in comparison to ordinary polynomial approximations is investigated by inspecting the corresponding error terms. Applications to approximations of derivatives and integrals are indicated. Two nonlinear extensions of the trapezoidal rule are introduced. Their significance for the numerical integration of improper integrals is demonstrated.

7. *Renewal polynomials*, by Mr. Harold Kaplan, U. S. Naval Academy.

The renewal equation $p=f-p^*f$ may be solved, in the continuous case, by expansion in terms of Laguerre polynomials, and, in the discrete case, by expansions in terms of polynomials introduced by Gottlieb in 1938. Semi-Markov processes may also be done this way.

HERTA T. FREITAG, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-eighth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 13 and 14, 1962, at the Nebraska Center for Con-

tinuing Education, Lincoln, Nebraska, in conjunction with the seventy-second annual meeting of the Nebraska Academy of Sciences. Professor W. E. Mientka, Chairman of the Section, presided. There were 51 persons present including 46 members of the Association.

The following officers were elected for 1962–1963: Chairman, Professor P. T. Rygg, State University of South Dakota; Vice-Chairman, Professor W. E. Mientka, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *Sieve methods and some applications*, by Mrs. Mildred L. Gross, Doane College and the University of Nebraska.

A discussion of the sieve methods of Brun and Selberg (upper and lower bounds) was presented and an indication of how these methods could be applied to some of the unsolved problems in number theory was given. In particular, the Selberg method, upper bound, can be used to prove the theorem: The number of integers $m \leqq N$ such that m^2+1 is prime, is less than or equal to $CN/\log N$, where C is a positive constant. An indication of how the Selberg method, lower bound, can be applied to the problem of proving that there are infinitely many primes of the form $2p+1$, where p is a prime, was given.

2. *The lattice of equational classes of algebras with one unary operation*, by Mr. Eugene Jacobs, State University of South Dakota.

The notion of the lattice of equational classes of algebras of a given similarity type is due to H. B. Ribeiro. This paper describes the lattice of equational classes of algebras with one unary operation.

3. *Some results concerning two problems in the theory of numbers*, by Mr. Jon Froemke, University of Nebraska.

Some results were presented concerning solutions of Diophantine equations of the type $\sum_{i=1}^{n} x^i = y^{kn}$ and the problem of finding the decomposition of $S = \sum_{i=1}^{n} a_i b_{\pi(i)}$ where $a_i \in A$, $b_i \in B$ and $\pi(i)$ is a permutation of the integers $1, 2, \cdots, n$.

4. *Some new aspects of the restricted Burnside problem for groups, by* Professor R. H. Bruck, University of Wisconsin. (By invitation).

5. *Generalization of the Pythagorean theorem*, by Mr. P. P. Frederickson, University of Nebraska.

The theorem of Pythagoras is usually first proved as a theorem in plane Euclidean geometry concerning the hypotenuse and legs of a right triangle. It may be generalized to higher dimensions in more than one way; in fact, there are $n-1$ generalizations to n-dimensions. Thus, for example, in three dimensions not only is the square of the diagonal of a rectangular parallelepiped equal to the sum of the squares of the three perpendicular edges, but the square of the largest face of a rectangular tetrahedron is equal to the sum of the squares of the other three faces.

6. *The Apollonius circle contact problem*, by Professor C. N. Mills, Sioux Falls College.

In this paper is presented a discussion of the famous Apollonius problem. An elaborate fourth degree equation was developed for the radius of the inner contact circle of three mutually exterior nontangent circles. A similar method will give the radius of each of the other seven contact circles. Special mention was made of the recent article by N. A. Court in the *Mathematics Teacher*, Volume LIV, Number 6, October 1961, in which one finds an extensive history of the Apollonius problem.

7. *Results of the Nebraska Mathematics Contests*, by Professors J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska.

Registration in the fifth Nebraska (Thirteenth National) Mathematics Contest on March 8, 1962, increased for the fifth straight year to a record 2917 students from 145 high schools. Some 622 students participated in both the fourth and fifth contests; their median scores were 7 and 14, respectively ($r=0.56$). The names of two Nebraska contestants appear on the National Honor Roll. (The Nebraska report which can be obtained upon request includes the Nebraska Honor

Roll, tables showing distributions of the team and individual scores and an item analysis of the questions on the 1962 test.)

8. *An exchange property in a pure inseparable field extension*, by Professor P. T. Rygg, State University of South Dakota.

Let F be a pure inseparable extension of the field K. If x and y are in F, y not an element of K, y an element of $K(x)$, then x is an element of $K(y)$ or y is an element of $K(x^{p^f})$ where f is the exponent of x over $K(y)$. Some applications of this exchange property are shown in a discussion of p-basis of F.

9. *Algebraic aspects of the problem of kinematic similarity*, by Professor G. H. Meisters, University of Nebraska.

The paper formulates the problem of "kinematic similarity" (cf. L. Markus, Math. Zeitschr, Bd. 62, (1955) 310) in terms of abstract algebra (not ignoring topological notions) in such a way as (1) to include the special cases which gave rise to the problem in the first place, (2) to organize from a unified point of view the results of S. P. Diliberto, N. P. Erugin, C. E. Langenhop, A. Liapunov, J. C. Lillo, L. Markus, O. Perron, et. al., and (3) to indicate several related problems.

10. *An embedding problem for finite affine planes*, by Professor R. H. Bruck, University of Wisconsin. (By invitation).

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The Spring Meeting of the Oklahoma Section of the Mathematical Association of America was held at Northwestern State College, Alva, Oklahoma, April 13 and 14, 1962. Professor Kathrine C. Mires, Vice-Chairman of the Section, presided. There were 55 persons in attendance including 43 members of the Association.

At the business meeting the group voted to separate the Fall Meeting of the Oklahoma Section of the Association completely from the meetings of the Oklahoma Education Association.

Papers presented at this meeting were largely expository papers presented by students. The following papers were presented:

1. *General theory of natural equivalences by Samuel Eilenberg and Saunders MacLane. from Transactions of the American Mathematical Society, Vol. 58, 1945*, by Mr. R. L. Tennison, Oklahoma State University.

2. *Modern development of tensor analysis*, by Mr. Forrest Miller, University of Oklahoma.

3. *Introduction a la théorie des structures infinitesimales et des pseudogroupes de Lie by Charles Ehresmann, from Geométrie Différentielle, Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953*, by Mr. G. S. Reeves, Oklahoma State University.

4. *A matrix solution for n simultaneous congruences in n unknowns*, by Mr. G. J. Wimbish, University of Oklahoma.

5. *Maximal subadditive extensions*, by Mr. Richard Laatsch, Oklahoma State University.

6. *A transform for number theoretic functions*, by Mr. Roy Fuller, University of Arkansas.

7. *Sur les fondements du calcul opératoire, by Jan G. Mikusinski, Studia Mathematica, Vol. II, 1949*, by Mr. J. E. Allen, Oklahoma State University.

8. *Mathematics, fact or fancy*, by Dr. James Scroggs, University of Arkansas.

9. *Differentiability in conformal n-space*, by Mr. L. E. DeNoya, Oklahoma State University.

10. *Transfinite induction and Zorn's lemma*, by Mr. Gene Hall, University of Arkansas.

11. *A mathematical model of the human heart*, by Mr. P. M. Berry, University of Oklahoma.

12. *Some upper semi-continuous decompositions of E^3 in E^3, by Louis F. McAuley, Volume 73, Annals of Mathematics*, by Mr. M. R. Hagan, Oklahoma State University.

13. *Recurrent tensors*, by Mr. Bill Hodges, University of Oklahoma.

14. *On the rational canonical form of a matrix*, by Professor W. A. Rutledge, University of Tulsa.

15. *Mathematics education in France*, by Professor Roy Deal, Oklahoma State University.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held on April 6–7, 1962 at Rice University, Houston, Texas. There were 150 persons present, of whom 100 were members of the Association.

The following officers were elected: Chairman, Professor G. R. MacLane, Rice University; Vice-Chairman, Professor H. C. Parrish, North Texas State University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

By invitation of the Section, Professor A. D. Wallace, Tulane University, gave the main address entitled, *The structure of topological semigroups—recent advances.*

The following papers were presented:

1. *On the adjoint of a closed transformation*, by Professor Arlen Brown, Rice University.

Let E, F be Banach spaces and let T be a closed linear transformation from E to F with dense domain D. We write E^*, F^* for the dual spaces and D^* for the domain of the adjoint T^*. It is well known that T is bounded if and only if $D=E$. The purpose of the present note is to prove the dual, viz: T^* is bounded if and only if $D^*=F^*$, and to show that this fact can be used to derive, in full generality, the basic alternative theorem for closed operators.

2. *Certain p-primary groups*, by Professor E. R. Keown, Agricultural and Mechanical College of Texas.

An Abelian group P is p-primary if each element of P has order a power of the prime p. An element $x\in P$ has finite height $h(x)=a$ if there exists $y\in P$ with $x=p^a y$, but no $z\in P$ with $x=p^{a+1}z$, and infinite height $h(x)$ if for all $a\geqq 0$ there exists $y\in P$ such that $x=p^a y$. P is called divisible if each of its elements has infinite height. Let P be a p-primary group such that $h(x+y)\leqq h(x)+h(y)$ for all x, $y\in P$ with $x+y\neq 0$. Then P is divisible or is the direct sum of cyclic groups of order p.

3. *A structure theorem for a certain class of primary Abelian groups*, by Mr. H. E. Heatherly, Agricultural and Mechanical College of Texas.

If G is a primary Abelian group such that for each x, $y\in G$, with $x+y\neq 0$, $h(x+y)\leqq h(x)+h(y)$, then G is divisible or is the direct sum of cyclic groups of order p. This lemma is a convenient tool to use in establishing some of the standard theorems of Abelian group theory. One such result is: If G is a primary Abelian group of bounded height, then G is the direct sum of cyclic groups. Since G is of bounded height if and only if G is of bounded order, the previous theorem is equivalent to Prüfer's First Theorem.

4. *Matrices over the ring of continuous complex valued functions on a Stonian space*, by Messrs. Don Deckard and Carl Pearcy, Humble Oil and Refining Company.

Let X be an extremely disconnected compact Hausdorff space, and let M_n be the full ring of $n\times n$ complex matrices under the operator norm. Denote by $M_n(X)$ the *-algebra of continuous functions from X to M_n. Any finite W^*-algebra of type I is a direct sum of algebras each of which has a faithful representation onto some $M_n(X)$. Thus properties of operators can be obtained by studying $M_n(X)$. The authors derive several structure theorems and obtain in particular: Every invertible element in $M_n(X)$ has roots of all orders in $M_n(X)$.

5. *Proof of theorems on Cauchy product by a theorem of Toeplitz*, by Professor L. L. Silverman, University of Houston.

The theorems of Mertens and Cesàro on the Cauchy-product of infinite series can, with the introduction of an appropriate notation, be expressed in a form which naturally suggests for the proof the application of the well-known theorem of Toeplitz on the regular transformation of sequences. This fact is well known. In this note the author merely wishes to call attention to the possibility of using this idea in courses in infinite series and advanced calculus.

6. *On functional representation by a certain type of generalized power series and some of its applications*, by Professor M. A. Al-Bassam, Texas Technological College.

This paper deals with the study and investigation of properties of functions that may be represented by a generalized power series of the form $S=\sum_{n=0}^{\infty} a_n(x-a)^{n\alpha}$, $\alpha>0$, which is convergent on $0\leqq a\leqq x\leqq b<r$, where r is the radius of its convergence. It is shown that if $f(x)=S$, then $a_n=[f'(a)]^{n\alpha-1}/\Gamma(n\alpha+1)$, where the numerator is the $(n\alpha-1)$-derivative of f' at $x=a$. Furthermore, the theory of this representation was used to establish the existence of S-solution of the differential equation of noninteger order: $y^{(\alpha)}+p(x)y=0$, where $p(x)$ admits of S representation.

7. *A generalization of Holder's inequality*, by Mr. Richard O'Neil, Rice University.

If A is a Young's function (nondecreasing, convex, and $A(0)=0$) the norm $\|f\|_A$, of a function f, in the Orlicz space L_A is defined as the greatest lower bound of all $k>0$ such that $\int A(|F(x)|/k)dx\leqq 1$. If A, B, C are Young's functions such that for all $x\geqq 0$, $A^{-1}(x)B^{-1}(x)\leqq C^{-1}(x)$, then the following generalization of an inequality of W. H. Young is valid: $C(\frac{1}{2}xy)\leqq\frac{1}{2}A(x)+\frac{1}{2}B(y)$. It follows that for $f\in L_A$, $g\in L_B$ their product $h(x)=f(x)g(x)$ is in L_C and $\|h\|_C\leqq 2\|F\|_A\|g\|_B$.

8. *Asymptotic values of holomorphic functions of slow growth*, by Professor G. R. MacLane, Rice University.

Let $f(z)$ be holomorphic in $|z|<1$ and let $\int_0^1(1-r)\log|F(re^{i\theta})|dr<\infty$ for $\theta\in\Theta$, where Θ is a dense subset of $[0, 2\pi]$. Then there exists a dense subset, Z, of $|z|=1$, such that for each $J\in Z$ there is a curve Γ in $|z|<1$, ending at J, with $f(z)\to c$ as $z\to J$ on Γ. Here c may be finite or infinite. The modular function shows that Z may be only countable. It has been shown [MacLane, Michigan Math. J. 9(1962)21–24] that functions exist for which none of the curves Γ can be radii. Sundry related results are derived.

9. *On the geometry of meromorphic functions with small characteristic and no asymptotic values*, by Mr. K. F. Barth, Rice University.

Nevanlinna has shown that if $f(z)$ is meromorphic in $|z|<1$ and of bounded characteristic, then $f(z)$ possesses radial limits almost everywhere. G. R. MacLane has constructed a function, $f(z)$, meromorphic in $|z|<1$ with characteristic of arbitrarily slow growth that has no asymptotic values, in particular no radial limits. The object of this paper is to construct such a function geometrically. A simply connected, hyperbolic Riemann surface, S, was constructed using Caratheodory's theory of kernels, and $f(z)$ was defined as the function which maps $|z|<1$ onto S. The growth condition obtained was not quite that of MacLane.

10. *An interesting meromorphic function*, by Mr. J. E. McMillan, Rice University.

Given $\mu(r)$ with $0<\mu(r)\uparrow\infty$, G. R. MacLane has constructed a function ϕ, meromorphic in $|z|<\infty$, with the properties that (1) the image under $w=\phi(z)$ of any unbounded curve in $|z|<\infty$ is dense on the sphere and (2) the Nevanlinna characteristic of ϕ satisfies $\tau(r)<\mu(r)\log r$. ϕ was constructed geometrically by specifying the Riemann surface of its inverse as a covering of the sphere. The object of the present paper is to give an explicit formula for a meromorphic function with properties (1) and (2).

11. *The problem of type for a certain class of simply connected Riemann surfaces*, by Mr. G. S. Innis, Jr., University of Texas.

Let an open simply connected Riemann surface F be formed by joining copies of the Z-sphere

together along simple Jordan arcs, each symmetric with respect to the real axis, in such a way as to form first order branch points at the ends of the arcs. The surface F is shown to be parabolic. If a finite number of the arcs do not satisfy the symmetry condition, it is shown that the surface F is still parabolic.

12. *Some classes of integral solutions to* $a^2+b^2+c^2=d^2$, by Dr. J. F. Gray, S. M., St. Mary's University.

A Pythagorean quadruple or an integral solution to $a^2+b^2+c^2=d^2$ extends the concept of Pythagorean triple and has obvious usefulness in courses ranging from basic spatial geometry through advanced calculus and vector analysis. Various widely differing classes of solutions are presented—e.g., $(n, n+i, n+j, n+i+j)$ for all integral i, j, n such that $ij=n^2$.

13. *Total differential equations in the light of dimensional analysis*, by Professor Louis Brand, University of Houston.

The total differential equation $Pdx+Qdy+Rdz=0$ or $f\cdot dr=0$ is integrable when $f\cdot\text{rot } f=0$. When f is homogeneous in x, y, z the multiplier $1/r\cdot f$ will make the equation exact. When f is isobaric with the nonzero weights a, b, c assigned to x, y, z, the substitution $x=X^a$, $y=Y^b$, $z=Z^c$ will make f homogeneous in X, Y, Z. If $c=0$, put $z=Z/X$; if $b=c=0$, put $y=Y/X$, $z=Z/X$. In *all* cases the given equation admits the multiplier $1/(axP+byQ+czR)$ if the denominator is not zero. An analogous result applies to a total equation with four or more variables.

14. *Modification of Bairstow iteration*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

Let $f(z)$ be a polynomial for which a polynomial factor of degree m is desired. Let $f(z)=g(z)h(z)+q(z)$ where $g(z)$ is a polynomial of degree m while $h(z)$ and $q(z)$ are the polynomials resulting from division of $f(z)$ by $g(z)$. Bairstow iteration consists in applying Newton-Raphson methods to the coefficients of $q(z)$ considered as functions of the coefficients of $g(z)$. It is shown that a simpler technique—one easily adapted to modern computing methods—can be found using linear combinations of the coefficients of $q(z)$.

15. *A finite sequentially compact process for the adjoints of matrices over arbitrary integral domains*, by Mr. L. G. Guseman, Jr. and Professor H. A. Luther, Agricultural and Mechanical College of Texas.

A method is described whereby the adjoint of a square matrix is built, the elements of the matrix being those of an arbitrary integral domain. The method is compact in the sense that only n^2+n locations are used for an $n\times n$ matrix. The technique is well adapted for use with high speed digital computers.

16. *The symmetric method for compactly inverting a real symmetric matrix*, by Mr. J. E. Strub, Agricultural and Mechanical College of Texas.

This paper presents a method which required only $\frac{1}{2}(n^2+3n)$ computational entries to invert any nonsingular real symmetric matrix. Using symmetric row and column operations, the method takes the given matrix A through a finite sequence of triangular arrays and thereby obtains a triangular matrix T and a permutation matrix P having the relation $T^tAT=P$; the identities $P=P^{-1}$ and $TP^{-1}T^t=A^{-1}$ then yield the required inverse. The method includes special operations for the case when, in the main diagonal of any kth array, the $(k+1)$th element vanishes. The IBM 709/7090 program of this method also detects singularity and solves linear systems $X=A^{-1}B$.

17. *The growth of Sylvester's cyclotomic numbers*, by Professor L. K. Durst, Rice University.

Suppose L and M are integers, L positive and M nonzero, α and β are the roots of $z^2-L^{1/2}z+M$, $C_n(z)$ is the nth cyclotomic polynomial, and $Q_n=\beta^{\phi(n)}C_n(\alpha/\beta)$. Then $Q_n\geqq\gamma^{\phi(n)-2^m}$ if $M<0$ or $L>4M>0$, where $\gamma=\frac{1}{2}(1+\sqrt{5})$, m is one less than the number of distinct primes dividing n, and $n>2$. This result affords completely elementary proofs of the theorems of Lekkerkerker, Ward and Durst on the exceptional indices of real Lucas and Lehmer sequences.

18. *A representation theorem for the Laplace transform*, by Mr. J. T. White, University of Texas.

In finding a necessary and sufficient condition that a function $f(s)$ be the Laplace-Stieltjes transform of a function $\beta(t)$, where $\beta(t)$ is nondecreasing and attains its upper bound, the following theorem is needed. A necessary and sufficient condition that the sequence C_0, C_1, C_2, $\cdots$ should have the form $C_n=\int_a^1 x^n d\alpha(x)$, $n=0, 1, 2, \cdots$, where $0<a<1$, and $\alpha(x)$ is nondecreasing, is that for each choice of nonnegative integers m and k with $0\leqq m\leqq k$, $A_{k,m}\geqq 0$,

$$A_{k,m}=(-1)^{k-m}\sum_{r=0}^{m}(-a)^r\binom{m}{r}\Delta^{k-m}C_{m-r} \text{ and } \Delta^k C_n=\sum_{r=0}^{k}(-1)^r\binom{k}{r}C_{n+k-r}.$$

19. *A theorem on nonconstant, nonunivalent analytic functions*, by Mr. P. J. Knopp, University of Texas.

Let A be the set of functions analytic in the unit disk. If f is in A and $f(z)=\sum_{n=0}^{\infty}a_n z^n$, then define ϕ by $\phi(f)=\sup\{|a_n|\}$ if it exists, and $\phi(f)=\infty$, if $\sup\{|a_n|\}$ does not exist. Induce a topology on A using ϕ.

THEOREM: *The set of nonconstant, nonunivalent functions in A form an open set in A.*

20. *A model for an atmospheric jet-stream*, by Professor J. T. Hurt, Agricultural and Mechanical College of Texas.

Preliminary results from a simple mathematical model of a jet-stream indicate some factors that may explain the appearance and disappearance of a jet-stream in the upper atmosphere of the earth.

21. *A characterization of the gamma distribution*, by Mr. V. Seshadri, Southern Methodist University.

Suppose that (1) X, Y, are two independent random variables, and $X/(X+Y)$ is a beta variate with parameters X_1, X_2. A necessary and sufficient condition for X and Y to be distributed as gamma variates with the same scale is that $E[X/(X+Y)]^n=E(X)^n/E(X+Y)^n$, for all n. This condition yields the moments of both X and Y from which the moment generating functions can be easily deduced.

22. *Approximate confidence intervals for the product of two binomial parameters*, by Mr. B. S. Shannon and Professor P. D. Minton, Southern Methodist University.

For a system consisting of two elements operating in parallel interval estimation of the probability of failure of the system by failure of either element is considered. Buehler (J.A.S.A., *52* (1957) 482) gave a one-sided confidence interval based on a Poisson approximation. In this paper, a normal approximation is given following Buehler's approach.

23. *Distribution of estimates of parameters from incomplete bivariate samples*, by Professor P. D. Minton, Southern Methodist University, and Dr. Manus Foster, Socony Mobil Company, Inc.

Given a sample of n paired observations $(x_1, y_1), \cdots, (x_n, y_n)$ plus some extra observations on y alone, say $y_{n+1}, \cdots, y_{n+m}$, with the assumption that (x, y) is bivariate normally distributed, the problem is the estimation of the parameters of the distribution, particularly the mean of x. This would have practical applications where observations on x are costly and difficult and those on y are easy. Nicholson (J.A.S.A., 52 (1957) 523) provides maximum likelihood estimators for the parameters; the present paper appeals to the asymptotic normality of maximum likelihood estimators and shows the asymptotic means, variances and covariances of the estimators.

24. *On a class of truncated sequential sampling plans*, by Mr. W. T. Tucker, Southern Methodist University.

In this paper is presented a general method of attack to the problem of truncated sequential analysis which is applicable to a class of distributions. The sampling process is not as efficient as the Wald method for untruncated sampling plans. However, exact solutions for the operating characteristic curve and the average sample number curve may be obtained for the case of truncated sampling.

25. *On sequence to sequence transformations*, by Mr. J. P. Brannen, University of Texas.

Let $\{s_n\}$ be an absolutely convergent complex number sequence, i.e., $\sum_{p=0}^{\infty}|s_p-s_{p+1}|$ converges, and if the function g is Riemann integrable on $[0, 1]$, let H^g denote the Hausdorff matrix determined by g and let $\{a_n\}$ denote the moment sequence determined by g. The sequence $\{t_n\}$ where $t_n=\sum_{p=0}^{n}H^g_{n,p}s_p$ converges for all absolutely convergent $\{s_n\}$ if and only if for each nonnegative integer $P\{\binom{n}{p}\Delta^{n-p}a_p\}_{n=p}^{\infty}$ converges.

26. *Mathematics training for prospective elementary school teachers*, by Mr. H. H. Walbesser, University of Texas.

Recent content changes in the elementary school mathematics curriculum, initiated by experimental projects such as the School Mathematics Study Group, have created the need for a corresponding change in the training of elementary school teachers. Given the opportunity, the average elementary education major is able to successfully study the foundations of elementary mathematics from a rigorous, axiomatic point of view. One course of study which has proved effective begins with the development of number systems and progresses through studies of nonmetric geometry, the language of sets, and symbolic logic.

27. *An evaluation of the effectiveness of a collegiate general mathematics course*, by Professor Lillian K. Bradley, Texas Southern University.

In order to evaluate the effectiveness of the collegiate "General Mathematics" course, it was necessary to determine: how the course fitted into the program of the college; the students' mathematical background and vocational background and aim; the objectives and content of the "General Mathematics" course as well as the objectives and content of the general mathematics textbooks; and the extent to which the objectives of the "General Mathematics" course were being achieved. The results of this study indicated that the "General Mathematics" course was more effective for the poorer student concerning "Proof-deductive and inferential reasoning."

28. *Mathematical truth in lending*, by Mr. R. A. Hassell, Southern Methodist University.

The principal objection raised by the opponents of the "Truth in Lending Bill" (Federal Bill 1740, currently in committee) is that it is an almost impossible task to list the simple annual interest on a declining principal. This paper presents a new approach with a formula from which a table has been compiled. It explains how this table makes it an uncomplicated task for merchants and lenders to list the simple annual interest rate charged in transactions with consumers.

29. *A mathematics in-service education project in Dallas Elementary Schools*, by Professor Roger Osborn, University of Texas.

A project for in-service education of elementary school teachers of mathematics was carried out in Dallas during the 1960–61 school year. Lectures, TV programs, consultation services, and printed materials were all used in the project. In connection with the project, studies were made of the effectiveness of various media of presentation of the materials. More immediately of importance than the statistical results of the study are the implications of such in-service education programs for the stimulation of teacher interest and professional growth.

30. *A proposed new mathematics course for business majors*, by Professor L. A. Colquitt, Texas Christian University.

31. *CUPM recommendations are what now?* by Professor W. T. Guy, Jr., University of Texas.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The forty-first annual meeting of the Illinois Section of the Mathematical Association of America was held at North Central College, Naperville, Illinois, on May 11–12, 1962. Professor T. E. Rine, Chairman of the Section, presided at all sessions. There were 97 persons in attendance, including 78 members of the Association.

The following officers were elected to serve for the coming year: Chairman, Professor Anice Seybold, North Central College; Vice-Chairman, Professor John Olmsted, South-

ern Illinois University; Secretary-Treasurer, Professor Arnold Wendt, Western Illinois University.

During the business meeting Professor Douglas Bey, in reporting for the Committee on Secondary School Lecturers, stated that 27 individuals had served 50 high schools as lecturers. However, twice this many schools had completed applications for lecturers.

Professor J. M. Sachs reported for the Committee on Undergraduate Membership. One recommendation was as follows: That the Mathematical Association of America be asked to consider establishing a committee to recommend respected mathematicians as lecturers to students at the Section meetings to present some ideas of the current fields of interest to research mathematicians, the presentation being concrete and as elementary as possible, concluding with suggestions for further investigation by undergraduates. This recommendation was approved by the Section.

The report of the Committee on the Strengthening of the Teaching of Mathematics, presented by Professor Gertrude Hendrix, contained several recommendations, each recognizing that the CUPM recommendations for the training of the teachers of mathematics for all five levels constitute a reasonable minimal goal toward which we should strive in the near future. The strongest recommendation was one to urge the State Legislature to pass a bill to strengthen considerably the requirements for certification of elementary school teachers. The report was unanimously approved by the Section.

The Friday evening banquet speaker was Professor E. C. Kiefer, former Secretary-Treasurer and former Governor of the Section. He gave a very interesting description of the founding of the Illinois Section in 1917, of the men involved, and some of the topics discussed at the early meetings. He continued by pointing out some of the high-lights of the work of the Section to the present time.

Following a brief welcome by Dr. Arlo L. Schilling, President of North Central College, the following program was presented.

1. *Mathematics applied to business operations*, by Mr. Donald Schiller, Peat, Marwick, Caywood, Schiller and Co., Chicago, Illinois. (By invitation.)

A discussion of some of the fields in which mathematics is applied to business with examples and case histories from the files of PMCS and Co. Some differentiation between the present applications of mathematics to business operations from the more usual fields of engineering are given.

2. *Report from CUPM*, by Dr. R. J. Wisner, Executive Director, CUPM.

A progress report on the activities of the CUPM, the areas in which it is working to strengthen the teaching of mathematics, and the recommendations that have been made at this date.

3. *An advanced proof of an elementary theorem*, by Professor L. J. Rodabaugh, Southern Illinois University, introduced by Professor W. C. McDaniel.

4. *Randomness, absolute and relative*, by Professor C. E. Langenhop, Southern Illinois University. (By invitation.)

Let H_0 and H_1 denote two stochastic operations on a finite set and let H_2 denote H_1 followed by H_0. Let H_i be characterized by the probability measure P_i on a corresponding finite set S_i where $S_2=S_1\times S_0$ and P_2 is an amalgam of P_1 and P_0 appropriate to the specific interpretation. DEFINITION: *H_i is absolutely random if P_i is uniform on S_i and H_1 is random relative to H_0 if* $P_0(x)=\sum_{t\in S_1}P_2(t,x)$ for all $x\in S_0$. Various interpretations and corresponding theorems relating relative randomness of H_1 to absolute randomness of H_0 and, or, H_1 are given.

5. *The invariant* B^2-4AC, by Professor Rose Lariviere, University of Illinois, Navy Pier.

This report exhibits an unusually brief proof of the invariance of B^2-4AC under rotation of coordinate axes to simplify the conic $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

6. *Summation of factorials*, by Professor Michael Skalsky, Southern Illinois University.

By use of a generating function and Leibnitz' formula, the following identity has been proved: $\sum_{k=0}^{n}(m+k)_m=(m+n+1)_{m+1}/(m+1)$ where $(x)_m=x(x-1)\cdots(x-m+1)$.

7. *A proof of the extremal value theorem*, by Professor Gerald Jungck, Bradley University, introduced by the secretary.

It is shown that if $f(x)$ is continuous on $[a, b]$ and if $f(a)$ is not maximum, then the l.u.b. of the set $S=\{c\in(a, b]\mid \exists\ a \text{ point } d\in[c, b]\ni f(d)>f(x) \text{ for } x\in[a, c)\}$ yields the maximum value of $f(x)$ on $[a, b]$.

8. *Note on the existence of the definite integral*, by Professor C. E. Langenhop, Southern Illinois University and Professor John M. H. Olmsted, Southern Illinois University, presented by Professor Olmsted.

Techniques for establishing existence of mathematical entities based on limit processes are relatively advanced when their values are unknown. For example, the integrability of products involves much greater sophistication than does that of sums. For beginning classes in calculus there is no elementary proof of integrability for continuous functions. However, the two additional assumptions of a bounded derivative and existence of a primitive on the closed interval of integration (both clearly true for the most frequently encountered functions) make a proof of integrability, with a simultaneous proof of the fundamental theorem of integral calculus, practical for consideration in a beginning course.

A. W. MCGAUGHEY, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, on Saturday, May 5, 1962, under the aegis of the Indiana School and College Committee on Mathematics. One hundred forty persons attended the meeting, including fifty-one members of the Association.

Professor John Yarnelle of Hanover College, Chairman of the Section, presided at the meeting. New officers of the Section, elected at a brief business meeting, are Professor Ernst Snapper, Indiana University, Chairman; Professor Kaj Nielsen, Butler University, Vice-Chairman; and Professor Paul Mielke, Wabash College, Secretary-Treasurer.

The program for the meeting centered on the study of geometry and consisted of three one-hour lectures, each followed by forty-five minutes of discussion, as follows:

1. *The high school curriculum in geometry or Is Euclid on the skids?*, by Professor R. A. Rosenbaum, Wesleyan University.

2. *The college curriculum in geometry*, by Professor R. H. Bruck, University of Wisconsin.

3. *Lines in projective space*, by Professor H. S. M. Coxeter, University of Toronto.

P. T. MIELKE, *Secretary*

THE MAY MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on May 5, 1962. Professor A. W. Goodman, Chairman of the Section, presided at both the morning and afternoon sessions. Fifty-seven persons attended the meeting including forty-nine members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor W. H. Spragens, University of Louisville; Secretary-Treasurer, Professor W. C. Royster, University of Kentucky.

By invitation Professor B. J. Pettis delivered an hour address at the afternoon session. The title of his address was "Some Comments on the Structure of Mathematics."

The following papers were presented:

1. *The 1961 Curriculum Study in Mathematics*, by Professor A. S. Howard, Eastern Kentucky State College.

2. *A geometric definition of an analytic function*, by Professor A. W. Goodman, University of Kentucky.

3. *The* (F_n, a_{nk}) *topological space*, by Mr. T. R. Westbrook, University of Louisville.

4. *Eigenvalue problems of ordinary differential equations*, by Mr. James Rolf, University of Kentucky.

V. F. COWLING, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-fifth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at The South Dakota School of Mines and Technology, Rapid City, South Dakota, May 4 and 5, 1962. The meeting was divided into several sessions with Professors F. M. Carpenter, P. O. Steen, and Lawrence Fearnley, presiding. There were 68 persons registered for the meeting.

Officers elected at the meeting for 1962–1963 were: Chairman, Professor H. J. Fletcher, Brigham Young University; Vice-Chairman, Col. J. W. Ault, United States Air Force Academy; Secretary-Treasurer, Professor Leota C. Hayward, Colorado State University; and Director of High School Mathematics Contest, Professor D. C. B. Marsh, Colorado School of Mines.

The following papers were presented:

1. *Quasi-resolutions of the identity*, by Professor E. R. Deal, Colorado State University.

For certain nonspectral operators, if the conditions that $\{E(\delta)\}$ be a resolution of the identity be weakened to the condition that $\{E(\delta)\}$ be an operator measure, it is still true that T may be represented in the form $T=\int_{\sigma(T)} \lambda E(d\lambda)+N$ where N is a generalized nilpotent operator. Such an operator T is called a quasi-spectral operator. Examples of quasi-spectral operators are given, and sufficient conditions for an operator to be quasi-spectral are given.

2. *Convergence and stability in the numerical integration of ordinary differential equations*, by Professor R. A. Hansen, Brigham Young University.

The use of the class of difference equations $y_{n+k}+a_{k-1}y_{n+k-1}+\cdots+a_0y_n=h(b_ky'_{n+k}+\cdots+b_0y_n')$ for the numerical solution of the initial value problem for an ordinary differential equation $y'=f(x, y)$, $y(a)=y_0$, is considered. Sufficient conditions are indicated which guarantee the convergence of the solution of the difference equation to the solution of the differential equation as the tabular interval h approaches zero. Stability of solution is defined and sufficient conditions are given which insure stability.

3. *Some n-dimensional coverage problems*, by Professor W. C. Guenther, University of Wyoming.

The center of a sphere is aimed at a point target in an n-dimensional coordinate system, with aiming errors being governed by a p.d.f. $f(X)$. Before the sphere arrives the point selects a new position according to a probability law whose p.d.f. is $g(X')$. The probability that the sphere covers the point target when the sphere comes to rest is computed for several choices of $f(X)$ and $g(X')$.

4. *A linear congruence with side conditions*, by Professor David Rearick, University of Colorado.

For positive integral r and n, and integral m, denote by $\phi_r(n, m)$ the number of distinct solutions of the congruence $x_1+x_2+\cdots+x_r=m \pmod n$ with x_i relatively prime to n for all i. It is shown that $\phi_r(n, m)$ and the rth power of Ramanujan's exponential sum $C_n(m)$ form a Fourier transform pair. From this is deduced a formula for $\phi_r(n, m)$ in terms of the Euler ϕ-function.

5. *An integral transform*, by Professor R. H. Niemann, Colorado State University.

The Riemann Stieltjes integral from zero to infinity of $g(z+t)/g(z)$ with respect to $c(t)$ has several interesting special cases. Here $g(z)$ is assumed to be analytic and $e^{pz^q}g(z)$ can be represented in an asymptotic series in a sector of the complex plane that includes the positive real axis. The exponents p and q are polynomials in z and $c(t)$ is a function of bounded variation. If $g=\exp(-z^2/2)$ the integral reduces to the Laplace transform. If g is the reciprocal of the gamma function the integral reduces to the factorial transform and the factorial series if $c(t)$ is chosen properly.

6. *Concave functions and points of inflection*, by Professor L. C. Barrett, South Dakota School of Mines and Technology.

In this note we give an analytic definition for concavity of a function at a point and then extend the definition to concavity over an interval. Generalizations of the concept are noted and point of inflection is also defined. These ideas are enlarged upon by means of theorems and illustrative examples. It is pointed out that the definition of concavity may be formulated in terms of a determinant, or in terms of second order central differences.

7. *Matrices of basis vectors*, by Captains R. L. Eisenman and D. R. Barr, United States Air Force Academy, presented by Captain Barr.

A linear combination of basis vectors can be written as a formal product RC_R, where R is a row matrix of basis vectors and C_R is a column matrix of coefficients. This notation has been used at the Air Force Academy, and has advantages in unifications of concepts and notations of vector analysis and linear algebra, and in formulation and testing of conjectures. These advantages are illustrated by the finding of the relations between the matrices of a linear transformation of a vector space into itself in two different bases and by a generalization of the Coriolis theorems, respectively.

8. *The Fibonacci matrix modulo m*, by Professor D. W. Robinson, Brigham Young University.

The periodic properties of the Fibonacci sequence modulo m (see D. D. Wall, this MONTHLY 67 (1960) 525–532) are studied by considering integral powers of the 2-by-2 matrix with first row (0, 1) and second row (1, 1).

9. *Some convergence results for continued fractions*, by Professor K. L. Hilliam, University of Colorado.

10. *On the convergence criterion of Du Bois Reymond and the theory which has evolved from it*, by Professor Alexander Peyerimhoff, University of Utah.

The theory of convergence—and summability factors is discussed as a generalization of the convergence criterion of Du Bois Reymond. This generalization is obtained by weakening one of the assumptions of the criterion through the idea of summability. Complete results are obtained if the method of summability is connected with a certain mean value theorem—as was observed first by L. S. Bosanquet in the case of the Cesáro method.

11. *Some observations regarding binomial coefficients*, by Dr. T. C. Fry, Consultant to the Director, National Center for Atmospheric Research.

12. *A multiphase diffusion problem*, by Lt. C. F. Lutz, U. S. Air Force Academy.

The diffusion equation is $D_i\partial^2 C_i/\partial X^2 = \partial C_i/\partial t$ where the diffusivity, D, is considered a constant. The solution of the diffusion equation is derived for the experimental situation of an infinite bar of constant cross section, where the bar may be regarded as extending along the x-axis. After time $t=0$ it is found that the diffusion causes a separation of the bar into n-segments corresponding to n pure phases. Thus, there are phase boundaries at points, X_i, where discontinuous changes in C_i are observed. The solution is generalized for the case where D is variable.

13. *A mathematical treatment of the eutectoid in the W-C system*, by Professors G. W. Orton and Rudolph Speiser, United States Air Force Academy and The Ohio State University, presented by Lt. Col. George Orton.

An analysis is made of the three univariant curves about the W-C eutectoid to relate temperature, enthalpy and the activity of carbon in the reactions. The activity of carbon determined experimentally is related to the free energy change in the reaction and the free energy is expressed in terms of enthalpy and entropy. Values are presented for ΔF^0, ΔH^0 and ΔS^0 of each reaction.

14. *The college training of high school teachers*, by Professor W. R. Orton, University of Arkansas.

LEOTA C. HAYWARD, *Secretary*

DISTRIBUTION OF MAA FILMS

Effective September 1, 1962, the Association will discontinue the free distribution of the films by Henkin, McShane, and Hewitt, which were produced by the MAA Committee on Production of Films. These films may be rented from Modern Learning Aids, 3 East 54th Street, New York 22, N. Y.

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CALENDAR OF FUTURE MEETINGS

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

Forty-fourth Summer Meeting, University of Colorado, Boulder, August 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State University, University Park, May 4, 1963.

ILLINOIS, Northern Illinois University, De Kalb, May 10–11, 1963.

INDIANA, Evansville College, October 5, 1962.

IOWA, Iowa State University, Ames, April 19–20, 1963.

KANSAS, Kansas State University, Manhattan, April 20, 1963.

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1963.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D.C., December 1, 1962.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University, East Lansing, March 23, 1963.

MINNESOTA, Bemidji State College, November 3, 1962.

MISSOURI

NEBRASKA, University of Nebraska, Lincoln, May 3–4, 1963.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, Connecticut General Life Insurance Company, Bloomfield, Connecticut, November 24, 1962.

NORTHERN CALIFORNIA, University of California, Berkeley, January 1963.

OHIO, Ohio State University, Columbus, May 4 1963.

OKLAHOMA, Oklahoma City University, November 10, 1962.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.

ROCKY MOUNTAIN, Brigham Young University, Provo, Utah, Spring, 1963.

SOUTHEASTERN, University of Chattanooga, Chattanooga, Tennessee, March 29–30, 1963.

SOUTHERN CALIFORNIA, University of California, Riverside, March 9, 1963.

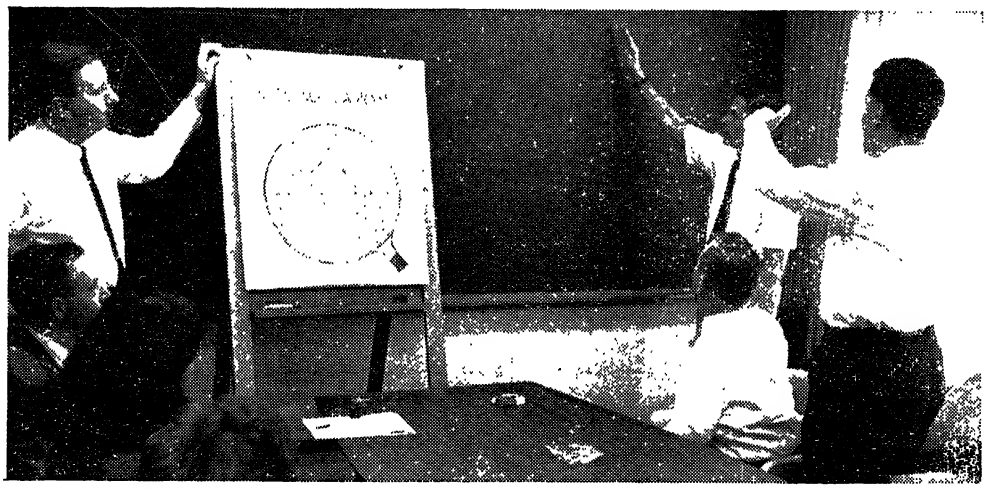
SOUTHWESTERN, Arizona State College, Flagstaff, April, 1963.

TEXAS, North Texas State University, Denton, April 19–20, 1963.

UPPER NEW YORK STATE, University of Buffalo April 27, 1963.

WISCONSIN, Carroll College, Waukesha, May 4, 1963.

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1. The purpose of this paper is mainly pedagogical. While there are, in general, comparatively few opportunities, on the elementary level, to show the student how the arsenal of mathematical weapons he has acquired can be mobilized for an attack on a problem arising outside of mathematics, probability theory abounds in them. The problem we present here is one of the vast class of waiting line problems. It is simple to state and not too difficult to solve. It furnishes an excellent illustration of the elementary theory of Markov chains and it leads to instructive analytical byplays. Above all, it is an example of "live" mathematics in action.

Suppose that a shuttle, for example a helicopter, leaves a certain station periodically, say every twenty minutes. Prospective passengers appear at the point of departure and wait in line. For reasons of safety, the vehicles in question can carry only a fixed number of passengers, say n. Thus at departure time the first n passengers in line board, and those remaining await a subsequent departure. We assume that the shuttle always departs on schedule, with some or all seats empty if there are fewer than n passengers in line at the scheduled time of departure. We suppose known, for each m, the probability ρ_m that exactly m passengers arrive in the interval between consecutive departures, and in particular we suppose that this probability is independent of the time of day. We should like to find the average number of passengers waiting at any given departure time: this is a question which for instance would interest an architect planning a waiting room, or an engineer seeking to determine the frequency with which the shuttle should depart.

Let us suppose that the system goes into operation at a certain time, and that subsequent departures are numbered consecutively. Let X_k be the number of passengers arriving between the $(k-1)$st and kth departures, and let M_k be the number of passengers waiting in line just before the shuttle is boarded for the kth departure; then

$$M_{k+1} = X_{k+1} + \max\{M_k - n, 0\},$$

that is, the passengers waiting at the $(k+1)$st boarding are just those who arrived subsequent to the kth departure together with any who were left in line after the kth departure. Formally we imagine a 0th departure at which time no passengers are waiting

$$M_0 = 0;$$

we are supposing then that $X_1, X_2, \cdots, X_k, \cdots$ are independent random variables subject to the distribution ρ and that $M_1, M_2, \cdots, M_k, \cdots$ are defined by this initial condition and the above recursion formula. We are in effect asking about the distribution of the variables M; we will prove that under certain reasonable assumptions the average of their means over the first N departures tend to a certain limit as N becomes large.

2. Let $P(\lambda|\eta)$ be the probability that if $M_k=\eta$ then $M_{k+1}=\lambda$. Evidently if $M_k=\eta$, then

$$M_{k+1} = X_{k+1} + (\eta - n) \qquad \text{if } \eta \geqq n,$$
$$M_{k+1} = X_{k+1} \qquad \text{if } \eta < n;$$

therefore if $M_k=\eta$ then $M_{k+1}=\lambda$ only if X_{k+1} is $\lambda-(\eta-n)$ when $\eta\geqq n$, and only if X_{k+1} is λ when $\eta<n$. It follows that

$$P(\lambda \mid \eta) = \begin{cases} \rho_{\lambda-(\eta-n)} & \text{if } \eta \geqq n, \\ \rho_\lambda & \text{if } \eta < n. \end{cases}$$

We see that we are dealing with a Markov chain whose transition probabilities are the quantities $P(\lambda|\eta)$.

Let $P^{(k)}(\lambda)$ be the probability that $M_k=\lambda$. Evidently

$$P^{(k+1)}(\lambda) = \sum_{\eta=0}^{\infty} P(\lambda \mid \eta) P^{(k)}(\eta)$$

and therefore on making use of the above values for $P(\lambda|\eta)$ we obtain the recursion formula

$$P^{(k+1)}(\lambda) = \sum_{\eta=0}^{n-1} \rho_\lambda P^{(k)}(\eta) + \sum_{\eta=n}^{\infty} \rho_{\lambda-(\eta-n)} P^{(k)}(\eta).$$

Since $M_0=0$, it follows that

$$P^{(0)}(\lambda) = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

and these equations serve in principle to determine all the distributions $P^{(k)}$.

We will study the distributions $P^{(k)}$ by studying their generating functions. Define for each z in the open unit disk of the complex plane

$$f(z) = \sum_{\lambda=0}^{\infty} \rho_\lambda z^\lambda,$$

$$F^{(k)}(z) = \sum_{\lambda=0}^{\infty} P^{(k)}(\lambda) z^\lambda;$$

evidently these functions are all analytic in this disc. It is immediate that

$F^{(0)}(z)$ is identically 1; the relation between $P^{(k)}$ and $P^{(k+1)}$ leads naturally to a relation between the functions $F^{(k)}$ and $F^{(k+1)}$: indeed as one verifies without difficulty

$$z^n F^{(k+1)}(z) = f(z)\left(F^{(k)}(z) + \sum_{\eta=0}^{n-1}(z^n - z^\eta)P^{(k)}(\eta)\right).$$

Now define, for $0 \leqq w < 1$,

$$G(z, w) = \sum_{k=0}^{\infty} F^{(k)}(z)w^k.$$

The relation between $F^{(k)}$ and $F^{(k+1)}$ just given may be translated into an equation involving G, as follows. Since

$$G(z, w) = F^{(0)}(z) + w\sum_{k=0}^{\infty} F^{(k+1)}(z)w^k,$$

we see that on multiplying this equation by z^n and then replacing the function $z^n F^{(k+1)}(z)$ by its equivalent in terms of $F^{(k)}(z)$,

$$z^n G(z, w) = z^n + wf(z)\left(G(z, w) + \sum_{\eta=0}^{n-1}(z^n - z^\eta)A_\eta(w)\right),$$

where we have introduced the abbreviation

$$A_\eta(w) = \sum_{k=0}^{\infty} P^{(k)}(\eta)w^k.$$

We will further abbreviate by setting

$$\sigma(z, w) = \sum_{\eta=0}^{n-1}(z^n - z^\eta)A_\eta(w),$$

and using this terminology, we conclude that

$$G(z, w) = \frac{z^n + wf(z)\sigma(z, w)}{z^n - wf(z)}.$$

It turns out to be possible on the one hand to obtain considerable information about the function G, and on the other to draw conclusions about the behavior of the distributions $P^{(k)}$ from the behavior of G. We will undertake the latter in the next section, and then the former in Section 4.

3. Since $P^{(k)}(\eta)$ is the probability that there are η passengers waiting in line just prior to the kth departure, the average number of passengers waiting at the kth departure is the mean of the distribution $P^{(k)}$; we will call this mean μ_k:

$$\mu_k = \sum_{\eta=0}^{\infty} \eta P^{(k)}(\eta).$$

We will show in a moment that this sum is finite if the distribution ρ had a finite mean in the first place. It will follow that we may take the average number of passengers waiting in line at the first N departures to be

$$\frac{1}{N} \sum_{k=0}^{N} \mu_k$$

and it is appropriate to call

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \mu_k$$

the *average number of passengers waiting in line*, provided however that this limit exists.

Now let z be restricted to the open unit interval. Since

$$\left.\frac{\partial G}{\partial z}\right|_{z,w} = \sum_{\eta=0}^{\infty} \sum_{k=0}^{\infty} \eta P^{(k)}(\eta) w^k z^{\eta-1},$$

one computes using the more or less "closed" expression for G obtained in Section 2 that

$$\lim_{z\to 1^-} \left.\frac{\partial G}{\partial z}\right|_{z,w} = \frac{w}{(1-w)^2}\left[(1-w)\left.\frac{\partial \sigma}{\partial z}\right|_{1,w} - (n - f'(1))\right],$$

where

$$\left.\frac{\partial \sigma}{\partial z}\right|_{1,w} = \sum_{\eta=0}^{n-1} (n-\eta) A_\eta(w).$$

Thus so long as $f'(1)$ exists, i.e., so long as ρ has a finite mean, $\lim_{z\to 1-} \partial G/\partial z$ also exists and is finite. But now since z is real and $\eta P^{(k)}(\eta) w^k$ is always non-negative

$$\lim_{z\to 1^-} \sum_{\eta=0}^{\infty} \sum_{k=0}^{\infty} \eta P^{(k)}(\eta) w^k z^{\eta-1} = \sum_{\eta=0}^{\infty} \sum_{k=0}^{\infty} \eta P^{(k)}(\eta) w^k.$$

The order of summation is of course immaterial and therefore in particular $\sum_{\eta=0}^{\infty} \eta P^{(k)}(\eta)$ exists and is finite for each k. We have shown, then, that *if ρ has a finite mean so does each $P^{(k)}$; and if μ_k is the mean of $P^{(k)}$ then*

$$\sum_{k=0}^{\infty} \mu_k w^k = \frac{w}{(1-w)^2}\left[(1-w) \sum_{\eta=0}^{n-1} (n-\eta) A_\eta(w) - (n - f'(1))\right].$$

Thus we have obtained certain information about the generating function of the sequence $\mu_1, \mu_2, \cdots, \mu_k, \cdots$.

Now according to a well-known Tauberian theorem

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \mu_k = \lim_{w\to 1-} (1 - w) \sum_{k=0}^{\infty} \mu_k w^k,$$

where the existence of either limit is assured by the existence of the other (for an elementary treatment of this proposition one might consult for instance Hobson's book [1]). We find then that *the average number of passengers waiting in line is*

$$\lim_{w\to 1-} \frac{w}{1 - w} \left[(1 - w) \sum_{\eta=0}^{n-1} (n - \eta) A_\eta(w) - (n - f'(1)) \right]$$

provided this limit exists. Thus the problem essentially becomes to calculate the functions $A_\eta(w)$.

4. Our analysis of the functions $A_\eta(w)$ is based on the following observation. It is evident directly from its definition that for each value of w between 0 and 1 the function $G(z, w)$ is analytic in z throughout the open unit disk. On the other hand we have seen that

$$G(z, w) = \frac{z^n + wf(z) \sum_{\eta=0}^{n-1} (z^n - z^\eta) A_\eta(w)}{z^n - wf(z)}.$$

An argument based on Rouché's theorem will show that the denominator of this fraction has for each w exactly n roots within the unit circle; let us call them $\theta_1, \theta_2, \cdots, \theta_n$, where of course each θ_i depends on w. But the analyticity of $G(z, w)$ requires that the numerator of the fraction vanish whenever the denominator does. Thus there arises a system of linear equations.

$$\theta_i^n + wf(\theta_i) \sum_{\eta=0}^{n-1} (\theta_i^n - \theta_i^\eta) A_\eta(w) = 0, \qquad i = 1, 2, \cdots, n$$

involving the n unknown functions $A_0(w), A_1(w), \cdots, A_{n-1}(w)$. We will prove below that for w sufficiently near 1 the n roots $\theta_1, \theta_2, \cdots, \theta_n$ are distinct and do not vanish; thus we actually have n equations, which, on noting that $wf(\theta_i)$ may be replaced by θ_i^n, may be written as

$$(1 - \theta_i^n) A_0(w) + (\theta_i - \theta_i^n) A_1(w) + \cdots + (\theta_i^{n-1} - \theta_i^n) A_{n-1}(w) = 1$$

with i running from 1 through n.

This system of equations turns out to be quite easy to handle. Let D be the determinant of the coefficients:

$$D = \begin{vmatrix} 1-\theta_1^n & \theta_1-\theta_1^n & \cdots & \theta_1^{n-1}-\theta_1^n \\ 1-\theta_2^n & \theta_2-\theta_2^n & \cdots & \theta_2^{n-1}-\theta_2^n \\ \cdots & \cdots & \cdots & \cdots \\ 1-\theta_n^n & \theta_n-\theta_n^n & \cdots & \theta_n^{n-1}-\theta_n^n \end{vmatrix}.$$

On subtracting the first column from each other column and rearranging the result we find that

$$D = \begin{vmatrix} 1-\theta_1 & 1-\theta_1^2 & \cdots & 1-\theta_1^n \\ 1-\theta_2 & 1-\theta_2^2 & \cdots & 1-\theta_2^n \\ \cdots & \cdots & \cdots & \cdots \\ 1-\theta_n & 1-\theta_n^2 & \cdots & 1-\theta_n^n \end{vmatrix}.$$

On factoring $1-\theta_j$ from the jth row for each j and then subtracting each column from the one to its right, this operation being begun on the right, we find

$$D = \prod_{j=1}^{n}(1-\theta_j)\cdot \begin{vmatrix} 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^{n-1} \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \theta_n & \theta_n^2 & \cdots & \theta_n^{n-1} \end{vmatrix}.$$

One recognizes this as the Vandermonde determinant, and it follows that

$$D = \prod_{j=1}^{n}(1-\theta_j)\cdot \prod_{i<j}(\theta_j-\theta_i).$$

We see in particular that D is not zero and we are thus assured that the functions $A_\eta(w)$ are in fact determined.

One now writes down each $A_\eta(w)$ using Cramer's rule and then combines the resulting quantities into a single determinant:

$$\sum_{\eta=0}^{n-1}(n-\eta)A_\eta(w) = -\frac{1}{D}\begin{vmatrix} 0 & n & n-1 & \cdots & 1 \\ 1 & 1-\theta_1^n & \theta_1-\theta_1^n & \cdots & \theta_1^{n-1}-\theta_1^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1-\theta_n^n & \theta_n-\theta_n^n & \cdots & \theta_n^{n-1}-\theta_n^n \end{vmatrix}$$

$$= (-1)^n \frac{1}{D}\begin{vmatrix} 0 & 1 & 2 & \cdots & n \\ 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \theta_n & \theta_n^2 & \cdots & \theta_n^n \end{vmatrix}$$

$$= (-1)^n \frac{1}{D} \frac{d}{d\xi} \begin{vmatrix} 1 & \xi & \xi^2 & \cdots & \xi^n \\ 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \theta_n & \theta_n^2 & \cdots & \theta_n^n \end{vmatrix}$$

where this derivative is to be evaluated at $\xi = 1$. Thus

$$\sum_{\eta=0}^{n-1} (n - \eta) A_\eta(w) = \frac{1}{D} \frac{d}{d\xi} \prod_{j=1}^{n} (\xi - \theta_j) \Big|_{\xi=1} \cdot \prod_{j>i} (\theta_i - \theta_j) = \sum_{j=1}^{n} \frac{1}{1 - \theta_j} \cdot$$

Therefore it has been proved that *the average number of passengers waiting in line is*

$$\lim_{w \to 1^-} \frac{w}{1 - w} \left[(1 - w) \sum_{j=1}^{n} \frac{1}{1 - \theta_j} - (n - f'(1)) \right];$$

and thus we are led to a detailed discussion of the behavior of the roots $\theta_1, \theta_2, \cdots, \theta_n$ for w less than but near 1.

5. The analysis of the behavior of the roots $\theta_1, \theta_2, \cdots, \theta_n$ of the equation

$$z^n - wf(z) = 0,$$

which lie inside the unit circle requires two additional assumptions, the first being that the mean of the distribution ρ be less than n,

$$n - f'(1) > 0,$$

and the second being that there be a positive probability that no passengers will arrive between two consecutive departures:

$$\rho_0 \neq 0.$$

The first of these assumptions is intuitively very reasonable: it merely means that the number of new passengers arriving between two consecutive departures is on the average less than the capacity of the shuttle. The second involves no essential limitation: if ρ_0 were zero we would replace the distribution ρ with a new distribution ρ' where

$$\rho_i' = \rho_{i-m},$$

ρ_m being the first nonzero entry of ρ, and of course simultaneously decrease the capacity of the shuttle by m, i.e., replace n by $n - m$.

Now let

$$p_w(z) = z^n - wf(z)$$

for $0 < w < 1$ and $|z| < 1$. As already noted, it is a consequence of Rouché's theorem that, for each value of w less than 1, p_w has exactly n zeros in the interior of the unit disk: the point here is that the function z^n has an n-fold zero at

the origin, and if $w<1$ and r is any number such that $w^{1/n}<r<1$ then on the circle centered at the origin and with radius r one has

$$| wf(z) | < | z^n |$$

so that the difference $z^n - wf(z)$ also has exactly n zeros inside this circle. Furthermore, none of these zeros can vanish because

$$p_w(0) = w\rho_0$$

and we have already assumed that ρ_0 is not zero.

We will now prove that for w sufficiently near 1 *the n zeros* $\theta_1, \theta_2, \cdots, \theta_n$ *of* p_w *are distinct and may be regarded as tracing out n smooth curves in the disk as w varies.* Indeed, the implicit function theorem assures us that if w_0 and θ_0 satisfy the relations

$$\theta_0^n - w_0 f(\theta_0) = 0,$$
$$n\theta_0^{n-1} - w_0 f'(\theta_0) \neq 0,$$

then there is a function $\theta(w)$ defined and differentiable for w near w_0 such that $\theta(w_0)=\theta_0$ and satisfying

$$\theta(w)^n - wf(\theta(w)) = 0$$

identically. Now if we already knew that for values of w sufficiently near 1 the roots $\theta_1, \theta_2, \cdots, \theta_n$ were distinct and each could be so written locally as a function of w, then it would be an easy matter to piece these functions together into n differentiable curves $\theta_1(w), \theta_2(w), \cdots, \theta_n(w)$, all defined on the same open interval whose right end point is 1, as required. The essential point, then, is to be sure that the above-stated hypothesis of the implicit function theorem is satisfied. In other words, we must prove that the functions p_w and p'_w do not have common zeros for w sufficiently near 1. The question of the distinctness of $\theta_1, \theta_2, \cdots, \theta_n$ will be taken care of at the same time, of course, because p'_w must vanish at any multiple zero of p_w.

We will in fact prove a little more, namely that there are only a finite number of points w in the open unit interval such that p_w and p'_w have a common zero; it is then evident that our curves can be defined to the right of the largest of these. Suppose, to the contrary, that there are an infinite number of points $w_1, w_2, \cdots, w_i, \cdots$ such that p_{w_i} and p'_{w_i} have a common zero z_i. It is immediate that the points z_i are all distinct; and furthermore one finds that for each i

$$p'_{w_i}(| z_i |) \leqq 0,$$

because

$$n | z_i |^{n-1} = w_i \left| \sum_{k=0}^{\infty} k\rho_k z_i^{k-1} \right| \leqq w_i \sum_{k=0}^{\infty} k\rho_k | z_i |^{k-1}.$$

Take b to be the least upper bound of the numbers t between 0 and 1 such that

$$nt^{n-1} - f'(t) \leqq 0;$$

evidently $b<1$, since

$$\lim_{t\to 1^-} nt^{n-1} - f'(t) = n - f'(1) > 0.$$

But furthermore, for any w less than 1, if

$$p_w'(t) \leqq 0$$

then also

$$nt^{n-1} - f'(t) < 0,$$

so $t<b$; thus all the points z_i lie inside the circle centered at the origin and with radius b, and have therefore an accumulation point a which is interior to the unit disk. But now from the equations

$$z_i^n - w_i \sum_{k=0}^{\infty} \rho_k z_i^k = 0,$$

$$nz_i^{n-1} - w_i \sum_{k=0}^{\infty} k\rho_k z_i^{k-1} = 0,$$

one deduces that for each i

$$\sum_{k=0}^{\infty} (n-k)\rho_k z_i^k = 0;$$

and since a is a point of analyticity for the function

$$nf(z) - zf'(z) = \sum_{k=0}^{\infty} (n-k)\rho_k z^k,$$

it follows that this function must vanish identically, i.e., that for each k

$$(n-k)\rho_k = 0,$$

which is impossible since $\rho_0 \neq 0$. Therefore p_w and p_w' can vanish simultaneously for at most a finite number of values of w.

It now follows that, *as w approaches 1 from the left, each of the functions $\theta_i(w)$ converges to a certain complex number c_i which is either interior to the unit disk or else is one of the nth roots of unity.* Indeed, since the closed unit disk is compact, as w tends to 1 each function $\theta_i(w)$ clusters at some point c_i, and we will see that these cluster points are in fact limit points. Let us introduce the notation

$$\phi(z) = z^n - f(z);$$

since p_w converges uniformly to ϕ it is evident that $\phi(c_i)=0$ for each i. If now c_i is in the interior of the unit disk, then ϕ is analytic at c_i and therefore there

is a neighborhood U of c_i such that ϕ has no zero other than c_i in U; whereas if $\theta_i(w)$ failed to converge to c_i then, being continuous, it would have a second cluster point in U at which ϕ would also have to vanish. If on the other hand c_i is on the boundary of the unit disk then already $c_i^n = 1$, because the conditions $|c_i| = 1$ and $\phi(c_i) = 0$ entail $|f(c_i)| = 1$, whereas this is impossible unless all the vectors $\rho_k c_i^k$ point in the same direction, namely along the real axis. Therefore if in this case $\theta_i(w)$ failed to converge to c_i then it would have another nearby cluster point, which is evidently impossible.

Finally we will show that, *for each w, p_w has exactly one zero in the unit interval, and that this and no other zero of p_w approaches 1 as w approaches 1 from the left.* We begin by looking at polynomials which approximate p_w for some fixed choice of w, namely we let

$$q_n(t) = nt^{n-1} - w \sum_{k=0}^{N} k\rho_k t^{k-1}$$

for t real. Using Descartes' Rule of Signs, one proves that either $q_n(t)$ is positive for $0 < t \leqq 1$, or else there is a point t_n such that $0 < t_n < 1$ and

$$q_n(t) < 0 \qquad \text{for } 0 < t < t_n,$$
$$q_n(t) > 0 \qquad \text{for } t_n < t \leqq 1;$$

these two cases occur according as all of $\rho_1, \rho_2, \cdots, \rho_{n-1}$ vanish, or one or more do not. In either case, however, one finds that $q_n(t)$ has at most two positive roots, and that if there are two then one of them must be greater than 1. It follows that either $p_w(t)$ is nonnegative for $0 < t \leqq 1$, or else there is a point, namely $\lim_{N\to\infty} t_n$, to the left of which p_w is nonpositive and to the right of which p_w is nonnegative. Accordingly, p_w is either monotone increasing throughout the unit interval, or else p_w decreases to the left of $\lim_{N\to\infty} t_N$ and then increases to the right of this point. In either event, if $0 < w < 1$ then

$$p_w(0) = -w\rho_0 < 0 \quad \text{and} \quad p_w(1) = 1 - w > 0,$$

so p_w vanishes exactly once in the open unit interval; we let $\theta_1(w)$ be this particular root of p_w. Now

$$\phi(0) = -\rho_0 < 0$$

and ϕ also either increases, or else first decreases and then increases, throughout the unit interval; but $\phi(1) = 0$, and therefore if $0 \leqq t < 1$ then $\phi(t) \neq 0$. Evidently on the other hand c_1 lies in the (closed) unit interval; we conclude that $c_1 = 1$, i.e., that

$$\lim_{w\to 1-} \theta_1(w) = 1.$$

To see that if $i \neq 1$ then $c_i \neq 1$, let us write

$$p_w(z) = (z - \theta_1(w))\alpha_w(z).$$

We know that there is a function which satisfies this relation and is continuous throughout the closed unit disk, and indeed we find that

$$\alpha_w(z) = \sum_{j=0}^{n-1} z^j \theta_1(w)^{n-1-j} - w \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \rho_{j+k+1} \theta_1(w)^k z^j.$$

Similarly, define

$$\beta(z) = \frac{\phi(z)}{z-1},$$

except for $z=1$; there is the analogous series representation for β, and in particular it turns out on setting $\beta(1)=n-f'(1)$ that β will be continuous throughout the closed unit disk. Comparison of the series for α_w and β reveals that, as w approaches 1 from the left, α_w tends uniformly to β throughout the closed disk. If now $c_j=1$ for a value of j other than 1, we would have

$$\beta(1) = \lim_{w \to 1-} \alpha_w(\theta_j(w)) = 0,$$

which would contradict the hypothesis that $n-f'(1)>0$. Thus none of the curves $\theta_j(w)$ other than $\theta_1(w)$ approach 1 as w tends to 1 from the left.

6. From the relations

$$[\theta_1(w)]^n - wf(\theta_1(w)) = 0,$$

$$\lim_{w \to 1-} \theta_1(w) = 1,$$

one calculates that, if $f''(1)$ is finite,

$$\lim_{w \to 1-} \theta_1'(w) = \frac{1}{n - f'(1)},$$

$$\lim_{w \to 1-} \theta_1''(w) = \frac{f''(1) + 2f'(1)(n - f'(1)) - n(n-1)}{(n - f'(1))^3}.$$

We find therefore that

$$\lim_{w \to 1-} (1-w) \sum_{j=1}^{n} \frac{1}{1 - \theta_j(w)} = \lim_{w \to 1-} \frac{1-w}{1 - \theta_1(w)} = n - f'(1).$$

Returning now to the expression obtained in Section 4 for the average number of passengers waiting in line, namely

$$\lim_{w \to 1-} \frac{w}{1-w} \left[(1-w) \sum_{j=1}^{n} \frac{1}{1 - \theta_j(w)} - (n - f'(1)) \right],$$

we see that the quantity in brackets tends to zero as w tends to 1 from the left.

Therefore, we can apply l'Hospital's Rule and we find that the limit in question becomes

$$\lim_{w\to 1^-} -\frac{d}{dw}(1-w)\sum_{j=1}^{n}\frac{1}{1-\theta_j(w)} = \frac{1}{2}\,\frac{f''(1)+2f'(1)(n-f'(1))-n(n-1)}{n-f'(1)} + \sum_{j=2}^{n}\frac{1}{1-c_j},$$

where as above $c_2, c_3, \cdots, c_n$ are those zeros of $z^n - f(z)$ which are distinct from 1 and lie inside or on the unit circle. Now the quantities $f''(1)$ and $f'(1)$ occur in the variance of the distribution ρ, in that

$$\mathbf{Var}\,(\rho) = f''(1) + f'(1) - f'(1)^2;$$

on using this we find the above limit to be

$$\frac{1}{2}\,\frac{\mathbf{Var}\,(\rho)}{n-f'(1)} + \frac{1}{2}\,(\mathbf{M}\rho) + \left(\sum_{j=2}^{n}\frac{1}{1-c_j} - \frac{n-1}{2}\right).$$

We note that the quantity in parenthesis is real; and if $n>1$ it is also positive, unless all of $c_2, c_3, \cdots, c_n$ lie on the unit circle, which occurs only if $\rho_k = 0$ except for k an integral multiple of n. *This, then, is the average number of passengers waiting in line under the assumptions that ρ_0 is not zero, the mean of ρ is less than n, and the variance of ρ is finite.*

A treatment of the continuous analogue to this problem is to be found in the dissertation of Lakshmi Venkataraman written recently at Columbia University.

Reference

1. E. W. Hobson, The Theory of Functions of a Real Variable, Cambridge, Cambridge University Press, 1927.

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1. Introduction. The topics of homogeneous and barycentric coordinates are treated quite cavalierly in the current college curriculum. For the nongeometrician, the study of these and related topics often awaits a rigorous course in combinatorial topology at which time it is pointed out that simplices are best described (or defined) by their barycentric coordinates. It is the purpose of this note to introduce this subject axiomatically, to motivate it physically, and to interpret it geometrically.

* I am indebted to Mrs. Peter Di Paola, whose remarks gave impetus to the writing of this note, and to Professor W. Prenowitz for his considerable help in clarifying some of the original manuscript.

AREA-PERIMETER RELATIONS FOR TWO-DIMENSIONAL LATTICES

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1. Introduction. Minkowski's theorem for the two-dimensional unit square lattice states that any convex domain of area greater than four which is symmetric about a lattice point, contains at least one other lattice point. In this note we shall prove that any convex domain whose area is greater than half its perimeter, contains a lattice point. Thus, if D is a convex domain containing *no* lattice points, we have the inequality

$$A(D) \leqq \tfrac{1}{2}P(D), \tag{1.1}$$

where $A(D)$ is the area of D and $P(D)$ its perimeter. Inequality (1.1) may be stated in the following equivalent form: The supremum of $A(D)/P(D)$ over all convex domains D containing no lattice points is less than or equal to one half. We shall, in fact, prove a stronger and more general result.

THEOREM. *Let $\Lambda=\Lambda(\omega_1, \omega_2)$ be a lattice generated by two noncollinear vectors ω_1 and ω_2 making an angle θ with one another, where $0<\theta<\pi$, and let $c(\Lambda)$ be the supremum of $A(D)/P(D)$ over all convex domains D containing no points of Λ. Then if $|\omega_1| \leqq |\omega_2|$ we have the inequalities*

$$\tfrac{1}{2}|\omega_2| \sin\theta \leqq c(\Lambda) \leqq \tfrac{1}{2}\max\{|\omega_1|, |\omega_2| \sin\theta\}. \tag{1.2}$$

Note that for rectangular lattices the inequalities in (1.2) determine $c(\Lambda)$ exactly and give us $c(\Lambda)=\frac{1}{2}\max\{|\omega_1|, |\omega_2|\}$.

The lower bound in (1.2) is easily derived by considering the parallelogram

$$D_n = \{(x, y) \mid 0 < x < n, 0 < y < 1\},$$

the coordinates (x, y) being relative to the basis ω_1, ω_2. As $n\to\infty$ it is clear that $A(D_n)/P(D_n)\to\frac{1}{2}|\omega_2| \sin\theta$, and this implies $c(\Lambda)\geqq\frac{1}{2}|\omega_2| \sin\theta$.

2. The upper bound for $c(\Lambda)$. The upper bound for $c(\Lambda)$ will be established by reducing the problem to rectangular lattices and symmetric domains.

Let $\omega_1'=\omega_1$, and let ω_2' be a vector of length $|\omega_2| \sin\theta$, perpendicular to ω_1. (There are two such vectors, but they differ only in sign.) Let $\Lambda'=\Lambda(\omega_1', \omega_2')$ denote the rectangular lattice determined by the basis vectors ω_1', ω_2'. We shall prove the following lemma.

LEMMA 1. *If D is a convex domain containing no points of Λ, there exists another convex domain D' containing no points of Λ', such that*

(a) $P(D')\leqq P(D)$, $A(D')=A(D)$,

(b) *D' is symmetric about the lines $x'=\frac{1}{2}$, $y'=\frac{1}{2}$, the coordinates x' and y' being relative to the basis ω_1', ω_2'.*

Proof. Let D^0 be the region obtained from D by symmetrization with respect to the line $x'=\frac{1}{2}$. Symmetrization preserves convexity and areas and does not increase perimeters. Therefore D^0 is convex, $A(D^0)=A(D)$, and $P(D^0)\leqq P(D)$.

We shall show now that D^0 contains no points of the lattice Λ'.

If D^0 contained a lattice point of Λ', say the point $m\omega_1' + n\omega_2'$, then the line $y' = n$ (perpendicular to ω_2) would intersect D^0 in a line segment of length greater than 1. The same line would also intersect D in a line segment of the same length and this line segment, in turn, would contain a lattice point of Λ, contradicting the hypothesis of the lemma. Therefore D^0 contains no lattice points of Λ'.

We can argue now with D^0 as we did with D, except that we symmetrize D^0 with respect to the line $y' = \frac{1}{2}$. The symmetrization of D^0 gives us a domain D' with the properties described in the lemma.

In view of Lemma 1, to deduce the upper bound in (1.2) it suffices to prove

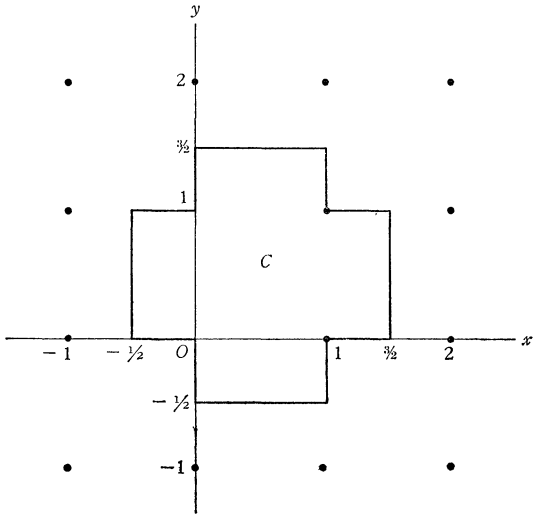


FIG. 1

LEMMA 2. *Let* $\Lambda = \Lambda(\omega_1, \omega_2)$ *be a rectangular lattice. If* D *is a convex domain, symmetric about the lines* $x = \frac{1}{2}$ *and* $y = \frac{1}{2}$ *(coordinates relative to the basis* ω_1, ω_2*) and if* D *contains no points of* Λ*, then*

$$\frac{A(D)}{P(D)} \leqq \frac{1}{2} \max \{ |\omega_1|, |\omega_2| \}.$$

Proof. Let $a = \sup\{x | (x, \frac{1}{2}) \in D\}$, and $b = \sup\{y | (\frac{1}{2}, y) \in D\}$, where the coordinates x and y are relative to the basis ω_1, ω_2. We consider three cases, depending on the sizes of a and b.

Case 1. $a \leqq \frac{3}{2}$ and $b \leqq \frac{3}{2}$. By the isoperimetric inequality we have

$$\frac{A(D)}{P(D)} \leqq \sqrt{\frac{A(D)}{4\pi}}. \tag{2.1}$$

We now obtain an upper bound for $A(D)$. In this case, D lies within the cross-shaped region C shown in Figure 1. Therefore $A(D) \leqq A(C) = 3|\omega_1|\,|\omega_2|$, and (2.1) becomes

$$\frac{A(D)}{P(D)} \leqq \frac{1}{2}\sqrt{\frac{3}{\pi}}\sqrt{(|\omega_1|\,|\omega_2|)} < \frac{1}{2}\sqrt{(|\omega_1|\,|\omega_2|)} \leqq \frac{1}{2}\max\{|\omega_1|, |\omega_2|\}.$$

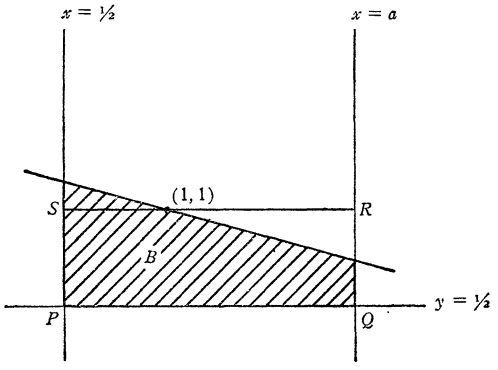


FIG. 2

Case 2. $a > \frac{3}{2}$. We consider D', the quarter of D with $x \geqq \frac{1}{2}$ and $y \geqq \frac{1}{2}$. We now obtain an upper bound for $A(D')$. Since D is convex, the region D' lies below some line through (1, 1) with nonpositive slope as shown in Figure 2. Therefore D' is a subset of the shaded region B shown in Figure 2. Since $a > \frac{3}{2}$ the region B has an area not exceeding that of the rectangle $PQRS$. Therefore $A(B) \leqq \frac{1}{2}|\omega_2| \cdot (a - \frac{1}{2})|\omega_1|$. Thus

$$A(D) = 4A(D') \leqq 4A(B) \leqq 2|\omega_1|\,|\omega_2|\,(a - \tfrac{1}{2}).$$

But since $P(D) \geqq 4|\omega_1|\,(a - \frac{1}{2})$ we have the inequality

$$\frac{A(D)}{P(D)} \leqq \frac{1}{2}\,|\omega_2|\,.$$

This proves Lemma 2 when $a > \frac{3}{2}$.

Case 3. $b > \frac{3}{2}$. The argument in this case is similar to Case 2 and we obtain the inequality

$$\frac{A(D)}{P(D)} \leqq \frac{1}{2}\,|\omega_1|\,.$$

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A NON-APPROXIMATION THEOREM

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We recall the classical approximation theorem due to Weierstrass. In its simplest form it may be expressed as follows: *If $f(x)$ is a real continuous function on the interval $0 \leqq x \leqq 1$ and $\epsilon > 0$, then there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ on $0 \leqq x \leqq 1$.* Choosing therefore for each natural number n a polynomial $p_n(x)$ such that $|f(x) - p_n(x)| < 1/n$ on $0 \leqq x \leqq 1$, one may say: if $f(x)$ is continuous on the interval $0 \leqq x \leqq 1$, then there is a sequence of polynomials which converges uniformly to $f(x)$.

There are many proofs of the above theorem in the literature. Of interest is, for instance, the one given by L. M. Graves, [2], who has shown that if $f(x)$ is also assumed to possess all the derivatives up to and including that of order k (the last being continuous), then one may choose a sequence of polynomials converging uniformly to $f(x)$ for which also the corresponding sequence of the ith derivatives converges uniformly to $f^{(i)}(x)$ for $i = 1, 2, \cdots, k$.

Availing ourselves of some of the terminology used in normed linear spaces, the above approximation theorem may be translated into the following version:

Let $\mathfrak{C}$ denote the set of all continuous functions on $0 \leqq x \leqq 1$, and $\mathfrak{P}$ stand for its subset of all polynomials. With the obvious definition of addition and scalar multiplication, $\mathfrak{C}$ (as well as $\mathfrak{P}$) is a linear space. Defining

$$\|f\| = \max_{0 \leqq x \leqq 1} |f(x)|$$

for f in $\mathfrak{C}$, one obtains a norm on $\mathfrak{C}$. Any norm, of course, generates a metric; the distance of a pair of elements f and g in $\mathfrak{C}$ being given by

$$\|f - g\| = \max_{0 \leqq x \leqq 1} |f(x) - g(x)| .$$

Clearly, in the resulting normed linear space, a sequence of elements $\{f_n\}$ converges to an element f if and only if the sequence of functions $\{f_n(x)\}$ converges to $f(x)$ uniformly on $0 \leqq x \leqq 1$. Weierstrass' approximation theorem may thus be translated as follows: Given $f \in \mathfrak{C}$ and $\epsilon > 0$, then there exists a $p \in \mathfrak{P}$ such that $\|f - p\| < \epsilon$. Equivalently: Given an $f \in \mathfrak{C}$ then there exists a sequence $\{p_n\}$ of elements of $\mathfrak{P}$ for which $\|f - p_n\| \to 0$. Of course this means that $\mathfrak{P}$ *is a dense subset of the metric space* $\mathfrak{C}$. Again, the last may be also expressed as follows: The sequence of functions $1, x, x^2, x^3, \cdots$ is *fundamental* in $\mathfrak{C}$. We mean hereby that the finite linear combinations of those functions (polynomials) form a dense set in $\mathfrak{C}$.

Thus whenever there is given a subset $\mathfrak{P}$ of a normed linear space $\mathfrak{N}$ one has always an approximation problem: Is $\mathfrak{P}$ fundamental in $\mathfrak{N}$? That is, do the finite linear combinations of the elements in $\mathfrak{P}$ form a dense set in $\mathfrak{N}$, or what amounts to the same thing, does the closed linear manifold spanned by $\mathfrak{P}$ coincide with $\mathfrak{N}$? (An analogous question may be asked of any subset in an arbitrary metric

space: Is that subset dense in the whole space?). If the answer is "yes," we have an approximation theorem. On the other hand, a negative answer yields equally interesting information. It should be emphasized that it is the least of our intentions to state that every approximation problem is of the above type. We do assert, however, that *whenever a subset of a metric space is specified an approximation problem arises.*

In particular one may ask: Is the sequence of functions $1, x^2, x^4, x^6, \cdots$ fundamental in $\mathfrak{C}$? More generally: Given an increasing sequence of natural numbers $p_1<p_2<p_3<\cdots$ when is the corresponding sequence of functions $1, x^{p_1}, x^{p_2}, x^{p_3}, \cdots$ fundamental in $\mathfrak{C}$? The answer to the above question was already given in 1914 by Müntz [3] who proved that the last sequence of functions is fundamental in $\mathfrak{C}$ if and only if the series $\sum_i 1/p_i$ diverges.

All the above (with the exception perhaps of some of the terminology) has been well known for half a century. An elementary but quite important example of a *non-approximation* theorem is described as follows (compare [4]):

Consider the set of all bounded sequences $x=(x_1, x_2, \cdots)$ of real numbers. With the obvious definitions of addition (by adding the corresponding coordinates) and scalar multiplication one obtains a linear space. There

$$\|x\| = \sup_i |x_i|$$

defines a norm. The last generates a metric; the distance of two elements $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ being given by

$$\|x - y\| = \sup_i |x_i - y_i| .$$

In the resulting normed linear space commonly denoted by (m), we distinguish the subset $\mathfrak{L}$ consisting of all those sequences which have only a finite number of nonzero terms (coordinates). Since $\mathfrak{L}$ is obviously linear to start with, the corresponding approximation problem reads: Is $\mathfrak{L}$ dense in (m)? It is easy to see that the answer to the above question is "no", since the closure of $\mathfrak{L}$ (which because $\mathfrak{L}$ is linear, coincides with the closed linear manifold spanned by $\mathfrak{L}$) consists precisely of all those elements $x=(x_1, x_2, \cdots)$ whose coordinates satisfy the additional condition $x_i \to 0$. The argument follows:

Let $x_i \to 0$. For $n=1, 2, \cdots$ define

$$x^{(n)} = (x_1, x_2, \cdots, x_n, 0, 0, 0, \cdots).$$

Then $x^{(n)}$ is in $\mathfrak{L}$ and

$$\|x - x^{(n)}\| = \sup_{i>n} |x_i| .$$

The right side can be made as small as we please for sufficiently large n. Thus, x belongs to the closure of $\mathfrak{L}$.

Conversely, suppose that $x=(x_1, x_2, \cdots)$, is an element of (m) having the property that given an $\epsilon>0$, there is an element $\bar{x}=(\bar{x}_1, \bar{x}_2, \cdots)$ in $\mathfrak{L}$ for which

$$\|x - \bar{x}\| = \sup_i |x_i - \bar{x}_i| < \epsilon.$$

Since beginning with some place, say for all $i \geqq N(\epsilon)$ we have $\bar{x}_i = 0$, it follows that $|x_i| < \epsilon$ for all $i \geqq N(\epsilon)$. Thus, necessarily $x_i \to 0$.

The main purpose of this expository note is to discuss a not so well-known nonapproximation theorem. As its setting, it has the space of all Lebesgue measurable and essentially bounded functions $K(s, t)$ on the square $0 \leqq s, t \leqq 1$. We state that the set of all products of the form $f(s)g(t)$, where both $f(s)$ and $g(s)$ are measurable and essentially bounded on $0 \leqq s \leqq 1$, is not fundamental in the above space. The details follow; the dimensionality argument employed below is basically the one to be found in [1].

One recalls that a function $f(s)$ defined and Lebesgue measurable on the interval $0 \leqq s \leqq 1$ is termed essentially bounded if for some constant $A \geqq 0$ the set of all points s for which $|f(s)| > A$ is of measure 0, that is, if $|f(s)| \leqq A$ almost everywhere (a.e.) on $0 \leqq s \leqq 1$. For an essentially bounded $f(s)$, the least of such constants A (which exists since the union of a sequence of sets of measure 0 is also of measure 0) is termed its essential least upper bound and denoted by ess. l.u.b.$_{0 \leqq s \leqq 1} |f(s)|$.

Let $\mathfrak{M}_1$ stand for the set of all real-valued functions $f(s)$ on $0 \leqq s \leqq 1$, which are Lebesgue measurable and essentially bounded. The sum of two functions in $\mathfrak{M}_1$, and a scalar multiple of a function in $\mathfrak{M}_1$ are obviously in $\mathfrak{M}_1$. "Identifying" in $\mathfrak{M}_1$ two functions $f(s)$ and $g(s)$ if and only if they differ on a set of measure 0, we see that $\mathfrak{M}_1$—or to be more precise the quotient space $\mathfrak{M}_1/\mathfrak{N}$ where $\mathfrak{N}$ is the set of all functions equal to 0 almost everywhere—is a linear space. There,

$$\|f\| = \text{ess. l.u.b.}_{0 \leqq s \leqq 1} |f(s)|$$

defines a norm. The last generates a metric, the distance of two elements $f(s)$ and $g(s)$ in $\mathfrak{M}_1$ being given by

$$\|f - g\| = \text{ess. l.u.b.}_{0 \leqq s \leqq 1} |f(s) - g(s)|.$$

Together with the normed linear space $\mathfrak{M}_1$ (which incidentally is a Banach space, since its norm generates a complete metric space) we consider the normed linear space $\mathfrak{M}_2$ of all Lebesgue measurable and essentially bounded functions $K(s, t)$ on the square $0 \leqq s, t \leqq 1$; again, two functions are identified if and only if they differ on a set of planar Lebesgue measure 0. In the last

$$\|K\| = \text{ess. l.u.b.}_{0 \leqq s,\, t \leqq 1} |K(s, t)|$$

represents the norm and thus $\|H - K\|$ stands for the distance of the two elements H and K in $\mathfrak{M}_2$.

It is easy to see that whenever both $f(s)$ and $g(s)$ are in $\mathfrak{M}_1$, then their "product" $f(s)g(t)$ is in $\mathfrak{M}_2$. It is not true however, that every function in $\mathfrak{M}_2$ can be written in the form of such a product, or even as a finite linear combination of

such products. In fact the problem which we propose to consider inquires about a more modest condition: Is the set of all functions of the form $f(s)g(t)$ fundamental in $\mathfrak{M}_2$? In other words, is it true that given $K(s, t)$ in $\mathfrak{M}_2$ and $\epsilon>0$, there is always a finite sum of the form $\sum_{i=1}^{p} f_i(s)g_i(t)$ with $f_i(s)$, $g_i(s)$ in $\mathfrak{M}_1$, for which

$$\operatorname*{ess.\,l.u.b.}_{0\leqq s,\ t\leqq 1}\left|K(s,t)-\sum_{i=1}^{p}f_i(s)g_i(t)\right|<\epsilon.$$

We shall see that the answer to the above problem is "no." In particular, we shall prove the following:

THEOREM. *The function $F(s, t)$ defined on $0\leqq s,\ t\leqq 1$ as follows:*

$$F(s,t)=1 \qquad \text{if } s\leqq t,$$
$$F(s,t)=0 \qquad \text{if } s>t,$$

is not approximable in the metric space $\mathfrak{M}_2$ by finite sums of "products", that is, does not lie in the closed linear manifold spanned by all the above "products."

The proof of the above theorem is based on a few elementary lemmas:

LEMMA 1. *Let E be a set in the square $0\leqq s, t\leqq 1$ of planar Lebesgue measure 0. Then there exists a set T in $0\leqq t\leqq 1$ of linear Lebesgue measure 1, having the following property: for each $t\in T$ there is a set S_t in $0\leqq s\leqq 1$, again of linear Lebesgue measure 1, such that if $t\in T$ and $s\in S_t$ then $(s, t)\notin E$.*

Proof. Let $\chi_E(s, t)$ stand for the characteristic function of E. Its integral with respect to the planar Lebesgue measure, is obviously 0. Fubini's theorem permits us to replace this integral by an "iterated" integral. Hence,

$$\int_0^1\left[\int_0^1\chi_E(s,t)ds\right]dt=0.$$

Thus, there exists a set T of linear measure 1 such that for $t\in T$ the integral $\int\chi_E(s, t)ds=0$. Repeating the same argument: For each $t\in T$ there is a set S_t of linear measure 1, such that for $s\in S_t$ we have $\chi_E(s, t)=0$. Thus, if $t\in T$ and $s\in S_t$ then $\chi_E(s, t)=0$, and therefore $(s, t)\notin E$.

LEMMA 2. *Let $f_1(s), \cdots, f_n(s)$ belong to $\mathfrak{M}_1$ and $c_1, \cdots, c_n$ be real constants. Given $\epsilon>0$ one can find rational numbers $r_1, \cdots, r_n$ such that the linear combinations $\sum_{i=1}^{n} c_if_i$ and $\sum_{i=1}^{n} r_if_i$ have (in $\mathfrak{M}_1$) a distance $<\epsilon$.*

Proof. It is sufficient to observe that

$$\left\|\sum_{i=1}^{n}c_if_i-\sum_{i=1}^{n}r_if_i\right\|=\operatorname*{ess.\,l.u.b.}_{0\leqq s\leqq 1}\left|\sum_{i=1}^{n}c_if_i(s)-\sum_{i=1}^{n}r_if_i(s)\right|$$
$$\leqq\sum_{i=1}^{n}|c_i-r_i|\operatorname*{ess.\,l.u.b.}_{0\leqq s\leqq 1}|f_i(s)|.$$

LEMMA 3. *If a metric space $\mathfrak{M}$ is separable (that is, contains a denumerable subset $\mathfrak{D}$ which is dense in $\mathfrak{M}$) then any collection $\mathfrak{K}$ of elements of $\mathfrak{M}$ whose distance from each other is $\geqq 1$, is at most denumerable.*

Proof. To each $y \in \mathfrak{K}$ assign an element $x \in \mathfrak{D}$ at a distance $[y, x] < \frac{1}{2}$. To different elements in $\mathfrak{K}$ must then be assigned different elements in $\mathfrak{D}$. For suppose that $y_1 \neq y_2$ and $[y_1, x] < \frac{1}{2}$ as well as $[y_2, x] < \frac{1}{2}$. Then $[y_1, y_2] \leqq [y_1, x] + [x, y_2] < \frac{1}{2} + \frac{1}{2} = 1$, contrary to our assumption that $[y_1, y_2] \geqq 1$.

We are now ready to prove our theorem. Suppose that for every natural number n there is a finite sum $\sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t)$ such that

$$\operatorname*{ess.\,l.u.b.}_{0 \leqq s,\ t \leqq 1} \left| F(s, t) - \sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t) \right| < \frac{1}{n} .$$

The set E_n of points (s, t) for which

$$\left| F(s, t) - \sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t) \right| \geqq \frac{1}{n}$$

is thus of planar Lebesgue measure 0. Consequently $E = E_1 \cup E_2 \cup E_3 \cup \cdots$ is also of Lebesgue measure 0, and if $(s, t) \notin E$ we have

$$\left| F(s, t) - \sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t) \right| < \frac{1}{n} \qquad (n = 1, 2, \cdots) .$$

By Lemma 1, there is a set T of linear measure 1 having the following property: For each $t_0 \in T$ there is a set S_{t_0} of linear measure 1 such that if $s \in S_{t_0}$ then $(s, t_0) \notin E$ and hence

$$\operatorname*{ess.\,l.u.b.}_{0 \leqq s \leqq 1} \left| F(s, t_0) - \sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t_0) \right| \leqq \frac{1}{n} \quad (n = 1, 2, \cdots).$$

The last sequence of inequalities implies that for each $t_0 \in T$ the function $F(s, t_0)$ belongs to the closed linear manifold $\mathfrak{K}$ (in $\mathfrak{M}_1$) spanned by the denumerable set of functions $f_i^{(n)}(s)$ (where $n = 1, 2, \cdots$, and for a particular n the subscript $i = 1, 2, \cdots, p_n$). It is a simple consequence of Lemma 2 that the finite linear combinations of these functions $f_i^{(n)}(s)$ with rational coefficients form a dense set in $\mathfrak{K}$. Thus $\mathfrak{K}$ is separable and, therefore, any collection of elements in $\mathfrak{K}$ any two of which have a distance $\geqq 1$ is necessarily denumerable (Lemma 3). On the other hand if t_1 and t_2 are two different points of the obviously non-denumerable set T, then the functions $F(s, t_1)$ and $F(s, t_2)$, which as was proven above are in $\mathfrak{K}$, have the distance

$$\operatorname*{ess.\,l.u.b.}_{0 \leqq s \leqq 1} \left| F(s, t_1) - F(s, t_2) \right| = 1.$$

We thus have a contradiction.

Remark. If we dispense with measurability and consider the analogous prob-

lem in the space of all bounded functions, then our argument may be considerably simplified. The details follow.

Let $\mathfrak{B}_1$ denote the linear set of all bounded functions $f(s)$ on $0 \leqq s \leqq 1$. Clearly,

$$\|f\| = \sup_{0 \leqq s \leqq 1} |f(s)|$$

defines a norm in $\mathfrak{B}_1$. Similarly, let $\mathfrak{B}_2$ stand for the linear set of all bounded functions $K(s, t)$ on the square $0 \leqq s, t \leqq 1$ with the norm

$$\|K\| = \sup_{0 \leqq s,\ t \leqq 1} |K(s, t)| .$$

Again, the function $F(s, t)$ defined above is not approximable by finite sums of "products" of functions in $\mathfrak{B}_1$. Otherwise, for every natural n one could find a finite sum such that

$$\sup_{0 \leqq s,\ t \leqq 1} \left| F(s, t) - \sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t) \right| < \frac{1}{n} .$$

Again, the closed linear manifold $\mathfrak{K}$ in $\mathfrak{B}_1$ spanned by all the functions $f_i^{(n)}(s)$ is separable. Moreover, for each t_0 in $0 \leqq t \leqq 1$ the sequence of inequalities

$$\sup_{0 \leqq s \leqq 1} \left| F(s, t_0) - \sum_{i=1}^{p_n} f_i^{(n)}(s) g_i^{(n)}(t_0) \right| < \frac{1}{n}$$

implies that $F(s, t_0)$ is in $\mathfrak{K}$. The separable $\mathfrak{K}$ would then contain a nondenumerable collection of functions $F(s, t_0)$ whose distance from each other is 1. The last can not happen. Thus *the products* $f(s)g(t)$ *(with* $f(s)$ *and* $g(s)$ *in* $\mathfrak{B}_1$*) do not form a fundamental set in* $\mathfrak{B}_2$.

We may add that in the case of $\mathfrak{B}_2$, a non-approximable is also the function $\phi(s, t)$ defined as follows: $\phi(s, t) = 0$, if $s \neq t$, $\phi(s, s) = 1$, for all s.

The preceding argument remains valid verbatim if throughout it $F(s, t)$ is replaced by $\phi(s, t)$.

Moreover, if $\mathfrak{B}^{(1)}$ stands for the Banach space of all bounded functions on any *nondenumerable* set S, and $\mathfrak{B}^{(2)}$ denotes the Banach space of bounded functions on the set of all ordered pairs (s, t) with s and t in S, then again the function $\phi(s, t)$ defined as above, is not approximable in $\mathfrak{B}^{(2)}$ by finite sums of "products" of functions in $\mathfrak{B}^{(1)}$.

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EXISTENCE AND UNIQUENESS OF INTERPOLATING RATIONAL FUNCTIONS

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I. Introduction. The classical problem of interpolation is that of selecting from a given class of functions $\alpha(x)$ a limited number $\alpha_0(x)$, $\alpha_1(x)$, $\cdots$, $\alpha_n(x)$ and attempting to determine constants a_i such that the function

$$y(x) = a_0\alpha_0(x) + a_1\alpha_1(x) + \cdots + a_n\alpha_n(x)$$

satisfies the $n+1$ conditions

$$y(x_i) = y_i, \qquad i = 0, 1, \cdots, n, \tag{1}$$

where (x_i, y_i) are $n+1$ points with the x_i distinct. A particularly simple choice of functions $\alpha_i(x)$ for most computational purposes is $\alpha_i(x) = x^i$, so that $y(x)$ is a polynomial of degree n. Other choices for $\alpha(x)$ include classes of orthogonal polynomials (so that $y(x)$ is again a polynomial), trigonometric functions, and exponential functions. A method for determining the a_i is yielded by Cramer's rule. In particular, there is a unique polynomial of degree at most n satisfying the $n+1$ conditions (1), since the Vandermonde determinant formed from the distinct set of x_i, $i=0, 1, \cdots, n$, does not vanish.

A natural extension of the problem of interpolation would be to seek constants a_i, $i=0, 1, \cdots, k$, $k \leqq n$, and b_j, $j=0, 1, \cdots, n-k$, such that the function

$$y(x) = \frac{a_0\alpha_0(x) + a_1\alpha_1(x) + \cdots + a_k\alpha_k(x)}{b_0\alpha_{k+1}(x) + b_1\alpha_{k+2}(x) + \cdots + b_{n-k}\alpha_{n+1}(x)}$$

satisfies the conditions (1). The choice

$$\alpha_i(x) = \begin{cases} x^i, & i = 0, 1, \cdots, k \\ x^{i-k-1}, & i = k+1, \cdots, n+1 \end{cases}$$

leads to an interpolating function $y(x) = p(x)/q(x)$ for which the coefficients in both numerator and denominator must be determined.

A great deal of material on interpolating rational functions has been published by Hildebrand, Milne, and others, yet each individual contribution seems to lack a certain amount of generality. In this paper, we extend this work.

Hildebrand considered a method of rational interpolation based on an analogue of the Newton interpolation formula with divided differences. The use of successive divided differences is replaced by the use of successive inverted differences. A rational interpolating function $y(x) = p(x)/q(x)$ is obtained formally as a continued fraction, the polynomials $p(x)$ and $q(x)$ appearing as the numerator and denominator of the nth convergent to the continued fraction. The numerator and denominator are of degree p and $p-1$, respectively, when

$n=2p-1$, and are each of degree p when $n=2p$. Specifically, the continued fraction has the following form:

$$y(x) = a_0 + \cfrac{x - x_0}{a_1 + \cfrac{x - x_1}{a_2 + \cfrac{x - x_2}{\ddots \cfrac{}{a_{n-1} + \cfrac{x - x_{n-1}}{a_n}}}}}, \tag{2}$$

where

$$a_k = \phi_k[x_0, x_1, \cdots, x_{k-1}, x_k] = \frac{x_k - x_{k-1}}{\phi_{k-1}[x_0, x_1, \cdots, x_{k-2}, x_k] - \phi_{k-1}[x_0, x_1, \cdots, x_{k-2}, x_{k-1}]},$$

for $1 \leqq k \leqq n$, and $a_0 = \phi_0[x_0] = y_0$.

It is important to note that, although the method does not yield a spurious rational function satisfying (1) when none exists, the continued fraction may fail to exist in the form assumed even though a rational function having the desired property does exist. For example, consider the set of points (0, 1), (1, 1), (2, 3/5), (3, 2/5), (4, 5/17), (5, 3/13), which correspond to the rational function $y(x)=(x+1)/(x^2+1)$. For this arrangement, $\phi_1[0, 1]=\infty$, so there is no continued fraction of the form above. This difficulty can be averted here by reordering the points in such a way that the equal ordinates are not consecutive. Yet, the question remains: How can one arrange $n+1$ points (x_i, y_i), $i=0, 1, \cdots, n$, which satisfy $y(x)=p(x)/q(x)$, in such a way that no ϕ_k is undefined, $1 \leqq k \leqq n$?

A similar technique was used by Milne to construct a rational function of the form

$$y(x) = \frac{a_0 + a_1x + \cdots + a_px^p}{b_0 + b_1x + \cdots + b_{p-1}x^{p-1}} = \frac{a(x)}{b(x)}, \qquad p = \tfrac{1}{2}(n+1),\ n \text{ odd},$$

$$= \frac{a_0 + a_1x + \cdots + a_px^p}{b_0 + b_1x + \cdots + b_px^p} = \frac{a(x)}{b(x)}, \qquad p = \tfrac{1}{2}n,\ n \text{ even}.$$

The equations (1) yield the $n+2$ conditions

$$a(x_i) - b(x_i)y_i = 0, \qquad i = 0, 1, \cdots, n,$$

$$a(x) - b(x)y = 0,$$

and the condition that nonzero a's and b's exist satisfying these equations is that the determinant

$$\left| 1 \quad x_1 \quad x_2^2 \cdots x_p^p \quad y_{p+1} \quad x_{p+2}y_{p+2} \cdots x_n^{p-\gamma-1}y_n \quad x^{p-\gamma}y \right|, \qquad \gamma = 0 \text{ or } 1,$$

vanishes. This, in turn, leads to the equation

$$\begin{vmatrix} 1 & \pi_n(x_n, x_{n-1}, \cdots, x_1, x_0) \\ 1 & \pi_n(x, x_{n-1}, \cdots, x_1, x_0) \end{vmatrix} = 0, \tag{3}$$

where

$$\pi_{k+1}(x_{k+1}, x_k, \cdots, x_1, x_0) = \frac{x_{k+1} - x_k}{\pi_k(x_{k+1}, x_{k-1}, \cdots, x_0) - \pi_k(x_k, \cdots, x_0)} + \pi_{k-1}(x_{k-1}, \cdots, x_0),$$

$$\pi_0(x_0) = y_0, \qquad \pi_1(x_1, x_0) = (x_1 - x_0)/(y_1 - y_0).$$

But from the definition of the π's, it follows that

$$\begin{aligned} y &= y_0 + (x - x_0)/\pi_1(x, x_0), \\ \pi_1(x, x_0) &= \pi_1(x_1, x_0) + (x - x_1)/[\pi_2(x, x_1, x_0) - y_0], \\ \pi_2(x, x_1, x_0) &= \pi_2(x_2, x_1, x_0) + (x - x_2)/[\pi_3(x, x_2, x_1, x_0) - \pi_1(x_1, x_0)], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

and by eliminating in succession $\pi_1(x, x_0)$, $\pi_2(x, x_1, x_0)$, $\cdots$, and finally using equation (3) to eliminate $\pi_n(x, x_{n-1}, \cdots, x_1, x_0)$, $y(x)$ appears as the continued fraction

$$y(x) = \cfrac{x - x_0}{\pi_1(x_1, x_0) + \cfrac{x - x_1}{\pi_2(x_2, x_1, x_0) - y_0 + \cfrac{x - x_2}{\pi_3(x_3, x_2, x_1, x_0) - \pi_1(x_1, x_0)}}}$$

terminating with

$$\frac{x - x_{n-1}}{\pi_n(x_n, x_{n-1}, \cdots, x_1, x_0) - \pi_{n-2}(x_{n-2}, x_{n-3}, \cdots, x_1, x_0)}.$$

The technique indicated by Milne yields the desired result (1), provided no difficulty occurs in carrying out the indicated operations. The following are some exceptional cases which may occur in using this method:

1. The determinant set up to determine the rational interpolating function factors into two rational factors; i.e., the two polynomials $a(x)$ and $b(x)$ contain common factors. Milne proceeds to prove that no rational interpolating function exists satisfying (1) if this is the case.

2. The determinant which is set up to obtain the rational function vanishes identically. According to Milne, this will occur if and only if the $n+1$ pairs of

given values arise from a rational function $y=R(x)$ of order not exceeding $n-2$. (The order of the rational function is just the sum of the degrees of the numerator and denominator.) But, if we consider the set of points (0, 1), (1, 0), (2, 0), (3, 0), (4, 0), the above determinant vanishes identically, and we will show in subsequent results that no rational interpolating function of the form

$$y(x) = \frac{a_0 + a_1 x}{b_0 + b_1 x}$$

exists satisfying (1).

3. The continued fraction defines a function $y(x)$ which fails to satisfy (1). In connection with this case, Milne defines a degenerate set to be a set of $n+1$ points which belong to a rational function of order less than n. He then proves that if a set of $n+1$ points is not degenerate and contains no degenerate subset there exists an irreducible rational fraction $y=R(x)$ of order n which is satisfied by the given points. The following question arises: How can we determine when the set of points is degenerate or contains a degenerate subset? An answer to this question is implicit in the following results.

II. Existence. The problem may be stated as follows: Given $n+1$ points (x_i, y_i), $i=0, 1, \cdots, n$, with the x_i distinct, and a nonnegative integer $k \leqq n$, under what conditions does there exist an irreducible rational function $y(x) = p(x)/q(x)$ satisfying (1), where $p(x)$ is of degree $r \leqq k$ and $q(x)$ is of degree $s \leqq n-k$?

We propose the following results:

THEOREM 1. *Let (x_i, y_i), $i=0, 1, \cdots, n$, be $n+1$ points with the x_i distinct and let k be a nonnegative integer, $k \leqq n$. If there exists a nontrivial solution of the system of equations*

$$a(x_i) - b(x_i)y_i = a_0 + a_1 x_i + \cdots + a_k x_i^k - (b_0 + b_1 x_i + \cdots + b_{n-k} x_i^{n-k})y_i = 0,$$

then there exist relatively prime polynomials $t(x)$ of degree $r \leqq k$ and $v(x)$ of degree $s \leqq n-k$ such that $y_i = t(x_i)/v(x_i)$, $i=0, 1, \cdots, n$, if and only if the pair $A(x)$, $B(x)$ obtained by dividing out all common factors in any nontrivial solution $a(x)$, $b(x)$ of the system is again a solution of the system.

Proof. Let $a(x)$, $b(x)$ be any nontrivial solution of the system above; i.e., $a(x_i) - b(x_i)y_i = 0$, $i=0, 1, \cdots, n$. If $y_i = t(x_i)/v(x_i)$, $0 \leqq i \leqq n$, where the degree of $t(x) = r \leqq k$ and the degree of $v(x) = s \leqq n-k$, we may write $a(x_i)v(x_i) - b(x_i)t(x_i) = 0$. This polynomial of degree at most n has $n+1$ distinct zeros $x = x_i$. Hence $d(x)[A(x)v(x) - B(x)t(x)]$ vanishes identically, where $d(x)$ is the g.c.d. of $a(x)$ and $b(x)$, and so $A(x)v(x) - B(x)t(x)$ vanishes identically. Therefore, we have $A(x_i)v(x_i) - B(x_i)t(x_i) = 0$, $i=0, 1, \cdots, n$. Now $v(x_i) \neq 0$ for all i, since $t(x)$ and $v(x)$ are relatively prime, and $y_i = t(x_i)/v(x_i)$. Thus we have $A(x_i) - B(x_i)y_i = 0$; i.e., $A(x)$, $B(x)$ is a solution of the system of equations.

If $A(x)$, $B(x)$, as defined above, is a solution of the system, we have $A(x_i) - B(x_i)y_i = 0$, $i = 0, 1, \cdots, n$. Now $B(x_i) \neq 0$ for all i, for otherwise we would have $A(x_i) = 0$, for some i, contrary to the assumption that $A(x)/B(x)$ is irreducible. Thus, we have $y_i = A(x_i)/B(x_i)$, $i = 0, 1, \cdots, n$, and $y(x) = A(x)/B(x)$ is an irreducible rational function satisfying (1).

If existence conditions of more general applicability than those of the previous theorem are desired, we offer the following results:

Define the determinant β and the matrix A_i, $i = 0, 1, \cdots, n$, as follows:

$$\beta = \begin{vmatrix} 1 & x_1 & x_2^2 \cdots x_k^k & y_{k+1} & x_{k+2}y_{k+2} \cdots x_n^{n-k-1} y_n & x^{n-k} y \end{vmatrix},$$

$$A_i = \begin{pmatrix} 1 & x_0 & x_0^2 \cdots x_0^{k-1} & y_0 & x_0 y_0 \cdots x_0^{n-k-1} y_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{i-1} & x_{i-1}^2 \cdots x_{i-1}^{k-1} & y_{i-1} & x_{i-1}y_{i-1} \cdots x_{i-1}^{n-k-1} y_{i-1} \\ 1 & x_{i+1} & x_{i+1}^2 \cdots x_{i+1}^{k-1} & y_{i+1} & x_{i+1}y_{i+1} \cdots x_{i+1}^{n-k-1} y_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 \cdots x_n^{k-1} & y_n & x_n y_n \cdots x_n^{n-k-1} y_n \end{pmatrix}.$$

We consider first the case in which β does not vanish identically and propose the following theorem:

THEOREM 2. *Let* (x_i, y_i), $i = 0, 1, \cdots, n$, *be* $n+1$ *points with the* x_i *distinct. There exists an irreducible rational function* $y(x) = p(x)/q(x)$, $\deg[p(x)] \leqq k$, $\deg[q(x)] \leqq n-k$, *satisfying* (1) *if and only if the matrices* A_i, $i = 0, 1, \cdots, n$, *are nonsingular.*

If A_i is nonsingular for all i, we observe, then β does not vanish identically. For if β vanishes identically, then all cofactors of the last row vanish. In particular,

$$\begin{vmatrix} 1 & x_1 \cdots x_{k-1}^{k-1} & y_k & x_{k+1}y_{k+1} \cdots x_{n-1}^{n-k-1} y_{n-1} & x_n^k \end{vmatrix} \tag{4}$$

and

$$\begin{vmatrix} 1 & x_1 \cdots x_{k-1}^{k-1} & y_k & x_{k+1}y_{k+1} \cdots x_{n-1}^{n-k-1} y_{n-1} & x_n^{n-k} y_n \end{vmatrix} \tag{5}$$

vanish. Let the cofactors of the elements of some row, say the first row, of (4) be $C_0, C_1, \cdots, C_{k-1}, C_k, C_{k+1}, \cdots, C_{n-1}, C_n$ and the cofactors of the elements of the first row of (5) be $C_0', C_1', \cdots, C_{k-1}', C_k', C_{k+1}', \cdots, C_{n-1}', C_n$. Note that C_n is common in both sets of cofactors. Then

$$C_0 + C_1 x + \cdots + C_{k-1} x^{k-1} + C_k y + C_{k+1} xy + \cdots + C_{n-1} x^{n-k-1} y + C_n x^k = 0,$$

$$C_0' + C_1'x + \cdots + C_{k-1}'x^{k-1} + C_k'y + C_{k+1}'xy + \cdots + C_{n-1}'x^{n-k-1}y + C_nx^{n-k}y = 0$$

are both satisfied by the points (x_i, y_i), $i=0, 1, \cdots, n$ (cf. [2], pg. 5). Eliminating y between these two equations, we obtain an equation in x of the form

$$C_nx^n + (\text{lower powers of } x) = 0.$$

But this is an equation of degree at most n satisfied by $n+1$ distinct values of x_i, $i=0, 1, \cdots, n$. Hence, $C_n=0$.

In the same fashion we can show that all the cofactors of elements in the last column of (4) and (5) vanish. Therefore, the first n columns are linearly dependent, and A_i is singular for all i.

Before proving the theorem, we consider the following lemmas. We assume that the set of points (x_i, y_i), $i=0, 1, \cdots, n$, satisfies the conditions of the theorem.

LEMMA 1. *If there exists an irreducible rational function* $y(x)=p(x)/q(x)$ *where* $\deg[p(x)]=u$ *and* $\deg[q(x)]=w$, *satisfying* $y(x_{j_\alpha})=y_{j_\alpha}$, $\alpha=1, 2, \cdots, t$, *for* $\max[k+w, n-k+u]<t<n+1$, *and* $y(x_{j_\alpha})\neq y_{j_\alpha}$, $\alpha=t+1, \cdots, n+1$, *then* $A(x_{j_\alpha})/B(x_j)\neq y_i$, $\alpha=t+1, \cdots, n+1$, *for the pair* $A(x)$, $B(x)$ *obtained by removing all common factors in any solution* $a(x)$, $b(x)$ *of the system of equations in Theorem* 1.

Proof. The polynomial $a(x)q(x)-b(x)p(x)$ is a polynomial of degree less than t having t distinct zeros $x=x_{j_\alpha}$, $\alpha=1, 2, \cdots, t$, hence vanishes identically. We can write $a(x)q(x)-b(x)p(x)=T(x)[A(x)q(x)-B(x)p(x)]$, where $T(x)$ is the g.c.d. of $a(x)$ and $b(x)$. It must be that $A(x)q(x)-B(x)p(x)$ vanishes identically.

Now assume that $y_{j_\alpha}=A(x_{j_\alpha})/B(x_{j_\alpha})$ for some α, $t+1\leqq\alpha\leqq n+1$. Then $B(x_{j_\alpha})\neq 0$, and we have $p(x_{j_\alpha})-q(x_{j_\alpha})y_{j_\alpha}=0$. But $q(x_{j_\alpha})\neq 0$, for otherwise $p(x)/q(x)$ would be reducible. Therefore, we have for some α, $t+1\leqq\alpha\leqq n+1$, $y_{j_\alpha}=p(x_{j_\alpha})/q(x_{j_\alpha})$, a contradiction.

LEMMA 2. *If there exists an irreducible rational function*

$$y(x) = \frac{t_0 + t_1x + \cdots + t_rx^r}{v_0 + v_1x + \cdots + v_sx^s} = \frac{t(x)}{v(x)}, \qquad \begin{matrix} 0 \leqq r \leqq k-1 \\ 0 \leqq s \leqq n-k-1, \end{matrix}$$

such that $t(x_i)/v(x_i)=y_i$, $i=0, 1, \cdots, n$, *then* β *vanishes identically.*

Proof. The assumption that $y_i=t(x_i)/v(x_i)$, $i=0, 1, \cdots, n$, implies that

$$t(x_i) - v(x_i)y_i = t_0 + t_1x_i + \cdots + t_rx_i^r - v_0y_i - v_1x_iy_i - \cdots - v_sx_i^sy_i$$

vanishes. But the vanishing of this expression implies that $x_it(x_i)-x_iv(x_i)y_i$ vanishes for $i=0, 1, \cdots, n$. Using these two relations, we can combine columns in β so that two columns have only zeros in the first $n+1$ rows. Thus all cofactors of the last row vanish.

LEMMA 3. *If* $\beta = a(x) - b(x)y$ *factors into* $P(x)[A(x) - B(x)y] = 0$, *where* $A(x) - B(x)y$ *is irreducible and* $\deg[P(x)] = p \leqq \min[k, n-k]$, *then there exists no irreducible rational function* $t(x)/v(x)$, $\deg[t(x)] \leqq k$, $\deg[v(x)] \leqq n-k$, *satisfying* $t(x_i)/v(x_i) = y_i$, $i = 0, 1, \cdots, n$.

Proof. Since β does not vanish identically, there is a subset of the points (x_i, y_i), say $(x_{j_\gamma}, y_{j_\gamma})$, $\gamma = 1, 2, \cdots, t$, such that $A(x_{j_\gamma})/B(x_{j_\gamma}) \neq y_{j_\gamma}$, by Lemma 2. This implies that $A(x_{j_\gamma}) - B(x_{j_\gamma})y_{j_\gamma} \neq 0$. For if $A(x_{j_\gamma}) - B(x_{j_\gamma})y_{j_\gamma} = 0$ and $B(x_{j_\gamma}) = 0$, then $A(x_{j_\gamma}) = 0$, a contradiction; and if $B(x_{j_\gamma}) \neq 0$, then $A(x_{j_\gamma})/B(x_{j_\gamma}) = y_{j_\gamma}$, a contradiction. Then $P(x_{j_\gamma}) = 0$, $\gamma = 1, 2, \cdots, t$, and $p \geqq t$.

Now $\deg[A(x)] \leqq k-p$, $\deg[B(x)] \leqq n-k-p$, and $A(x_{j_\gamma})/B(x_{j_\gamma}) = y_{j_\gamma}$, $\gamma = t+1, \cdots, n+1$. Then $\max[k-p+n-k, n-k-p+k] = n-p < n+1-t < n+1$, and by Lemma 1, there exists no irreducible rational function satisfying (1).

LEMMA 4. *Let* $\beta = m(x) - n(x)y = 0$, *and let* $M(x)/N(x)$ *be the rational function obtained by removing all common factors in* $m(x)/n(x)$. *If* $M(x_{j_\gamma})/N(x_{j_\gamma}) \neq y_{j_\gamma}$, *for some* γ, $1 \leqq \gamma \leqq n+1$, *then* $(x - x_{j_\gamma})$ *is a factor of* $n(x)$.

Proof. Assume for some x_{j_γ}, we have $M(x_{j_\gamma})/N(x_{j_\gamma}) \neq y_{j_\gamma}$ and $n(x_{j_\gamma}) \neq 0$. Then, if $r_1, r_2, \cdots, r_w$ are the common zeros of $m(x)$ and $n(x)$,

$$y(x_{j_\gamma}) = m(x_{j_\gamma})/n(x_{j_\gamma}) = \frac{(x_{j_\gamma} - r_1)(x_{j_\gamma} - r_2) \cdots (x_{j_\gamma} - r_w)A(x_{j_\gamma})}{(x_{j_\gamma} - r_1)(x_{j_\gamma} - r_2) \cdots (x_{j_\gamma} - r_w)B(x_{j_\gamma})} = \frac{A(x_{j_\gamma})}{B(x_{j_\gamma})},$$

a contradiction.

To prove the theorem, assume A_i singular for at least one i, $0 \leqq i \leqq n$. Then $|A_i|$ vanishes. Now

$$\beta = m(x) - n(x)y =$$

$$\pm \begin{vmatrix} (x-x_0) & x_1(x_1-x) & \cdots & x_{k-1}^{k-1}(x_{k-1}-x) & y_k & y_{k+1}(x_{k+1}-x) & x_{k+2}y_{k+2}(x_{k+2}-x) & \cdots & x_n^{n-k-1}y_n(x_n-x) \end{vmatrix}$$

$$\pm y \begin{vmatrix} 1 & (x_1-x) & x_2(x_2-x) & \cdots & x_{k-1}^{k-2}(x_{k-1}-x) & y_k(x_k-x) & x_{k+1}y_{k+1}(x_{k+1}-x) & \cdots & x_n^{n-k-1}y_n(x_n-x) \end{vmatrix} = 0$$

is a solution of the system $a(x_i) - b(x_i)y_i = 0$, $i = 0, 1, \cdots, n$, where

$$n(x) = \sum_{i=0}^{n} \left[\prod_{j=0}^{n} (x_j - x) \, |A_i| \right]$$

and $(x - x_i)$ divides $n(x)$. Therefore, by Lemma 3, there exists no irreducible rational function $t(x)/v(x)$, where $\deg[t(x)] \leqq k$, $\deg[v(x)] \leqq n-k$, satisfying (1).

If A_i is nonsingular, $i = 0, 1, \cdots, n$, $|A_i|$ does not vanish and $(x - x_i)$ does

not divide $n(x)$. This follows since

$$n(x) = \sum_{i=0}^{n}\left[\prod_{j=0}^{n}(x_j - x)\,|\,A_i|\right],$$

and (x_i-x) divides all terms of the sum except

$$\prod_{\substack{j=0\\ j\neq i}}^{n}(x_j - x)\,|\,A_i|\,.$$

Therefore, by Lemma 4, $M(x_i)/N(x_i)=y_i$, $i=0, 1, \cdots, n$.

Now let us again consider the problem of fitting a rational function of the form

$$y(x) = \frac{a_0 + a_1x}{b_0 + b_1x}$$

to the points (0, 1), (1, 0), and (2, 0). We have

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

and, obviously, A_0 is singular. Hence, there exists no rational function of this form passing through all three points.

As we have previously noted, if β vanishes identically, then A_i is singular for at least one i, $0\leqq i\leqq n$. If this is the case, we offer the following generalization of Theorem 2.

THEOREM 3. *If the rank of A_i is constant for $i=0, 1, \cdots, n$, there exists an irreducible rational function $y(x)=p(x)/q(x)$ satisfying* (1).

Proof. Assume there exists no such $y(x)$. Then in any solution $a(x)$, $b(x)$ of the system $a(x_i)-b(x_i)y_i=0$, $i=0, 1, \cdots, n$, we must have $(x-x_i)\,|\,b(x)$ for at least one i. Hence,

$$a(x) - b(x)y = (x - x_i)^t[a'(x) - b'(x)y],$$

where $a'(x_i)-b'(x_i)y_i\neq 0$, but $a'(x_j)-b'(x_j)y_j=0$, $j=0, 1, \cdots, n$, $j\neq i$.

Now let $a'(x) = A_0 + A_1x + \cdots + A_{k-t}x^{k-t}$ and $b'(x) = B_0 + B_1x + \cdots + B_{n-k-t}x^{n-k-t}$. Then the matrix of the system $a'(x_j)-b'(x_j)y_j=0, j=0, 1, \cdots, n, j\neq i$, is an n row by $n-2t+2$ column matrix, $t\geqq 1$; hence, must have rank less than $n-2t+2$. Therefore, since A_i is just the matrix of this system with the columns $x^{k-t+1}, \cdots, x^{k-1}, x^{n-k-t+1}y, \cdots, x^{n-k-1}y$ augmented, we must have rank A_j less than $n-2t+2+t-1=n-t+1$. Moreover, since $a'(x_i)-b'(x_i)y_i\neq 0$, for some i, $0\leqq i\leqq n$, the matrix of the system $a'(x_\gamma)-b'(x_\gamma)y_\gamma=0$, $\gamma=0, 1, \cdots, n, \gamma\neq j, j\neq i$, must have rank equal to $n-2t+2$. But A_j is just the matrix augmented by the columns above; hence A_j has rank greater than that of A_i, $j\neq i$.

III. Uniqueness. Let $p(x)/q(x)$ be any irreducible rational function satisfying (1), with $\deg[p(x)] \leqq k$ and $\deg[q(x)] \leqq n-k$. If $\bar{p}(x)/\bar{q}(x)$ is any other irreducible rational function satisfying $\bar{p}(x_i)/\bar{q}(x_i) = y_i$, $i = 0, 1, \cdots, n$, where $\deg[\bar{p}(x)] \leqq k$ and $\deg[\bar{q}(x)] \leqq n-k$, then $p(x)\bar{q}(x) - \bar{p}(x)q(x)$ is a polynomial of degree at most n having $n+1$ distinct zeros $x = x_i$, and must vanish identically. Thus

$$p(x)\bar{q}(x) \equiv \bar{p}(x)q(x),$$

and since $p(x)$ and $q(x)$ are relatively prime, we have $p(x) \mid \bar{p}(x)$. Similarly, $\bar{p}(x) \mid p(x)$, and the uniqueness follows.

References

1. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill, New York, 1956.
2. William E. Milne, Numerical Calculus, Princeton University Press, Princeton, 1949.

THE 1961 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-second William Lowell Putnam Mathematical Competition held on December 2, 1961, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of Michigan State University, East Lansing, Michigan. The members of the team were Richard Freeman, Frederick J. Gilman, and Robert E. Greene; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Steven Orszag, Frank Rubin, and John C. Wells; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were William Richard Emerson, Roger Calvert Hill, and John Hathaway Lindsey; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Melvin Hochster, John Mather, and William C. Waterhouse; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Dartmouth College, Hanover, New Hampshire. The members of the team were George E. Cooke, Anthony Knapp, and Robert S. Strichartz; to each of these a prize of ten dollars is awarded.

The five individuals ranking highest in the examination, named in alphabetical order, are Edward Bender, California Institute of Technology; Elwyn R. Berlekamp, Massachusetts Institute of Technology; John Hathaway Lindsey, California Institute of Technology; William C. Waterhouse, Harvard University; and Barry Wolk, University of Manitoba. To each of these a prize of seventy-five dollars is awarded. The $3,000 William Lowell Putnam Scholarship at Harvard has been awarded to John Hathaway Lindsey.

The five individuals ranking second highest in the examination, named in alphabetical order, are Lawrence Corwin, Harvard University; William Richard Emerson, California Institute of Technology; Robert E. Greene, Michigan State University; John Mather, Harvard University; and David Sachs, New York University. To each of these a prize of thirty-five dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Brown University, Providence, Rhode Island, the members of the team being Richard N. Cain, Joel A. Shapiro, and Albion Dennis Taylor; University of California, Berkeley, California, the members of the team being George Bergman, Frank Henyey, and Robert Oberg; University of Toronto, Toronto, Ontario, the members of the team being Andrew Adler, Charles F. J. Dunkl, and Leonard C. Siebenmann; Yale University, New Haven, Connecticut, the members of the team being Rufus G. Clay, Richard S. Hamilton, and Stuart J. Sidney; and Yeshiva University, New York, New York, the members of the team being Stanley Boylan, Lawrence Schulman, and Benjamin Weiss.

Honorable mention is given to the following twenty individuals, named in alphabetical order: Steadman Tom Bagby, Jr., University of Kentucky; George Bergman, University of California, at Berkeley; Thomas Bloom, McGill University; James Dowling, Harvard University; Serge Dubuc, Université de Montréal; Frederick J. Gilman, Michigan State University; Christopher J. Henrich, Princeton University; Rodney Johnson, Stanford University; Richard R. Karch, Case Institute of Technology; Anthony Knapp, Dartmouth College; Sidney W. Marshall, Dartmouth College; Steven Orszag, Massachusetts Institute of Technology; Robert W. Robinson, Dartmouth College; Joel A. Shapiro, Brown University; Leonard C. Siebenmann, University of Toronto; Robert S. Strichartz, Dartmouth College; Walter F. Taylor, Swarthmore College; Benjamin Weiss, Yeshiva University; John C. Wells, Massachusetts Institute of Technology; and David Wigner, Harvard University.

A total of fourteen hundred and thirty-seven contestants from one hundred and ninety-seven colleges and universities (one hundred and eighty of these entering teams) entered the Competition. One thousand and ninety-four contestants from one hundred and ninety-two colleges and universities (one hun-

dred and sixty-five having teams) participated in the examination on December 2, 1961.

The following is a list, in alphabetical order, of all the colleges and universities which entered the Competition:

Acadia University, Agricultural and Mechanical College of Texas, Albright College, American University, Amherst College, Anna Maria College, Antioch College, Arizona State College, Arizona State University, Asbury College, Beloit College, Bethel College, Boston College, Boston University, Bowdoin College, Bowling Green State University, Brandeis University, Brooklyn College, Brown University, Butler University, California Institute of Technology, Carleton College, Carleton University, Carnegie Institute of Technology, Case Institute of Technology, Catholic University of America, Central Methodist College, Central Michigan University, Chatham College, Christian Brothers University, College Misericordia, College of Saint Thomas, College of the City of New York, College of the Holy Cross, Columbia College, Concordia College, Cornell University, Creighton University, Dartmouth College, Delaware State College, Dennison University, Doane College, Dominican College, Drew University, East Central State College, Eastern Baptist College, Eastern New Mexico University, Fairleigh-Dickinson University, Fordham University, Fresno State College, Georgetown University, George Pepperdine College, Georgia Institute of Technology, Goshen College, Goucher College, Grambling College, Grinnell College, Harvard University, Harvey Mudd College, Hastings College, Haverford College, Hofstra College, Holy Names College, Humboldt State College, Hunter College, Huntingdon College, Idaho State College, Illinois Institute of Technology, Iona College, Iowa State University, Iowa Wesleyan College, Kalamazoo College, Kent State University, Kenyon College, King College, Knox College, Lebanon Valley College, LeMoyne College, Macalester College, Madison College, Manhattan College, Marian College, Marquette University, Massachusetts Institute of Technology, McGill University, McMaster University, Memphis State University, Miami University, Michigan College of Mining and Technology, Michigan State University, Monmouth College, Mount Allison University, Muhlenberg College, Newark College of Engineering, New York University, Nicholls State College, Northeastern University, Northeast Louisiana State College, North Texas State University, Northwestern University, Oberlin College, Occidental College, Ohio State University, Ohio Wesleyan University, Oklahoma State University, Oregon State University, Pacific Lutheran University, Parsons College, Pennsylvania State University, Polytechnic Institute of Brooklyn, Pomona College, Portland State College, Princeton University, Purdue University, Queen's University, Radcliffe College, Reed College, Rosary College, Rose Polytechnic Institute, Saint Ambrose College, Saint Augustine's College, Saint Francis Xavier University, Saint Joseph's College (Brooklyn), Saint Joseph's College (Philadelphia), Saint Martin's College, Saint Olaf College, San Diego State College, San Francisco College, San Jose State College, Seattle University, Siena College, South Dakota School of Mines, Southern Illinois University, Southwestern at Memphis, Stanford University, State College (Mansfield, Pa.), Swarthmore College, Tulane University, United States Air Force Academy, Université de Montréal, University of Alberta, University of Arizona, University of Arkansas, University of British Columbia, University of Buffalo, University of California at Berkeley, University of California at Davis, University of California at Los Angeles, University of Colorado, University of Dallas, University of Hawaii, University of Idaho, University of Illinois, University of Kansas, University of Kentucky, University of Manitoba, University of Maryland, University of Massachusetts, University of Michigan, University of Minnesota, University of Nebraska, University of New Hampshire, University of Notre Dame, University of Oregon, University of Ottawa, University of Puerto Rico, University of Rochester, University of San Diego, University of San Francisco, University of Santa Clara, University of Southwestern Louisiana, University of Texas, University of the South, University of Toronto, University of Vermont, University of Washington, University of Waterloo, University of Western Ontario, University of Wisconsin, Ursinus College, Valparaiso University, Villanova University, Virginia Military Institute, Wake Forest College, Washington and Jefferson College, Washington State University,

Washington University, Wayland College, Wayne State University, Wesleyan University, Western Carolina College, Western Michigan University, West Virginia University, Xavier University, Yale University, and Yeshiva University.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the Competition were given the following problems to solve:

Part I

1. The graph of the equation $x^y = y^x$ in the first quadrant (i.e. the region where $x>0$ and $y>0$) consists of a straight line and a curve. Find the coordinates of the intersection point of the line and the curve.
2. For a real-valued function $f(x, y)$ of the two positive real variables x and y, define $f(x, y)$ to be *linearly bounded* if and only if there exists a positive number K such that $|f(x, y)| < (x+y)K$ for all positive x and y. Find necessary and sufficient conditions on the real numbers α and β such that $x^\alpha y^\beta$ is linearly bounded.
3. Evaluate
$$\lim_{n\to\infty} \sum_{j=1}^{n^2} \frac{n}{n^2+j^2}.$$
4. Define a function f over the domain of positive integers as follows: $f(1)=1$, and for $n>1$, $f(n)=(-1)^k$ where k is the total number of prime factors of n. For example $f(9)=(-1)^2$, $f(20)=(-1)^3$. Define $F(n)$ as $\Sigma f(d)$ where the sum ranges over all positive integer divisors of n. Prove that for every positive integer n, $F(n)=0$ or $F(n)=1$. For which integers n is $F(n)=1$?
5. Let Ω be a set of n points, where $n>2$. Let Σ be a nonempty subcollection of the 2^n subsets of Ω that is closed with respect to unions, intersections and complements (that is, if A and B are members of Σ, then so are $A\cup B$, $A\cap B$, $\Omega-A$ and $\Omega-B$, where $\Omega-B$ denotes all points in Ω but not in B). If k is the number of members of Σ, what are the possible values of k? Give a proof.
6. If $J_2=\{0, 1\}$ is the field of integers modulo 2, and if $J_2[x]$ is the integral domain of polynomials in one indeterminate with coefficients in J_2, prove that $p(x)=1+x+x^2+\cdots+x^n$ is reducible (factorable) in case $n+1$ is composite. Is the converse true? That is, if $n+1$ is prime, is $p(x)$ irreducible?
7. Let S be a nonempty closed set in the Euclidean plane for which there is a closed disk D (a circle together with its interior) containing S such that D is a subset of *every* closed disk that contains S. Prove that every point inside D is the midpoint of a segment joining two points of S.

Part II

1. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of positive real numbers; define s_n as $(\alpha_1+\alpha_2+\cdots+\alpha_n)/n$ and r_n as $(\alpha_1^{-1}+\alpha_2^{-1}+\cdots+\alpha_n^{-1})/n$. Given that $\lim s_n$ and $\lim r_n$ exist as $n\to\infty$ prove that the product of these limits is not less than 1.
2. Let α and β be given positive real numbers, with $\alpha<\beta$. If two points are selected at random from a straight line segment of length β what is the probability that the distance between them is at least α?
3. Consider four points in a plane, no three of which are collinear, and such that the circle through three of them does not pass through the fourth. Prove that one of the four points can be selected having the property that it lies inside the circle determined by the other three.

4. For a fixed positive integer n let $x_1, x_2, \cdots, x_n$ be real numbers satisfying $0 \leqq x_k \leqq 1$ for $k=1, 2, \cdots, n$. Determine the maximum value, as a function of n, of the sum of the $n(n-1)/2$ terms:

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} |x_i - x_j|.$$

5. Let k be a positive integer, and n a positive integer greater than 2. Define

$$f_1(n) = n, f_2(n) = n^{f_1(n)}, \cdots, f_{j+1}(n) = f^{f_j(n)}, \text{ etc.}$$

Prove either part of the inequality

$$f_k(n) < n!!! \cdots !! < f_{k+1}(n),$$

where the middle term has k factorial symbols.

6. Consider the function $y(x)$ satisfying the differential equation $y'' = -(1+\sqrt{x})y$ with $y(0)=1$ and $y'(0)=0$. Prove that $y(x)$ vanishes exactly once on the interval $0<x<\pi/2$, and find a positive lower bound for the zero.

7. Given a sequence $\{a_n\}$ of non-negative real numbers such that $a_{n+m} \leqq a_n a_m$ for all pairs of positive integers, m and n, prove that the sequence $\{\sqrt[n]{a_n}\}$ has a limit as $n \to \infty$.

Solutions. Part I

The following solutions are not from any of the contestants' papers, but generally embody ideas used by many contestants. They are published solely for the information of interested persons. Neither the editor, nor the director of the Competition, nor the paper graders will enter into any correspondence concerning them.

1. $x^y = y^x$ is equivalent to $(\ln x)/x = (\ln y)/y$ for the region specified. The set of points with coordinates satisfying this equation may be obtained with the aid of the graph of $u = (\ln v)/v$. Finding the horizontal tangent gives (e, e) as the desired intersection point.

2. To investigate boundedness of $(x^\alpha y^\beta)/(x+y)$ in the first quadrant, transform to polar coordinates with $0<r$ and $0<\theta<\pi/2$, and consider

$$g(r, \theta) = r^{\alpha+\beta-1}(\cos^\alpha \theta \sin^\beta \theta)/(\cos \theta + \sin \theta).$$

Consideration of limits as θ approaches 0 and $\pi/2$ and as r approaches 0 and ∞ yields $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta=1$ as necessary conditions. For these values of α and β, $g(r, \theta) \leqq 1$; thus, the conditions are also sufficient.

3.* For all positive integers n,

$$\int_1^{n^2+1} n/(n^2+j^2)dj \leqq \sum_{j=1}^{n^2} n/(n^2+j^2) \leqq \int_0^{n^2} n/(n^2+j^2)dj,$$

and thus it is clear that the limit as n approaches ∞ is $\pi/2$.

4. f is multiplicative and thus, so is F. Therefore, if $n = \prod_1^r p_i^{a_i}$ then $F(n) = \prod_1^r F(p_i^{a_i})$. If p is a prime then $F(p^a) = \sum_{i=0}^a f(p^i) = \sum_{i=0}^a (-1)^i$, which is 1 or 0 according as a is even or odd. Thus, $F(n)$ is always 0 or 1 and is 1 if and

* Cf. H. W. Gould and T. A. Chapman, Some curious limits involving definite integrals, this MONTHLY 69 (1962) 652.

only if n is a square.

5. Under the operation $\oplus$ defined by $A \oplus B = (A \cap B') \cup (A' \cap B)$ the collection of all subsets of Ω forms a group of order 2^n. Any Σ must also be closed under $\oplus$ and therefore be a subgroup. This means k has to be a power of 2 since the order of a subgroup must divide the order of the group.

By induction it can be shown that k actually assumes all powers of 2, that is, $k = 2^i$ for all i, $1 \leqq i \leqq n$. The main step involves showing that a Σ with 2^{i+1} elements for an Ω with $n+1$ elements can be obtained from a Σ with 2^i elements for an Ω with n elements by keeping the elements of the latter Σ and adding to it the union of the new $n+1$st point with each of its elements. The Σ so obtained has 2^{i+1} elements and can be shown to be closed under the operations.

6. Let $n+1 = rs$; then $p(x) = (1 + x + \cdots + x^{r-1})(1 + x^r + \cdots + x^{(s-1)r})$. The converse is false as is shown by the counter example

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = (1 + x + x^3)(1 + x^2 + x^3).$$

7. Suppose a point p on the circumference of D is in the (open) complement of S. Then an open circle about p also contains no point of S. This fact can be used to produce another disk containing S but not D. Thus, S must contain the circumference of D. Since each point on the circumference of D belongs to S, each point in D is the midpoint of a segment joining two points of S.

Solutions. Part II

1. Since $\lim r_n$ and $\lim s_n$ both exist, $\lim r_n s_n$ also exists and is equal to the product of $\lim r_n$ and $\lim s_n$. The product $n^2 r_n s_n$ is equal to the sum of n 1's plus $n(n-1)/2$ pairs of terms of the form $\alpha_i \alpha_j^{-1} + \alpha_j \alpha_i^{-1}$, formed for all pairs (i, j) with $i \neq j$. Now because $x + x^{-1} \geqq 2$ for all positive x, it follows that $n^2 r_n s_n \geqq n + (n^2 - n) = n^2$. Hence, $r_n s_n \geqq 1$ for all n. Thus $\lim r_n \lim s_n = \lim r_n s_n \geqq 1$. The result can also be obtained using the known fact that for any set of positive numbers the arithmetic mean is not less than the harmonic mean, so that $s_n \geqq r_n^{-1}$.

2. Let the distances from the two points from one end of the line segment be x and y. Since the positions of x and y are independent, a choice of two points corresponds to a choice of a random point in the square $0 \leqq x \leqq \beta$, $0 \leqq y \leqq \beta$. The region of this square corresponding to $|x - y| \geqq \alpha$ consists of two triangles at the corners, whose total area is $(\beta - \alpha)^2$. Hence the required probability is the ratio of this area to the total area of the square, or $(\beta - \alpha)^2/\beta^2$.

3. If the four points are such that one of them is interior to the triangle having the remaining three as vertices, the result holds for this special point, since the circle which circumscribes the triangle surely includes the entire interior of the triangle in its interior. If, on the other hand, no one of the points is interior to the triangle formed by the other three, the four points form the vertices of a convex quadrilateral. The sum of the interior angles of this quadrilateral is 360°. Since the four points are not concyclic, however, neither pair of opposite interior angles adds to 180°. Let the four points be A, B, C, and D,

and let A and C be opposite vertices of the quadrilateral with the sum of the angles at A and C greater than 180°. Then the circle passing through A, B, and D contains C in its interior and also the circle passing through B, C, and D contains A in its interior. To see this, draw a diagram. If E is any point on the circle ABD on the arc BD, the angle BED is 180° minus the angle at A, and the angle BCD is greater than this, so that C is inside the circle.

4. It is necessary to consider odd n and even n separately. Suppose that $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq 1$. To maximize the required sum of distances with respect to just x_1, it is clear that this is accomplished by making $x_1 = 0$. Similarly the sum is maximized with respect to x_n by making $x_n = 1$. Continuing this reasoning, it is seen that the maximum is obtained by setting the earlier x's equal to zero and the later x's equal to 1. If n is even, the maximum is attained with $n/2$ 0's and $n/2$ 1's, the sum of the distances being $n^2/4$. If n is odd, we put $(n-1)/2$ x's equal to zero, $(n-1)/2$ equal to 1, and allow the last x to be arbitrary. The sum of the distances is then $(n^2-1)/4$ independent of the position of the nth x.

5. (a) Proof of the lower estimate for the iterated factorial. Let $g_1(n) = n!$, $g_2(n) = n!!$, $\cdots$, $g_{k+1}(n) = g_k(n)!$. We are to show that $f_k(n) < g_k(n)$ for all positive integers k and all positive integers n, with $n > 2$. It is not difficult to establish by induction that (I) $m! > m^m e^{-m}$ for all positive integers m, and that (II) $g_k(n) > ne$ for all k and n as above except for the case that $k = 1$, $n = 3$. The key fact in establishing (I) is that $(1+1/n)^n < e$ for all n. (II) is proved by induction on k, noting that it is true for $k = 1$ for $n > 3$, and for $k = 2$, $n = 3$.

We now prove the desired result. First, it holds for $n = 3$, $k = 1$ since $f_1(3) = 3 < 6 = 3! = g_1(3)$. In general use induction on k. The result is clear for $k = 1$, $n > 2$. Assume now that $f_k(n) < g_k(n)$. Then

$$\begin{aligned} f_{k+1}(n) &= n^{f_k(n)} < g_k(n)^{f_k(n)} e^{-f_k(n)} && \text{(by II)} \\ &< g_k(n)^{g_k(n)} e^{-g_k(n)} && \text{(by induction hypothesis)} \\ &< g_k(n)! = g_{k+1}(n) && \text{(by I).} \end{aligned}$$

(b) Proof of the upper estimate for the iterated factorial. No contestant was able to obtain this result. The proof given is due to a member of the committee. Using the notation above, we are to show that $g_k(n) < f_{k+1}(n)$ for all positive integers k and all positive integers n, with $n > 2$.

Define the quantities $a_1(n)$, $a_2(n)$, $\cdots$ by the relation

$$f_{k+1}(n) = f_k(n)^{a_k(n)}.$$

Then $a_1(n) = n$, $a_2(n) = n^{n-1}$, and in general $a_{k+1}(n) = f_k(n)^{a_k(n)-1}$. To see this, note that

$$\begin{aligned} a_{k+1}(n) &= \frac{\log f_{k+2}(n)}{\log f_{k+1}(n)} = \frac{f_{k+1}(n) \log n}{f_k(n) \log n} = \frac{f_k(n)^{a_k(n)}}{f_k(n)} \\ &= f_k(n)^{a_k(n)-1} \end{aligned}$$

To prove the desired result we establish by induction on k that $g_{k-1}(n) < a_k(n)$ and that $g_k(n) < f_{k+1}(n)$. The second relation is clear for $k=1$. For $k=2$, we have

$$g_1(n) = n! < n^{n-1} = a_2(n) \qquad \text{for all } n > 2.$$

Then

$$g_2(n) = g_1(n)! < g_1(n)^{g_1(n)} < f_2(n)^{a_2(n)} = f_3(n).$$

Assuming the above relations proved through a certain k, we have

$$g_k(n) = g_{k-1}(n)! < [g_{k-1}(n)]^{g_{k-1}(n)-1} < f_k(n)^{a_k(n)-1} = a_{k+1}(n)$$
$$g_{k+1}(n) = g_k(n)! < g_k(n)^{g_k(n)} < f_{k+1}(n)^{a_{k+1}(n)} = f_{k+2}(n).$$

Hence for all k and all $n>2$ it follows that $g_k(n) < f_{k+1}(n)$.

6. To solve this problem we use the fact proved in intermediate level texts on differential equations that if $a^2 \leqq g(x) \leqq b^2$, solutions of the differential equation $y''+g(x)y=0$ have the property of oscillating more rapidly than the solutions of $y''+a^2y=0$, and less rapidly than the solutions of $y''+b^2y=0$. This means in particular that if $y(x)$ is a solution of $y''+g(x)y=0$, the distance from a maximum or minimum point of $y(x)$ to the next zero of $y(x)$ (or to the next preceding zero of $y(x)$) is at least $\pi/(2b)$ and at most $\pi/(2a)$, and that the distance between consecutive zeros of $y(x)$ is at least π/b and at most π/a. For the example of the problem, where $g(x)=1+\sqrt{x}$, we see that for $0 \leqq x \leqq \pi/2$, $a=1$, $b=\sqrt{(1+\sqrt{(\pi/2)})}$. Since there is a maximum of $y(x)$ at $x=0$, the first zero of $y(x)$ occurs between $\pi/(2b)$ and $\pi/2$. It does not occur at $\pi/2$ since $g(x)>1$ for $x>0$. Also, $\pi/(2b)$ is a positive lower bound for the zero. The spacing of zeros of $y(x)$ is at least π/b, and since $b<2$, there cannot be two zeros on $0<x<\pi/2$.

7. Since $a_{n+1} \leqq a_n a_1$, it follows by induction that $a_n \leqq a_1^n$ for all n, and hence that $0 \leqq a_n^{1/n} \leqq a_1$ for all n. The sequence $\{a_n^{1/n}\}$ is thus bounded above and below. Let $L=\limsup a_n^{1/n}$. We have $0 \leqq L \leqq a_1$. We shall show that $a_n^{1/n} \geqq L$ for all n. Once this is established, it will follow that $\liminf a_n^{1/n} \geqq L = \limsup a_n^{1/n}$, which will imply the existence of $\lim a_n^{1/n}$.

Let $m_1 < m_2 < m_3 < \cdots$ be such that

$$\lim_{k\to\infty} a_{m_k}^{1/m_k} = L.$$

Let n be any integer, and let the rational number m_k/n be given by q_k+d_k where q_k is an integer and $0 \leqq d_k < 1$. We note that $q_k/m_k = (1/n) - d_k/m_k$, so that as $k\to\infty$, $q_k/m_k \to 1/n$, since $d_k/m_k < 1/m_k \to 0$. Now by the given relation

$$a_{m_k} = a_{nq_k+nd_k} \leqq a_n^{q_k} a_1^{nd_k}.$$

Hence

$$a_{m_k}^{1/m_k} \leqq a_n^{q_k/m_k} a_1^{nd_k/m_k},$$

and letting $k\to\infty$ we obtain

$$L \leqq a_n^{1/n},$$

since $q_k/m_k\to 1/n$ and $nd_k/m_k\to 0$. It follows that $\lim a_n^{1/n}$ exists.

The sequence $\{a_n^{1/n}\}$ need not be monotonic. As an example let $a_n=e^{-n}$ if n is even, $a_n=e^{-n+1}$ if n is odd. For this sequence $\{a_n^{1/n}\}$ is an oscillating sequence converging to e^{-1}.

MATHEMATICAL NOTES

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Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida.

THE UNIFORMIZING OF PARTITIONS*

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It appears that the partition topology of a partition $\mathfrak{A}$ of a topological space X (namely, the largest topology of $\mathfrak{A}$ under which the projection $\pi: X\to\mathfrak{A}$ is continuous) is firmly established as the most natural and useful topology of $\mathfrak{A}$. However, what uniformity to take as the "partition uniformity" of a partition of a uniform space seems to depend on the particular situation. A good candidate for the "partition uniformity" of a given partition is often the well-known generalization of the Hausdorff metric for subsets of a metric space. That choice leads to a possible difficulty in that the topology r induced by this "partition uniformity" need not coincide with the partition topology p as simple examples show. (For example, there exists a pointwise periodic continuous flow on a compact annulus in the plane such that $p\neq r$ for the orbit partition. Take speeds on the concentric circles continuously decreasing to zero; there results a "quasirotation.") We here present a criterion for $p=r$ when every member of the partition is compact. We use the notation and terminology of [1].

Let $(X, \mathfrak{U})$ be a uniform space and let $\mathfrak{A}$ be a partition of X.

The *partition uniformity of* $\mathfrak{A}$ is defined to be the uniformity of $\mathfrak{A}$ generated by the uniformity base $\{\{(A,B)\in\mathfrak{A}\times\mathfrak{A} \mid A\subset B\alpha \text{ and } B\subset A\alpha\} \mid \alpha\in\mathfrak{U}\}$ of $\mathfrak{A}$. The partition $\mathfrak{A}$ is said to be a *decomposition of* X provided that each member of $\mathfrak{A}$ is a compact subset of X. If $E\subset X$, then the *star* $E\mathfrak{A}$ *of* E *in* $\mathfrak{A}$ is defined to be $\cup\{A\in\mathfrak{A} \mid A\cap E\neq\phi\}$. The partition $\mathfrak{A}$ is said to be [*star-open*] [*star-closed*] provided that $E\mathfrak{A}$ is [open] [closed] in X for every [open] [closed] subset E of X. If $A\in\mathfrak{A}$ and if $\alpha\in\mathfrak{U}$, then $N_\alpha(A)$ denotes $\{B\in\mathfrak{A} \mid B\cap A\alpha\neq\phi\}$. Let t denote

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the collection of all subsets $\mathfrak{B}$ of $\mathfrak{A}$ such that if $B\in\mathfrak{B}$, then $N_\alpha(B)\subset\mathfrak{B}$ for some $\alpha\in\mathfrak{U}$. Then t is a topology of $\mathfrak{A}$, although $N_\alpha(A)$ for $A\in\mathfrak{A}$ and $\alpha\in\mathfrak{U}$ is not necessarily a neighborhood of A in the topology t. Let p denote the partition topology of $\mathfrak{A}$ and let r denote the topology of $\mathfrak{A}$ induced by the partition uniformity of $\mathfrak{A}$.

The following statements are readily verified, except possibly for (5) whose proof is given in full:

(1) $t\subset p\subset r$.

(2) *The projection* $\pi: X\to(\mathfrak{A}, r)$ *is open.*

(3) *If* $\mathfrak{A}$ *is a decomposition of* X, *then the projection* $\pi: X\to(\mathfrak{A}, r)$ *is closed and* $t=p$.

(4, 5) *If* $\alpha\in\mathfrak{U}$, *if* $\mathfrak{A}$ *is a* [*star-closed*] [*star-open*] *decomposition of* X, *and if* $A\in\mathfrak{A}$, *then there exists* $\beta\in\mathfrak{U}$ *such that* $B\in N_\beta(A)$ *implies* $[B\subset A\alpha][A\subset B\alpha]$.

Proof of (5). Choose $\gamma\in\mathfrak{U}$ so that γ is open symmetric and $\gamma^2\subset\alpha$. There exists a finite subset F of A such that $A\subset F\gamma$. Since $x\gamma\mathfrak{A}$ is an open neighborhood of A in X for each $x\in A$, it follows that $A_\beta\subset\bigcap_{x\in F}x\gamma\mathfrak{A}$ for some $\beta\in\mathfrak{U}$. Let $B\in N_\beta(A)$. If $x\in F$, then $B\cap x\gamma\mathfrak{A}\neq\phi$, $B\cap x\gamma\neq\phi$, $x\in B\gamma$, and $x\gamma\subset B\gamma^2\subset B\alpha$. Hence $A\subset F\gamma=\bigcup_{x\in F}x\gamma\subset B\alpha$ and $A\subset B\alpha$.

THEOREM. *If* $\mathfrak{A}$ *is a decomposition of* X, *then* $p=r$ *if and only if* $\mathfrak{A}$ *is both star-open and star-closed.*

The proof of the theorem is easily constructed by use of the above statements.

The author wishes to thank the referee for helpful suggestions.

References

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ANTICONVEXITY IN THE PLANE

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Anticonvexity is, in a sense, the exact opposite of convexity. In this note, we examine a few implications of this property as applied to plane sets of points.

DEFINITION. *Let* M *be a point set in a linear space.* M *is said to be convex if it contains the straight line segment joining any two of its points.* M *is said to be anticonvex if it contains no other points of the segment joining any two of its points.*

Thus, for example, an ellipse is an anticonvex set, whereas its interior is convex. It is immediate from the definition that every subset of an anticonvex set is anticonvex; in particular, the intersection of two anticonvex sets is anticonvex.

The following properties of anticonvex sets are all well known and/or easily proven:

(1) *The interior of an anticonvex simple closed curve (in the plane) is convex.* The converse of this statement is false: a convex polygon, for instance, has a spectacularly nonanticonvex simple closed curve as its boundary. We have only the following partial converse:

(2) *If the boundary of a convex set contains no segments, it is anticonvex.*

(3) *Every anticonvex set which is dense in some open region of the plane is zero-dimensional in that region.* In particular, no anticonvex subset of the plane is two-dimensional. It is well known that there exist subsets of the plane which meet *every* straight line in exactly two points. These bizarre sets can be chosen to lie in the complement of any given bounded region of the plane, and they can all be proven zero-dimensional.

(4) *Let M be a plane anticonvex set, and suppose $M \supset N = S - F$, where S is a simple closed curve and F is a countable set. Then $M - N \subset F$.* Thus, as a special case, no simple closed curve is a proper subset of a plane anticonvex set.

DEFINITION. *M is said to be locally anticonvex if for every $p \in M$, there exists a neighborhood U (relative to M) of p such that U is anticonvex.*

Trivially, every anticonvex set is locally anticonvex. The Archimedean Spiral provides an example of a locally anticonvex set which is not anticonvex.

THEOREM. *Let M be the graph of a single-valued continuously differentiable function defined on the real line. Then the following statements are equivalent:*

(A) *M is anticonvex;*

(B) *M is locally anticonvex;*

(C) *M contains no segments or points of inflection.*

The proof is left to the reader. The proof of the implication $(B) \Rightarrow (C)$ is especially recommended as an interesting exercise in the elementary geometrical applications of calculus.

It is noteworthy that many interesting results in various branches of mathematics can be reformulated in terms of the notion of anticonvexity. Much work has been done, for instance, in studying anticonvex subsets of finite projective planes. The field of Combinatorial Analysis holds many unsolved problems involving this idea—for example, it is not known how many points comprise the largest anticonvex subset of an n by n square lattice. Another interesting question is (unsolved) problem 4774 from this MONTHLY: Find an anticonvex simple closed curve in 3-space, no four of whose points lie on a circle.

THE FUNCTIONAL EQUATION $f(x+y) = g(x) + h(y)$

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It is well known that any continuous map $f: R \to R$ with the property: $\forall x, y \in R$,

$$f(x + y) = f(x) + f(y) \tag{1}$$

must be of the form $f(x)=ax$, where $a\in R$. It is perhaps surprising that with very little additional work one can generalize this result to the class of all continuous maps $f: R\to R$ with the property:

There exist functions $g=g_f$, $h=h_f$, $g, h: R\to R$, such that $\forall x, y\in R$,

$$f(x+y) = g(x) + h(y). \tag{2}$$

In this paper we show that all continuous $f: R\to R$ with property (2) are of the form $f(x)=ax+b$, $a, b\in R$.

LEMMA 1. *Suppose that $f: R\to R$ is a function. Then f has property* (2)$\Leftrightarrow$ $\forall x, y\in R$,

$$f(x+y) = f(x) + f(y) - f(0). \tag{3}$$

Proof. Suppose that f has property (2). $\forall x, y\in R$, $f(x+y)=g(x)+h(y)$. Set $y=0$. We get $f(x)=f(x+0)=g(x)+h(0)$. Similarly, $f(y)=g(0)+h(y)$. Therefore, $f(x+y)=g(x)+h(y)=[f(x)-h(0)]+[f(y)-g(0)]=f(x)+f(y)-(g(0)+h(0))$ $=f(x)+f(y)-f(0)$.

For the converse, suppose that $\forall x, y\in R$, $f(x+y)=f(x)+f(y)-f(0)$. Then f has property (2) with (say) $g(x)=f(x)$, $h(x)=f(x)-f(0)$.

LEMMA 2. *If $f: R\to R$ is a function with property* (2), *then for all real x, and for any integer n,*

$$f(nx) = nf(x) - (n-1)f(0). \tag{4}$$

Proof. We use induction on n. For $n=0$, and any $x\in R$, (4) becomes $f(0)$ $=f(0\cdot x)=0\cdot f(x)-(0-1)f(0)=f(0)$. Suppose that (4) holds for $n=m-1$ (and all $x\in R$). For $n=m$, (and any $x\in R$) $f(mx)=f[(m-1)x+x]=f((m-1)x)+f(x)$ $-f(0)$ by Lemma 1. Hence, $f(mx)=(m-1)f(x)-(m-2)f(0)+f(x)-f(0)=mf(x)$ $-(m-1)f(0)$. Thus, for all real x, and for every nonnegative integer n, $f(nx)$ $=nf(x)-(n-1)f(0)$. Now suppose that $n>0$, and $m=-n$. We have $f(0)$ $=f(mx+nx)=f(mx)+f(nx)-f(0)$, so that $f(mx)=-f(nx)+2f(0)$ $=-[nf(x)-(n-1)f(0)]+2f(0)=(-n)f(x)-(-n-1)f(0)=mf(x)$ $-(m-1)f(0)$.

We can now prove our main result for rational arguments of f without imposing any continuity restrictions on f.

THEOREM 1. *If $f: R\to R$ is a function with property* (2), *then for all rational r,* $f(r)=(f(1)-f(0))r+f(0)$.

Proof. We use Lemma 2. Let p, q be integers ($q\neq 0$). Setting $x=p/q$, $n=q$ in (4), we get $f(p)=qf(p/q)-(q-1)f(0)$. Setting $x=1$, $n=p$ in (4) yields $f(p)$ $=pf(1)-(p-1)f(0)$. Equating the two expressions for $f(p)$ yields $f(p/q)$ $=(f(1)-f(0))p/q+f(0)$.

We now have $f(r)=ar+b$ for all rational r. As we shall see, this result cannot be extended to all arguments of f without first restricting f. We will assume f continuous at a single point.

LEMMA 3. *If $f: R \to R$ is a function with property* (2) *and f is continuous at a single point of R, then f is continuous at every point of R.*

Proof. Let f be continuous at $x_0 \in R$. Let x be any point of R. Choose any $\epsilon > 0$. Then we can find $\delta_\epsilon > 0$ such that $|f(x+u) - f(x)| = |f(u) - f(0)| = |f(x_0+u) - f(x_0)| < \epsilon$ whenever $|u| < \delta_\epsilon$. Hence f is continuous at x.

We can now prove our main theorem.

THEOREM 2. *If $f: R \to R$ is continuous at some point, then f has property* (2) *if and only if*

$$\forall x \in R, \qquad f(x) = (f(1) - f(0))x + f(0).$$

Proof. Suppose that f has property (2). By Theorem 1, $f(r) = (f(1) - f(0))r + f(0)$ for all rational r. Since f is continuous at some point, by Lemma 3, f is continuous at every point of R. Let x be any point of R. Choose a sequence $\{r_n\}$ of rationals such that $r_n \to x$. By continuity, $f(r_n) \to f(x)$ as $r_n \to x$, so $f(x) = \lim f(r_n) = \lim(ar_n + b) = a \lim(r_n) + b = ax + b$, where $a = (f(1) - f(0))$, $b = f(0)$.

For the converse, let $f(x) = (f(1) - f(0))x + f(0)$. Then f is continuous on R, and $f(x+y) = g(x) + h(y)$ for all x, yR, where $g(x) = f(x)$, $h(x) = f(x) - f(0)$.

There exist nowhere continuous functions $f: R \to R$ with property (2). We obtain such a function by a slight generalization of an argument used by Kamke, ([1] p. 118).

It follows from (4) by a simple argument that if f has property (2), then for any real x and rational r, $f(rx) = rf(x) - (r-1)f(0)$. We use this to find our function.

Let $B = \{b_1, b_2, \cdots\}$ be a Hamel basis for R, and suppose that we have defined f (arbitrarily) on $B \cup \{0\}$. Any nonzero $x \in R$ can be written uniquely as $x = \sum_i r_i b_i$ (finite sum) with the r_i rational. Now define f on R by

$$f(x) = \sum_i r_i f(b_i) - \left(\sum_i r_i - 1 \right) f(0).$$

This agrees with $f(b_j)$ if $x = b_j$ is a basis element. To see that f has property (2), let $x = \sum_i r_i b_i$, $y = \sum_i s_i b_i$. Then $x + y = \sum_i (r_i + s_i) b_i$, and

$$\begin{aligned} f(x+y) &= \sum_i (r_i + s_i) f(b_i) - \left(\sum_i (r_i + s_i) - 1 \right) f(0) \\ &= \left[\sum_i r_i f(b_i) - \left(\sum_i r_i - 1 \right) f(0) \right] \\ &\quad + \left[\sum_i s_i f(b_i) - \left(\sum_i s_i - 1 \right) f(0) \right] - f(0) \\ &= f(x) + f(y) - f(0), \end{aligned}$$

and so, by Lemma 1, f has property (2).

It is easy to define f on $B \cup \{0\}$ so that $f(x) = (f(1) - f(0))x + f(0)$ does not hold for all $x \in R$. Suppose $f(x) = ax + b$ for all $x \in R$. Let $b_1, b_2, b_3 \in B$ be distinct. Let $f(b_1) = f(b_2) = f(b_3) + 1$. Then $ab_1 + b = ab_2 + b$, so $ab_1 = ab_2$ which means $a = 0$. Then $f(x) = b$ for all $x \in R$, contradicting the fact that $f(b_3) \neq f(b_2)$. Hence, there exists a function $f: R \to R$ with property (2) which is not of the form $f(x) = ax + b$. Because of Theorem 2, f must be nowhere continuous.

Reference

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A NOTE ON CYCLOTOMIC POLYNOMIALS

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1. Let $\mu(n)$ denote Möbius's function, and let F_n be the n-th cyclotomic polynomial, i.e. the monic polynomial whose zeros are the primitive n-th roots of unity. As is well known,* the sum of all zeros of F_n is equal to $\mu(n)$. In the present note we shall be concerned with the question: how large can be the *partial* sums of the zeros of F_n? More precisely, we shall estimate the expression

$$s_n = \max_{P \subseteq P_0} \left| \sum_{k \in P} e^{2\pi i k/n} \right|, \tag{1}$$

where P_0 denotes the set of integers k such that $1 \leqq k \leqq n$ and $(k, n) = 1$. The following result will be established.

THEOREM. *We have*

$$s_n = \frac{1}{\pi} \phi(n) + O(d(n)).$$

Here $\phi(n)$ denotes Euler's function and $d(n)$ the number of positive divisors of n. The constants implied by the O-notation are assumed to be absolute throughout.

2. The proof will depend essentially on a simple identity given by the following lemma.

LEMMA 1. *Let N_0 denote the set $\{1, 2, \cdots, n\}$; let $z_1, \cdots, z_n$ be any complex numbers; and let $\mathfrak{H}$ be the set of all closed half-planes† passing through the origin of the complex plane. Then‡*

$$\max_{N \subseteq N_0} \left| \sum_{k \in N} z_k \right| = \sup_{H \in \mathfrak{H}} \left| \sum_{z_k \in H} z_k \right|. \tag{2}$$

* See, for example G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford (4th ed., 1960), p. 239.

† The lemma remains valid if closed half-planes are replaced by open half-planes or by half-planes of the form $\beta < \arg z \leq \beta + \pi$.

‡ The supremum in (2) can be replaced by a maximum since the expression $|\sum_{z_k \in H} z_k|$ can assume only a finite number of values.

For any $N \subseteq N_0$, we have

$$\Re\left(\sum_{k\in N} z_k\right) \leqq \Re\left(\sum_{\Re z_k \geqq 0} z_k\right) \leqq \left|\sum_{\Re z_k \geqq 0} z_k\right|.$$

This implies that, for any α in the range $0 \leqq \alpha \leqq 2\pi$,

$$\Re\left(\sum_{k\in N} z_k e^{i\alpha}\right) \leqq \left|\sum_{\Re(z_k e^{i\alpha}) \geqq 0} z_k e^{i\alpha}\right| = \left|\sum_{\Re(z_k e^{i\alpha}) \geqq 0} z_k\right|.$$

Maximizing with respect to α and using the fact that $\sup \Re(we^{i\alpha}) = |w|$, we infer the relation

$$\left|\sum_{k\in N} z_k\right| \leqq \sup_{0\leqq\alpha\leqq 2\pi} \left|\sum_{\Re(z_k e^{i\alpha}) \geqq 0} z_k\right| = \sup_{H\in\mathfrak{H}} \left|\sum_{z_k\in H} z_k\right|.$$

Hence

$$\max_{N\subseteq N_0} \left|\sum_{k\in N} z_k\right| \leqq \sup_{H\in\mathfrak{H}} \left|\sum_{z_k\in H} z_k\right|.$$

On the other hand, we have for any value of α,

$$\max_{N\subseteq N_0} \left|\sum_{k\in N} z_k\right| \geqq \left|\sum_{\Re(z_k e^{i\alpha}) \geqq 0} z_k\right|,$$

and therefore

$$\max_{N\subseteq N_0} \left|\sum_{k\in N} z_k\right| \geqq \sup_{0\leqq\alpha\leqq 2\pi} \left|\sum_{\Re(z_k e^{i\alpha}) \geqq 0} z_k\right| = \sup_{H\in\mathfrak{H}} \left|\sum_{z_k\in H} z_k\right|.$$

This completes the proof.

LEMMA 2. *For any positive integer m and any real number σ, we define*

$$E(m, \sigma) = \sum_{\sigma\leqq k\leqq \sigma+\frac{1}{2}\leqq m} e^{2\pi ik/m}.$$

Then

$$E(m, \sigma) = \frac{im}{\pi} e^{2\pi i\sigma/m}\left\{1 + O\left(\frac{1}{m}\right)\right\}.$$

We have

$$E(m, \sigma) = \sum_{k=r}^{r+s-1} e^{2\pi ik/m} = e^{2\pi ir/m} \frac{1 - e^{2\pi is/m}}{1 - e^{2\pi i/m}},$$

where $r = \sigma + O(1)$ and $s = \frac{1}{2}m + O(1)$. Hence

$$e^{2\pi ir/m} = e^{2\pi i\sigma/m}\left\{1 + O\left(\frac{1}{m}\right)\right\},$$

$$1 - e^{2\pi is/m} = 2\left\{1 + O\left(\frac{1}{m}\right)\right\},$$

$$1 - e^{2\pi i/m} = -2i\frac{\pi}{m}\left\{1 + O\left(\frac{1}{m}\right)\right\},$$

$$(1 - e^{2\pi i/m})^{-1} = \frac{im}{2\pi}\left\{1 + O\left(\frac{1}{m}\right)\right\}.$$

The assertion therefore follows.

3. We now come to the proof of the theorem. We shall write

$$G(n, \alpha) = \sum_k \left\{ \begin{matrix} (k, n) = 1 \\ \alpha \leqq \dfrac{2\pi k}{n} \leqq \alpha + \pi \end{matrix} \right\} e^{2\pi ik/n}. \tag{3}$$

By the principle of disjunction, it is then clear that

$$G(n, \alpha) = \sum_{d|n} \mu(d) \sum_k \left\{ \begin{matrix} d \mid k \\ \alpha \leqq \dfrac{2\pi k}{n} \leqq \alpha + \pi \end{matrix} \right\} e^{2\pi ik/n},$$

and therefore

$$\begin{aligned} G(n, \alpha) &= \sum_{d|n} \mu(d) \sum_h \left\{ \frac{\alpha n}{2\pi d} \leqq h \leqq \frac{\alpha n}{2\pi d} + \frac{n}{2d} \right\} e^{2\pi ihd/n} \\ &= \sum_{d|n} \mu(d) E\left(\frac{n}{d}, \frac{\alpha n}{2\pi d}\right). \end{aligned}$$

Hence, by Lemma 2,

$$\begin{aligned} G(n, \alpha) &= \sum_{d|n} \mu(d) \frac{in}{\pi d} e^{i\alpha}\left\{1 + O\left(\frac{d}{n}\right)\right\} \\ &= \frac{in}{\pi} e^{i\alpha} \sum_{d|n} \frac{\mu(d)}{d} + O\left(\sum_{d|n} 1\right) \\ &= ie^{i\alpha} \frac{\phi(n)}{\pi} + O(d(n)). \end{aligned} \tag{4}$$

But, in view of (1), (3), and Lemma 1, we have

$$s_n = \sup_{0 \leqq \alpha \leqq 2\pi} |\, G(n, \alpha) \,| \,;$$

and the theorem now follows by (4).

4. An analogous but simpler question which can be dealt with by the same method as that used above is concerned with the polynomial x^n-1. Let

$$s_n' = \max_{N\subseteq N_0}\left|\sum_{k\in N} e^{2\pi ik/n}\right|.$$

It then follows by Lemmas 1 and 2 that

$$s_n' = \frac{n}{\pi} + O(1).$$

The discussion of both problems we have mentioned depends ultimately upon the fact that the zeros of the polynomials in question are distributed uniformly, or almost uniformly, on the unit circle. This suggests the following more general problem.

Let $\{d_n\}$ be a sequence of positive integers tending to ∞. Suppose that, for any n, D_n is a set of d_n complex numbers of unit modulus. If

$$0 < \beta - \alpha \leqq 2\pi, \tag{5}$$

denote by $N_n(\alpha, \beta)$ the number of numbers in D_n whose arguments satisfy the inequalities $\alpha<\arg z\leqq\beta$; and suppose that, whenever (5) is valid, we have

$$\lim_{n\to\infty}\frac{N_n(\alpha,\beta)}{d_n} = \frac{\beta-\alpha}{2\pi}.$$

Finally, we write

$$t_n = \max_{D\subseteq D_n}\left|\sum_{\xi\in D}\xi\right|.$$

We may then conjecture that

$$\lim_{n\to\infty}\frac{t_n}{d_n} = \frac{1}{\pi}.$$

This relation was, in fact, proved by Professor P. T. Bateman who obtained it as a special case of a more comprehensive result on sequences of mass-distribution on the unit circle which tend to a uniform distribution.

NOTE ON OPERATOR ROOTS OF POLYNOMIALS

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By an operator we mean a linear transformation of a vector space into itself. A subspace U of a vector space V is said to be invariant under an operator T on V if $TU\subset U$. If p is a polynomial with scalar coefficients, T is said to be an operator root of p on V if $p(T)$ is the zero operator on V.

The purpose of the present paper is to establish the

THEOREM. *Let A be an operator on the vector space V over an algebraically closed field F. Then the following are equivalent.*

(1) *If $d(z)$ denotes the linear dimension of the subspace spanned by the vectors $A^i z$, $i=0, 1, 2, \cdots$, then $\sup_{z\in V} d(z)$ is finite.*

(2) *A is an operator root of a nonzero polynomial with scalar coefficients on V.*

Proof. (For an alternate proof using principal ideals in the ring of all polynomials over F see [7], pp. 5, 33–41; our argument employs more elementary means, primarily the uniqueness of the dimension of a vector space.) Assume (2) and let p be a nonzero polynomial with scalar coefficients for which $p(A)=0$ on V. Then for each vector z in V, $d(z)$ does not exceed the degree of p. The implication (2)$\Rightarrow$(1) is proved and it remains only to show that (1)$\Rightarrow$(2).

The remainder of the proof in part parallels the arguments given in [2]. Assume (1). For each scalar λ let V_λ denote the set of all vectors in V annihilated by a power of $A-\lambda$. Plainly V_λ is a subspace of V; we claim that all the V_λ together span V.

To prove this let V' be the subspace of V spanned by all the V_λ. We shall show by induction on $d(z)$ that every nonzero vector z in V is in V'. Suppose that $d(z)=1$. Then there exists a scalar λ such that $(A-\lambda)z=0$ and z is in $V_\lambda \subset V'$. Suppose that v is in V' if $d(v)=N-1$ and put $d(z)=N$. Then there exists a nonzero polynomial p with scalar coefficients of degree N such that $p(A)z=0$. Since F is algebraically closed there exist scalars $\lambda_1, \cdots, \lambda_N$ such that $p(A)=(A-\lambda_1)\cdots(A-\lambda_N)$ and $[(A-\lambda_1)\cdots(A-\lambda_N)]z=0$. Either all the λ_i are equal or two (or more) of them are different. In the former case z is in $V_{\lambda_i}\subset V'$. In the latter case suppose that $\lambda_i\neq\lambda_j$; then $d[(A-\lambda_i)z]=N-1$, $d[(A-\lambda_j)z]=N-1$ and by the induction hypothesis $(A-\lambda_i)z$ and $(A-\lambda_j)z$ are in V'. It follows that the vector $(\lambda_i-\lambda_j)^{-1}[(A-\lambda_j)z-(A-\lambda_i)z]=z$ is in V' and this completes the proof that $V'=V$.

Next we claim that the V_λ are linearly independent; the proof is by contradiction. Suppose z is a nonzero vector in V_λ and $z=\sum_1^m z_i$, z_i is in V_{λ_i} and $\lambda_i\neq\lambda$ for all i. Each V_{λ_i} is invariant under A and consequently under any polynomial in A. We can further suppose that $(A-\lambda)z=0$ and $z\neq 0$ by replacing z with $(A-\lambda)^e z$ and replacing z_i with $(A-\lambda)^e z_i$ for some appropriate power e if necessary. Then for powers n_i which satisfy $(A-\lambda_i)^{n_i}z_i=0$ we have $[(A-\lambda_1)^{n_1}\cdots(A-\lambda_m)^{n_m}]z=0$. But $(A-\lambda_1)z=(A-\lambda)z+(\lambda-\lambda_1)z=(\lambda-\lambda_1)z$. Repeated applications of this principle show that $[(\lambda-\lambda_1)^{n_1}\cdots(\lambda-\lambda_m)^{n_m}]z=0$, contrary to the fact that the product of a nonzero vector by a nonzero scalar is a nonzero vector. This completes the proof that V is the direct sum of all the V_λ. (See [1] p. 67, Theorem 1 for another argument employing modules.)

Now let $r=\sup_{z\in V} d(z)$; we claim that $(A-\lambda)^r V_\lambda=(0)$. Indeed if $(A-\lambda)^r z\neq 0$ for some vector z in V_λ, then by a simple argument ([5], p. 109) the vectors $(A-\lambda)^i z$, $i=0, 1, 2, \cdots, r$ are linearly independent and $d(z)>r$ contrary to the definition of r.

We claim that $V_\lambda\neq(0)$ for at most r distinct scalars λ. The proof is by con-

tradiction; suppose $\lambda_1, \cdots, \lambda_{r+1}$ are distinct scalars for which $V_{\lambda_i} \neq (0)$ and z_i is a nonzero vector in V_{λ_i} for which $(A-\lambda_i)z_i=0$. Then

$$\left[\prod_{j\neq i}(A-\lambda_j)\right]z = \left[\prod_{j\neq i}(\lambda_i-\lambda_j)\right]z_i,$$

where $z=\sum_1^{r+1} z_i$. It follows that the subspace spanned by $z, Az, A^2z, \cdots$ contains the linearly independent vectors $z_1, \cdots, z_{r+1}$ and $d(z)>r$ contrary to the definition of r.

Consequently $\prod_i (A-\lambda_i)^r=0$ on V where λ_i runs over all scalars λ for which $V_\lambda \neq (0)$. This concludes the proof.

We make one further observation on this theorem. Suppose A is an operator on V and $d(z)$ is finite for each vector z in V; then A need not be an operator root of a nonzero polynomial on V. Consider for example a vector space V with a countably infinite basis $\{z_1, z_2, z_3, \cdots\}$ and let A be the operator for which $Az_1=0$ and $Az_n=z_{n-1}$ for all indices $n>1$.

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THE ARITHMETICAL FUNCTION $\tau_k(N)$

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1. The most important multiplicative functions which are studied in elementary number-theory are: (a) the number of divisors of a given number N, denoted by $\tau(N)$, (b) the sum of the divisors of N, denoted by $\sigma(N)$, (c) the number of numbers a, with $a \leqq N$, $(a, N)=1$, denoted by $\phi(N)$. The following relations are well known:

(a) $$\tau(N) = \prod_{i=1}^{n}(a_i+1),$$

(b) $$\sigma(N) = \prod_{i=1}^{n}(p_i^{\alpha_i+1}-1)/(p_i-1),$$

(c) $$\phi(N) = N.\prod_{i=1}^{n}(1-1/p_i),$$

$N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ being the canonical factorization of N. While (b) and (c)

have been generalised in various ways, resulting in $\sigma_k(N)$ and $\phi_k(N)$, generalisations of (a) seem not to be well known. In this paper we will give one.

It is easy to see that $\tau(N)$ is equal to the number of all possible factorizations of N into a product of 2 factors. Also: $\tau(N)$ equals the sum of the numbers of all possible factorizations of the divisors of N into a product of $2-1$ factors. This suggests the following two generalisations:

(1) What is the number of all possible factorizations of N into a product of k factors?

(2) What is the sum of the numbers of all possible factorizations of the divisors of N into a product of $(k-1)$ factors?

We will show that these two problems have the same solution.

2. THEOREM 1. *The number of all possible factorizations of* $N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ *into a product of k factors equals*

$$\tau_k(N) = \frac{1}{[(k-1)!]^n}\cdot\prod_{i=1}^{n}\frac{(\alpha_i+k-1)!}{\alpha_i!}. \tag{1}$$

Proof. $\tau_k(N)$ is a multiplicative function, which means that

$$\tau_k(pq) = \tau_k(p)\cdot\tau_k(q), \tag{2}$$

where $(p, q)=1$. If we consider a factorization of p in k factors, and one of q:

$$p = f_1f_2\cdots f_k$$
$$q = g_1g_2\cdots g_k,$$

then the corresponding factorization $pq=(f_1g_1)(f_2g_2)\cdots(f_kg_k)$ exists. If we are given a factorization $pq=h_1h_2\cdots h_k$, then the equalities $h_j=f_jg_j$ $(j=1, 2, \cdots, k)$ determine the f's and g's uniquely. From this (2) follows.

So now we only have to determine $\tau_k(p_i^{\alpha_i})$, because then we also have

$$\tau_k(N) = \prod_{i=1}^{n}\tau_k(p_i^{\alpha_i}), \tag{3}$$

$\tau_1(p_i^{\alpha_i})=1$ and $\tau_2(p_i^{\alpha_i})=\alpha_i+1$. Thus, for $k=1$ and $k=2$ the formula

$$\tau_k(p_i^{\alpha_i}) = \frac{(\alpha_i+k-1)!}{\alpha_i!(k-1)!} \tag{4}$$

holds.

The induction is now as follows. Suppose that (4) is valid for $k-1$; then we will prove that it is also correct for k.

To obtain all factorizations of $p_i^{\alpha_i}$ into a product of k factors, the easiest way is to take each of the factorizations of $p_i^{\alpha_i}$ into $k-1$ factors, and to factorize in each case in all possible ways the first of these factors into two others.

Among the factorizations of $p_i^{\alpha_i}$ into k factors we distinguish these where the first factor is p_i^e, where e is arbitrary but fixed between 0 and $k-1$, the limits included. There are $\tau_1(p_i^e)\cdot\tau_{k-1}(p_i^{\alpha_i-e})$ factorizations of this kind. Giving to e the successive values 0, 1, 2, $\cdots$, α_i we find by summation all factorizations of $p_i^{\alpha_i}$ into a product of k factors. Thus:

$$\tau_k(p_i^{\alpha_i}) = \sum_{e=0}^{\alpha_i} \tau_1(p_i^e)\cdot\tau_{k-1}(p_i^{\alpha_i-e}),$$

and since $\tau_1(p_i^e)=1$:

$$\tau_k(p_i^{\alpha_i}) = \sum_{e=0}^{\alpha_i} \tau_{k-1}(p_i^{\alpha_i-e}). \tag{5}$$

From (2), (3) and (4) now follows:

$$\tau_k(N) = \prod_{i=1}^{n} \tau_k(p_i^{\alpha_i}) = \prod_{i=1}^{n} \frac{(\alpha_i + k - 1)!}{\alpha_i!(k-1)!},$$

or

$$\tau_k(N) = \frac{1}{[(k-1)!]^n}\cdot \prod_{i=1}^{n} \frac{(\alpha_i + k - 1)!}{\alpha_i!}. \tag{1}$$

Remarks. Instead of considering factorizations of $p_i^{\alpha_i}$ into a product of k factors where the first factor equals p_i^e, we can consider more generally the factorizations into a product of k factors where the product of the first r factors equals p_i^e, with $1 \leqq r \leqq k-1$. There are $\tau_r(p_i^e)\cdot\tau_{k-1}(p_i^{\alpha_i-e})$ factorizations of this kind. When e varies from 0 to α_i, we obtain all factorizations into a product of k factors of $p_i^{\alpha_i}$. This gives:

$$\tau_k(p_i^{\alpha_i}) = \sum_{e=0}^{\alpha_i} \tau_r(p_i^e)\cdot\tau_{k-r}(p_i^{\alpha_i-e}), \tag{6a}$$

and in connection with (4) and (1) we find the following not-too-trivial formula:

$$\sum_{e=0}^{\alpha_i} \frac{(e+r-1)!(\alpha_i+k-r-e-1)!}{e!(r-1)!(\alpha_i-e-1)!(k-r)!} = \frac{(\alpha_i+k-1)!}{\alpha_i!(k-1)!}. \tag{6b}$$

3. COROLLARY.

$$\tau_k(N) = \sum_{d/N} \tau_{k-1}(N/d). \tag{7}$$

Proof. Substitution of $i=1, 2, \cdots, n$ in (5) gives

$$\tau_k(p_i^{\alpha_i}) = \sum_{e=0}^{\alpha_i} \tau_{k-1}(p_i^{\alpha_i-e}),$$

or

$$\tau_k(p_i^{\alpha_i}) = \sum_{d_1/p_i^{\alpha_i}} \tau_{k-1}(p_i^{\alpha_i}/d_1) = \sum_{d_1/p_i^{\alpha_i}} \tau_{k-1}(d_1).$$

In analogy we have relations of this kind for $d_2, d_3 \cdots d_n$. After multiplication of these relations and application of (3) we find:

$$\tau_k(N) = \prod_{i=1}^{n}\left[\sum_{e=0}^{\alpha_i} \tau_{k-1}(p_i^{\alpha_i - e})\right];$$

application of (2) gives

$$\tau_k(N) = \sum_{0 \leqq e_1 \leqq \alpha_1 \cdots 0 \leqq e_n \leqq \alpha_n} \tau_{k-1}(p_1^{\alpha_1 - e_1} \cdots p_n^{\alpha_n - e_n}),$$

which is (7).

Remark. It is easy to see that

$$\tau_k(N) = \prod_{i=1}^{n}\left[\sum_{e=0}^{\alpha_i} \tau_r(p_i^{e}) \cdot \tau_{k-r-1}(p_i^{\alpha_i - e})\right] \tag{7a}$$

and

$$\tau_k(N) = \sum_{d_1/N} \sum_{d_2/(N/d_1)} \tau_{k-2}(N/d_1 d_2) = \sum\sum\sum \cdots \text{etc.}$$

By introducing Möbius' function $\mu(d)$ we find:

$$\tau_{k-1}(N) = \sum_{d/N} \mu(d) \cdot \tau_k(N/d). \tag{8}$$

4. Application. From the integers 1, 2, $3 \cdots N$ we choose the numbers t, with the properties: $t \mid N$, $(t, N) \neq 1$. The number of numbers of this kind is denoted by $\xi(N)$. Example: $\xi(12) = 3$. The sum of the numbers of $\xi(N)$ is $\chi(N)$.

THEOREM 2. *If* $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ *is the canonical factorization of* N, *then*:

$$\chi(N) = \sigma(N) + \tau(N) - (\tfrac{1}{2})^n \cdot \prod_{i=1}^{n} \frac{(\alpha_i + 2)!}{\alpha_i!} - N. \tag{9}$$

Proof. For each integer N we have: $N = \tau(N) + \phi(N) + \xi(N) - 1$. If $d \mid N$ then $d = \tau_2(d) + \phi(d) + \xi(d) - 1$. Summation gives:

$$\sum_{d/N} d = \sum_{d/N} \tau_2(d) + \sum_{d/N} \phi(d) + \sum_{d/N} \xi(d) - \tau_2(N). \tag{10}$$

By definition: $\sum_{d/N} \xi(d) = \chi(N)$, from which:

$$\sum_{d/N} \tau_2(d) = \tau_3(N) = (\tfrac{1}{2})^n \cdot \prod_{i=1}^{n} \frac{(\alpha_i + 2)!}{\alpha_i!},$$

and in combination with (10), formula (9) is established.

Remarks. 1. The function $\chi(n)$ is nonmultiplicative, which can be seen from (9), but also from an example as $\chi(4q)\neq 0$, for $q>2$, where $\chi(4)\cdot\chi(q)=0$ as $\chi(4)=0$.

2. From (8) we easily see that

$$\xi(N) = \sum_{d|N} \mu(d)\cdot\chi(N/d).$$

ON THE MODULE OF DOUBLY-CONNECTED REGIONS UNDER HARMONIC MAPPINGS*

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We consider one-to-one mappings $x=x(\xi, \eta)$, $y=y(\xi, \eta)$ of a given annulus $\rho<|\zeta|<1$ in the ζ-plane ($\zeta=\xi+i\eta$) onto annuli $r<|z|<1$ in the z-plane ($z=x+iy$) and ask which annuli $r<|z|<1$ are so obtainable. In general there exist no restrictions on the possible values of r, i.e. there are mappings for which r can have any value in $0\leqq r<1$. Of course, for any conformal mapping r must be equal to ρ, and for any quasi-conformal mapping with dilatation bounded by a number D the inequality $\rho^D\leqq r\leqq\rho^{1/D}$ must hold. It is the purpose of the present note to demonstrate that also harmonic mappings (i.e. mappings where $x(\xi, \eta)$ and $y(\xi, \eta)$ are harmonic, but not necessarily conjugate harmonic, functions) lead to a certain universal restriction for the possible values of r.

Denote by $\mathfrak{C}(\rho)$ the class of one-to-one harmonic mappings of the annulus $\rho<|\zeta|<1$ onto an annulus $r<|z|<1$. Special examples of the form

$$x = f(\sigma)\cos\theta, \qquad y = f(\sigma)\sin\theta, \qquad \zeta = \sigma e^{i\theta}, \tag{1}$$

where the function $f(\sigma)$ must be equal to $a\sigma+b/\sigma$, show that the given annulus $p<|\zeta|<1$ can be mapped harmonically onto any annulus $r<|z|<1$ in the z-plane, as long as its inner radius r does not exceed the value $2\rho/(1+\rho^2)$. A mapping onto a punctured disk, for instance, is given by

$$f(\sigma) = \frac{1}{1-\rho^2}\left(\sigma - \frac{\rho^2}{\sigma}\right).$$

In other words: Whereas there are harmonic transformations of the form (1) which map a given annulus onto a punctured disk, there are none which would map it onto too thin an annulus. We shall prove that the same is true for all mappings of class $\mathfrak{C}(\rho)$.

Let $R(\rho)$ be the least upper bound of all inner radii r obtainable by mappings of class $\mathfrak{C}(\rho)$: $R(\rho)=\text{l.u.b.}_{\mathfrak{C}(\rho)}\, r$. We note that $\rho=0$ implies $r=0$, i.e. $R(0)=0$. This follows from the removability of isolated singularities for bounded harmonic functions.

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THEOREM. $R(\rho)<1$, *i.e. for all one-to-one harmonic mappings of the given annulus* $\rho<|\zeta|<1$ *onto an annulus* $r<|z|<1$ *the inequality* $r\leqq R(\rho)<1$ *holds.*

Proof. Choose a closed subannulus A of $\rho<|\zeta|<1$, for instance $(1+3\rho)/4 \leqq |\zeta| \leqq (3+\rho)/4$. By Harnack's inequality for positive harmonic functions $u(\xi,\eta)$ in $\rho<|\zeta|<1$ there exists a constant $k>1$, depending only on ρ, such that the inequalities

$$k^{-1}u(\xi_2, \eta_2) \leqq u(\xi_1, \eta_1) \leqq ku(\xi_2, \eta_2)$$

are valid for any pair of points ζ_1 and ζ_2 in A. The function $u(\xi, \eta)=1+x(\xi, \eta)$ is harmonic and positive in $\rho<|\zeta|<1$. Furthermore there must be a point ζ_1 in A where $x(\xi_1, \eta_1)>r$ and a point ζ_2 where $x(\xi_2, \eta_2)<-r$. Then

$$1-r>1+x(\xi_2, \eta_2)=u(\xi_2, \eta_2)\geqq k^{-1}u(\xi_1, \eta_1)=k^{-1}(1+x(\xi_1, \eta_1))>k^{-1}(1+r).$$

From these inequalities we conclude $(1+r)/(1-r)<k$ or $r<(k-1)/(k+1)$. This implies $R(\rho)\leqq(k-1)/(k+1)<1$, *q.e.d.*

Clearly the proof applies to the higher dimensional case as well. It would be of interest to determine the exact value of $R(\rho)$ and to see whether it is identical with the number $2\rho/(1+\rho^2)$.

ON THE NUMBER OF PARTITIONINGS OF A SET OF n DISTINCT OBJECTS

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Introduction. By partitioning a set S we mean dividing the set into mutually exclusive subsets. Two partitionings are considered the same if and only if every subset in one of them is also a subset in the other. In this paper we shall investigate the number of different partitionings of a set of n distinct objects.

The Difference Equation. Let us denote by $P(n)$ the number of partitionings of a set S of n distinct objects. Let us choose one object of S and name it a. Consider the case that a and i other objects form one subset, and the remaining $n-i-1$ objects are partitioned in one way or another. The result is a partitioning of S. After a has been chosen, the additional i objects may be chosen in

$$\binom{n-1}{i}$$

ways. Since there are $P(n-i-1)$ ways to partition the remaining $n-i-1$ objects, there are

$$\binom{n-1}{i}\cdot P(n-i-1)$$

with

$$a(n, k) = ka(n-1, k) + a(n-1, k-1), \tag{6}$$
$$a(1, 1) = 1.$$

But Eq. 6 is precisely the defining relation for Stirling numbers of the second kind, completing the proof of the lemma.

The proof of Eq. 5 now follows directly, for

$$\left.\frac{d^{n-1}E(x)}{dx^{n-1}}\right|_{x=0} = \sum_{k=1}^{n} a(n, k) = P(n).$$

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CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

AN ALGEBRAIC ALGORITHM FOR THE REPRESENTATION PROBLEMS OF THE AHMES PAPYRUS

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The *Ahmes* (or *Rhind*) *Papyrus*, a famous Egyptian mathematical document described in [1], was largely concerned with the following problem: Given a rational number, e.g., 5/7, express it as a sum of reciprocals of distinct integers (thus, $5/7 = 1/2 + 1/6 + 1/21$). Apparently, the Egyptians had a convenient notation for reciprocals, but not for fractions in general. The picture was somewhat complicated by the fact that they also possessed a special symbol for 2/3; but in our treatment, we will not allow 2/3 to be used as an "admissible component." We prove the following.

THEOREM. *If $0<p<q$ where p and q are integers, then p/q can be represented as a sum of reciprocals of p or fewer distinct integers.*

Proof. We proceed by induction. The first case, $0<1<2$, has the representation $\frac{1}{2}=\frac{1}{2}$, and is the only instance where the denominator q is 2. Suppose the theorem is proved for all denominators less than some $q>2$. If p/q is not in lowest terms, it reduces to a fraction whose denominator is less than q, for which the theorem is known to hold. If p/q is in lowest terms, then $(p, q)=1$, and p has a multiplicative inverse p' modulo q, such that $pp'=qr+1$ with $0<p'<q$. Thus

$$\frac{p}{q}=\frac{qr+1}{p'q}=\frac{1}{p'q}+\frac{r}{p'}.$$

From $pp'=qr+1$ it follows that $(r, p')=1$ and that $0\leqq r<\min(p, p')$. If $r=0$, (the case $p=p'=1$), there is nothing more to do. Otherwise, since $p'<q$, it follows from the inductive assumption that r/p' is a sum of reciprocals of $r\leqq p-1$ distinct integers. It remains only to show that none of these is the reciprocal of $p'q$. This is accomplished by iterating the algorithm and observing that the successive denominators are strictly decreasing. Thus if $r\neq 0$, pick r' such that $rr'=sp'+1$, with $0<r'<p'$. Hence

$$\frac{r}{p'}=\frac{sp'+1}{r'p'}=\frac{1}{r'p'}+\frac{s}{r'},$$

with $0\leqq s<r'$. Since both $r'<p'$ and $p'<q'$, we observe that $p'q'>r'p'$ (*each* of the corresponding factors being bigger), and the successive denominators decrease.

Note 1. The number of terms allowed by the theorem is not best possible in all cases. For one thing, the algorithm itself may take fewer than p steps. The proof just given actually shows the number of steps does not exceed $r+1\leqq\min(p, p')$, so that if $p'<p$ this gives a sharper bound. (Thus $\frac{3}{5}=\frac{1}{10}+\frac{1}{2}$, since $p=3$, $p'=2$.) Moreover, the algorithm does not always lead to the minimal representation. For example, the algorithm leads to

$$\frac{q-1}{q}=\frac{1}{q(q-1)}+\frac{1}{(q-1)(q-2)}+\frac{1}{(q-2)(q-3)}+\cdots+\frac{1}{2\cdot 1}$$

for a full $q-1$ steps, whereas $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$, and $\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$.

Note 2. If the restriction $p<q$ is dropped, it is still possible to represent p/q as a sum of reciprocals of distinct integers [2], [3], but the essential aspect of the present theorem, viz. the restriction to at most p terms, is lost. For example, $101/2$ cannot be a sum of 101 or fewer reciprocals of distinct integers, because $\sum_{k=1}^{101} 1/k$ is much too small.

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SOME NONEQUIVALENT DEFINITIONS OF INFLECTION POINTS

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A standard definition of an inflection point states that x_0 is a point of inflection of a function f provided that sufficiently close to x_0, the function is concave-up on one side of x_0 and concave-down on the other. (The function f is concave-up on (a, b) provided that for any two points x, y in (a, b) and any number α, $0 \leqq \alpha \leqq 1$, the inequality

$$f(\alpha x + (1 - \alpha)y) \leqq \alpha f(x) + (1 - \alpha)f(y)$$

is satisfied. For differentiable functions f, this is equivalent to having a derivative f' which is monotone increasing.) Many calculus texts state this definition for twice-differentiable functions by calling x_0 a point of inflection of f provided that, sufficiently close to x_0, the function f'' is nonpositive on one side of x_0 and nonnegative on the other. However, many other texts give definitions which require a first derivative of f to exist in a neighborhood of x_0, but not necessarily a second derivative. The definitions of this latter type are not equivalent. It is the purpose of this note to point out the relationships which exist among three such definitions.

Let f be a differentiable function defined on an open interval (a, b) containing the point x_0. Following are three statements which can be taken as (first derivative) requirements for x_0 to be an inflection point of f (see [4], [2], [5]):

(A) There exists an open interval $I \subset (a, b)$, $x_0 \in I$, such that on I the function f' is increasing on one side of x_0 and decreasing on the other.

(B) There exists an open interval $I \subset (a, b)$, $x_0 \in I$, such that on I the function f' attains a maximum or minimum at x_0.

(C) There exists an open interval $I \subset (a, b)$, $x_0 \in I$, such that on I we have $T \geqq f$ on one side of x_0 and $T \leqq f$ on the other side. Here T is the tangent to the graph of f at the point x_0.

Condition (A) is, for differentiable functions, equivalent to the definition given at the beginning of this paper.

The three theorems below show the relationships among these three conditions.

THEOREM 1. *Let f be differentiable on (a, b) and let $x_0 \in (a, b)$. If f satisfies condition* (A) *at x_0 then f satisfies condition* (B) *at x_0 and if f satisfies condition* (B) *at x_0, then f satisfies condition* (C) *at x_0.*

The proof of this theorem is immediate and will be omitted.

THEOREM 2. *There exist functions with infinitely many derivatives in a neighborhood of x_0 which satisfy condition* (C) *at x_0 but not condition* (B), *and there exist functions with infinitely many derivatives in a neighborhood of x_0 which satisfy condition* (B) *but not condition* (A).

Proof. To show that condition (C) does not imply condition (B) we define a function f by the equations

$$f(x) = \begin{cases} (e^{-x^{-2}} \sin 1/x)^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -(e^{-x^{-2}} \sin 1/x)^2 & \text{if } x < 0. \end{cases}$$

This function is infinitely differentiable and all derivatives vanish at the origin. In addition, f is nonpositive for $x<0$ and nonnegative for $x>0$ so that condition (C) is satisfied at the origin. On the other hand f' takes on positive and negative values in any interval having the origin as an end point so that f' does not attain a relative maximum or minimum at the origin.

To prove the second part of this theorem, we need only exhibit a function g which is infinitely differentiable in a neighborhood of the origin, such that $g' \geqq 0$ in this neighborhood, $g'(0)=0$, but g'' takes on both positive and negative values in any interval having the origin as an end point. The function g given below is such a function. Define g by the equation

$$g(x) = \int_0^x h(t)dt,$$

where

$$h(0) = 0$$

and

$$h(x) = (e^{-x^{-2}} \sin 1/x)^2 \qquad \text{for } x \neq 0.$$

Theorem 2 shows that no amount of differentiability makes the three conditions equivalent. We have however the following result.

THEOREM 3. *Let f be analytic on an interval (a, b) containing x_0. Then conditions* (A), (B) *and* (C) *are equivalent.*

Proof. We first note that the hypothesis implies that f'' cannot vanish on a set of points having x_0 as an accumulation point unless f is linear. For f'' is also analytic and hence determined by its values on such a set. Thus for $x_0 \in (a, b)$, there exists an open interval $I \subset (a, b)$, $x_0 \in I$, such that on I, f' is monotonic on each side of x_0. If condition (C) is satisfied by f at x_0, then it is clear that this monotonicity of f' must be of opposite kinds. This shows that condition (*C*) implies condition (A) and, from Theorem 1, we see that all three conditions are equivalent in this case.

According to a standard test for inflection points, a function f having n continuous derivatives in a neighborhood of x_0, the first $n-1$ of which vanish at x_0, has an inflection point at x_0 if the nth derivative does *not* vanish at x_0 and n is odd, $n \geqq 3$. The proof given often leads to the result that condition (C) is satisfied by f at x_0. (See Fulks [3]). However, for $n=3$ it is easy to check that condition

(A) is also satisfied and for $n>3$ we can see that condition (A) is satisfied by observing that the nth derivative of f is the $(n-2)$nd derivative of f'' and if condition (C) is satisfied by f'' then condition (A) is satisfied by f.

It is interesting to note, in conclusion, that it is possible for a function differentiable on (a, b) to have no inflection points, at least by definition (A), without being concave up or concave down on (a, b). Denjoy [1] discusses differentiable functions f having the property that f' takes on positive and negative values in every interval. The derivative f' of such a function must obviously be discontinuous at all points for which f' does not vanish. Thus the function f cannot be concave (up or down) in any interval, since the derivative of a concave function is continuous on any interval on which it exists. (The derivative of a concave function, being monotone, can have discontinuities of *only* the first kind, and, being a derivative can have *no* discontinuities of the first kind. See Rudin [6].)

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RATIONAL TABULATED VALUES OF TRIGONOMETRIC FUNCTIONS

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In the usual tables are entered values of the trigonometric functions for rational multiples of their periods. Such a tabulated value, if finite, is here called an entry. Although doing so may be repetition, it seems worth while to point out that (1) every entry is an algebraic number; (2) double every entry of cosine is an algebraic integer and so is double every entry of sine; (3) an entry of cosine or sine is an algebraic integer if and only if it is $0, \pm 1$; (4) for each x at least one of the two entries $\tan x$ and $\cot x$ is an algebraic integer; (5) both of the entries $\tan x$ and $\cot x$ are algebraic integers if $\cos 2kx = 0$ for some positive integer k; (6) entry $\sec x$ is an algebraic integer if and only if $\tan x$ is and $\csc x$ if and only if $\cot x$ is; (7) the only rational entries of $\cos x$, $\sin x$ are $0, \pm .5, \pm 1$, of $\tan x$, $\cot x$ are $0, \pm 1$, and of $\sec x$, $\csc x$ are $\pm 1, \pm 2$.

Every mathematician teaching identities is aware, or at least vaguely conscious, of the ideas involved in these statements. There is perhaps in the problem literature frequent mention of them. Some references are given by I. Niven [2], whose discussion Olmsted called to our attention. Statement (1) is obvious; (2) is proved by D. H. Lehmer [1], whose primary interest is in the degree;

(7) is proved in elementary fashion by J. H. M. Olmsted [3]; (3), (4), (5), (6) are not in the literature cited by Niven.

The purpose of this paper is to call attention to these results and to give for them elementary proofs based on recursion formulas obtained from the addition theorems for the trigonometric functions.

Put

$$P_n = 2\cos nx, \qquad Q_n = \csc x \sin nx, \qquad u = 2\cos x$$

for positive integral n. The identity

$$\cos nx + \cos (n-2)x = 2\cos (n-1)x \cos x$$

and its like for the sum of sines show that P_n, Q_n are roots of the difference equation

$$y_n - uy_{n-1} + y_{n-2} = 0.$$

The initial conditions are $P_0=2$, $P_1=u$; $Q_0=0$, $Q_1=1$. In terms of Chebyshev polynomials T_n (see [4])

$$P_n(u) = 2T_n(\cos x).$$

The first few of the polynomials are:

$$P_0 = 2 \qquad Q_0 = 0$$
$$P_1 = u \qquad Q_1 = 1$$
$$P_2 = u^2 - 2 \qquad Q_2 = u$$
$$P_3 = u^3 - 3u \qquad Q_3 = u^2 - 1$$
$$P_4 = u^4 - 4u^2 + 2 \qquad Q_4 = u^3 - 2u$$
$$P_5 = u^5 - 5u^3 + 5u \qquad Q_5 = u^4 - 3u^2 + 1$$
$$P_6 = u^6 - 6u^4 + 9u^2 - 2 \qquad Q_6 = u^5 - 4u^3 + 3u$$
$$P_7 = u^7 - 7u^5 + 14u^3 - 7u \qquad Q_7 = u^6 - 5u^4 + 6u^2 - 1$$
$$P_8 = u^8 - 8u^6 + 20u^4 + 16u^2 + 2 \qquad Q_8 = u^7 - 6u^5 + 10u^3 - 4u.$$

The coefficients of P_n, Q_n clearly are integers and their initials (=leading coefficients) are 1.

If a value for nonzero x is an entry, $nx = m \cdot 360°$, where m, n are relatively prime positive integers. For an entry, therefore, $\cos nx = 1$ and the corresponding u is a root of $P_n - 2$. Consequently, $2\cos x$ is an algebraic integer for every entry $\cos x$. The same is true of $2\sin x$ since $\sin x = \cos(x - 90°)$.

If $\cos x$ is rational, so is u. If u for an entry is rational, it is integral and since $|\cos x| \leqq 1$ it is $0, \pm 1, \pm 2$ and the rational entries for $\cos x$ are $0, \pm .5, \pm 1$.

For $\cos x \neq 0$ the expressions for $\cos 2x$, $\sin 2x$ in terms of $\tan x$ show that $\tan x$ is rational only if $\cos 2x$ and $\sin 2x$ both are. Hence either $\cos 2x$ or $\sin 2x = 0$ and the only rational entries for $\tan x$ are $0, \pm 1$.

Further properties of polynomials P_n-2 are needed to prove (3). If z is an nth root of unity and z^* its conjugate, $u=z+z^*$ is a root of P_n-2. Call u *primitive* if z is. The roots of P_n-2 are $P_1(u)$, $P_2(u)$, $\cdots$, $P_n(u)=2$ if u is a primitive root of P_n-2.

Except for $n=1$ and $n=2$, a primitive nth root of unity is imaginary. Two conjugate imaginary primitive roots of unity give equal primitive roots of P_n-2. Hence each root of P_n-2 has multiplicity 2 except the roots 2, -2, each of which is simple when it occurs. All P_n-2 have the simple root 2 and all $P_{2k}-2$ have the simple root -2 as well. Let f_n be the polynomial with initial 1 and the primitive roots of P_n-2 for simple roots. Then P_n-2 is the product of f's, whose subscripts are the set of all divisors of n. The exponent on f_1 and on f_2, when it occurs, is 1 and that on the others is 2. For example,

$$P_9-2=f_1f_3^2f_9^2, \qquad P_{12}-2=f_1f_2f_3^2f_4^2f_6^2f_{12}^2.$$

Polynomials f_n can be computed recursively. To find f_3, for example, write

$$P_1-2=f_1=u-2, \qquad P_3-2=f_1f_3^2=u^3-3u-2$$

and have f_3 by division and extraction of the square root of a square.

Polynomial f_n is irreducible in the rational field, a fact not used here, but proved by Lehmer [1].

The polynomial f_n is linear if and only if the number of primitive nth roots does not exceed 2, that is, for $n=1$, 2, 3, 4, 6 and the corresponding f's are $f_1=u-2$, $f_2=u+2$, $f_3=u+1$, $f_4=u$, $f_6=u-1$. The first quadratic f is $f_5=u^2+u-1$.

Removing $u-2$ from $P_{2k+1}-2$ and u^2-4 from $P_{2k+2}-2$ gives a polynomial F_n which has even degree $2k$, where $n=2k+1$ or $n=2k+2$, and is in fact a square.

The constant term in F_{4r} is 0. Polynomial $u^{-2}F_{4r}$ has constant term $-r^2$. If 4 does not divide n, the constant term in F_n is ± 1.

Let a_n be the absolute value of the constant term of f_n. Then

$$a_1=a_2=2, \qquad a_3=1, \qquad a_4=0.$$

For larger n not divisible by 4 the integer a_n divides 1 and is therefore 1.

From $F_{2^j}=F_{2^{j-1}}f_{2^j}^2$, $2<j$, follows

$$a_n=2, \qquad n=2^j, \qquad 2<j.$$

Likewise, from $F_{4p^j}=F_{2p^j}f_4^2f_{4p^j}^2$ follows

$$a_n=p^j, \qquad n=4p^j, \qquad p=\text{odd prime}, \qquad 0<j.$$

If product p^jq^k, where p, q are unequal odd primes, $0<jk$, divides r in $n=4r$, the constant term $-r^2$ of $u^{-2}F_n$ is divisible by $(p^jq^ka_n)^2$. If p^j, q^k are the highest powers of p, q dividing r, then $a_n=1$.

Similarly, if $n=2^ip^k$, $2<j$, by induction $a_n=1$. Hence the constant term of f_n divides 2 or is odd for all n except $n=4$.

Let entry cos x be an algebraic integer which is root of irreducible polynomial g of degree d with initial 1. Then $2^dg(\cos x)$ equals a polynomial in u dividing f_n for some n. Hence $a_n=0$, $n=4$ or $d=1$, $n=1$, 2 and the cosines are 1, -1, 0.

Write $v=\tan x$, $nx=(n-2)x+2x$, apply the addition theorem for tangent, and find the equations

$$R_n \tan nx - S_n = 0, \qquad R_n - S_n \cot nx = 0, \qquad 1 < n, \tag{8}$$

where

$$R_2 = -v^2 + 1, \quad R_3 = -3v^2 + 1, \quad R_4 = v^4 - 6v^2 + 1, \quad R_5 = 5v^4 - 10v^2 + 1,$$
$$S_2 = 2v, \quad S_3 = -v^3 + 3v, \quad S_4 = -4v^3 + 4v, \quad S_5 = v^5 - 10v^3 + 5v.$$

The polynomials R_n, S_n are roots of the difference equation

$$y_n - 2(1 - v^2)y_{n-2} + (1 + v^2)^2y_{n-4} = 0,$$

and

$$(-1)^kS_{2k+1}(v^{-1}) = v^{-2k-1}R_{2k+1}(v). \tag{9}$$

The following are also useful:

Polynomial	Degree	Initial	Constant term
R_{2k}	$2k$	$(-1)^k$	1
S_{2k}	$2k-1$	$(-1)^{k-1}2k$	
R_{2k+1}	$2k$	$(-1)^k(2k+1)$	
S_{2k+1}	$2k+1$	$(-1)^k$	

Since tan nx is determinate, R_n, S_n have no common root. One or both polynomials in (8) exist for all n and v is a root of all of them.

For an entry, $nx=m\ 180°$, where m, n are relatively prime positive integers. Hence v is root of S_n.

If n is odd, v is an algebraic integer since the initial of S_{2k+1} is $(-1)^k$. Although v^{-1} is root of R_{2k+1}, it is merely proved an algebraic number because the initial of R_{2k+1} is not ± 1.

If $n=2(2k+1)$, then $(2k+1)x$ is an odd multiple of 90°. From the second of (8) v is root of $R_{2k+1}-S_{2k+1}$ cot $(2k+1)x$, that is, of R_{2k+1} and by (9) v^{-1} is root of S_{2k+1} so that cot x is an algebraic integer.

If $n=4r$, since rx is an odd $(2s+1)$ multiple of 45°, v is a root of $(-1)^s R_r-S_r$ whose initial is ± 1 for all r. Hence v is an algebraic integer.

It is easily proved by induction that if u is a root of R_{2k}, so also is u^{-1}. The corresponding x's satisfy cos $2kx=0$, that is, kx is an odd multiple of 45°. In this case both tan x and cot x are units. For example, the roots of R_4 are

$3\pm 2\sqrt{2}$ which are respectively tan 22°30′ and cot 22°30′.

Since all of a, $1+a$, a^2 are algebraic integers or none is, sec x or csc x is algebraic integer according as tan x or cot x is (whether entry or not).

An alternative method for finding f_n can be obtained as follows. Polynomial Q_{2k+1} is a polynomial of degree k in u^2. Put

$$Q_{2k+1} = q_k(u^2).$$

Write $Q_n[ax]$ to show that the u is 2 cos ax. Then

$$F_{2k+1}(x) = Q^2_{2k+1}[\tfrac{1}{2}x], \qquad F_{2k+2}(x) = Q^2_{k+1}[x],$$

whence

$$F_{2k+1} = q^2_k(u+2), \qquad F_{2k+2} = Q^2_{k+1},$$

the indeterminate being $u=2\cos x$ in the last two equations. Hence the f's can be found recursively from the Q's without extracting a square root.

The assumption $|\cos x| \leqq 1$ can be avoided as follows.

For odd n the constant term of P_n-2 is -2 and integral u divides 2.

For $n=4k+2$ the constant term of P_n-2 is -4 but P_n-2 is a polynomial in u^2 so that again integral u divides 2.

For $n=4r$ only u divides $2r$ can be inferred directly. If $\cos 2^i(2k+1)x=1$, however, $\cos(2k+1)(2^ix)=1$ and it has been shown that $2\cos 2^ix$ has one of the values 0, ± 1, ± 2. The formula $2\cos^2 x=1+\cos 2x$ shows that halving an angle with cosine 0, $\pm .5$, ± 1 can lead to an angle with rational cosine only if the original cosine is ± 1, $-.5$ and when it does the resulting angle has cosine 0, ± 1, $\pm .5$.

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ORTHOGONALITY OF BESSEL FUNCTIONS

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In textbooks the orthogonality property $\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx=0$ if $J_n(\alpha)=J_n(\beta)=0$ and $\alpha\neq\beta$, is usually obtained by considering the differential equations having $J_n(\alpha x)$ and $J_n(\beta x)$ as solutions and by integrating the equation obtained from these after eliminating the terms in n^2.

An alternative method, however, uses the relations

$$\frac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x), \qquad \frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$$

and integration by parts. For,

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx$$

$$= \left[x^{-n}J_n(\beta x)\frac{x^{n+1}}{\alpha}J_{n+1}(\alpha x)\right]_0^1 - \int_0^1 \frac{x^{n+1}}{\alpha}J_{n+1}(\alpha x)\{-\beta x^{-n}J_{n+1}(\beta x)\}dx$$

$$= \frac{\beta}{\alpha}\int_0^1 xJ_{n+1}(\alpha x)J_{n+1}(\beta x)dx,$$

provided that α and β are zeros of $J_n(x)$ or of $J_{n+1}(x)$. However, interchanging α and β in the above equation we see that, provided $\alpha \neq \beta$, both integrals must be zero.

This shows that the functions $J_n(\lambda x)$ are orthogonal with respect to the weighting function x on the interval (0, 1) not only when $\lambda_1, \lambda_2, \cdots$ are the zeros of $J_n(x)$ but also when they are the zeros of $J_{n\pm1}(x)$. This fact can be deduced from the Sturm-Liouville theorem on orthogonal functions since two solutions $\phi_1(x)$ and $\phi_2(x)$ of an equation of the Sturm-Liouville type are orthogonal with respect to the appropriate weighting function on the interval (a, b) if

$$\phi(a) + A\frac{d\phi}{dx}\bigg|_a = 0, \qquad \phi(b) + B\frac{d\phi}{dx}\bigg|_b = 0,$$

where A and B are constants. In our example these conditions follow from the recurrence relations (equivalent to the differentiation relations quoted above)

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x),$$
$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x).$$

THE COMPLETE INDEPENDENCE OF SOME COMMON AXIOM SYSTEMS

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1. Introduction. In a first course in Modern Algebra, it is not uncommon to discuss the basic properties of axiom systems in general, for example, consistency, independence, etc. These properties are most frequently illustrated in the literature by axiom systems which a student will never encounter again, or by very specialized axiom systems such as those for finite projective planes which appear much later in a student's career (cf. [2]). The axioms for an equivalence relation and the group axioms provide simple and instructive examples for all these concepts, but a complete discussion of the details seems lacking in standard texts on both Modern Algebra and Foundations.

Following the discussion of the properties of axiom systems in [1], we say that an axiom system S is *consistent* if it is satisfiable, that is, if it has a model. If the axiom system S is satisfiable, then an axiom A in S is *independent* if the system $(S-A)+\overline{A}$ is satisfiable, where $\overline{A}$ denotes the denial of the axiom A.

The system S is then called *independent* if each of its axioms is independent. The notion of independence can be extended by saying that an axiom system S is *completely independent* if, for any subset S_1 of axioms in S, the system $(S-S_1)+\overline{S}_1$ is satisfiable. Finally, a satisfiable axiom system is *complete* if there is no S-statement A, which is not in S, such that A is independent in $S+A$. The test for completeness is usually found in the notion of a *categorical* system, that is, an axiom system S for which there is a unique model with respect to S-statements. It is easy to show that categorical implies complete ([1], p. 36).

2. The axioms for a group. The group axioms are commonly stated as follows. Given a set G and a mapping $\circ: G\times G\to G$, we say that $(G, \circ)$ is a *group* if:

1) $(a\circ b)\circ c=a\circ(b\circ c)$ for all a, b, c in G,
2) there is an element e in G such that $e\circ a=a$ for all a in G,
3) for each a in G there is an element a^{-1} in G such that $a^{-1}\circ a=e$.

In discussing the independence of these axioms, the appearance of the left identity in 3) presents a problem. That is, what does 3) mean if 2) does not hold? This is remedied by replacing 3) with

3*) for every pair a, b in G there is an element c in G such that $c\circ a=b$.

The axioms 1), 2), 3*) are obviously equivalent to 1), 2), 3).

We now prove the complete independence of the axioms 1), 2), 3*) by exhibiting multiplication tables which serve as models for the eight possible combinations of 1), $\overline{1}$), 2), $\overline{2}$), 3*), $\overline{3}$*). We remark here that complete independence clearly implies independence. It also implies consistency by taking S_1 as the empty set.

1), 2), 3*):

	e
e	e

1), 2), $\overline{3}$*):

	e	a
e	e	a
a	e	a

1), $\overline{2}$), 3*):

	e	a
e	e	e
a	a	a

$\overline{1}$), 2), 3*):

	e	a	b
e	e	a	b
a	a	e	b
b	b	a	e

1), $\overline{2}$), $\overline{3}$*):

	e	a
e	e	e
a	e	e

$\overline{1}$), 2), $\overline{3}$*):

	e	a
e	e	a
a	e	e

$\overline{1}$), $\overline{2}$), 3*):

	e	a
e	a	a
a	e	e

$\overline{1}$), $\overline{2}$), $\overline{3}$*):

	e	a
e	a	e
a	a	e

Observe that if G has two elements then 2) and 3*) together imply 1). Hence the necessity for three elements in the model for $\overline{1}$), 2), 3*).

The above tables show that the group axioms 1), 2), 3*) are completely independent and hence, as remarked above, they are consistent and independent.

Since there are groups of order n for every positive integer n, these axioms are not categorical. To demonstrate the notion of a categorical system we can add the following axiom to 1), 2), 3*),

4) the set G contains exactly three elements.

The unique model for the axiom system 1), 2), 3*), 4) is then the cyclic group of order three. Thus this system is completely independent, categorical and so complete. The fact that it is completely independent requires the extension of the above tables in the obvious fashion to a set with three elements. We remark that in 4) the integer three may be replaced by any positive integer n with the property that the greatest common divisor of n and $\phi(n)$ is one, since there is only one group of each such order. However, proving complete independence becomes more complicated with an increase in the number of elements.

3. The axioms for an equivalence relation. The notion of equivalence relation pervades all of mathematics. For our purposes we make the following definitions.

DEFINITION 1. *A relation on a set A is a subset R of $A \times A$.*

DEFINITION 2. *An equivalence relation on a set A is a subset R of $A \times A$ satisfying:*

1) $(a, a) \in R$ *for all* a *in* A,
2) *if* $(a, b) \in R$, *then* $(b, a) \in R$,
3) *if* $(a, b) \in R$ *and* $(b, c) \in R$, *then* $(a, c) \in R$.

We now show that the axioms 1), 2), 3) are completely independent by defining relations on a set A which satisfy the eight possible combinations of 1), $\bar{1}$), 2), $\bar{2}$), 3), $\bar{3}$). For the set A we take the collection of all subgroups (proper and improper) of S_4, the symmetric group on four letters (cf. [2]). We are particularly interested in the following subgroups whose elements we write in cycles, A_4, the alternating subgroup of S_4, $K = \{(1), (12)(34), (13)(24), (14)(23)\}$, and $H = \{(1), (12)(34)\}$. It is known that K is normal in S_4 and hence in A_4. Clearly H is normal in K since K is Abelian, but H is not normal in A_4 since A_4 is non-Abelian and the center of A_4 consists of the identity alone. The relations on this collection of subgroups of S_4 which demonstrate the complete independence of 1), 2), 3) are the following. Unless it is obvious we indicate the reasons why the above axioms fail to hold.

1), 2), 3) $(a, b) \in R$ if b is conjugate to a,
$\bar{1}$), 2), 3) $(a, b) \in R$ if a and b are Abelian,
1), $\bar{2}$), 3) $(a, b) \in R$ if a is a subgroup of b,
1), 2), $\bar{3}$) $(a, b) \in R$ if $a \cap b$ is normal in both a and b, H is normal in K, K is normal in A_4, but H is not normal in A_4,
$\bar{1}$), $\bar{2}$), 3) $(a, b) \in R$ if a is a proper subgroup of b,
$\bar{1}$), 2), $\bar{3}$) $(a, b) \in R$ if $a \cap b$ is Abelian, for $\bar{3}$) take A_4, K and A_4 in that order,

$\bar{1}$), $\bar{2}$), $\bar{3}$) $(a, b) \in R$ if a is a normal subgroup of b, for $\bar{3}$) take H, K and A_4 in that order,

$\bar{1}$), $\bar{2}$), $\bar{3}$) $(a, b) \in R$ if a is a proper normal subgroup of b.

For further discussion on the axioms for an equivalence relation and their connection with independence, see [3].

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MATHEMATICAL EDUCATION NOTES

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THE AMHERST FRESHMAN PROGRAM IN MATHEMATICS AND PHYSICS

ALFRED B. WILLCOX, Amherst College

As a part of its freshman-sophomore required curriculum, Amherst has offered for the past fifteen years a course, called Science 1, 2, in mathematics and physics. Science 1, 2 is administered and taught jointly by the departments of mathematics and physics with the help of the chemistry department and attempts to present an introduction to the physical sciences and to the calculus which will explore and exploit the historical connections between the two subjects in a way which will not obscure the essential independence of mathematics from any particular model. To put it another way, Science 1, 2 attempts to be honestly interdisciplinary, at the same time recognizing that it *is* dealing with two distinct disciplines; it represents a marriage between two partners who refuse to become one and yet cannot be finally divorced.

Can this marriage be successful? The answer depends, of course, on how one defines success, and there are probably as many distinct answers coming from the experience of Science 1, 2 as there are members of the staff. In the context of a program which has been in existence for such a long time, as educational ventures go, it is hard to separate this question from a host of others generated out of the local situation: Should such a course be required of all students? Is the present format the most effective method of achieving the stated goals? Does this introduction favor one partner over the other in its tendency to attract students into a major program? If I, as close as I am to Science 1, 2, could stand apart enough to separate the first question from the clutter of the rest, I believe

I would have to answer, "yes." I would claim that the marriage can be and is, in the balance, successful. This claim needs both documentation and a bit of moderation, and it is the purpose of this note to present a measure of both.

Perhaps it would be best to begin with a description of the mechanics of the course. The freshman class is divided into ten sections of approximately 26 students. Once each week (twice in the second semester) the ten sections are assembled in one lecture room for a physics lecture. The sections meet individually once each week for a discussion session in physics and for a two-hour laboratory period. The mathematics is taught entirely in section meetings of which there are two each week. Over the years there has been an effort to maximize the carry-over between the two halves of the course by encouraging individual instructors to teach "across-the-board," though commitments to the major programs, difficulties in scheduling, and other considerations have kept the number of instructors who so participate down to four—two physicists and two mathematicians.

The physics side of the course has been described elsewhere (Arons, A., Structure, Methods, and Objectives of the Required Freshman Calculus—Physics Course at Amherst College, American Journal of Physics, 9, 658, Dec. 1959). For our purposes it should suffice to say that it presents an introduction to the physical sciences culminating in and climaxed by a development of the Bohr model of the hydrogen atom. The approach to the subject is similar to the PSSC high school course, being appropriately more sophisticated and making much use of the concepts of the calculus as soon as they have been developed in the mathematics half of the course.

In mathematics we present what might be described as a fairly standard first course in analytic geometry and the calculus somewhat shortened to fit into four semester-hours. Very few decisions concerning content or pedagogy have been dictated entirely by the demands of physics. We have arranged to introduce both the derivative and the definite integral early in the game so that they may be used in discussion of the appropriate physical concepts, but no other important aspect of the treatment could be described as tailored especially for our spouse. The natural walls created by limitations in time, by the keen competition for the student's time and attention in a crowded freshman year, and by the fact that the course is required for every (nonadvanced placement) student we find far more confining than is our marriage to physics.

The advantages to physics in such a combined course are obvious and need not be catalogued here. I might point out, however, just one benefit beyond the fact that physics cannot be done decently without the calculus. The presence of a few needling mathematicians in the course—even sitting in on the physics lectures—has measurably reduced the heresy which mathematicians have grown to expect a physics teacher to commit in the name of mathematics. It has also been an education for us mathematicians to be in a position where it is either "put up or shut up" in many of the areas where physicists are well known to murder the King's mathematics. Not all of our stereotyped admonitions to the

physicist offer workable alternatives.

Most mathematicians to whom I describe the course find no difficulty in imagining the disadvantages to mathematics—more accurately, perhaps, to a mathematics department—in such close ties to physics. Is it possible, to mention only one, to present an introductory course in the calculus tied so closely to a physics course without leaving the student with an impression of mathematics as a tool, indispensable, conceivably, but still a tool? No mathematician could be happy with a course, especially one which is terminal for a significant number of students, which was so cast as to make it virtually impossible to present convincingly the freedom of mathematics from any necessary interpretation. We have tried very hard to avoid this trap, with the full support and cooperation of the physicists. I have mentioned above that the pedagogical demands of the physics side of the course on the mathematics are minor, and we have avoided any relaxation in the rigor of the mathematics which would not be necessary in any required freshman course. In addition, we have been able to open up a six-week period at the end of the second semester, devoted to any mathematics which we desire to teach, independently of any direct application to physics. We have experimented with topics in set theory, propositional calculus and Boolean algebra and with additional topics in the calculus treated in somewhat greater depth. None of us feel that we have found the most useful way to spend this period, but we do have this opportunity to explore a topic designed either to broaden the student's concept of mathematics or to deepen his penetration into at least one significant topic.

There are significant advantages to mathematics in this marriage too, and these are so obvious that they are perhaps easily overlooked in the weighing. No one would conceive of offering an introductory course in the calculus without including discussions of several physical problems either as motivation for definitions or as applications of results. This consumes a nontrivial amount of the precious time allotted for such a course; if it doesn't, then it can be questioned whether any part of such a discussion "sinks in" enough to pass for motivation at all. In a sense, the physics half of Science 1, 2 assumes this burden and makes it possible for us to teach more mathematics per hour in our allotted four semester-hours than we could otherwise. In a curious sort of reverse twist this close alliance often makes it easier to point out the unique flavor of mathematics and to wage the friendly holy war between the mathematician and the physical heathen. When I heap abuse on the physicist who misuses a mathematical concept, the student knows that it is my responsibility not to make the same mistake the next day when I put on my physicist cloak. I sometimes fail in this —in small ways, I insist—and the student can perhaps see that it is not a difference in personality or mentality that separates the mathematician from the physicist but a divergence of purpose.

Perhaps a brief catalogue of specific areas of cross-fertilization will provide further illustrations of the benefits of the marriage to both partners and, at the same time, further description of the course itself.

1. From very early in the first semester we speak, in the physics side of the course, of velocity and acceleration as slopes of tangent lines to the s vs. t and v vs. t curves. By the time we arrive at the concept of derivative in the mathematics class no further discussion of the problems of *defining* a tangent line and *defining* velocity is necessary. We have been referring to them long enough so that the student has already begun to appreciate their significance; and we have been referring to them in the language which we would want to use to begin a mathematical discussion of the problems and the concepts to which they lead.

2. When the derivative has been introduced and the student has learned how to differentiate a polynomial function we turn to the definite integral and a heuristic discussion of the Fundamental Theorem. At about the same time the concept of work has been introduced in physics and the work-kinetic energy theorem is looming on the horizon. The necessity of proving this theorem without sophisticated substitution techniques provides an excellent opportunity to derive significant results from a combination of the definition of the integral and the mean value theorem, and, alternatively, as the chain rule is introduced, by an application of the fundamental theorem.

3. There are other occasions, less pivotal, perhaps, than the two mentioned above, when mathematical concepts and physical applications occur naturally in close proximity. In the mathematics the calculus of trigonometrical functions comes at just about the right time to be applied in discussions of centripetal force, simple harmonic motion, and wave phenomena. When the conic sections appear in the mathematics the student has long been prepared for their arrival by a discussion of the Newtonian synthesis, and hopefully, he is impressed by the fact that our discussion of the synthesis had been necessarily naive and incomplete at the time because of our lack of analytical knowledge of these classical curves. At about the same time that the hyperbola is being introduced, the discussion of the interference patterns formed by wave trains generated from two discrete sources provides an excellent application. The problem of exponential decay is an application of the analysis of the logarithm and exponential function which is of far more than casual interest to a student who is encountering the phenomenon of radioactivity as an important link in the chain which leads to the Bohr model of the hydrogen atom.

Thus far this report has been documentation without much moderation. Should our enthusiasm for Science 1, 2 be tempered by moderation? Let it suffice to say that it is. Like all teachers each of us finds himself continually evaluating this venture both from the point of view of its contribution to the liberal education of the students who take it and from the point of view of its effectiveness as a first course for a potential major in one of the two fields. On the latter point neither a physicist nor a mathematician would say that Science 1, 2 represents the most efficient first technical course in either field. A student who majors in mathematics at Amherst is, at the end of his sophomore year, somewhat behind

his contemporaries at some of our sister institutions. This places a relatively heavier burden on our upper class curriculum if we wish to prepare our students adequately for graduate school. In spite of our best efforts there is an impression left in the minds of some students that mathematics has played the role of honored and trusted servant to physics. But the alternative, for most of these, I suspect, is no impression at all, and we have the satisfaction of knowing that even the budding novelist and the flowering historian knows, when he leaves Amherst, something of what calculus is about and something of its historical and logical connections with the physical sciences.

In summary, Science 1, 2 is conceived of as more than a technical introduction into two subject matters; in offering it we are willing to sacrifice some things to the larger goals of liberal studies. We do not sacrifice rigor, we sacrifice coverage and our privilege of standing completely apart from the physicist. We would be neither human nor mathematicians if we didn't admit to feeling keenly the sacrifice at times, but as educators who are more interested in helping critical minds develop themselves than in serving up information, we are convinced that the sacrifice is worth making.

SUMMER WORKSHOP IN MATHEMATICS, ENTEBBE, UGANDA JUNE 18–AUGUST 10, 1962

W. T. MARTIN, Massachusetts Institute of Technology

During the summer of 1961 a group of approximately 15 Africans, a few persons from the United Kingdom and around 25 persons from the United States met for six weeks at M.I.T.'s Endicott House in Dedham, Massachusetts to consider various matters related to education in various countries of English speaking tropical Africa. The group, which was under the chairmanship of Professor Jerrold R. Zacharias of the Department of Physics, M.I.T., devoted most of its attention to primary and secondary school curricula, including preparation of material and training of teachers. The conference was under the auspices of the nonprofit institution, Educational Services, Inc. of Watertown, Massachusetts, and had financial support from the International Cooperation Administration and from the Ford Foundation.

As an outgrowth of the Endicott Conference groups of mathematicians and persons concerned with mathematics teaching from Africa, the United Kingdom and the United States have met twice during recent months in Africa to consider specific problems relating to mathematics. Small projects growing out of these meetings are now underway in Ghana, Liberia, Nigeria, Sierra-Leone and Uganda. Some of these consist of experimental classes in mathematics, others consist of studies of existing material from the United States. Groups there have taken active interest in SMSG materials and in the work of the Stanford Arithmetic Project, the Syracuse Madison Project and the University of Illinois Arithmetic Project.

For eight weeks during the summer of 1962 a group of approximately twenty African mathematics teachers and university mathematicians will be

joined in Entebbe, Uganda, by ten expatriate English and American mathematics teachers and university mathematicians now resident in Africa, and by a group from the United States with a few persons from the United Kingdom. They will consider the primary, secondary and sixth form programs in mathematics. It is hoped that these considerations will lead to a structure for the experimental development of material, probably with several different approaches. Testing, and training of teachers will also be considered. It is not anticipated that in the time available at the summer workshop any body of material will be brought to the stage where it will be ready for experimental use in schools. The principal goal of the workshop is to establish firm guidelines, and to create enough material to serve as prototype for further work. The experience of the past few months indicates that emphasis is likely to be placed upon materials for the age groups 6–8 and 14–16; the workshop, of course, may decide otherwise.

A tentative list of the persons from the United States who are expected to participate during part or all of the eight-week workshop is as follows:

Dr. Marvin L. Bender, Greater Cleveland Research Council, Cleveland, Ohio.
Professor David Blackwell, Department of Statistics, University of California, Berkeley.
Professor Samuel Eilenberg, Department of Mathematics, Columbia University.
Professor Wade Ellis, Department of Mathematics, Oberlin College.
Dr. Ross L. Finney III, Department of Mathematics, Massachusetts Institute of Technology.
Professor Helen Garstens, Department of Mathematics, University of Maryland.
Professor E. W. Hamilton, Department of Mathematics, State College of Iowa.
Professor Clarence Hardgrove, Department of Mathematics, Northern Illinois University.
Dr. Shirley Hill, Arithmetic Project, Stanford University.
Professor Paul B. Johnson, Department of Mathematics, University of California, Los Angeles.
Miss Eunice Lewis, Laboratory High School, University of Oklahoma.
Miss Mary McDermott, Mount Diablo Unified School District, Concord, California.
Dr. Sheldon S. Myers, Educational Testing Service, Princeton, New Jersey.
Professor Walter Prenowitz, Department of Mathematics, Brooklyn College.
Professor Merrill Shanks, Department of Mathematics, Purdue University.
Mr. Nicholas Sterling, Madison Project, Webster College.
Mr. Henry Swain, New Trier Township High School, Winnetka, Illinois.

(Submitted in May 1962)

EVALUATING THE EFFECTS OF COLLEGE ON THE TALENTED STUDENT

ALEXANDER W. ASTIN, National Merit Scholarship Corporation

For the past four years the National Merit Scholarship Corporation's Research Division has conducted a series of studies aimed at comparing the effects of different undergraduate institutions on the educational aspirations of talented students. Two recently completed projects were concerned with re-evaluating the earlier evidence on "Ph.D. productivity." These earlier studies had shown that undergraduate colleges differ markedly in the proportions of their B.A. and B.S. recipients who eventually obtain the Ph.D. Since it had also been shown that these differences in output of Ph.D.'s were highly related to the intellectual climate or "environment" of the institutions, it was thought earlier that the

"highly productive" college actually fosters or stimulates higher academic attainment among its students more than the "less productive" college.

The Research Division at NMSC became interested in these studies when it was observed that the finalists in the scholarship competition overwhelmingly preferred to attend the highly productive institutions [5]. Obviously, if these institutions were attracting more able or more highly motivated students to begin with, their higher "productivity" may be more a function of their recruitment practices, rather than any special motivating "effect" on the student. Following this same line of reasoning, other input variables, such as the sex-ratio or major fields of study of the incoming student body, might also be expected to have some effect on the institution's eventual output of Ph.D.'s. To test these ideas, the actual Ph.D. outputs of 265 institutions were compared with "expected" Ph.D. outputs based on the ability level, sex-ratio and intended fields of study of their student bodies. As hypothesized, there was a high relationship between these expected Ph.D. output rates and the actual Ph.D. output rates. Furthermore, many of the previously "highly productive" colleges turned out to be producing far less than their expected proportions of Ph.D.'s. Specifically, it was found that men's institutions located in the northeastern states tend to be underproductive, whereas public institutions and technological institutions tend to be overproductive. No consistent relationships were found between the college's "environment" and its tendency to be over- or underproductive [3].

Several other projects at NMSC have been concerned with improving methods for assessing the college "environment." One study [4] demonstrated that the characteristics of the college environment are, to a large extent, dependent on the characteristics of the student body. This relationship was, in fact, shown to be largely responsible for the high correlations found in the earlier work between Ph.D. output and the college environment [1]. Another project attempted to develop an empirical typology of colleges by factor-analyzing 335 institutions [2]. The 33 variables used in the analyses included measures of each college's financial resources, curriculum, faculty and student body, and "environment." The factor accounting for the largest proportion of variation among institutions was identified as "Affluence" ("Wealth"). Institutional characteristics most descriptive of this factor included financial resources (endowment, capital income, budget, scholarship and research funds), tuition, relative library size, ability level of the student body, faculty-student ratio, and level of training of the faculty. The environments of the highly affluent institutions tended to be more "intellectual," whereas the environments of the less affluent institutions tended to be more "conventional."

Nearing completion are several longitudinal studies of the effects of colleges on samples of Merit Scholars and Finalists. One purpose of these projects is to compare the effectiveness of various colleges in encouraging talented students to pursue careers in scientific or teaching fields. The design of the studies is similar to the input-output model used in evaluating Ph.D. productivity. For example, the proportion of graduating students who actually enter careers in a scientific

field will be compared with an "expected" proportion based on the personal and intellectual characteristics of the same students at the time they entered college. If significant differences between expected and actual outputs occur, an attempt will be made to account for them on the basis of differences in the college environment.

Another college study in progress was designed to assess more comprehensively the differences among colleges in their student "inputs." In Fall, 1961, complete freshman classes entering 248 representative colleges and universities were administered a short information blank to obtain data about each student's potential for later achievement in academic, scientific, artistic, and leadership fields. Analyses of these student input data will be useful in showing how student talent is distributed among higher educational institutions, and should also provide more information regarding the relationships between the college environment and the student input.

A detailed summary of NMSC's complete research program, which in addition to the college projects includes studies of vocational choice, talent loss, identification of talent, and prediction of achievement, is contained in the Research Division's *Technical Report No. 6.* Copies of this report, as well as individual copies of completed studies, can be obtained by writing Dr. John L. Holland, Director of Research, National Merit Scholarship Corporation, 1580 Sherman Avenue, Evanston, Illinois.

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4. A. W. Astin and J. L. Holland, The environmental assessment technique: a way to measure college environments, J. Educ. Psychol., **52** (1961) 308–316.

5. J. L. Holland, Undergraduate origins of American scientists, Science, **126** (1957) 433–437.

THE UICSM PROGRAMMED INSTRUCTION PROJECT

HERBERT WILLS, Programer, UICSM, University of Illinois

In April 1961 the U. S. Office of Education issued a grant to Professors Max Beberman (UICSM) and Lawrence Stolurow (Dept. of Psychology) of the University of Illinois in support of a study to compare principles of programming mathematics. The content used for this purpose is that contained in the existing experimental textbooks produced by UICSM. Since September 1961 over 200 students have studied from the programmed texts prepared by the Programed Instruction Project (PIP). The information received from this trial of the materials will be helpful in revising them for use next year. In fact, some improvements have already been implemented in later preliminary programs as a result of experience gained from the use of the earlier ones.

In the course of our work during 1961–62, we have developed certain points of view and guidelines for programming mathematics. One of these is concerned with the selection of programming personnel. Recently, considerable discussion has been devoted to determining the relative roles of a "subject-matter specialist" and a "learning-theory specialist" in programming. These discussions have apparently overlooked a third contributor. This individual has the experience and ability to put correct mathematical ideas across to children, and this is the kind of job by which he earns his living. I am referring, of course, to the creative classroom teacher. This person has had much experience in removing students' difficulties, in preparing motivating lessons, in observing individual differences in action, in acquiring a sensitivity to what is easy and what is difficult for children, in asking those questions which enable a student to "see for himself," in pointing to the proper mark on a chalkboard in order to clear up a student's misunderstanding, and in knowing when most students have acquired the desired concept or skill. The creative classroom teacher has been working miracles for years in the face of such discouragements as crowded classrooms, teaching overloads, and a host of nonacademic duties. What could the best of these teachers do if all of their time were devoted to just the one duty of preparing an elaborate lesson [that is, a program] on a single topic?

The creative teacher who has this opportunity faces new problems. Not the least of these is the fact that all of his teaching must be done on paper. In trying to write a program, he soon discovers how much he had relied upon inflections of voice and body movements in "live" teaching. In the UICSM programming work, we try to translate some of these paralinguistic and kinesic components of communication through the medium of type-face changes and layout of material on the printed page. The judicious use of arrows may do something of what finger-pointing does in the live classroom. It may very well turn out to be the case that the creative programmer will need to consult with the commercial artist as well as with the subject-matter and learning-theory experts. A second example of a new teaching-on-paper problem for programmers is that of making sure that students learn correct pronunciations of new terms.

The creative classroom teacher knows how important it is to awaken the student's interest and maintain it. So, the creative programmer will devote considerable effort toward making his presentation interesting and exciting. There is no doubt that immediate knowledge of results lends interest and stimulation, but it is also the case that students need variety (even though pigeons may not). Consequently, instead of searching for a single best mode of presentation (scrambled book, linear, branched, etc.), we deliberately vary our presentations. A given programmed text may incorporate several formats. We are encouraged to use that format which appears to be most suited to the particular topic; if no standard format exists, we invent one. A cooperating teacher in one of our experimental centers reported, "The experimental class clapped when they saw the new books. I must have become very boring." Our mode of operation was to have live teaching for a few days alternate with programmed teaching. [It was

a practical expedient since we did not have the time to convert our entire course into programmed materials.] This procedure had interesting results. It happened that by the time the students had finished working in a programmed text, they were eager to get back to live teaching, and after a period of live teaching, they started asking for more programmed texts. So, instead of programmed texts replacing teachers, it turns out that they make students more appreciative of their teachers! This experience suggests that programmed texts may be used to give teachers adequate time for preparing their own live presentations.

There are, of course, many other uses of programmed materials in addition to alternating live and programmed instruction. These include the use of programmed texts for homework, thereby reducing paper-grading; make-up work for students who have missed several days of classwork; instruction for homebound students; and remedial (or, perhaps, enrichment) work to help take care of individual differences. At the present time, we do not advocate using programmed materials, exclusively, for an entire course. Programmed materials should be regarded as a teaching aid similar to those now available in a well-equipped classroom or audio-visual aids department.

Another goal for our programmers is to encourage students to discover the desired concepts by providing them with sequences of developmental exercises. In this respect we capitalize on the vast experience gained by UICSM writers and teachers during the decade of UICSM's existence. Although many programmers subscribe verbally to the "Socratic" method of teaching, one would think from looking at many of the available programs that their authors believed that Socrates taught by telling his students small bits of information and that his famous questioning technique consisted merely of asking, "Now what did I just tell you?" Our programming technique is to start with performance-type questions which gradually reveal a pattern. Our test for discovery is to give more performance-type questions but on a much higher difficulty level. Verbalization of the discovery is withheld at least until students give evidence that the discovered principle is an old friend. We feel that a student who can *do* a problem doesn't need a rule whereas merely having a rule does not guarantee successful application of it.

Our study should lead to the development of several principles of programming. These should be useful to teachers who wish to create programs of their own or who wish just to improve their own lesson-planning.

THE MAINE MATHEMATICS FAIR

R. L. PAGE, Nasson College, Springvale

The Association of Teachers of Mathematics in Maine (ATOMIM) sponsored the first Annual Mathematics Fair at the University of Maine on May 12, 1962. Sixty-four secondary students from public and parochial schools competed in two classifications (Freshman-Sophomore and Junior-Senior) with two types of projects, visual displays or exhibits and oral presentations.

The Fair served not only as a demonstration of how similar fairs could be undertaken by individual schools, but also as a stimulus to mathematical exploration and achievement by the participants.

Seven teams of judges, each composed of one representative of high schools, colleges and industry, rated the students on such qualities as creative ability, scientific thought, clarity and dramatic value. The highest 20% determined by each judging team received paperbound sets of *The World of Mathematics*. The next 30% received copies of mathematical tables. All entrants received ribbons as mementos of their efforts.

Financial aid for administrative costs came from Title III funds under NDEA while money for prizes was furnished by four member companies of Associated Industries of Maine.

Surprisingly, about one-third of the students gave oral presentations, many of which were illustrated. Typical subjects of the talks and displays were "Binary Computer," "Associated Magic Squares," "Vector Products" and "Rubber-sheet Geometry."

Comments from some of the hundreds of spectators who viewed the exhibits and heard the talks ranged from surprise at the degree of accomplishment displayed by the students to complaints of "Why couldn't mathematics have been this much fun when I was in school?" The students themselves were in unanimous agreement concerning the value of the Fair in arousing their mathematical curiosity and in providing them with a feeling of satisfaction for their hard work.

As usual, many people were responsible for the smooth-running and well-planned affair. At its business meeting, ATOMIM voted to extend next year's fair to include elementary students.

NEWS OF PUBLICATIONS

A second printing of *Undergraduate Research in Mathematics* (report of a conference held in June 1961) is now available for free distribution to libraries and interested individuals. *Delta-Epsilon*, a journal published for undergraduate work by Carleton students, is being printed in sufficient quantity for free distribution to individuals on other campuses. For either of these publications write Prof. K. May, Carleton College, Northfield, Minnesota.

Readers of this department will be interested in the article entitled "The Math Wars" by Benjamin DeMott, Professor of English at Amherst College, who also studied the history of science at Harvard University. The article appears in the Spring 1962 issue, page 296, of *The American Scholar*, official publication of Phi Beta Kappa.

In the March 11, 1962 issue of *The National Observer*, Mark Kac, in his article "Can High Schools Handle 'Abstract' Math?," discusses the changing mathematics curriculum.

Lauren G. Woodby, specialist in mathematics, U. S. Office of Education; Lawrence P. Bartnick, vice principal, Natick High School, Natick, Massachu-

setts; and Aubrey W. Calvert, supervising field representative, Bureau of School Planning, State Department of Education, Los Angeles, California, wrote a special report, "Planning the Mathematics Classroom," for the May, 1962 issue of *The American School Board Journal*. The article points out increasing enrollments in mathematics, and increasing needs for specialization in mathematical education. Five design factors of great importance are indicated as space itself, thermal control, visual efficiency, sonic adequacy, and suitable aesthetics.

The *Journal of Engineering Education* for December, 1961 has a special section of four articles on "Mathematics and Engineering," as well as four articles constituting a *Mechanics Division Bulletin*, considering vector analysis and tensor calculus in particular.

Audio-visual Instruction for March, 1962 has about a dozen articles in an issue devoted to "The New Mathematics" of both elementary and secondary schools, with special references to programmed learning materials for mathematics. The March, 1962 issue of *Educational Leadership* is also devoted to "Mathematics in the School." *The Grade Teacher* for April, 1962 has a section devoted to the elementary program, called "A Mathematics Roundup."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1533. *Proposed by D. J. Newman and W. E. Weissblum, Yeshiva University*

Six circular areas are given in the plane with the property that none contains the center of any other. Prove that they have no point in common.

E 1534. *Proposed by J. H. Edmonston, Federal Power Commission, Washington, D. C.*

A circular park is planned, with center O and radius r. Construction of radial and concentric paths is planned. Heavy pedestrian traffic is anticipated joining two points A and B on the circumference. It is desired to construct radial paths AO, BO, and a concentric path with radius r', meeting AO and BO at A' and B', respectively, such that the composite path

$$p = \overline{AA'} + \overset{\frown}{A'B'} + \overline{B'B}$$

is minimized (so as partly to eliminate motivation for walking on the grass). Determine r'.

E 1535. *Proposed by S. W. Golomb and A. W. Hales, Jet Propulsion Laboratory, California Institute of Technology*

An $n \times n$ array (matrix) of nonnegative integers has the property that for any zero entry, the sum of the row plus the sum of the column containing that entry is at least n. Show that the sum of all elements of the array is at least $n^2/2$.

E 1536. *Proposed by R. L. Syverson, University of Minnesota*

Find functions $y=f(x) \in C'$ such that x, $y>0$, $y' \neq 0$, and $dY/dX = dy/dx$, where X and Y exist at each point satisfying $y=f(x)$ and are respectively the x- and y-intercepts of the tangent to the function.

E 1537. *Proposed by Roy Dowling, University of Manitoba*

If the square root of an integer is not an integer, then it is not a rational number. This suggests a conjecture concerning an element a of an integral domain D: If $\sqrt{a}$ is not in D, then it is not in the field of quotients of D. Is this a valid conjecture?

E 1538. *Proposed by M. S. Klamkin and Jerry Yos, Avco Corporation*

A simple closed curve has the property that there exist inscribed squares of the same dimension in every direction. Must the curve be a circle?

E 1539. *Proposed by Leo Moser, University of Alberta*

Let $a_1 < a_2 < \cdots < a_n$ and $\epsilon_i = \pm 1$. Prove that $\sum_{i=1}^{n} \epsilon_i a_i$ assumes at least $\binom{n+1}{2}$ distinct values as the ϵ_i range over the 2^n possible combinations of signs.

E 1540. *Proposed by Azriel Rosenfeld, Yeshiva University*

Prove that every polynomial has a nonzero polynomial multiple whose exponents are all divisible by 1,000,000.

SOLUTIONS

A Triangle of Maximum Area and Perimeter

E 1498 [1962, 57]. *Proposed by Joseph Hammer, University of Sydney, Australia*

The six rectangular coordinates of two points P and Q in space are the integers 1 to 6 taken in some order. If O is the origin, which ordering will maximize (1) the area, (2) the perimeter, of triangle OPQ?

Solution by C. F. Pinzka, University of Cincinnati. (1) Denoting the coordinates of P and Q by (a, b, c) and (d, e, f) and the area by A, we have

$$4A^2 = (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2.$$

It is easily seen that $ad+be+cf$ has a minimum value of $(6)(1)+(5)(2)+(4)(3)$

$=28$. Assigning values so as to make the factors of the first term as nearly alike as possible, we get $a=6$, $b=2$, $c=3$, $d=1$, $e=5$, $f=4$. This gives $4A^2=(49)(42)-28^2$, which we assert is a maximum. For the first term cannot exceed $(45)(46)=(49)(42)+12$; but any change in the second term would increase the squared expression by at least $29^2-28^2=57>12$. Thus the maximum area is $7\sqrt{26}/2$.

(2) We wish to maximize

$$p=(a^2+b^2+c^2)^{1/2}+(d^2+e^2+f^2)^{1/2}+[(a-d)^2+(b-e)^2+(c-f)^2]^{1/2}$$

and note that the sum of the first two terms is a maximum when the radicands are 46 and 45. Assigning values so as to maximize the third radical, but keeping the first two radicals as nearly alike as possible, we get $a=6$, $b=2$, $c=3$, $d=1$, $e=5$, $f=4$ as before. These values give $p=\sqrt{49}+\sqrt{42}+\sqrt{35}$. Since $\sqrt{49}+\sqrt{42}+\sqrt{35}>\sqrt{46}+\sqrt{45}+\sqrt{34}$, this is the maximum perimeter.

The minimum area is $3/2$, realized with $P(6, 2, 5)$ and $Q(4, 1, 3)$. The minimum perimeter is $\sqrt{35}+\sqrt{56}+\sqrt{3}$, realized with $P(6, 2, 4)$ and $Q(5, 1, 3)$.

Also solved by E. G. Andrews, Kandiah Balachandran, Merrill Barnebey, Walter Bluger, Brother U. Alfred, D. I. A. Cohen, D. M. Danvers, Jane Evans, Michael Goldberg, Ralph Greenberg, D. L. Linfield, D. C. B. Marsh, Walter Penney, L. A. Ringenberg, and D. V. Somaroo.

Editorial Note. Of course the six permutations of the coordinates of one of the points with corresponding changes in the coordinates of the other point yield triangles of the same area and the same perimeter.

Limit of a Sequence of Points

E 1499 [1962, 57]. *Proposed by José Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela*

Denote the vertices of a triangle by A_1, B_1, C_1, and the midpoints of the respective opposite sides by A_2, B_2, C_2. Continue the process and let A_n, B_n, C_n be the vertices of the nth triangle. Let I_n be the incenter of triangle $A_nB_nC_n$. Prove that the sequence $\{I_n\}$ has a limit point and find its position.

Solution by Michael Goldberg, Washington, D. C. The sequence of similar triangles converges to the common centroid of all the triangles. Hence, *any* point homothetically associated with these triangles converges to this same point.

Also solved by E. G. Andrews, Kandiah Balachandran, Leon Bankoff, Merrill Barnebey, Walter Bluger, Robert Bowen, Brother U. Alfred, Stephen Coffin, D. I. A. Cohen, Romae J. Cormier, K. W. Crain, L. J. Dixon, Serge Dubuc, Jane Evans, N. V. Glick, L. D. Goldstone, Ralph Greenberg, A. R. Hyde, Erwin Just and Norman Schaumberger (jointly), Steven Keményffy, Jiang Luh, D. C. B. Marsh, G. J. Michaelides, R. J. Oberg, Walter Penney, Paul Pennock, C. F. Pinzka, L. A. Ringenberg, Allen Rubenstein, R. Sibson, Jr., H. P. Smith, D. V. Somaroo, John Stout, Julius Vogel, Sherwood Washburn, W. C. Waterhouse, Dale Woods, J. E. Yeager, and the proposer.

Editorial Note. The problem has nice pedagogical value in illustrating the Cantor nested set theorem, the Bolzano-Weierstrass theorem, and Cauchy sequences.

Related Tetrahedra

E 1500 [1962, 57]. *Proposed by N. A. Court, University of Oklahoma*

Show that a given point M and its three harmonic conjugates with respect to the three pairs of opposite edges of a given tetrahedron (T) are the vertices of a tetrahedron harmonic with respect to the cevian tetrahedron of M for (T).

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* For convenience we use the notation of Altshiller-Court's *Modern Pure Solid Geometry*, 1935, Figures 67 and 68, pp. 232 and 234. The line of the harmonic set P, S, U, U'', cuts MM' in U'' and $M''M'''$ in U, whence S is the harmonic conjugate of P with respect to lines MM' and $M''M'''$; similarly, Q and R are harmonic conjugates of P with respect to the other two pairs of opposite edges of tetrahedron $MM'M''M'''$, making QRS the polar plane of P relative to this tetrahedron. Again, by symmetry, each face of tetrahedron $PQRS$ may be seen to be the polar plane of the opposite vertex with respect to tetrahedron $MM'M''M'''$, whence these tetrahedra are harmonic with respect to each other.

II. *Solution by Sahib Ram Mandan, University of Sydney, Australia.* It is known that the cevian tetrahedron (T') of M for (T) is the transform of (T) in the homology $(M, m, -3)$, m being the plane of perspectivity of (T) and (T') (Court, "On the cevian tetrahedron," this MONTHLY, 43 (1936), 89–91; Mandan, "Cevian simplexes," *Proc. Amer. Math. Soc.*, 11 (1960), 837–845). M and its three harmonic conjugates, lying in m, with respect to the three pairs of opposite edges of (T) form a tetrahedron (T'') harmonic with respect to (T) (Court, *Modern Pure Solid Geometry*; Mandan, "Properties of mutually selfpolar tetrahedra," *Bul. Cal. Math. Soc.*, 33 (1941), 147–155). Cross-ratios, and harmonic relations in particular, are unaltered by homological transformations. (T'') transforms into itself in the homology. Hence follows the present proposition.

Also solved by the proposer, who called attention to Court, "Sur la géométrie du tetraèdre," *Mathesis*, 51 (1937), 307–313.

A VESGUEARY

A topology shaped like an esgue
Which is pseudo-compact and quite vesgue
Is treated with scorn
By adherents of Zorn
If its measures are a.e. Lebesgue

ANON.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5044. *Proposed by P. M. Cohn, Manchester, England*

Show that an algebraic integer whose conjugates are all of absolute value one is necessarily a root of unity. Does the conclusion hold for any algebraic integer of absolute value one?

5045. *Proposed by J. V. Whittaker, University of British Columbia*

Show that no group G of homeomorphisms of the circle C has the property that, for any $u, v, x, y \in C$ with $u \neq v$ and $x \neq y$, there is exactly one $g \in G$ such that $g(u) = x$ and $g(v) = y$.

5046. *Proposed by Ronald C. Read, University College of the West Indies*

Prove the formal identity

$$\prod_{m=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[H_n\left(\frac{i}{2\sqrt{m}}\right) \right]^2 (-\tfrac{1}{2}t^m)^n \right\} = (1-t)^{-1/2}(1-t^2)^{-1}(1-t^3)^{-1/2}(1-t^4)^{-1} \cdots$$

where $H_n(x)$ is the Hermite polynomial of degree n.

5047. *Proposed by M. S. Knebelman, Washington State University*

Given a plane curve C_0 and a fixed point P. C_n is the pedal of P with respect to C_{n-1}, $n = 1, 2, \cdots$. Prove that as $n \to \infty$, $\lim C_n$ consists of all circles with center at P, each tangent to C_0. (Circles of zero or infinite radius not excluded.)

5048. *Proposed by Hans Schwerdtfeger, McGill University*

Let $m \geqq n$ and let A be a complex $m \times n$ matrix of rank n. Let A^* denote the conjugate transpose of A. Then the Gram matrix $G = A^*A$ is a positive definite hermitian $n \times n$ matrix. Show that the hermitian matrix $B = I_m - AG^{-1}A^*$ is nonnegative if I_m denotes the $m \times m$ unit matrix.

5049. *Proposed by G. D. Chakerian, California Institute of Technology*

Let K be a compact, convex subset of R^3, and let F be the area of the surface of K. Then K contains a triangle with area greater than or equal to $(3\sqrt{3}/16\pi)F$.

5050. *Proposed by G. D. Chakerian, California Institute of Technology*

Let K be a convex subset of R^3 with constant width. Let F be the area of the surface of K and let M be the total mean curvature. Then

$$1 \leqq \frac{M^2}{4\pi F} \leqq \frac{\pi}{2(\pi - \sqrt{3})} \approx 1.114.$$

SOLUTIONS

Subsets of the Primes

4970 [1961, 511]. *Proposed by S. W. Golomb, Jet Propulsion Laboratory, California Institute of Technology*

Let $Q = \{q_i\}$ be an infinite subset of the primes, and let $g(x)$ denote the number of members of Q which do not exceed x. Call Q rare if $\sum 1/q_i$ converges, and call Q sparse if $g(x) = o(x/\log x)$ as $x \to \infty$. What causal relation, if any, exists between rarity and sparsity?

Solution by P. T. Bateman, University of Illinois. Neither of the two properties, rarity and sparsity, implies the other. In discussing the following examples we need Tchebychef's theorem:

$$\frac{3x}{4 \log x} < \pi(x) < \frac{4x}{3 \log x}$$

for all sufficiently large positive x, where $\pi(x)$ denotes the number of primes not exceeding x. A corollary is that, for a sufficiently large positive integer n, we have

$$\pi(e^{n+1}) - \pi(e^n) > \frac{3e^{n+1}}{4(n+1)} - \frac{4e^n}{3n} > \frac{e^n}{2n}. \tag{1}$$

Let Q consist of those primes q such that $e^{n^2} < q < e^{n^2+1}$ for some positive integer n. Then for positive integral N we have by (1)

$$g(e^{N^2+1}) > \pi(e^{N^2+1}) - \pi(e^{N^2}) > \frac{e^{N^2}}{2N^2} > \frac{e^{N^2+1}}{6(N^2+1)}.$$

Thus $g(x)$ is not $o(x/\log x)$ and Q is not sparse. But Q is rare because

$$\sum_{i=1}^{\infty} q_i^{-1} < \sum_{n=1}^{\infty} e^{-n^2}\{\pi(e^{n^2+1}) - \pi(e^{n^2})\} < \sum_{n=1}^{\infty} e^{-n^2}\pi(e^{n^2+1}),$$

and the latter series converges, since

$$e^{-n^2}\pi(e^{n^2+1}) < e^{-n^2}\frac{4e^{n^2+1}}{3(n^2+1)} < \frac{4}{n^2+1}$$

for large n. Thus rarity does not imply sparsity.

On the other hand choose Q as follows. In each interval of the form (e^n, e^{n+1}), where n is a positive integer, choose every $[1+\log n]$-th prime beginning with the first prime in the interval, $[t]$ denoting the greatest integer not exceeding t. (By Bertrand's postulate there are primes in every such interval.) Now the number of primes in (e^n, e^{n+1}) which are assigned to Q, namely $g(e^{n+1})-g(e^n)$, satisfies the relation

$$\frac{\pi(e^{n+1}) - \pi(e^n)}{[1 + \log n]} \leqq g(e^{n+1}) - g(e^n) < \frac{\pi(e^{n+1}) - \pi(e^n)}{[1 + \log n]} + 1.$$

Hence, if $e^{2N} \leqq x < e^{2N+2}$, where N is a sufficiently large positive integer, we have

$$\begin{aligned} g(x) &\leqq g(e^{2N+2}) = g(e^N) + \sum_{m=N}^{2N+1} \{g(e^{n+1}) - g(e^n)\} \\ &\leqq g(e^N) + \sum_{m=N}^{2N+1} \left\{\frac{\pi(e^{n+1}) - \pi(e^n)}{[1 + \log N]} + 1\right\} \\ &\leqq e^N + \frac{\pi(e^{2N+2})}{\log N} + N + 2 \leqq e^N + \frac{4e^{2N+2}}{3(2N + 2) \log N} + N + 2 \\ &\leqq x^{1/2} + \frac{4e^2 x}{3 \log x \log (\frac{1}{2} \log x - 1)} + \frac{1}{2} \log x + 2. \end{aligned}$$

Thus $g(x)$ is $o(x/\log x)$ and Q is sparse. But Q is not rare because for all sufficiently large positive integers n we have by (1)

$$\sum_{e^n < q_i < e^{n+1}} \frac{1}{q_i} > \frac{g(e^{n+1}) - g(e^n)}{e^{n+1}} > \frac{\pi(e^{n+1}) - \pi(e^n)}{e^{n+1}(1 + \log n)} > \frac{1}{2en(1 + \log n)}.$$

Since $\sum n^{-1}(1+\log n)^{-1}$ diverges (by the integral test) we infer that $\sum q_i^{-1}$ diverges. Thus sparsity does not imply rarity.

The set of all primes is of course neither rare nor sparse. It is easy to give examples of sets of primes which are both rare and sparse.

Also solved by Robert Breusch and by the proposer, both of whom cited examples different from the above.

An Unbounded Operator

4971 [1961, 576]. *Proposed by D. J. Newman, Yeshiva University*

Does the following infinite matrix take l^2 vectors into l^2 vectors?

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & \cdots \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Solution by N. G. de Bruijn, Technological University, Eindhoven, Netherlands. The answer is in the negative. We shall construct a real vector $(x_1, x_2, \cdots)$ with $\sum_1^\infty x_i^2 < \infty$, whose image $(y_1, y_2, \cdots)$ under the given transformation has infinite norm, *i.e.*, $\sum_1^\infty y_i^2 = \infty$.

Let $p_1, p_2, \cdots$ be the consecutive primes. We put $P_t = p_1 p_2 \cdots p_t$; let S_t be the set of all integers of the form

$$p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_t^{\lambda_t}, \quad \text{with} \quad t^2 < \lambda_\nu \leqq (t+1)^2, \qquad (\nu = 1, \cdots, t)$$

and let W_t be the subset of all $i \in S_t$ for which iP_t also belongs to S_t. The sets $S_1, S_2, \cdots$ are obviously disjoint. S_t has $(2t+1)^t$ elements, and W_t has $(2t)^t$ elements.

We now take $x_i = 0$ if i does not belong to any of the S_t, and for all i in S_t we give x_i one and the same value, *viz.*, $\{(2t+1)^t t \log^2 (t+1)\}^{-1/2}$. So $\sum_1^\infty x_i^2 = \sum_{t=1}^\infty t^{-1} \log^{-2} (t+1) < \infty$.

We next show that $\sum y_i^2 = \infty$. We have

$$\sum_{i=1}^\infty y_i^2 = \sum_{k=1}^\infty \sum_{m=1}^\infty \sum_{n=1}^\infty m^{-1} n^{-1} x_{km} x_{kn}.$$

For each t $(t = 1, 2, \cdots)$ we consider the set of terms which satisfy $m \mid P_t$, $n \mid P_t$, $k \in W_t$. For these terms we have $km \in S_t$, $kn \in S_t$. Therefore, the contribution of this set of terms is

$$\sum_m \sum_n (mn)^{-1} \{2t/(2t+1)\}^{-t} \{t \log^2 (t+1)\}^{-1},$$

where both m and n run through the divisors of P_t. We have $\sum_m m^{-1} = \prod_{\nu=1}^t (1 + p_\nu^{-1})$, and there is a constant C such that this is at least $C \log (t+1)$ for all t. Since $\{2t/(2t+1)\}^{-t} \to e^{-1/2}$, it follows that

$$\sum_{i=1}^\infty y_i^2 \geqq \sum_{t=1}^\infty C^2 \{2t/(2t+1)\}^{-t} t^{-1} = \infty$$

as desired.

Also solved by the proposer.

Series with All Coefficients Positive

4972 [1961, 576]. *Proposed by P. T. Bateman, University of Illinois*

If $P(x)$ is a polynomial with real coefficients but no zeros in (0, 1), it is known that the power series for $(1-x)^{-r-1} P(x)$ has positive coefficients for all sufficiently large positive integers r. How large must r be taken if $P(x) = 1 - 20x + 101x^2$?

Solution by F. W. Owens, University of Notre Dame. We have

$$(1 - x)^{-r-1}(1 - 20x + 101x^2) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_0 = 1$, $c_1 = r - 19$, and for $n \geqq 2$,

$$\frac{r!c_n}{(n+1)\cdots(n+r-2)} = 82n^2 - (82 + 18r)n + r^2 - r \equiv f(n, r).$$

Since $f(x, y) = 0$ is an ellipse and for any point inside the ellipse $f(x, y) < 0$, we have to find the smallest integer $r \geqq 19$ such that for no positive integer n the point (n, r) lies inside the ellipse. Now, if x_1 and x_2 are the solutions of $f(x, y) = 0$ for x in terms of y, we find

$$x_1 - x_2 = \frac{1}{41}\sqrt{\{41^2 + y(820 - y)\}} > 1$$

for $19 \leqq y \leqq 820$, which implies that there are positive integers n such that (n, r) is inside the ellipse for $19 \leqq r \leqq 819$. For $r = 820$ we have $x_1 = 91$, $x_2 = 90$, and $x_1 - x_2 = 1$, so that the points (91, 820), (90, 820) are on the ellipse. Thus 821 is the required smallest value of r.

Also solved by Robert Breusch, W. G. Brown, L. Carlitz, Ragnar Dybvik, N. J. Fine, S. H. Greene, V. Linis, D. C. B. Marsh, R. W. Means, R. C. Mullin, R. G. Nath, Walter Penney, W. C. Waterhouse, David Zeitlin, and the proposer.

Topologically Free Sequence

4973 [1961, 577]. *Proposed by Albert Wilansky, Lehigh University*

The sequence $\{\cos nx\}$ is topologically free on $[0, \pi]$, *i.e.*, no one of its members is uniformly approximable by a linear combination of the others. Is the same true on $[0, a]$ if $0 < a < \pi$?

Solution by N. J. Fine, University of Pennsylvania. It is false for every such a. Choose h so that $a < h < \pi$, and define

$$f(x) = 1, \quad (|x| \leqq h),$$
$$= -\frac{h}{\pi - h}, \qquad (h < |x| \leqq \pi),$$
$$f(x + 2\pi) = f(x).$$

Then f has the Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx.$$

In particular, $a_0 = 0$. The series converges uniformly to $f(x)$ in $[0, a]$, so

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n \cos nx = 1 \quad \text{uniformly in} \quad [0, a].$$

Also solved by K. A. Post and W. C. Waterhouse.

Field in which Multiplicative Group is a Free Group?

4974 [1961, 577]. *Proposed by P. M. Cohn, The University, Manchester, England*

Does there exist a field (commutative or not) whose multiplicative group is a free group?

Solution by W. S. Martindale, 3rd., Smith College. We prove that the multiplicative group D^* of a field D cannot be free. For suppose D^* is free. Since no element of D^* except 1 can have finite order, we have $-1=1$, *i.e.*, the prime field P of D is the field of two elements. As $P \neq D$, we may choose a commutative subfield M of D, $M \neq P$. Since every subgroup of a free group is free, the commutative multiplicative group M^* of M is the infinite cyclic group with generator t. It follows that M is the subfield generated by t. Since $t+1 \in M^*$, we must have $t+1=t^k$ for some integer k. Hence t is algebraic over P, and therefore M and M^* are finite, a contradiction.

We remark that in the commutative case, if we interpret free group to mean free abelian group, there does exist a field whose multiplicative group is free, viz., the field $P(x)$ of rational functions over the field P of two elements. The set of all irreducible polynomials serves as a free basis.

Also solved by P. R. Chernoff and W. C. Waterhouse, K. A. Post, John Wilkinson, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Mathematics; A Cultural Approach. By Morris Kline. Addison Wesley, Reading, Mass., 1961. 679 pp. $7.75.

According to the author, this book is intended as a textbook for a one year terminal course in mathematics for liberal arts students. Mathematics is approached historically through descriptions of scientific, artistic, and philosophi-

cal milieux from which mathematical concepts and procedures have arisen. Important questions of sciences such as earth measure, cosmology, gravitation, and electromagnetism, and of arts, such as perspective drawing and musical composition, are described in detail sufficient to motivate discussions of mathematical notions they generate: geometry, algebra, trigonometry, calculus, and so on. Throughout these discussions what the author calls "the ways of mathematics" recur as a theme. These have to do with precision, notation, number, notions of order, and other attributes of mathematical predisposition. Non-euclidean geometries are introduced and serve as a barrier that separates from the classical notions the more modern, such as abstract algebra, statistics and probability, and mathematics directed toward the social sciences.

A reading of Prof. Kline's text leaves, at least to this reader, the impression that all significant mathematical advances had their origin in questions raised by scientists outside mathematics, and that they developed only after the intellectual climate was appropriately conditioned by philosophers. The author's presentation of this material appears somewhat like a sketch for a novel. The important events are listed and many important relations drawn. What is missing is the excitement that comes with a full delineation of character, as well as a tantalizing quality some authors convey by leaving to their readers a number of open questions and unresolved conflicts. Much of this could have been provided by investigating the influence of mathematics on itself and on intellectual developments in other scientific fields. The reader is barred from a view of much of abstract mathematics and from some of the open questions of mathematics by the limited extent to which the notions introduced in this book are developed. He could be carried further into a view of mathematical influences on other sciences by a fuller description of contemporary mathematical endeavors.

One is tempted to raise the question as to whether the limitations of the mathematical material presented here represent a judgment about the mathematical acumen of prospective students in such a course. Yet, just as the excellent historical material here helps to motivate the student to accept the mathematical notions introduced, these notions in turn could move the student to consider and enjoy the more abstract mathematical ideas without which a true picture of the role of mathematics in our culture is all but impossible to attain.

EDWARD J. COGAN
Sarah Lawrence College

Numerical Methods in Engineering, Second Edition. By M. G. Salvadori and M. L. Baron. Prentice-Hall, New Jersey, 1961. 302 pp. $8.00.

Lacking in this new edition is any practical recognition of the existence of digital computers. However, for those whose interests and/or convictions gravitate toward desk calculators, this text is excellent.

In Chapter I on solution of equations, derivations are generally omitted, enabling the authors to include a large variety of methods which are carefully illustrated by calculation schemes. Serious omissions are brief statements de-

scribing what the Gauss Reduction and the Simplex Method are, basically, and how to deal with zero diagonal coefficients in the solution of linear systems.

Differencing operators are developed in Chapter II. Although short and neat, the treatment is amazingly complete. The notation y_r, y_l, y_a, y_b for y right, left, above and below when writing difference operators at the point y_i is a convenient and mnemonically useful choice.

In the application of these operators to problems in O.D.E. and P.D.E. (all of which are interesting engineering problems) the authors force the reader to deduce in each case the difference equation from the differential equation, thus insuring the acquisition of good manipulative skills. Also reduction of all problems to dimensionless form is logically proposed and always done.

Excellent figures for difference operator molecules are included along with derivations and applications of operators in nonrectangular systems.

Unsurpassed problem sections with answers are included in all chapters. The authors' explanations are always clear and their style is attractive.

Minor complaints are the tardy and cursory treatment with poor notation of the nth order O.D.E. as a system of n first order O.D.E.'s and no mention of "over-relaxation" in the heavy use made here of relaxation techniques.

The only errors noted are on page 113 (should read $y_0^{(n-1)}$) and on page 114 (where the example is linear, not nonlinear).

ROBERT H. OWENS
University of New Hampshire

A History of Mathematics. By H. A. Freebury. Macmillan, New York, 1961. 198 pp. $3.95.

It should be noted at once that this is a history of elementary mathematics, in fact, very elementary mathematics. It is presumably intended for students on a high school level, and even some of these may find it a little too unsophisticated. To illustrate the style let us quote: "But knowledge was surrounded by all kinds of fantastic beliefs and superstitions, and further hampered by the scarcity of writing materials and of books, the latter, especially, being so highly expensive that if you had three or four you were considered a very wealthy man. It is not surprising, then, that so little progress was made in those times."

The material is that which is found in the general histories of mathematics, with brief references to the works of various mathematicians, but space limitations precluding any real discussion of their ideas. The presentation goes as far as trigonometry and logarithms; the last mathematician to be discussed is Leibniz. The final part includes articles on the history of algebra, trigonometry, of π (the latest computations not included), calculating devices, the calendars, time reckoning.

There is a brief discussion of the history of English weights and measures and an introduction to the metric system. This may be in anticipation of the British entry into the European Common Market, which, as it is widely predicted, may result in a change to decimal coinage and to the metric system in England. It

may give some food for thought that this would leave the United States as the only prominent nation struggling under the handicap of a medieval system of weights and measures.

But back to our book. Its emphasis lies on number systems, algebraic notations and their developments, all good topics for high school students. One may not be satisfied with the author's presentation, but nevertheless, it should be a useful book for the school library.

OYSTEIN ORE
Yale University

Dynamic Optimization and Control. By W. Kipiniak. M.I.T. Press, Cambridge, and Wiley, New York, 1961. x+233 pp. $4.95.

In the first of the three main chapters of this book (Chapter 2) the author derives necessary conditions and some sufficient conditions associated with optimal control in various types of control problems. Some related topics such as stability of the Euler equations are also discussed. The treatment is, to a large extent, heuristic and formal. Many of the results are restatements of known theorems concerning the Bolza problem in the calculus of variations. The reason (such as possible computational convenience) for treating problems with constraints by means of penalty functions when exact methods are available is not specified.

In Chapter 3, several computational procedures are discussed. This contribution to the pool of experience should interest those engaged in developing effective computational procedures for control problems. Chapter 4 treats stochastic control problems; the treatment, again, is largely heuristic and formal. Nevertheless, some interesting ideas are advanced.

This book, which is an outgrowth of the author's Ph.D. thesis, is not recommended as an introduction to control problems or as a text. There is no clear delineation among statements of theorems, proofs (rigorous or formal), conjectures, discussions, and passing remarks. The notation is complicated and leads to an unsightly format.

LEONARD D. BERKOVITZ
The RAND Corporation

Characteristic Functions. By Eugene Lukacs. Griffin's Statistical Monographs and Courses, Number 5. Hafner Publ. Co., New York, 1960. 216 pp. $6.50.

In this monograph the author gives a comprehensive account of the theory of characteristic functions with emphasis on their intrinsic mathematical interest rather than on their possible applications. The first four chapters deal with the fundamental properties of characteristic functions, uniqueness theorems, inversion theorems, convolution and continuity theorems, and necessary and sufficient conditions that a given function shall be a characteristic function. The theory of infinitely divisible distributions and stable distributions is discussed in Chapter 5. Analytic characteristic functions are considered in Chapter 7. The

theory of decomposition of probability distributions is discussed in Chapters 6 and 8. In Chapter 9, mixtures of distribution functions and some transformations of characteristic functions are discussed. For the first time many important results which heretofore have been available only in Russian and French journals are presented in English within the pages of this excellent volume. For example, theorems of Bochner and Khintchine on the necessary and sufficient conditions for a function to be positive-definite are given in Chapter 4. Results of Bergström, Linnik, Skorohod, and Zolotarev on the analytical properties of the stable distributions are given in Chapter 5. The fundamental general theorems of Khintchine on the decomposition of distribution functions [Bjull. Moskov. Univ. Sect. A 1 (1937), no. 1, 6–17] are included in Chapter 6. Results of Lévy, Raikov, and Dugué on analytic characteristic functions are given in Chapters 7 and 8. More recent theorems of Linnik on the decomposition of infinitely divisible distributions are stated without proof.

This monograph should serve admirably as a text book for a course on characteristic functions. In addition, it will certainly constitute a valuable source of information on this subject for the research worker who is interested in analytical probability. The author is to be particularly commended for the invaluable service which he has rendered to the English speaking statistical community in compiling this work.

CLIFFORD COHEN
University of Georgia

BRIEF MENTION

Human Values and Science, Art, and Mathematics. By L. R. Lieber and H. G. Lieber. Norton, New York, 1961. 149 pp. $3.95.

"This little book is intended to show that 'Science' and 'The Humanities' are NOT two separate domains at war with each other, but, on the contrary, are one and the same!"

Industrial Control Electronics. By Matthew Mandl. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961. 344 pp. $8.00.

A qualitative, nonmathematical description of basic electronic circuits and their uses, ranging as far as servomechanisms, counters, and digital computers.

Curricula in Solid Mechanics. By H. Liebowitz and J. M. Allen, Editors. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961. 147 pp. $1.50.

Consists of the transactions of the "First Study Session in Theoretical and Applied Mechanics Curricula" held at Boulder, Colorado, June 19–24, 1961.

Modern Physics (2nd ed.). By F. W. Van Name, Jr. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962. 319 pp. $8.25.

The second edition of this well-known junior level survey text has been augmented by descriptions of some of the recent conspicuous discoveries in physics, such as the maser and the Van Allen radiation belts, and by a number of new problems. As before, this edition assumes little background other than elementary physics and calculus.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

North Dakota State University: Associate Professor F. J. Arena has been promoted to Professor; Assistant Professor D. E. Eustice has been promoted to Associate Professor.

University of California, Riverside: Professor F. B. Jones, University of North Carolina, has been appointed Professor; Dr. W. A. Kirk, University of Missouri, has been appointed Assistant Professor; Dr. L. M. Young, Harvard University, has been appointed Instructor; Assistant Professor H. G. Tucker has been promoted to Associate Professor; Assistant Professor R. C. Gilbert is on sabbatical leave at the Mathematical Research Center, U. S. Army, University of Wisconsin.

University of Illinois, Chicago Undergraduate Division: Associate Professor M. C. Hartley has been promoted to Professor; Mr. N. C. Scholomiti has been promoted to Assistant Professor.

Wesleyan University: Dr. E. K. Blum, Space Technology Laboratories, Los Angeles, California, has been appointed Professor and Director of the Computer Laboratory; Dr. E. F. Assmus, Jr., Columbia University, has been appointed Lecturer.

Mr. E. B. Abbott, Texas Agricultural and Mechanical College, has accepted a position as Engineer in the Computing Department of North American Aviation, Los Angeles, California.

Mr. G. D. Adams, Cornell University, has accepted a position as an Officer in the Public Health Service of the National Institutes of Health, Bethesda, Maryland.

Dr. G. H. Andrews, University of Michigan, has been appointed Assistant Professor at Oberlin College.

Mr. W. G. Andrews, University of Delaware, has accepted a position as Mathematician in the Computer Services Section of the Thiokol Chemical Corporation, Elkton, Maryland.

Assistant Professor J. D. Arrison, Iowa Wesleyan College, has been appointed Assistant Professor at Monmouth College.

Mr. G. C. Arrowsmith, Stanford University, has accepted a position as Research Engineer with the Boeing Airplane Company, Seattle, Washington.

Dean W. L. Ayres, Purdue University, has been appointed Vice-President and Provost at Southern Methodist University.

Associate Professor A. G. Azpeitia, University of Massachusetts, has been promoted to Professor.

Miss Ann H. Beckman, St. John's University, has been promoted to Instructor.

Dr. M. P. Berri, University of California, Los Angeles, has been appointed Visiting Assistant Professor at Tulane University.

Mr. J. H. Boots, Jr., Dakota Wesleyan University, has accepted a position as Senior Analyst with the South Dakota Department of Highways, Pierre, South Dakota.

Professor R. D. Boswell, Jr., Mississippi State University, has been appointed Professor and Head of the Department of Mathematics at Monmouth College.

Mr. J. W. Brown, Stanford University, has been appointed Instructor at the University of Michigan.

Mr. R. L. Brueck, University of Colorado, has accepted a position on the technical staff of Texas Instruments, Dallas, Texas.

Mr. C. K. Burge, University of Kansas, has been appointed Associate Professor and Registrar at Greenville College.

Mr. H. T. Chase, Jr., San Jose State College, has accepted a position as Claims Representative with the Social Security Administration, Fresno, California.

Mr. D. H. Clanton, Oak Ridge Institute of Nuclear Studies, has been appointed Assistant Professor at Furman University.

Dr. S. D. Conte, Aerospace, Los Angeles, California, has been appointed Director of the Computation Center at Purdue University.

Mr. J. W. Counts, University of Missouri, has accepted a position as Programming Analyst with Pratt and Whitney Aircraft, Palm Beach, Florida.

Mr. G. T. Crocker, Auburn University, has been appointed Associate Professor at the University of Southern Mississippi.

Mr. C. G. Cullen, Case Institute of Technology, has been appointed Assistant Professor at the University of Pittsburgh.

Mr. L. E. De Noya, Oklahoma State University, has been appointed Instructor at Ohio Wesleyan University.

Mr. Paul Donis, University of Notre Dame, has accepted a position as Mathematician with Naval Avionics Facility, Indianapolis, Indiana.

Mr. K. W. Dundas, Fort Hays Kansas State College, has been appointed Instructor at Hutchinson Junior College.

Mr. L. W. Ehrlich, University of Texas, has accepted a position as Mathematician in the Applied Physics Laboratory of Johns Hopkins University.

Professor H. H. Ferns, University of Saskatchewan, has been appointed Visiting Professor at Victoria University.

Mr. P. C. Fischer, Massachusetts Institute of Technology, has been appointed Assistant Professor at Harvard University.

Mr. C. H. Gardner, White Plains High School, White Plains, New York, has accepted a position as Associate Editor with the Silver Burdett Publishing Company, Morristown, New Jersey.

Mr. W. R. Garner, Jr., St. Bonaventure University, has accepted a position as Engineer with Martin Marietta, Middle River, Maryland.

Assistant Professor A. M. Garsia, University of Minnesota, has been appointed Associate Professor at the California Institute of Technology.

Mr. D. M. Girard, University of Detroit, has accepted a position as Mathematician with the NASA Lewis Research Center, Cleveland, Ohio.

Mr. Anton Glaser, Abington High School, Abington, Pennsylvania, has been appointed Instructor at Pennsylvania State University, Ogontz Campus.

Mr. E. P. Graney, Westinghouse, Kansas City, Missouri, has accepted a position as Mathematician in the Research Analysis Department of National Cash Register, Dayton, Ohio.

Mr. H. E. Grenard, Dade County Junior College, has accepted a position as Research Meteorologist with Allied Research Associates, Boston, Massachusetts.

Assistant Professor J. S. Growdon, Texas College of Arts and Industries, has been appointed Assistant Professor at Western Carolina College.

Mr. L. S. Henderson, University of Toronto, has accepted a position as Programmer with Manufacturers Life Insurance, Toronto, Canada.

Mr. B. D. Hendrix, California Institute of Technology, has accepted a position as Computing Analyst with Aerojet-General, Sacramento, California.

Mr. T. L. Herring, East Carolina College, has accepted a position as Mathematician with the Naval Weapons Laboratory, Dahlgren, Virginia.

Dr. Joseph Hershenov, Massachusetts Institute of Technology, has been appointed Assistant Professor at Queens College.

Mr. H. W. Hickey, Convair Astronautics, San Diego, California, has accepted a position as Computer Programmer with United Electrodynamics, Alexandria, Virginia.

Associate Professor J. E. Houle, Georgetown University, has been appointed Associate Professor at Seton Hall University.

Dr. H. F. Hunter, Knolls Atomic Power Laboratory of General Electric, Schenectady, New York, has been appointed Assistant Professor at Renssalaer Polytechnic Institute.

Mr. H. G. Junghans, Kent State University, has accepted a position as Associate Mathematician with the Applied Physics Laboratory of the Johns Hopkins University.

Mr. J. H. Kelly, Johns Hopkins University, has accepted a position as Programmer Trainee with the Federal Systems Division of International Business Machines, Bethesda, Maryland.

Professor L. D. Kovach, Acting Head of the Department of Mathematics and Physics at Pepperdine College, has been appointed Head.

Mr. J. M. Laible, University of Minnesota, has been appointed Assistant Professor at Western Illinois University.

Mr. H. A. Larson, University of North Dakota, has accepted a position as Mathematician-Master Scientist with the Phillips Petroleum Company, Idaho Falls, Idaho.

Mr. Robert Leaf, Bendix Computer Division, Oakland, California, has accepted a position as Application Analyst with the Control Data Corporation, Palo Alto, California.

Mr. C. P. Lecht, Industrial Data Agency, Snowville, Virginia, has accepted a position as Executive Vice-President and Director of the Advanced Computer Techniques Corporation, New York, New York.

Mr. D. A. Leiti, Union Carbide Nuclear Company, Oak Ridge, Tennessee, has accepted a position as Mathematician with C. E. I. R., Arlington, Virginia.

Mr. Bruce Lercher, University of Rochester, has been appointed Acting Assistant Professor at Harpur College.

Dr. P. E. Long, Southern Illinois University, has been appointed Assistant Professor at Oklahoma State University.

Mr. T. R. McCarthy, Hercules Powder Company, Magna, Utah, has accepted a position as Computer Programmer with the RCA Service Company, Cherry Hill, New Jersey.

Mr. K. A. MacDonald, San Diego State College, has been appointed Assistant Professor at California State Polytechnic College.

Mr. P. N. MacDonald, University of Maine, has accepted a position with the Peerless Insurance Company, Keene, New Hampshire.

Professor Nathaniel Macon, Auburn University, has accepted a position as Chief of the Computing Section of the SHAPE Air Defense Technical Center, The Hague, Netherlands.

Dr. J. C. Mathews, University of Oklahoma, has been appointed Assistant Professor at Iowa State University.

Mr. J. P. Menard, Syracuse University, has been appointed Assistant Director of the Computer Science Center of the University of Maryland.

Dr. R. E. Messick, California Institute of Technology, has been awarded a Fulbright Grant and will be at the University of Sydney, Sydney, Australia for the academic year 1962–63.

Assistant Professor R. C. Mjolsness, Reed College, has accepted a position as Theoretical Physicist with the Space Sciences Laboratory of General Electric, Philadelphia, Pennsylvania.

Mr. M. L. Moeschberger, Taylor University, has been appointed Instructor.

Mr. K. E. Morris, Long Beach State College, has accepted a position on the technical staff of Aerospace, El Segundo, California.

Professor P. B. Patterson, University of Florida, is on leave at Appalachian State Teachers College as Professor.

Mr. Presley Peek, Bemidji State College, has been appointed Instructor at Fort Kent State Teachers College.

Mr. W. R. Quigley, Pepperdine College, has accepted a position as Systems Engineer with International Business Machines, Los Angeles, California.

Mrs. Mary L. Ramsey, University of Detroit, has been promoted to Instructor.

Associate Professor Philburn Ratoosh, University of California, Berkeley, has been appointed Professor at San Francisco State College.

Professor C. B. Read, University of Wichita, has been appointed Professor at Central Michigan University.

Mr. Thomas Robertson, Occidental College, has been promoted to Assistant Professor.

Mr. J. H. Rolf, University of Kentucky, has accepted a position as Systems Engineer Trainee with International Business Machines, Cincinnati, Ohio.

Mr. W. B. Rundberg, Bowdoin College, has been appointed Instructor at Arizona State University.

Dr. J. V. Ryff, Stanford University, has been appointed Pierce Research Instructor at Harvard University.

Professor H. J. Ryser, Ohio State University, has been appointed Professor at Syracuse University.

Mr. J. K. Seymour, Lebanon Valley College, has accepted a position as Actuarial Student Trainee with the Security Mutual Life Insurance Company, Binghamton, New York.

Dr. Seymour Sherman, General Precision, Pleasantville, New York, has accepted a position with the Paul Moore Research and Development Center of Republic Aviation, Farmingdale, New York.

Mr. R. D. Simpson, University of Idaho, has accepted a position as Computer Analyst with Aerojet-General, Azusa, California.

Associate Professor W. A. Small, Tennessee Polytechnic Institute, has been appointed Professor and Coordinator of the Mathematics Department of the State University of New York, College of Education at Geneseo.

Dr. J. R. Smart, Michigan State University, has been appointed Assistant Professor at the University of Wisconsin.

Professor N. E. Steenrod, Princeton University, is on leave for the academic year 1962–63 as Visiting Professor at the University of Chicago.

Mr. C. J. Stuth, University of Kansas, has been appointed Assistant Professor at East Texas State College.

Mr. S. C. Suda, North Dakota State University, has accepted a position as Research Engineer with Autonetics, Anaheim, California.

Mr. C. M. Walker, University of Kentucky, has been appointed Assistant Professor at Charlotte College.

Mr. R. P. Winter, College of St. Thomas, has accepted a position as Senior Development Engineer with Minneapolis-Honeywell, Hopkins, Minnesota.

Mr. Richard Wiseman, Idaho State College, has accepted a position with Aerojet-General, Sacramento, California.

Associate Professor W. S. Wunch, Arizona State University, has accepted a position as Technical Assistant to the President with the Talley Industries, Mesa, Arizona.

Dr. P. W. Zehna, C. E. I. R., Palo Alto, California, has been appointed Associate Professor at the U. S. Naval Postgraduate School.

Professor Charles Hatfield, Sr., Georgetown College, died on June 5, 1962. He was a member of the Association for 35 years.

Professor R. L. Swain, Rutgers, The State University, died on February 4, 1962. He was a member of the Association for 13 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT ON NONTEACHING MATHEMATICAL EMPLOYMENT

The MAA Advisory Committee for a Survey of Nonteaching Mathematical Employment, in conjunction with the U. S. Bureau of Labor Statistics, has conducted a study of employment in professional mathematical work in industry and government.

About 10,000 persons have provided data on their personal characteristics and the nature of their positions. Information was obtained about their education, salary, type of employer, and mathematical education required for their current position.

The results of this study are contained in an 82-page booklet "Employment in Professional Mathematical Work in Industry and Government," NSF-62-12, which may be obtained from the Superintendent of Documents, Government Printing Office, Washington 25, D.C., for 55 cents.

THE APRIL MEETING OF THE KANSAS SECTION

The forty-seventh annual meeting of the Kansas Section of the Mathematical Association of America was held at Bethel College, North Newton, Kansas, on April 28, 1962, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Professor L. E. Fuller, Chairman of the Section, presided at the joint meeting in the morning and at the Section meeting in the afternoon. There were 275 persons registered, including 95 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor A. M. Wedel, Bethel College; Vice-Chairman, Professor R. H. Thompson, Sterling College; Secretary-Treasurer, Professor Helen F. Kriegsman, Kansas State College of Pittsburg.

At the joint session, Professor L. M. Blumenthal, University of Missouri, delivered a paper entitled *The golden age of mathematics—today.*

The following short papers were presented at the afternoon session:

1. *Ratings of college mathematics courses by applied mathematicians*, by Professor R. W. Hart, Kansas State College of Pittsburg.

This paper gave the findings of two investigations that were made to determine what courses in undergraduate college mathematics are considered to be most valuable to applied mathematicians, and also what topics in advanced calculus are considered most important. The head mathematicians of more than twenty-five large companies engaged in research filled out the questionnaire that was sent to them, so the results would appear to be reliable. These findings should be important to advisors of college students who intend to go into industry, and they should also be a guide to those college teachers of advanced calculus.

2. *Some algebraic structures in analysis*, by Professor M. F. Ruchte, University of Wichita.

3. *The structure of the orbits and their limit sets in continuous flows*, by Professor N. E. Foland, Kansas State University.

Consider a continuous flow $\mathfrak{F}$ on a 2-cell X with set of rest points A. For each $x \in X$, let $\omega(x)(\alpha(x))$ denote the set of all ω-limit (α-limit) points of the orbit $0(x)$ of x under $\mathfrak{F}$. The following results are obtained: (1) If x is a nonperiodic point of X under $\mathfrak{F}$, then $\omega(x) \cap \alpha(x) \subset A$. (2) If X is a closed 2-cell, A a totally disconnected set and z a nonperiodic point in $\overline{0(x)} - 0(x)$, then $0(z)$ is an open arc whose closure, $\overline{0(z)}$, is either a closed arc or a simple closed curve.

4. *A glimpse of Turing machines through the Busy Beaver game*, by Mr. L. D. Nelson, Kansas State University.

The principle of an n-state binary Turing machine is exhibited with the aid of the Busy Beaver game invented by Professor Tibor Rado and played with a set of n cards on a both ways infinite tape.

HELEN F. KRIEGSMAN, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 36th annual meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Chatham College, Pittsburgh, Pennsylvania on May 5, 1962. Professor Evan Johnson, Jr., Chairman of the section, presided at the morning session and Professor W. A. Beck, Secretary-Treasurer of the section, presided at the afternoon session. Registered were 101 persons, including 78 members of the Association.

During the business meeting reports of the High School Contest were presented by the representatives: Professor I. D. Peters for West Virginia and Professor M. J. Mansfield for Western Pennsylvania. The change in section boundaries for the Allegheny Mountain Section was ratified and the by-laws amended accordingly.

Following remarks of welcome by David Henderson, Executive Dean of Chatham College, the film "What is an Integral?" with Professor Edwin Hewitt was shown. The following papers were presented:

1. *Properties of real numbers*, by Friedrich Kasch, Distinguished Visiting Professor, Pennsylvania State University. (By invitation).

2. *A representation theorem for linear lattices*, by Professor Fernando Bertolini, University of Pittsburgh.

The following representation theorem for linear lattices is proved. Let: (i) A be any boolean σ-lattice, (ii) M be the linear lattice of all the extended real valued, almost everywhere locally finite σ-measures on A, (iii) B be the boolean lattice of all the bands of M, (iv) L any linear lattice, (v) C be the boolean lattice of all the bands of L. (I) For every A, M is a conditionally complete, "fat" linear lattice, valued, almost everywhere locally finite σ-measure; conversely, (II) if L is a conditionally complete, "fat" linear lattice, and C is a complete boolean lattice carrying a strictly positive, extended real valued, almost everywhere locally finite σ-measure, then there exists a boolean σ-lattice A such that the linear lattice M is an isomorphic image of the linear lattice L.

3. *Sierpinski's criterion for rational and irrational numbers*, by Professor H. W. Gould, West Virginia University.

In 1909 Sierpinski made a study of the limit $L=\lim_{n\to\infty}(1/n)\sum_{k=1}^{n}(kx-[kx])$ where $[x]$ means the integral part of x. He proved that a real number x is rational or irrational according as $L<\frac{1}{2}$ or $L=\frac{1}{2}$. In fact, when $x=b/a$ is a positive rational, it is easily shown that $L=(a-d)/2a$, where d is the greatest common divisor of a and b. In the present paper summation formulas are developed from which the theorem may be proved, and certain upper and lower bounds are obtained.

4. *Matrix representations of n-person games*, by Mrs. Katherine K. McKelvey, Chatham College.

The representation of n-person games by n-way matrices provides a new approach to the theory of games presented by John von Neumann and Oskar Morgenstern. Through the use of partitions and compositions of n-way matrices the methods for solving two-person games can be extended to n-person games.

5. *Extension of a theorem of Levine on closure and interior operators*, by Mr. T. A. Chapman, West Virginia University.

Following Norman Levine's result that $A^{0-}=A^{-0}$ if and only if $A=(M-P)\cup(P-M)$, for M both open and closed and P nowhere dense, similar characterizations are given of all possible equalities that may occur between any two of the 14 Kuratowski sets formed by successive application of the closure and complement operators.

W. A. BECK, *Secretary*

THE MAY MEETING OF THE METROPOLITAN NEW YORK SECTION

The twenty-first annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the University Heights Center of New York University on May 12, 1962. Dr. John W. Knedler, Jr., Dean, University College of Arts and Sciences of New York University, welcomed the gathering. Professor Abraham Schwartz, Collegiate Vice-Chairman of the Section, presided at the morning session and Professor J. P. Russell, Chairman of the Section, presided at the business meeting and afternoon session. One hundred and twenty-five people, including eighty-five members of the Association, were registered at the meeting.

Professor C. T. Salkind presented the awards to local winners in the mathematics contest sponsored by the Mathematical Association of America and the Society of Actuaries. He then gave reports as Chairman of the Contest Committee and as Governor of the Section. Reports were also presented by the Treasurer, Mr. Aaron Shapiro, and by Professor J. N. Eastham, Chairman of the Speakers' Bureau.

The following papers were presented at the meeting:

1. *Power of a real number with integral difference*, by Professor H. J. Cohen, College of the City of New York.

2. *Anomalies of satellite orbits as viewed by a mathematician*, by Dr. Paul Savet, Staff Scientist, ARMA Division, American Bosch ARMA Corporation.

In classical astronomy one usually considers point masses at great distances. In satellite dynamics, short distances and finite mass distributions are considered; the particular shape of the earth contributes additional complexities. Finite expansions of two prime integrals are derived, and results extended in adapting this technique to various satellite trajectory problems. An up-to-date summary of the author's recent investigations in this area was presented. A preliminary paper on this subject has been published by the Pergamon Press, *Ballistic Missiles and Space Technology*, Vol. III, 1961.

3. The afternoon session consisted of a panel discussion entitled *Preparation for the professional mathematician*. The moderator was Dean Mina S. Rees, City University of New York. Included in the panel were: Dr. H. O. Pollak, Bell Telephone Laboratories; Dr. Leonard Procita, Downstate Medical Center; Mr. Harrison Givens, Jr., Equitable Life Assurance Society; Dr. L. W. Cohen, University of Maryland.

MARY P. DOLCIANI, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the Mathematical Association of America was held on May 12, 1962, at the College of Saint Teresa, Winona, Minnesota. Sister Thomas à Kempis, O.S.F., College of Saint Teresa, presided at the morning session, and the Section Governor, Professor Warren Loud, University of Minnesota, presided at the afternoon session. There were 99 persons registered for the meeting, of whom 66 were members of the Association.

At the business meeting Sister Seraphim, C.S.J., College of St. Catherine, briefly described the 1962 Minnesota High School Mathematics Contest, for which the Minnesota Section was a sponsoring agency. The following officers were elected to serve during the school year of 1962–1963: Chairman, Professor W. J. Thomsen, Mankato State College; Secretary-Treasurer, Professor Murray Braden, Macalester College; Members of the Executive Committee, Professor Charles Hatfield, University of North Dakota, Sister Thomas à Kempis, College of Saint Teresa, and Professor Lawrence Markus, University of Minnesota.

By invitation of the Section, Professor Lawrence Markus gave the main address of the morning which was entitled: *The influence of the differential equations of mechanics on modern mathematics.*

One feature of the afternoon program was a panel discussion dealing with the training of high school mathematics teachers, both while they are in college, and later, when they are engaged in teaching. Professor Paul Rosenbloom, University of Minnesota, and Sister Seraphim, College of Saint Catherine, discussed in-service institutes for high school teachers, and Professor David Storvick, University of Minnesota, and Professor Frank Wolf, Carleton College, discussed the present undergraduate training of high school mathematics teachers in the Minnesota colleges, measuring it against the C.U.P.M. recommendations. A general discussion followed.

The following papers were presented:

1. *A changing view of mathematics, and some of its effects*, by Dr. R. E. Smith, Control Data Corporation, Minneapolis, Minnesota.

Recent technological procedures portend a philosophy of *nonabsolutism* as well as *absolutism* in mathematics. Increasing use of the digital computer in two environments: those demanding more accuracy, those accepting less accuracy, have influenced this trend. In the former, theoretical accuracies often deteriorate at digital output unless algorithms are redesigned after analysis of experimental digital results. In the latter, uncertainties of *principle* require system structures, built upon initial approximations, and redesigned after analysis of experimental results. Both require a philosophy which recognizes the interdependence between traditional logic and experimental result. The college curriculum in particular, should examine this transitional concept.

2. *Minimal vector fields and minimal flows*, by Professor Alfred Aeppli, University of Minnesota.

A flow on a compact differentiable manifold M is called minimal if every streamline is everywhere dense. A continuous vector field (without zeros) on M is called minimal if through every point there is at least one streamline which is everywhere dense. *Theorem: There is a minimal vector field on* $S^2 \times S^1$, *everywhere differentiable except along a single fibre* S^1. The same is true for $P^2 \times S^1$ and for a manifold M^3 fibered by S^1 over S^2. *Conjecture:* There is no minimal flow on $S^2 \times S^1$ (and more generally on $M \times S^1$ for any compact manifold M with $\chi(M) \neq 0$).

3. *The inter-departmental honors program at the College of Saint Teresa*, by Professor A. C. Nydegger, College of Saint Teresa.

The Teresan Honors Program is a sequence of courses (one per year) which the honor student takes along with her regular college courses. It aims to provide her with an integrated background in the social sciences, the humanities, science and mathematics, and philosophy. Participation is by invitation, based upon scholastic record, and is purely voluntary. Each participating faculty member endeavors to present a general picture of his own field and to relate it to the other fields in the sequence.

4. *On the calculation of interest rates*, by Dr. P. M. Treuenfels, Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota.

In order to introduce students to the basic ideas underlying the numerical treatment of scientific and engineering problems by means of computing machines it is suggested that, in lieu of formal courses, informal extracurricular discussion be conducted within, say, the framework of presently existing mathematics clubs. A sample problem is presented. The basic mathematical equation is derived from the statement of the problem, a numerical iterative technique is discussed and illustrated by graphical sketches, the analysis of the method of solution into its logical components is carried out, and a flow chart is constructed. It is felt that this sample problem may serve to illuminate the analysis of scientific computing problems which must be carried out prior to the actual programming.

5. *Modern contributions to classical problems*, by Professor Seymour Schuster, Carleton College.

A description was given of a two-term course offered at Carleton College for students who wish to take advanced mathematics but for whom highly specialized advanced courses do not form a completely satisfactory program. This course is advocated for prospective teachers who finish the standard four-term calculus sequence. Focusing on certain problems that were implicit in Greek mathematics, many 19th and 20th century developments are presented. Topics include: completion of the real numbers, cardinal number theory, field extensions, constructibility, non-Euclidean geometry, coordinatization, and the Fundamental Theorem of Algebra.

6. *The influence of the differential equations of mechanics on modern mathematics*, by Professor Lawrence Markus, University of Minnesota. (By invitation)

The Newtonian differential equations of motion define a vector field and a flow in the phase space of Hamiltonian mechanics. The attempt to locate and describe equilibrium states and periodic oscillations of dynamical systems leads to the study of vector fields on manifolds. Through this study one introduces the concepts of the degree of a map of spheres, cross-sections in fiber bundles, and characteristic classes in cohomology groups with homotopy group coefficients.

7. *Unification of the two types of solutions to the general cubic, by use of complex angles*, by Brother L. W. Ringwelski, De La Salle High School, Minneapolis, Minnesota.

If the cubic equation $Z^3+aZ+b=0$ has real roots, we set $\phi = \cos^{-1}[-\frac{1}{2}b/(-\frac{1}{3}a)^{3/2}]$ and $f=2(-\frac{1}{3}a)^{1/2}$ which are necessary for the answers $Z=f\cos\frac{1}{3}(\phi+2k\pi)$. This solution is completely general if we allow complex angles, for which $\cos\phi = \cos u\cosh v - i\sin u\sinh v$, where $\phi=u+iv$. If a is positive, we set $u=\frac{1}{2}(1+2k\pi)$ and proceed to a solution through use of a table of hyperbolic sines. If the absolute value of $\cos\phi$ is greater than 1 and a is negative, we set $u=0$ and solve for v by use of a hyperbolic cosine table.

8. *Numerical evaluation of the Bode integral*, by Professor C. B. Germain, College of Saint Thomas.

In electrical networks in which the voltage curve is composed of straight line segments rather than a sine wave the current lag can be expressed in terms of the Bode integral.

$$\beta = \frac{1}{\pi}\int_0^x \log\frac{(1+x)}{(1-x)}\,\frac{dx}{x} = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2} \qquad 0 \leqq x \leqq 1. \tag{1}$$

It is shown that

$$\beta = \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1-(2k+1)\log\dfrac{(1-x)}{(1+x)}}{(2k+1)^2}\left(\frac{1-x}{1+x}\right)^{2k+1}. \tag{2}$$

For $x>.44$, (2) converges faster than (1). For $x=.9$, (1) requires 55 terms for 9 figure accuracy, (2) requires only 4 terms. For $x=.1$, (1) requires 5 terms, (2) requires 38 terms.

MURRAY BRADEN, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-sixth annual meeting of the Ohio Section of the Mathematical Association of America was held at Muskingum College, New Concord, Ohio, on Saturday, May 5, 1962. Professor W. E. Restemeyer, Chairman of the Program Committee, presided at the morning session and Professor R. L. Wilson, Chairman of the Section, presided at the afternoon session. Sixty-two persons registered in attendance, including 54 members of the Association.

Officers elected for the coming year are: Chairman, Professor W. G. Johnson, Hiram College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor F. C. Ogg, Bowling Green State University. The Program Committee is: Professor Clarence Heinke, Capital University, Chair-

man; Mr. J. W. Warner, College of Wooster; and Professor R. A. Roberts, Denison University.

The following program was presented:

1. *The exponential representation of unitary matrices,* by Professor R. F. Rinehart, Case Institute of Technology.

The classical result that a matrix is unitary if and only if it is of the form $\exp S$, where S is skew hermitian, is shown to be essentially obvious from the standpoint of the general properties of matrix functions and particular properties of the logarithmic and exponential functions. Knowledge of special properties of the eigenvalues or of the diagonability of unitary matrices is not required.

2. *The remainder terms in numerical integration formulas,* by Professor L. R. Bragg, Case Institute of Technology.

Let $F(x)\in C^{n+2}$ on an interval of real numbers containing the sequence $a, x_0, \cdots, x_n, b$. This paper is concerned with the remainder terms $R(F)$ associated with a numerical integration formula for estimating $\int_a^b F(t)dt$ in terms of the values of F at $x_0, x_1, \cdots, x_n$. Let $m=n$ or $n-1$, whichever is even. Then the results of Steffensen and Walter relating to the Newton-Cotes formulas can be extended as follows: $R(F)=AF^{(m+2)}(\xi)$ where A is a constant independent of F and $a<\xi<b$ if (a) $x_0\leqq x_1\leqq\cdots\leqq x_m$, (b) the points x_1 in (a) are symmetrically distributed about $x_{m/2}$, and (c) the lengths $x_{i+1}-x_i$ of the intervals are a nonincreasing function of i for $i\leqq m/2-1$.

3. *Relations between the Lototsky and certain other methods of summability,* by Professor B. J. Yozwiak, Youngstown University.

A discussion of the Lototsky method of summability and its relation with certain other methods, in particular, the Z_k method of Silverman and Szasz. It was pointed out that the Lototsky method was stronger than the Z_k-method, that the Z_k method did not include the Lototsky method, and that the Lototsky and Z_k are consistent methods. An iteration of the Lototsky and Z_k methods led to a method LZ_k which is stronger than either the Lototsky or the Z_k method. Certain identities needed in working with the Lototsky method were also presented.

4. *On logic and algebra* by Professor Wade Ellis, Oberlin College.

All binary connectives on logical propositions can be represented by sixteen distinct ordered quadruples $S=\{F_i(p, q)\}=\{\langle a_{1,i}, a_{2,i}, a_{3,i}, a_{4,i}\rangle\}$, $i=0, \cdots, 15$, where $a_{j,i}=1$ or 0 and 1, 0 are symbols for truth values truth and falsity, respectively. Hence any logical proposition built up out of two (only) propositions p, q, and binary connectives must be reducible to one of these. $\langle 1, 1, 0, 0\rangle$ and $\langle 1, 0, 1, 0\rangle$, for example, may be taken to represent p and q respectively. If other quadruples are substituted for these, permutations on the other F_i's are induced. The Symmetric and Alternating Groups on four letters arise, together with their subgroups.

A $1-1$ correspondence is set up between S and $GF(2^4)$ in such fashion that addition in $GF(2^4)$ can be interpreted as $(F_i\Leftrightarrow F_j)$. The multiplication in $GF(2^4)$ seems to have no coherent interpretation in S.

5. The afternoon session was devoted to a discussion of the recommendations of the Committee on the Undergraduate Program in Mathematics (CUPM). The discussion was led by a panel consisting of Professors R. L. Wilson, I. A. Barnett, Clarence Heinke, and the Secretary.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, May 11–12, 1962. Professor I. I. Kolodner, Chairman of the Section presided. Dr. Jim Douglas, Jr., Rice Institute, gave the invited address, "Mathematical Programming and Functional Equations." There were 82 persons in attendance including 62 members of the Association.

The following officers were elected: Chairman, Professor Deonisie Trifan, University of Arizona; Vice-Chairman, Professor Elbert Walker, New Mexico State University; Secretary-Treasurer, Professor George Baldwin, New Mexico State University.

The following papers were presented:

1. *Construction of the dominant eigenfunction for a sub-linear Sturm-Liouville problem*, by Professor I. I. Kolodner, University of New Mexico.

2. *Repeated independent trials and a class of dice problems*, by Professor E. O. Thorp, New Mexico State University.

3. *Some remarks on commutators in finite symmetric groups*, by Professor A. B. Gray, Jr., Arizona State College.

Since every element of the commutator subgroup of a symmetric group is a commutator it is reasonable to ask if there is an element a of the symmetric group such that every element of the commutator subgroup is a commutator of a or conjugate to a commutator of a. Since these questions have already been answered in the infinite cases only the finite cases will be considered here. Every element of the alternating group on three objects is a commutator of any two-cycle, there is no such element, however, in the larger finite symmetric groups. Every element in a finite alternating group is conjugate to a commutator of an n-cycle or $(n-1)$-cycle.

4. *An isomorphism theorem*, by Professor Oswald Wyler, University of New Mexico.

A generalization of the First Isomorphism Theorem is proved for general algebraic structures. For groups, the theorem reads as follows. Let G be a group, H a subgroup of G, N a normal subgroup of G. Then $H\cap N$ is a normal subgroup of H, and HN is a subgroup of G. If M is a normal subgroup of H containing $H\cap N$, then MN is a normal subgroup of HN, and the quotient groups H/M and HN/MN are isomorphic. Putting $M=H\cap N$, the First Isomorphism Theorem is obtained. The Lemma of Zassenhaus follows immediately from the general theorem.

5. *The mathematical theory of a family of combination locks*, by Mr. G. J. Simmons, Sandia Corporation.

The locks which were the origin of this paper are made up of n switches, which may be activated individually or in any combination whatsoever; however, each one can be closed, at most, only once. The sequencing of the discrete closures constitutes the "combination" of the lock. It was shown that the number of combinations available to an n switch lock is given by $p_n=\sum_{j=1}^{n}G(n, j)$, where $G(n, j)$ is defined by the partial difference equation $G(n, j)=jG(n-1, j-1)+(j+1)G(n-1, j)$. The solution to the above system is $p_n=\sum_{j=1}^{n}j!\mathcal{S}_{n+1}^{j+1}$, where $\mathcal{S}$ is a Stirling number of the second kind.

6. *On methods of defining topologies from convergence functions*, by Mr. D. C. Kent, University of New Mexico (introduced by Professor Kolodner).

A function which maps the set of all filters on a set S into the set of all subsets of S and satisfies certain other conditions is called a *convergence function*. Any topology on S uniquely defines a convergence function; a convergence function definable in this way is called *topological*. A *linkage function* is a mapping of the set of all convergence functions on S into the set of all topological convergence functions on S, where members of the latter set remain fixed points under the mapping. This paper defines and compares several such linkage functions.

7. *On the (C, 1) transform of a moment sequence*, by Mr. P. G. Carr, University of New Mexico.

8. *On certain commutative rings*, by Mr. E. Poluianov, New Mexico State University.

9. *Right-subinverses*, by Professor Berthold Schweizer, University of Arizona.

10. *Quasi-inverses*, by Professor Abe Sklar, University of Arizona and Illinois Institute of Technology.

11. *Injective and projective Banach spaces*, by Mr. H. Cohen, New Mexico State University, (introduced by Professor Kolodner).

12. *Conics through notable points of a triangle*, by Professor J. H. Butchart, Arizona State College.

When directly similar isosceles triangles are drawn with their bases on the sides of a given triangle, the three lines joining the new vertices with the corresponding vertices of the given triangle are concurrent on an equilateral hyperbola through the vertices of the triangle, its centroid, orthocenter, and both Fermat points. The points of this conic are isogonal conjugates of the points of the circumdiameter passing through the symmedian point. The transform by isogonal conjugates of any line in the plane is a conic through the vertices of the given triangle.

13. *Some applications of homological algebra to Abelian groups*, by Professor E. A. Walker, New Mexico State University.

14. *On certain subgroups of extension*, by Mrs. C. Peercy, New Mexico State University.

15. *On high subgroups of Abelian groups*, by Professor J. M. Irwin, New Mexico State University.

16. *Coprime chains*, by Mr. R. C. Entringer, University of New Mexico, (introduced by Professor Kolodner).

A sequence $A = \{a_1, \cdots, a_k\}$ of distinct integers greater than 1 is called a *coprime chain* if and only if A contains exactly one multiple of each prime equal to or less than a_k. Asymptotic results concerning the sum of powers of the members of coprime chains with maximal member n are obtained.

17. *Matrix method for orthogonalization and complementation of vector spaces*, by Professor M. McConnon, New Mexico Highlands University.

18. *Discussion: Curriculum in mathematics*, by Professor J. R. Foote, University of New Mexico and Holloman Air Force Base.

GEORGE BALDWIN, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Clarkson College of Technology, Potsdam, New York on May 5, 1962. Professor D. E. Kibbey, Chairman of the section, presided at the morning session and Dr. Frank Hawthorne, Vice-Chairman of the section, presided at the afternoon session. There were 72 persons in attendance, including 59 members of the Association.

At the business meeting, the following officers were elected: Chairman, Dr. Frank Hawthorne, New York State Department of Education; Vice-Chairman, Professor R. L. Beinert, Hobart College; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. The Chairman of the Contest Committee, Professor Nura D. Turner of the State University College of Education at Albany, reported on the 1962 High School Mathematics Contest. The question of the geographical grouping of participants for award purposes was discussed, and motions to the effect that the Section request the National Committee to study the question were passed. Note was taken of the growth of Community Colleges. A committee was authorized which would seek out ways to cooperate with these colleges and which would invite staff members at these colleges to become members of the M.A.A.

The program was as follows:

1. *A note on vibrations with heavy damping*, by Professor E. E. Haskins, Clarkson College.

2. *Accessibility in pair-homogeneous cut-point graphs*, by Professor M. W. Pownall, Colgate University.

Let A be the vertex set and N the adjacency relation of a locally finite, pair-homogeneous graph in which the vertex O is a cut-point. Let a relation M be defined by the equation

$$M = (N \cup I)^k, \tag{1}$$

where k is a fixed positive integer and I is the identity relation. Two vertices P and Q are *accessible* to each other at O provided there exists an M-path from P to Q in which no two consecutive members are M-related to O. This equivalence relation can be applied to prove the uniqueness of N and k in (1), and to provide a procedure for defining N in terms of M.

3. *Analogue of Cauchy's formula in non-Archimedean fields*, by Professor Viktors Linis, University of Ottawa.

Fields with non-Archimedean valuation can be studied as topological spaces with non-Archimedean metric. When algebraically closed these fields are noncompact and totally disconnected; however, certain analogues to classical Cauchy integrals can be defined by using appropriate limits of sums over zeros of cyclotomic polynomials. Formulas corresponding to residue theorem and fundamental theorem of algebra are obtained by this method.

4. *A self-determining sequence leading to Markov chains*, by Mr. Alexander Nagel, Bronx High School of Science; introduced by Dr. Frank Hawthorne.

A game of chance is defined in which there are n possible outcomes at each play, the probability of a given outcome at any play depending on the outcome of the previous play. Using some theorems about a certain self-determining sequence, a serial ordering of all the plays of this Markov chain can be achieved, and the following question can be answered: If a play is about to be made, but neither the outcome of the previous play nor the number of previous plays is known, what is the probability that a specified outcome will turn up?

5. *An obvious theorem with some nonobvious consequences*, by Professor Norman Miller, Queen's University.

A function $f(x)$, continuous for $a \leq x \leq b$ and such that $f(a) < 0, f(b) > 0$, assumes the value 0 for some x between a and b. This theorem of Bolzano (1817) is obvious to the intuition but many of its direct consequences are not obvious. This fact is illustrated by a number of questions which arise in various physical situations.

6. *Programmed instruction in college and high school mathematics*, by Professor B. H. Gere, Hamilton College, and Mr. A. L. Buchman, N. Y. State Education Department.

7. *Studies resulting from the results of the 1961 annual high school mathematics contest in the Upstate Section of New York*, by Professor Nura D. Turner, State University College at Albany.

Four statistical studies associated with the 1961 Mathematics Contest in Upstate New York Section provide: 1) a meaningful list of career information wanted and when wanted by students; 2) facts supporting contentions that (a) the proportion of boys among top-ranking students is significantly greater than that of girls; (b) the proportion of top-ranking students from schools with graduating size 200 and over is significantly greater than that of schools of less size, with the same being true of schools with graduating size 400 and over; (c) the proportion of top-ranking students from enlarged city districts is significantly greater than proportion from all other types of schools.

8. *The mathematics enrichment program for secondary school students at Clarkson College of Technology*, by Professor R. D. Larsson, Clarkson College of Technology.

The Program involved 86 students from 9 different high schools in 1961–1962. The program consisted of ten three hour lectures and was supported in the main by the Boards of Education and in part by the National Science Foundation. The necessary theory of groups, rings, integral domains and fields was developed to study conditions for solutions of equations with one and two operations, including the Gauss-Jordan process. Properties of inequalities followed, leading to lim-

its of area under curves and then to polynomial approximations. Three sessions were devoted to the programming and operation of the IBM 1620 computer for those problems previously considered.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The thirtieth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Marquette University, Milwaukee, Wisconsin, on May 12, 1962. Professor E. W. Swokowski, Chairman of the Section, presided. This meeting was held jointly with the May meeting of the Wisconsin Mathematics Council and there were 149 persons present, including 65 members of the Association and 84 members of the Wisconsin Mathematics Council.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor G. L. Bullis, Wisconsin State College and Institute of Technology, Platteville; Vice-Chairman, Professor P. W. Gillet, Extension Division of the University of Wisconsin, Wausau; Secretary-Treasurer, Professor E. F. Wilde, Beloit College.

The following papers were presented:

1. *A current model in automobile insurance*, by D. C. Weber, Wisconsin State College and Institute of Technology, Platteville.

When the Poisson frequency function is compounded with the Pearson Type III curve, the result is the negative binomial distribution. Recent papers appearing in the Casualty Actuarial Society Proceedings make use of this fact, where the Poisson reflects the way in which the number of accidents occur and the Pearson Type III represents the distribution of accident proneness, m, in the population. The latter distribution has the form $T(m)=a^r\Gamma^{-1}(r)m^{r-1}e^{-am}$, $m\geq 0$ and $a, r>0$. By applying Bayes' Rule, it is possible to derive a function, $T(m|c, s)$, which is the distribution of inherent accident potential for those individuals who have had c accidents in the past s years. The probability of x accidents in the next t years for this subgroup follows by compounding the Poisson (parameter mt) with $T(m|c, s)$. The resulting distribution has a mean, $t(r+c)/(a+s)$. Comparing this with the mean accident frequency for all risks, rt/a, an experience modification factor is obtained which may be used for merit rating on the basis of past accident records.

2. *Measures and metric spaces*, by Dr. C. J. Vanderlin, Jr., University of Wisconsin, Extension Division.

If (X, m, μ) is a measure space, the measurable sets of finite measure form a metric space M if the distance between two sets is defined by $\rho(E, F)=\mu(E\Delta F)$, where $E\Delta F$ is the symmetric difference of the two sets, and if two sets are considered "equal" if their symmetric difference is of measure zero. It was shown that M is complete. Various connections between M and (X, M, μ) were discussed. It was shown that $\bigcup, \bigcap$, and μ were uniformly continuous on M. If ν is an absolutely continuous measure on M, then ν is continuous on M. A necessary and sufficient condition that M be convex is that M be nonatomic, modulo sets of measure zero.

3. *Theory of braids on planes and spheres*, by Professor Edward Fadell, University of Wisconsin.

Let G_n denote the geometric braid group of Artin and B_n the group given by generators $\sigma_1, \cdots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $|i-j|\geq 2$. The speaker sketched a proof of Artin's result that G_n and B_n are isomorphic. Braids in G_n are placed between parallel planes. If we consider braids between concentric spheres, we obtain a new group G_n' and if B_n' is B_n with the additional relation $\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1=1$, then a proof that G_n' and B_n' are isomorphic was also sketched. It was also indicated why G_n' had elements of finite order for each n.

4. *In-service project in modern college geometry*, by Dr. R. D. Wagner, University of Wisconsin, Extension Division.

Under a grant from the National Science Foundation the Department of Mathematics of the University of Wisconsin Extension Division designed a special correspondence course in geometry

for teachers. Wisconsin participants met regularly in study groups and an instructor visited each group about six times. The text materials used included the SMSG Geometry and Teacher's Commentary; Studies in Mathematics, Volume 2, by Curtis, Daus, and Walker; and six chapters entitled "An Alternate Approach to Modern School Geometry for Teachers" by Prof. J. Marshall Osborn. Professor Marion B. Smith, Jr. was the author of the study guide and the visiting instructor. Most of the five hundred teachers who enrolled are making satisfactory progress.

5. *An alternate approach to geometry*, by Professor J. M. Osborn, Jr., University of Wisconsin.

This paper is a discussion of the supplementary material written by the author for use in the in-service course in college geometry for teachers given by the University of Wisconsin Extension Division under a grant from the National Science Foundation during the last year. This material deals primarily with models, independence of axioms, incidence geometry, affine geometry, order geometry, non-Euclidean geometry, and the position of geometry in modern mathematics. The author's motivation and purposes in selecting these topics to supplement the School Mathematics Study Group geometry material in this course are also discussed.

E. F. WILDE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

Forty-fourth Summer Meeting, University of Colorado, Boulder, August 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State University, University Park, May 4, 1963.

ILLINOIS, Northern Illinois University, De Kalb, May 10–11, 1963.

INDIANA

IOWA, Iowa State University, Ames, April 19–20, 1963.

KANSAS, Kansas State University, Manhattan, April 20, 1963.

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1963.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 1, 1962.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University, East Lansing, March 23, 1963.

MINNESOTA, Bemidji State College, November 3, 1962.

MISSOURI

NEBRASKA, University of Nebraska, Lincoln, May 3–4, 1963.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, Connecticut General Life Insurance Co., Bloomfield, Connecticut, November 24, 1962.

NORTHERN CALIFORNIA, University of California, Berkeley, January 1963.

OHIO, Ohio State University, Columbus, May 4, 1963.

OKLAHOMA, Oklahoma City University, November 10, 1962.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.

ROCKY MOUNTAIN, Brigham Young University, Provo, Utah, Spring, 1963.

SOUTHEASTERN, University of Chattanooga, Chattanooga, Tennessee, March 29–30, 1963.

SOUTHERN CALIFORNIA, University of California, Riverside, March 9, 1963.

SOUTHWESTERN, Arizona State College, Flagstaff, April 1963.

TEXAS, North Texas State University, Denton, April 19–20, 1963.

UPPER NEW YORK STATE, University of Buffalo, April 27, 1963.

WISCONSIN, Carroll College, Waukesha, May 4, 1963.



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LINEAR OPERATIONS ON MATRICES

MARVIN MARCUS,* University of British Columbia and National Bureau of Standards

I. Introduction. In this article we survey some of the problems that arise in studying linear operations on finite matrices. As we shall see, these problems come from several interesting sources, e.g. multilinear algebra, combinatorial analysis, commutativity. As might be expected, there are a large number of unanswered questions that appear immediately and we will describe some of these in the course of this article.

The usual elementary linear operations on matrices are:

(i) multiplying a row (column) by a fixed nonzero constant;
(ii) interchanging rows (columns);
(iii) adding a multiple of one row (column) to another;
(iv) taking the transpose.

The reason that these operations are studied is that they leave fixed some important property of the matrix and at the same time bring the matrix into manageable form. For example, we know that a square matrix with entries in a field may be brought to diagonal form with only 1 and 0 appearing on the main diagonal, without altering its rank. This gives us a simple algorithm for computing the rank, $\rho(A)$, of any matrix A.

Let T denote a fixed linear operation obtained as a composite of linear operations of the kind described above. Then the important property that T has insofar as ρ is concerned is embodied in the equation: $\rho(T(A)) = \rho(A)$ for all matrices A. A composite of linear operations of types (i), (ii), (iii) can be accomplished by pre- and post-multiplication by fixed nonsingular matrices. By this is meant that there exist fixed U and V such that

$$T(A) = UAV \qquad \text{for all } A. \tag{1}$$

This statement is false for (iv). That is, there exist no fixed U and V such that $UAV = A'$ for all A. We know of course that A and A' have the same similarity invariants and hence are similar, but the transforming matrix S_A satisfying $A' = S_A A S_A^{-1}$ depends on A. To see that U and V must depend on A first set $A = I$ so that $V = U^{-1}$. Then $UA = A'U$ and by varying A over diagonal matrices we conclude that U is a diagonal matrix. By next letting $A = E_{ij}$ (E_{ij} is the matrix with 1 in position (i, j), 0 elsewhere) we conclude that $U = I$. But then $A' = UAU^{-1} = A$ would imply that every matrix is symmetric.

We next consider a somewhat more involved example of a linear operation on matrices that holds fixed an important scalar valued function of the matrix. Let R_4 be the ordinary four dimensional vector space of real 4-tuples. Let $f(x)$

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be a function on R_4 to the real numbers defined by

$$f(x) = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

If L is a linear mapping of R_4 into itself satisfying

$$f(L(x)) = f(x)$$

for all x, then L is called a *Lorentz transformation*. Let ϕ be a mapping of R_4 into the set H_2 of complex 2-square Hermitian matrices defined by

$$\phi(x) = \begin{bmatrix} x_1 + x_2 & x_3 + ix_4 \\ x_3 - ix_4 & x_1 - x_2 \end{bmatrix}.$$

It is simple to verify that ϕ is an isomorphism of R_4 onto H_2, regarded as a vector space over the real numbers with $E_{11}+E_{22}$, $E_{11}-E_{22}$, $E_{12}+E_{21}$ and $i(E_{12}-E_{21})$ as a basis. We check immediately that

$$d(\phi(x)) = f(x),$$

where d denotes the determinant. Hence, if we define $T=\phi L\phi^{-1}$, a mapping on H_2 to itself, we have

$$d(\phi L\phi^{-1}(H)) = f(L\phi^{-1}(H)) = f(\phi^{-1}(H)) = d\phi(\phi^{-1}(H)) = d(H),$$

for all $H\in H_2$. Thus if $f(Lx)=f(x)$ we have $d(T(H))=d(H)$ and conversely. Therefore, determining the structure of the mappings L is the same as determining the structure of the mappings T that send 2-square Hermitian matrices into such and preserve the determinant.

II. Invariance. The general problem can be stated as follows. Let $M_n(F)$ denote the vector space of n-square matrices over a field F. Let Ω be a subspace of $M_n(F)$ and let f be a function on Ω with values in F. Let T be a linear transformation of Ω into itself with the property

$$f(T(A)) = f(A) \qquad \text{for all } A \in \Omega. \tag{2}$$

Problem. Determine the structure of the set $\Omega(f)$ of all T satisfying (2).

Sometimes the problem is modified by assuming that Ω is just a subset of some subspace R of $M_n(F)$ and that T is defined on R. For example, we can ask for all T that map the unitary group Ω into itself [11]. Ω is not a subspace of $M_n(F)$ and we must assume that T is defined on all of $M_n(F)$.

A natural choice for $f(A)$ is some fixed coefficient in the characteristic polynomial of A. Let $E_r(A)$ denote the sum of all principal r-square subdeterminants of A; if F is the complex field then $E_r(A)$ is the rth elementary symmetric function of the characteristic roots of A; e.g. $E_1(A)=tr(A)$, $E_n(A)=d(A)$. The problem of finding all T such that $d(T(A))=d(A)$ was solved originally by Kantor [6] and Frobenius [3]. These results were extended and generalized by Schur [26], Hua [4], and Dieudonné [2], and very recent results in this direction may

be found in [16]. On the other hand not much can be said about T if we assume only that $E_1(T(A))=E_1(A)$. Next suppose $f(A)=E_2(A)$. Let T_{ij} be the linear map that interchanges the entries in positions (i, j) and (j, i) and leaves all other entries fixed. Clearly $E_2(T_{ij}(A))=E_2(A)$ for all A but T_{ij} cannot be represented (unless $n=2$) in the form

$$T(A) = UAV \quad \text{or} \quad T(A) = UA'V. \tag{3}$$

There are several other types of linear operations that hold E_2 invariant and are also not representable in the form (3):

(a) multiply the (i, j) element by λ and the (j, i) element by λ^{-1} T_{ij}^{λ};

(b) interchange the (i, j), (j, i) pair with the (k, l), (l, k) pair of entries, T_{ijkl}.

We can easily prove that $\Omega(E_r)$ is a multiplicative group for $r\geqq 2$ and it is an open question whether or not $\Omega(E_2)$ is generated by the transformations T_{ij}, T_{ij}^{λ}, T_{ijkl} together with operations of type (3) with $U=V^{-1}$ (i.e. the *similarity transformations*). Clearly $\Omega(E_2)$ is closed under multiplication so it suffices in showing that $\Omega(E_2)$ is a group to prove simply that $E_r(T(A))=E_r(A)$ implies that T is nonsingular. Suppose then that $T(A)=0$. We have

$$\begin{aligned} E_r(A+X) &= E_r(T(A+X)) = E_r(T(A)+T(X)) \\ &= E_r(T(X)) \\ &= E_r(X), \end{aligned}$$

for all X. For $r\geqq 2$ one can make judicious choices of X to show that $A=0$. For $n>r\geqq 4$, $\Omega(E_r)$ is entirely generated by the similarity transformations and the transpose operation. That is, $T\in\Omega(E_r)$ if and only if T has the form (3) with $U=V^{-1}$ [12]. For $r=n$, $\Omega(E_n)$ consists of transformations of the type (3) with $d(UV)=1$. The structure of the group $\Omega(E_3)$ is an open question.

In 1925 I. Schur proved that if $r\geq 2$ and the rth order subdeterminants of $T(A)$ are fixed nonsingular linear homogeneous functions of the rth order subdeterminants of A then T must have the form (3). We may restate this result in terms of compound matrices. Recall that if A is an n-square matrix then $C_r(A)$, the rth compound of A, is the $\binom{n}{r}$-square matrix whose entries are the rth order subdeterminants of A arranged in doubly lexicographic order. Then Schur's result can be stated as follows: If there exists a nonsingular mapping S of the space $M_{\binom{n}{r}}(F)$ into itself such that $C_r(T(A))=SC_r(A)$ for all $A\in M_n(F)$ then T has the form (3). Here F is assumed to be the field of complex numbers.

An important fact that underlies many of the above results is the following: *If T maps $M_n(F)$ into itself and satisfies*

$$\rho(T(A)) = \rho(A) \tag{4}$$

for all $A\in M_n(F)$ then T must have the form (3). Hua [4] proved this for a somewhat more general class of transformations in which, however, it is assumed

initially that T^{-1} exists and also has the property (4). Jacob [5] proved the same type of result when the conditions on F are weakened. In [9] the hypothesis is only that T preserves rank 1. That is, (4) is replaced by the assumption that *if* $\rho(A)=1$ *then* $\rho(T(A))=1$. Of course, if we know *a priori* that T is nonsingular and maps the set of rank 1 matrices *onto* itself then this constitutes no improvement. For then, if $\rho(A)=r$ then

$$A = \sum_{i=1}^{r} A_i, \qquad \rho(A_i) = 1 \quad \text{and} \quad \rho(T(A)) \leqq \sum_{i=1}^{r} \rho(T(A_i)) \leqq r.$$

The same argument can be applied to T^{-1} to conclude (4) for any A. The point of these remarks is that we can determine the structure of $\Omega(f)$ in many cases by showing that *if* $T\in\Omega(f)$ *then* T *preserves rank* 1. To illustrate this idea suppose that f is the determinant function. It is not difficult to show that *if* $A\neq 0$ *then* deg $d(xA+B)\leqq 1$ *for all* $B\in M_n$ *if and only if* $\rho(A)=1$. Thus if $T\in\Omega(d)$ and $\rho(A)=1$, it follows

$$d(xA + B) = d(xT(A) + T(B))$$

must have degree 1 for all $B\in M_n(F)$. But since $\Omega(d)$ is a multiplicative group, $T(B)$ ranges over $M_n(F)$ as B does. Hence the above remark applies to allow us to conclude that $\rho(T(A))=1$. We assume here that the underlying field has characteristic zero.

The theorem that determines the rank 1 preservers leads naturally into a structure problem for linear maps on spaces of tensors.

III. Mappings on tensor product spaces. Let U be an n-dimensional vector space over a field F. The space $U^{(2)}=U\otimes U$ is defined to be the dual space of the space of all bilinear functionals on U to the field F. We call $U^{(2)}$ the *tensor product* of U with itself. If y_1 and y_2 are in U we define $\beta=y_1\otimes y_2$ to be the element of $U^{(2)}$ satisfying $\beta(\phi)=\phi(y_1, y_2)$ for any bilinear functional ϕ. It is easy to show that if $x_1, \cdots, x_n$ is a basis of U then $x_i\otimes x_j$, $i, j=1, \cdots, n$ is a basis of $U^{(2)}$. If $w\in U^{(2)}$ we define $\omega(w)$ to be the least number of terms of the form $y_1\otimes y_2$ used in any representation of w as a sum. We call $\omega(w)$ the *length* of w. Next define an isomorphism θ between $M_n(F)$ and $U^{(2)}$ as follows:

$$\theta(E_{ij}) = x_i \otimes x_j$$

and extend linearly to all of $M_n(F)$. It is then easy to show that θ has the important property:

$$\rho(A) = r \quad \text{if and only if} \quad \omega(\theta(A)) = r.$$

If T is a linear transformation on $M_n(F)$ to itself then we may define S on $U^{(2)}$ to itself by

$$S = \theta T \theta^{-1}.$$

Then T will preserve rank 1 if and only if S preserves length 1. We can then

proceed to show that *if* S *satisfies* $\omega(S(\beta)) = \omega(\beta)$ *for all* $\beta \in U^{(2)}$ *then* S *must have the form*

$$S(x \otimes y) = Ux \otimes Vy \quad or \quad S(x \otimes y) = Uy \otimes Vx$$

for fixed matrices U *and* V *in* $M_n(F)$. It is an open problem to determine the structure of those T that preserve rank k for some fixed k, $1 < k < n$. A question that does not have a simple matrix analogue involves higher order tensor spaces $U^{(k)}$. Let $U^{(k)}$ be the dual space of the space of all multilinear functionals of k vectors in U. Again, if $x_1, \cdots, x_k$ are in U we may define $x_1 \otimes \cdots \otimes x_k \in U^{(k)}$ by $x_1 \otimes \cdots \otimes x_k(\phi) = \phi(x_1, \cdots, x_k)$ for all multilinear functionals ϕ. The elements $x_{i_1} \otimes \cdots \otimes x_{i_k}$, $1 \leqq i_t \leqq n$, $t = 1, \cdots, k$ form a basis for $U^{(k)}$ and we may define the length of an element of $U^{(k)}$ in precisely the same way as we define it for $k = 2$. Tensors of length 1 are sometimes called *pure vectors* in $U^{(k)}$ and it would be interesting to find all linear maps S on $U^{(k)}$ that map pure vectors into pure vectors. In a recent doctoral dissertation, R. Westwick considered a similar problem for mappings on Grassmann product spaces [29]. The question treated there is the following: Let $G_k(U)$ denote the dual space of the space of all alternating multilinear functionals on sets of k vectors in U, and for $x_j \in U$, $j = 1, \cdots, k$ let $x_1 \wedge \cdots \wedge x_k \in G_k(U)$ be defined by

$$x_1 \wedge \cdots \wedge x_k(\phi) = \phi(x_1, \cdots, x_k)$$

for all alternating multilinear functionals ϕ. The problem then is to determine all linear S mapping $G_k(U)$ into itself and having the property that any nonzero vector of the form $x_1 \wedge \cdots \wedge x_k$ (a *pure vector of grade* k) is mapped into such a vector. In this case the map S turns out to be the kth compound of a matrix in $M_n(F)$ except when $n = 2k$. This result is close to one of Chow [1] in which both S and S^{-1} are assumed to exist and to map the entire set of nonzero pure vectors of grade k *onto* itself. In [8] this result is used to determine $\Omega(f)$ when $f(A) = E_{2r}(A)$ and Ω is the space of skew-symmetric matrices over the real numbers. This is related to some work of Morita [19] who showed that if Ω is the space of skew hermitian matrices and $f(A)$ is the Hilbert norm of A (i.e. the nonnegative square root of the largest characteristic root of A^*A) then to within scalar multiples every transformation in $\Omega(f)$ is of the form (3) with U and V unitary except for $n = 4$. Quite recently H. Minc and the present author extended this to symmetric functions of the eigenvalues of A^*A.

IV. The permanent function. A considerably less tractable matrix function to deal with is the permanent of A, defined by

$$p(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all $n!$ permutations of $1, \cdots, n$. Recent interest in the permanent stems from its application to certain combinatorial problems [24] and an unresolved conjecture of van der Waerden [18]. We will

give a short description of the v, k, λ problem and indicate where the permanent makes its appearance there. Let $0<\lambda<k<v$ and suppose we have v items $x_1, \cdots, x_v$ to be distributed into v sets $T_1, \cdots, T_v$ in such a way that $T_i \cap T_j$ has exactly λ of the x's in it for $i \neq j$ and each T_i has exactly k of the x's in it. We construct the *incidence matrix* A of this configuration as follows:

$$a_{ij} = 0 \quad \text{if } x_i \notin T_j,$$
$$a_{ij} = 1 \quad \text{if } x_i \in T_j.$$

Clearly the configuration is completely determined by the matrix A, and it can be proved [23] that

(a) A is normal and satisfies $AA' = A'A = (k-\lambda)I + \lambda J$ where J is the v-square matrix of 1's,
(b) $\lambda = k(k-1)/(v-1)$,
(c) $|d(A)| = k(k-\lambda)^{(v-1)/2}$.

Suppose that $T_1, \cdots, T_v$ and $R_1, \cdots, R_v$ are the sets used in two v, k, λ configurations of the items $x_1, \cdots, x_v$. We say that the two configurations are *isomorphic* if there is a permutation σ of $1, \cdots, n$ and a 1-1 correspondence θ between the sets T_j and R_j such that $\theta(T_i) = R_i$ and $x_i \in T_j$ if and only if $x_{\sigma(i)} \in R_j$.

A *system of distinct representatives* for a v, k, λ configuration is just a particular ordering of the items $x_{\mu(1)}, \cdots, x_{\mu(v)}$ such that

$$x_{\mu(i)} \in T_i, \qquad i = 1, \cdots, v.$$

That is, a system of distinct representatives corresponds uniquely to a term with value 1 in the permanent expansion of the incidence matrix A. Hence $p(A)$ is precisely the number of systems of distinct representatives for the v, k, λ configuration with incidence matrix A. In (c) above we noted that $|d(A)|$ is a function of v, k, λ only and does not depend on which one of the set of nonisomorphic v, k, λ configurations is represented by A. The same question may be asked about the number of distinct representatives, $p(A)$. Very recently Nikolai [20], using an identity of Ryser, described some interesting machine computations in which nonisomorphic $(v, k, \lambda) = (15, 7, 3)$ configurations with different permanents are exhibited. (There are no nonisomorphic v, k, λ configurations for $v < 15$ and in the case of finite projective planes with $(v, k, \lambda) = (n^2+n+1, n+1, 1)$ the first case of nonisomorphism arises for $n = 9$, $v = 91$.)

It is thus of interest to know in general what kinds of linear operations may be performed on a class Ω of matrices in such a way that the permanent of each $A \in \Omega$ is held fixed. In other words the problem is to determine the structure of $\Omega(p)$. Very recently it was proved [17] that except for $n = 2$ the only linear T that satisfy

$$p(T(A)) = p(A)$$

for all $A \in M_n(F)$ are of the type (3) where U and V both have the special form

$$\sum_{i=1}^{n} a_i E_{i\sigma(i)}, \qquad \prod_{i=1}^{n} a_i = 1, \tag{5}$$

where σ is a permutation of $1, \cdots, n$. (The field F is required to have enough elements.) The same result can be proved, with the further consequence that each $a_i = 1$ in (5), when it is assumed that T maps the polyhedron of doubly stochastic matrices into itself and preserves the permanent [15]. It might be conjectured that there exists some uniform linear operation that converts the permanent of A into the determinant:

$$p(T(A)) = d(A). \tag{6}$$

In this way we would be able to use the extensive results available concerning determinant preservers. An old result of G. Polya [22] shows that no affixing of $\pm$ signs to the elements of A can (except for $n = 2$) convert the permanent into the determinant uniformly; and very recently (again except for $n = 2$) it was proved that *no* linear T can satisfy (6) for all $A \in M_n(F)$ [14]. In these theorems the problem was in every case reduced to a consideration of rank 1 preservers.

V. Commutativity. We indicate briefly how the idea of a linear mapping on a space of matrices can be very fruitful in studying commutativity. A comprehensive survey and bibliography of the subject of commutativity in finite matrices is contained in the excellent account of Taussky [27].

Suppose that A is a fixed n-square matrix and we consider a corresponding linear mapping T_A on $M_n(F)$ defined by

$$T_A(X) = AX - XA. \tag{7}$$

It can be checked [7] that with respect to the basis E_{ij}, $1 \leqq i, j \leqq n$, ordered lexicographically, the matrix representation of T_A is $A \otimes I - I \otimes A'$. Here $C \otimes D$ is the Kronecker product of C and D; i.e. the n^2-square matrix which has $c_{ij}D$ as its (i, j) block when it is partitioned into n^2 nonoverlapping n-square submatrices. To see how this can be used we assume for example that A has the same minimum and characteristic polynomial. Then A is similar over F to the companion matrix C of its characteristic polynomial and it is trival to check that A may be replaced by C in examining the null space of T_A. Thus we may take

$$C = A = \begin{bmatrix} 0 & 1 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & 0 \\ 0 & & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}$$

and hence

$$C \otimes I - I \otimes C' = \left[\begin{array}{c|c|c|c|c} -C' & I & & & 0 \\ \hline 0 & -C' & I & & \cdot \\ \hline \cdot & 0 & \cdot\ \cdot & & \cdot \\ \hline \cdot & \cdot & \cdot & & 0 \\ \hline 0 & 0 & & -C' & I \\ \hline a_0 I & a_1 I & & a_{n-2} I & a_{n-1} I - C' \end{array}\right]$$

Clearly we may bring $C \otimes I - I \otimes C'$ by elementary row operations to the form

$$\left[\begin{array}{c|c|c|c|c} * & I & 0 & \cdot\ \cdot\ \cdot\ \cdot & 0 \\ \hline & & & & \cdot \\ & 0 & \cdot & & \cdot \\ \hline & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & \cdot & 0 \\ \hline \cdot & & & \cdot & I \\ \hline * & 0 & & & 0 \end{array}\right]$$

a matrix of rank at least $n(n-1)$. Hence the nullity of T_A is at most $n^2 - n(n-1) = n$. On the other hand the matrices A^s, $s = 0, \cdots, n-1$ are in the null space of T_A (they commute with A) and they are linearly independent (the degree of the minimum polynomial is n). Thus the null space of T_A is exactly n-dimensional and must therefore be generated by $I, A, \cdots, A^{n-1}$. In other words, the only matrices that commute with A are polynomials in A. Of course, this well-known result may be proved in other ways, but this technique has been applied successfully to the much more complicated study of higher order commutators [10], [25].

We indicate one more example of the approach. Suppose that A is a given linear transformation on a finite dimensional vector space U to itself, where U is equipped with a fixed nonsingular symmetric bilinear functional $\{u, v\}$. A problem considered in a recent paper by Osborn [21] is the following: Find the structure of the set Ω of all bilinear functionals α with respect to which A is self-adjoint:

$$\alpha(Au, v) = \alpha(u, Av), \qquad \text{all } u, v. \tag{8}$$

It is elementary that there exists a unique linear transformation X on U such that

$$\alpha(u, v) = \{Xu, v\}, \qquad \text{all } u, v.$$

Thus the equation (8) takes the form

$$\alpha(Au, v) = \{XAu, v\} = \alpha(u, Av) = \{Xu, Av\} = \{A^*Xu, v\}, \tag{9}$$

where A^* is the unique linear transformation satisfying

$$\{A^*u, v\} = \{u, Av\} \qquad \text{all } u, v.$$

By selecting a basis for U, and letting A_1 and X_1 denote the matrix representations of A and X in this basis respectively, we conclude from (9) and the non-singularity of $\{u, v\}$ that

$$X_1A_1 - A_1'X_1 = 0.$$

Thus Ω is known as soon as we know the null space of the mapping

$$T_{A_1}(X_1) = X_1A_1 - A_1'X_1. \tag{10}$$

In interesting papers Osborn [21] and Taussky and Zassenhaus [28] examined the null space of T_{A_1} from considerably different points of view, and still further results along these lines are found in [13].

VI. Symmetry operators. Returning to the space of tensors $U^{(k)}$ introduced in III we define a *permutation operator* on $U^{(k)}$ by the equation

$$\sigma(x_1 \otimes \cdots \otimes x_k) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(k)}. \tag{11}$$

The meaning in (11) is this: σ is a permutation in the symmetric group of degree k, S_k. We can then regard σ as a linear transformation on $U^{(k)}$ by defining its action on pure vectors (which span $U^{(k)}$) by the equation (11) and extending linearly. If λ_σ is a scalar for each $\sigma \in S_k$ then we can define a *symmetry operator* T_λ on $U^{(k)}$ by

$$T_\lambda = \sum_{\sigma \in S_k} \lambda_\sigma \sigma. \tag{12}$$

Associated with T_λ is a scalar valued function on $M_k(F)$:

$$d_\lambda(A) = \sum_{\sigma \in S_k} \lambda_\sigma \prod_{i=1}^{k} a_{i\sigma(i)}, \tag{13}$$

$A = (a_{ij}) \in M_k(F)$. For example, if $\lambda_\sigma = \epsilon(\sigma) = \pm 1$ according as σ is even or odd, then T_λ is the skew-symmetry operator and the corresponding d_λ is the determinant. Again, if $\lambda_\sigma = 1$ for all $\sigma \in S_k$ then d_λ specializes to the permanent function. Another example of interest in certain combinatorial problems is obtained by setting $\lambda_\sigma = 1$ for σ any power of the full cycle $(1, \cdots, k)$, $\lambda_\sigma = 0$ otherwise. In general, the invariance problem appears as follows: *for a given choice of the*

λ_σ *investigate the structure of the set of linear transformations on* $M_k(F)$ *into itself which hold fixed the corresponding* d_λ. This problem, of course, subsumes both the permanent and determinant preservation problems. There are various choices for the λ_σ and research is currently under way on those symmetrizers T_λ that arise from the Young diagrams ([30] p. 120). We close by describing the bridge that allows us to move freely between the symmetrizers T_λ and its associated matrix function d_λ. Let $e_1, \cdots, e_k$ be a linearly independent set in U and suppose $h_1, \cdots, h_k$ is a dual set of linear functionals, $h_i(e_j) = \delta_{ij}$. Suppose $A \in M_k(F)$ and set

$$x_i = \sum_{j=1}^{k} a_{ij} e_j.$$

If ϕ is the multilinear functional whose value is defined by $\phi(y_1, \cdots, y_k) = h_1(y_1) \cdots h_k(y_k)$, we compute that

$$\begin{aligned} T_\lambda(x_1 \otimes \cdots \otimes x_k)(\phi) &= \sum_{\sigma \in S_k} \lambda_\sigma x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(k)}(\phi) \\ &= \sum_{\sigma \in S_k} \lambda_\sigma \phi(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(k)}) \\ &= \sum_{\sigma \in S_k} \lambda_\sigma \prod_{t=1}^{k} h_t(x_{\sigma^{-1}(t)}) = \sum_{\sigma \in S_k} \lambda_\sigma \prod_{t=1}^{k} a_{\sigma^{-1}(t),t} \\ &= \sum_{\sigma \in S_k} \lambda_\sigma \prod_{t=1}^{k} a_{t,\sigma(t)} = d_\lambda(A). \end{aligned}$$

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CERTAIN SYSTEMS OF LINEAR EQUATIONS OVER A RING WITH AN APPLICATION TO POLYNOMIAL RINGS

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1. Introduction. Let $C=(c_{ij})$ be a square matrix of order n over an arbitrary ring R. We define the determinant $|C|$ of the matrix C by the same sum of terms used to define a determinant in the commutative case, except that we now insist that the factors in each term be written in the order of their row indices. That is, each term is to be written in the form $\pm c_{1i_1}c_{2i_2}\cdots c_{ni_n}$.

Although we now have a definition of determinant, it is not obvious that the concept has any usefulness; certainly, very little of the usual theory will carry over to the noncommutative case. The following generalization of a familiar theorem may therefore be of some interest.

THEOREM 1. *Let the a_{ij} and the x_j be elements of an arbitrary ring R such that*

$$\sum_{j=1}^{n} a_{ij}r_ix_j = 0 \qquad (i = 1, 2, \cdots, n) \tag{1}$$

for every choice of the r_i in R. Then for every choice of the r_i we have $|a_{ij}r_i|x_k = 0$, $(k = 1, 2, \cdots, n)$.

We shall prove this theorem in the next section, and then proceed to apply it to establish a theorem on annihilators in polynomial rings. Before stating this second theorem, we need to introduce some additional notation. Let $R[x]$ be the ring of all polynomials in an indeterminate x over R. If A is a right ideal in $R[x]$, we shall denote by A^r the set of all right annihilators of A, that is, the ideal consisting of all elements h of $R[x]$ such that $Ah = 0$. It is known [2] that if $A^r \neq 0$, then $A^r \cap R \neq 0$. The following theorem, which we shall establish in Section 3, gives some additional information about $A^r \cap R$ for the case in which A has a finite number of generators.

THEOREM 2. *Let R be an arbitrary ring and A a right ideal in $R[x]$ which has a finite number of generators. If $A^r \neq 0$, and b_0 is the leading coefficient of any nonzero element of A^r, then $b_0 \in A^r$ or $A^r \cap (Rb_0) \neq 0$.*

2. *Proof of Theorem* 1. First, we introduce some necessary notation. Let $C = (c_{ij})$ be a fixed matrix of order n over the arbitrary ring R. Let us denote by Z any product of elements c_{ij} which contains exactly $n-1$ of these elements, one from each of some $n-1$ rows of C, with the product so written that the row indices appear in their natural order. That is, for some integer k $(1 \leqq k \leqq n)$, we have

$$Z = c_{1s} \cdots c_{k-1,t}c_{k+1,u} \cdots c_{nv}.$$

If $a \in R$, we define

$$Z \circ a = c_{1s} \cdots c_{k-1,t}ac_{k+1,u} \cdots c_{nv},$$

it being understood that a appears as the first element of this product if $k = 1$ and as the last element if $k = n$. We also define $(-Z) \circ a = -(Z \circ a)$. Moreover, if Z_1 and Z_2 are such that $Z_1 \circ a$ and $Z_2 \circ a$ are defined, we define $(Z_1 + Z_2) \circ a = (Z_1 \circ a) + (Z_2 \circ a)$. Of course, if R is commutative, this operation "$\circ$" is just multiplication in R.

We use the definition of determinant given in the introduction, and let C_{ij} be the cofactor of c_{ij} in $|C|$. It follows that $C_{ij} \circ a$ is defined for each C_{ij} and each element a of R. Furthermore, for each choice of i, j, and k, $C_{ij} \circ c_{ik}$ is a sum of terms each of which is a product of n elements, one from each of the n rows of the matrix C, with the elements appearing in the order of their row indices. Accordingly, from the identities which hold in the commutative case

we have the following results which are important for our purposes:

$$|C| = \sum_{i=1}^{n} C_{ij} \circ c_{ij} \qquad (j = 1, 2, \cdots, n), \tag{2}$$

$$0 = \sum_{i=1}^{n} C_{ij} \circ c_{ik} \qquad (j \neq k). \tag{3}$$

It should be clear that the operation "$\circ$" has been defined in such a way that in these relations the possible noncommutativity of R plays no role.

We now consider Equations (1), and henceforth let r_i $(i=1, 2, \cdots, n)$ be fixed but arbitrary elements of R. We identify $(a_{ij}r_i)$ with (c_{ij}) and use the notation introduced above. We shall show that from Equations (1) it follows that for each $i=1, 2, \cdots, n$,

$$[C_{i1} \circ (a_{i1}r_i)]x_1 + \cdots + [C_{i1} \circ (a_{in}r_i)]x_n = 0. \tag{4}$$

Of course, in the commutative case, (4) can be obtained by simply multiplying the ith of Equations (1) by C_{i1}. In the general case we proceed as follows. If $i=1$, we may obtain (4) from the first of Equations (1) by replacing r_1 by r_1C_{11}. This is allowable since Equations (1) hold for arbitrary choices of the r_i. If $i=n$, we only need to multiply the nth of Equations (1) on the left by C_{n1}. If $i \neq 1, n$, we use the fact that C_{i1} is a sum of terms $\pm T$, where T is of the form

$$T = a_{1s}r_1 \cdots a_{i-1,t}r_{i-1}a_{i+1,u}r_{i+1} \cdots a_{nv}r_n.$$

In the ith of Equations (1) we multiply on the left by $a_{1s}r_1 \cdots a_{i-1,t}r_{i-1}$ and then replace r_i by $r_ia_{i+1,u}r_{i+1} \cdots a_{nv}r_n$. There results

$$[T \circ (a_{i1}r_1)]x_1 + \cdots + [T \circ (a_{in}r_i)]x_n = 0.$$

Adding these equations for each term T of C_{i1} we obtain (4). Now if we add Equations (4) for $i=1, 2, \cdots, n$ and use (2) and (3) it follows at once that $|a_{ij}r_i|x_1 = 0$. Similar arguments show that $|a_{ij}r_i|x_k = 0$ for $k=2, 3, \cdots, n$; and this completes the proof of the theorem.

We may remark that although in the definition of determinant we required the factors in each term to be written in the order of their row indices, we could just as well have specified some other fixed order of the row indices.

3. *Proof of Theorem* 2. In this section we consider the ring $R[x]$ of polynomials in an indeterminate x with coefficients in the arbitrary ring R, and begin by proving the following lemma from which our Theorem 2 will follow quite easily.

LEMMA. *Let g and f_i $(i=1, 2, \cdots, k)$ be nonzero elements of $R[x]$ such that $f_iRg=0$ $(i=1, 2, \cdots, k)$. If b_0 is the leading coefficient of g, then either $f_iRb_0=0$ $(i=1, 2, \cdots, k)$ or there exists an element c of R such that $cb_0 \neq 0$ and $f_iRcb_0=0$ $(i=1, 2, \cdots, k)$.*

The proof is by induction on k, so we begin by considering the case in which $k=1$. The method is a generalization of that outlined in [1] for the commutative case. For convenience, we shall write f in place of f_1 throughout this part of the proof.

Let us set

$$f = a_0x^n + \cdots + a_n \qquad (a_0 \neq 0),$$

$$g = b_0x^m + \cdots + b_m \qquad (b_0 \neq 0).$$

We assume that $n>0$ and $m>0$, since otherwise the result is obvious. Now, by hypothesis, we have

$$(a_0x^n + \cdots + a_n)R(b_0x^m + \cdots + b_m) = 0, \tag{5}$$

and it follows at once that $a_0Rb_0=0$. If $a_iRb_0=0$ for $i=0, 1, \cdots, n$, we have nothing to prove since this would imply that $fRb_0=0$. Accordingly, let us assume that

$$a_0Rb_0 = \cdots = a_{p-1}Rb_0 = 0,\ a_pRb_0 \neq 0, \tag{6}$$

where $0<p\leqq n$. Now by a consideration of the coefficients of $x^{n+m-p}, \cdots, x^{n-p}$ in (5), we find that the $m+1$ elements $b_0, \cdots, b_m$ of R satisfy the following $m+1$ equations in which $r_1, \cdots, r_{m+1}$ are arbitrary elements of R:

$$\begin{aligned} a_pr_1b_0 &+ a_{p-1}r_1b_1 + \cdots &&= 0,\\ a_{p+1}r_2b_0 &+ a_pr_2b_1 + \cdots &&= 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\cdots + a_pr_{m+1}b_m &&= 0. \end{aligned} \tag{7}$$

Thus, by Theorem 1, we have $Db_0=0$, where D is the determinant of the matrix of the coefficients of $b_0, \cdots, b_m$ in these equations. Now each nonzero element of this matrix to the right of the principal diagonal contains a factor a_i with $i<p$. Except for the diagonal term, each nonzero term in the expansion of D therefore contains such an a_i. Our assumptions (6) then show that the right multiplication by b_0 of any such term is necessarily zero. Since also $Db_0=0$, we conclude that the right multiplication of the diagonal term by b_0 is zero, that is, that $a_pr_1a_pr_2 \cdots a_pr_{m+1}b_0=0$, and hence that $(a_pR)^{m+1}b_0=0$. Let j be a positive integer so chosen that $(a_pR)^jb_0\neq 0$, whereas $(a_pR)^{j+1}b_0=0$. The existence of j is a consequence of our assumption that $a_pRb_0\neq 0$. Let c_1 be an element of $(a_pR)^j$ such that $c_1b_0\neq 0$. It follows that

$$a_pRc_1b_0 \subseteq (a_pR)^{j+1}b_0 = 0,$$

and hence from (6) we now have

$$a_0Rc_1b_0 = \cdots = a_pRc_1b_0 = 0.$$

If it happens that $a_iRc_1b_0=0$ $(i=0, 1, \cdots, n)$, we are through. Otherwise, we apply the above procedure to c_1g in place of g since we know that c_1b_0 is the lead-

ing coefficient of c_1g. Continuing in this way, we find that there must exist a finite number of elements $c_1, \cdots, c_l$ of R such that $c_l \cdots c_1b_0 \neq 0$ and $a_iRc_l \cdots c_1b_0 = 0$ $(i = 0, 1, \cdots, n)$. This means that $fRc_l \cdots c_1b_0 = 0$, and the proof for the case in which $k = 1$ is completed with $c = c_l \cdots c_1$.

We now prove the lemma for the case of k polynomials f_i $(i = 1, 2, \cdots, k)$ on the assumption that it holds for $k-1$ polynomials. We may just as well assume also that $f_iRb_0 \neq 0$ for some i, since otherwise there is nothing to prove. Suppose, for convenience of notation, that $f_1Rb_0 \neq 0$. By our induction hypothesis, there exists an element r of R such that $rb_0 \neq 0$ and $f_iRrb_0 = 0$ $(i = 1, 2, \cdots, k-1)$. If $f_kRrb_0 = 0$, the proof is completed. Otherwise, we apply the case already established (with f_k and rg taking the place of f and g, respectively) and find that there exists an element s of R such that $srb_0 \neq 0$ and $f_kRsrb_0 = 0$. Hence, if we set $c = sr$, we have $f_iRcb_0 = 0$ $(i = 1, 2, \cdots, k)$, and the lemma is proved.

It is now easy to establish Theorem 2 as follows. Suppose that the right ideal A is generated by the polynomials f_i $(i = 1, 2, \cdots, k)$. It is then clear that $h \in A^r$ if and only if

$$f_iRh = 0 \qquad (i = 1, 2, \cdots, k), \tag{8}$$

and also, in case R has no unit element,

$$f_ih = 0 \qquad (i = 1, 2, \cdots, k). \tag{9}$$

Let g be a nonzero element of A^r with leading coefficient b_0. Then the preceding lemma assures us that Equations (8) are satisfied for $h = b_0$ or for $h = cb_0 \neq 0$. We now complete the proof according to these two cases.

Suppose, first, that Equations (8) are satisfied for $h = b_0$. If $Rb_0 = 0$, then also $f_ib_0 = 0$ $(i = 1, 2, \cdots, k)$ and $b_0 \in A^r$. If $Rb_0 \neq 0$, say $db_0 \neq 0$, then clearly $db_0 \in A^r$ since $h = db_0$ satisfies Equations (9) as well as (8).

Suppose, now, that Equations (8) are satisfied for $h = cb_0 \neq 0$. The proof is essentially the same as in the preceding case. If it happens that $Rcb_0 = 0$, then $cb_0 \in A^r$. If $Rcb_0 \neq 0$, say $dcb_0 \neq 0$, we find that $dcb_0 \in A^r$. Hence both cases are disposed of, and the proof is completed.

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BINARY RING SEQUENCES*

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Introduction. In this paper a solution of the following combinatorial problem is derived:

Problem. Given two positive integers, k and n, can a ring sequence (ordered cycle) of k binary digits 0, 1 be constructed such that the k subsequences of n consecutive digits are all different?

A ring sequence of the required type will be called a *binary* (n, k) *ring sequence.* An example of a binary (4, 10) ring sequence is 0000101101, the 10 subsequences in question being 0000, 0001, 0010, 0101, 1011, 0110, 1101, 1010, 0100, 1000. Obviously, k can not exceed 2^n. If $k \leqq 2^n$, it will be shown that a binary (n, k) ring sequence can always be constructed.

This problem of binary ring sequences has important engineering applications, especially in connection with the logical design of digital computers ([1]–[5]). The problem of constructing maximal (i.e., $k=2^n$) sequences of the specified type has a long and rather interesting history [6].

Among the earlier technical applications of maximal sequences the "teleprinter problem" ([6], [7]) is worth mentioning. A *teleprinter* (*teletype*) *character* is a block of five 0's and 1's. With a maximal sequence ($n=5$) located around the circumference of a disc, one could print any teleprinter character by rotating the disc to the required position. A simple method of constructing a particular maximal sequence for any given n is due to M. H. Martin [8]. I. J. Good [7] solved the problem independently, using Euler's solution of the graph-tracing problem. N. G. de Bruijn [9] has shown that the number of different maximal sequences is given by $2^{2^{n-1}-n}$.

As regards nonmaximal sequences, a proof of their existence for any n and any $k \leqq 2^n$ is given in [10], according to [3]. Due to the difficult access to [10], however, the author is not aware of its exact contents.

The T_n-Net. Let $K=\{0, 1\}$, $V_1=K$, and $V_n=V_{n-1}\times K$; that is, any element $x\in V_n$ in an n-tuple $x=(x_1, \cdots, x_n)$, with components $x_i\in K$ $(i=1, \cdots, n)$. We define the relation $x\rightarrow y$ (read: "x precedes y" or "y follows x") in V_n by:

$$x \rightarrow y \text{ if and only if } (x_2, \cdots, x_n) = (y_1, \cdots, y_{n-1}).$$

Thus, if $x\rightarrow y$, then x (or y) determines all but one component of y (or x). We evidently have:

LEMMA 1. *Let w, x, y, z be elements of V_n, such that $w\rightarrow x$, $w\rightarrow y$ and $z\rightarrow x$; then $z\rightarrow y$.*

Let x, $y\in V_n$ and $x\rightarrow y$. We define $\langle x, y\rangle$ as the element $(x_1, \cdots, x_n, y_n) = (x_1, y_1, \cdots, y_n)$ of V_{n+1}. One immediately verifies:

* The research reported in this paper was performed at the IBM Development Laboratories, Poughkeepsie, New York.

LEMMA 2. *If $x \to y$ and $y \to z$ in V_n, then $\langle x, y\rangle \to \langle y, z\rangle$ in V_{n+1}.*

Following I. J. Good [7] and N. G. de Bruijn [9] it is convenient to associate with V_n a certain net (i.e., weighted, directed graph; see [11] or [12]). The net T_n associated with V_n is a directed graph with 2^n vertices, labeled (weighted) by the elements of V_n. T_n contains an oriented edge from the vertex labeled $x(\in V_n)$ to the vertex labeled $y(\in V_n)$, iff $x \to y$. The oriented edge representing $x \to y$ is labeled $\langle x, y\rangle(\in V_{n+1})$.

Graph-Theoretical Concepts. We present some of the definitions concerning directed graphs (see [12] and [13]) which we shall use later.

A chain (of length k) is a sequence of edges $(e_1, \cdots, e_k)$ without repetitions such that the terminal vertex v_{i+1} of e_i $(i=1, \cdots, k-1)$ is the initial vertex of e_{i+1}. The chain is *closed*, if the terminal vertex v_{k+1} of e_k coincides with v_1, the initial vertex of e_1; otherwise, *open*.

A nonintersecting open chain, that is, a chain $(e_1, \cdots, e_k)$ such that $(v_1, \cdots, v_{k+1})$ are all different, is a *path*.

A nonintersecting closed chain, that is, $(v_1, \cdots, v_k)$ are all different, but $v_{k+1}=v_1$, is a *cycle*. Let $(e_1, \cdots, e_k)$ be a cycle. Then the vertices $(v_1, \cdots, v_k)$ form a *vertex cycle*.

A cycle passing through all vertices of the graph is *maximal*.

A P-set of cycles is a set of vertex-disjoint cycles which includes all the vertices of the graph.

A directed graph is *strongly connected* if, for any two vertices v, v' $(v \neq v')$, there exists a path from v to v'.

A directed graph G is *regular of degree $2g$* if each vertex of G is the initial vertex of exactly g edges as well as the terminal vertex of exactly g edges.

Properties of T_n. There exists an obvious one-to-one correspondence between all binary (n, k) ring sequences and all cycles of length k of T_n. The task is: to show that for any $0 < k \leqq 2^n$ there exists in T_n a cycle of length k.

The following properties of T_n are easily verified:

LEMMA 3. *T_n is regular of degree 4.*

LEMMA 4. *T_n is strongly connected.*

LEMMA 5. *T_n contains 2^{n+1} edges. Each element of V_{n+1} appears exactly once as edge weight of T_n.*

In the lemma below, given any edge e of T_n, let us denote its weight by $\bar{e}(\in V_{n+1})$. We then have, by Lemma 2:

LEMMA 6. *The edge sequence $(e_1, \cdots, e_k)$ of T_n is a closed chain if and only if the vertices of T_{n+1}, labeled $\bar{e}_1, \cdots, \bar{e}_k$, form a vertex cycle.*

T_n-Cycles. The following two lemmas refer to P-sets of cycles of T_n.

LEMMA 7. *Let G_1 be a P-set of cycles of T_n, and let G_2 be the complementary subgraph of T_n (i.e., the subgraph formed by the remaining edges). Then G_2 also consists of a P-set of cycles of T_n.*

Proof. G_2 has the same vertices as T_n, and is regular of degree 2. Hence, G_2 forms a P-set of cycles of T_n. (See [13], Theorem I-28 and Theorem II-8.)

LEMMA 8. *Let G be a cycle of T_n. Then there exists a P-set of cycles of T_n which includes C.*

Proof. Let K be the closed chain of T_{n-1}, corresponding to C in accordance with Lemma 6, and let K' be the subgraph of T_{n-1} complementary to K. Let K' consist of r separate parts, $G_1, \cdots, G_r$ ("Zusammenhängende Bestandteile" in [13], p. 14). One easily can prove that Theorem II-7 of [13] applies to each separate part G_i of K'. Thus, the edges of G_i may be ordered to form a closed chain K_i $(i=1, \cdots, r)$. The cycles $C_1, \cdots, C_r$ of T_n, which correspond to $K_1, \cdots, K_r$ of T_{n-1}, together with C, obviously form a P-set.

Next, let C' and C'' be vertex-disjoint cycles of a directed graph G; let e be an edge of G, with initial vertex on C' and terminal vertex on C''. We shall call the edge e a *bridging edge* from C' to C''. We shall now prove:

LEMMA 9. *Let C' and C'' be vertex-disjoint cycles of T_n, and let e be a bridging edge from C' to C''. Then the vertices of C' and C'' may be ordered to form a single cycle C.*

Proof. Let C' be formed by the vertices $v_1', \cdots, v_k'$ and C'' by $v_1'', \cdots, v_k''$, where v_1' is the initial and v_1'' the terminal vertex of e.

Then, denoting by $\bar{v}$ the weight of v in T_n, we have,

$$\bar{v}_1' \rightarrow \bar{v}_1'', \ \bar{v}_1' \rightarrow \bar{v}_2', \ \bar{v}_k'' \rightarrow \bar{v}_1''.$$

Hence, by Lemma 1, $\bar{v}_k'' \rightarrow \bar{v}_2'$.

The cycle C in question may thus be formed by the following sequence of vertices: $v_1', v_1'', \cdots, v_k'', v_2', \cdots, v_k'$. We are now in a position to prove:

THEOREM A. *If there exists in T_n a cycle C of length k, then there also exists in T_n a closed chain of length $k+2^n$.*

Proof. By Lemmas 7 and 8, there exists a P-set of cycles G_1 which includes no edge of C. Let H_1 be the subgraph of T_n formed by the edges of G_1 and C. If H_1 consists of more than one separate part, there exists (by Lemma 4) an edge e in T_n connecting two separate parts of H_1. Evidently, e forms a bridging edge from a cycle D' of G_1 to another cycle D'' of G_1. Let D be the cycle obtained from D' and D'' by application of Lemma 9. By replacing D' and D'' of G_1 by D we obtain another P-set of cycles G_2. Then G_2 will also have no edges in common with C. In general, let H_i $(i \geqq 1)$ be the subgraph formed by the edges of G_i and C. The above procedure, applied to H_i $(i \geqq 1)$, will lead to a subgraph H_{i+1} with a reduced number of separate parts. Finally, one arrives at a subgraph, say H_r,

formed by the edge union of a P-set G_r and C, and which is connected (considered as a nonoriented graph). Theorem II-7 of Ref. [13] may now be applied to H_r, and the closed chain in question, of length $k+2^n$, may be constructed accordingly.

THEOREM B. *T_n contains a cycle of length k, for any $0<k\leqq 2^n$.*

Proof (by induction). For $n=1$, the theorem is obviously true. We now assume the theorem to hold for $n-1$. Accordingly, if $0<k\leqq 2^{n-1}$, a cycle of length k exists in T_{n-1} and, by Lemma 6, also in T_n. Now, let $k>2^{n-1}$, and $k'=k-2^{n-1}$. Obviously, $k'\leqq 2^{n-1}$, and by our assumption there exists a cycle C' in T_{n-1} of length k'. By Theorem A there also exists in T_{n-1} a closed chain of length $k'+2^{n-1}=k$ and, by Lemma 6, there also exists a cycle of length k in T_n. Theorem B is thus proved.

As mentioned previously, from any cycle of T_n of length k, one immediately derives, in an obvious way, a binary (n, k) ring sequence. We thus have the following

COROLLARY. *Let n and k be positive integers, such that $k\leqq 2^n$. Then there exists a binary (n, k) ring sequence.*

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TWO THEOREMS IN MATRIX THEORY

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Introduction. In this note we discuss two theorems in matrix theory which are detailed below.

A, B and C will denote hermitian matrices of order* $n(n \geqq 2)$; λ_r and Λ_r $(1 \leqq r \leqq n)$ will denote characteristic roots of such matrices.

If λ_r $(1 \leqq r \leqq n)$ are the characteristic roots of A then these roots† will be said to be *ordered* if $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{n-1} \leqq \lambda_n$.

The abbreviations p.d. and p.s-d. will be used to indicate positive definite and positive semi-definite respectively for hermitian matrices.

All the general theorems quoted on matrix theory will be found in the text by Mirsky [4].

THEOREM α (*the monotonic theorem*). *Let A be an hermitian matrix with ordered characteristic roots λ_r $(1 \leqq r \leqq n)$ and let B be a p. d. or p.s-d. hermitian matrix; define $C \equiv A + B$ and suppose C has ordered characteristic roots Λ_r $(1 \leqq r \leqq n)$; then*

$$\lambda_r \leqq \Lambda_r \qquad (1 \leqq r \leqq n)$$

with strict equality for all r if B is p.d.

THEOREM β (*the separation theorem*). *Let $A = [a_{rs}]$ $(1 \leqq r, s \leqq n)$ be an hermitian matrix with ordered characteristic roots λ_r $(1 \leqq r \leqq n)$; define‡ $A' = [a_{rs}]$ $(1 \leqq r, s \leqq n-1)$ and suppose A' has ordered characteristic roots λ_r' $(1 \leqq r \leqq n-1)$; then*

$$\lambda_r \leqq \lambda_r' \leqq \lambda_{r+1} \qquad (1 \leqq r \leqq n-1).$$

Most proofs of Theorem α given in textbooks seem to depend on mini-max arguments for continuous functions (see, for example, [1] page 115, [2] page 28 and [3] page 200); in this note we give a more direct matrix proof.

Theorem β is well known and a direct matrix proof will be found in [5] page 286; see also [1] page 115 and [6] page 101.

We show that these two theorems are related in that either theorem can be deduced from the other.

1. Proof of Theorem α. Since A is hermitian the characteristic vectors§ x_r $(1 \leqq r \leqq n)$ of A, which satisfy $Ax_r = \lambda_r x_r$, span the vector space B_n of order n over the complex field $\mathfrak{F}$ (see [4] Chapter X). Without loss of generality the x_r can be chosen to be an orthonormal basis for B_n (see [4] pages 65 and 300) i.e. $x_r^* x_s = \delta_{rs}$ for $1 \leqq r, s \leqq n$, where an asterisk will denote the transposed conjugate of a matrix.

* The proofs of all results stated here are quite straightforward if $n = 1$.

† All characteristic roots of an hermitian matrix are real.

‡ A' could be any $(n-1) \times (n-1)$ principal submatrix.

§ The x_r are $n \times 1$ column vectors.

If $y \in B_n$ then

$$y = \sum_{t=1}^{n} \alpha_t x_t \qquad (\alpha_t \in \mathfrak{F}),$$

where $\alpha_t = x_t^* y$ $(1 \leqq t \leqq n)$. By direct multiplication

$$y^*y = \sum_{t=1}^{n} |\alpha_t|^2 \quad \text{and} \quad y^*Ay = \sum_{t=1}^{n} \lambda_t |\alpha_t|^2.$$

Let the characteristic vectors of C be z_r $(1 \leqq r \leqq n)$ and suppose that, as above, $z_r^* z_s = \delta_{rs}$.

For any r satisfying $1 \leqq r \leqq n$ define

$$y_r \equiv \sum_{t=1}^{r} \beta_t z_t,$$

where the β_t are so chosen to satisfy the following conditions:

(i)† if $y_r = \sum_{t=1}^{n} \alpha_t x_t$ then $\alpha_t = 0$ for $1 \leqq t \leqq r-1$;

(ii) $y_r^* y_r = \sum_{t=1}^{r} |\beta_t|^2 = 1.$

Both (i) and (ii) can always be satisfied; (i) leads to the following set of $(r-1)$ homogeneous equations

$$\alpha_t = \sum_{p=1}^{r} \beta_p x_t^* z_p = 0, \qquad (1 \leqq t \leqq r-1),$$

which always has a nonnull solution (see [4] page 30 Theorem 1.6.2) and then (ii) can be satisfied on multiplication by a suitable positive scalar.

Thus, since the characteristic roots are ordered,

$$y_r^* C y_r = \sum_{t=1}^{r} \Lambda_t |\beta_t|^2 \leqq \Lambda_r \sum_{t=1}^{r} |\beta_t|^2 = \Lambda_r$$

and so

$$\begin{aligned} \Lambda_r \geqq y_r^* C y_r &= y_r^* A y_r + y_r^* B y_r \\ &\geqq y_r^* A y_r \text{ (with strict inequality if } B \text{ is p.d.)} \\ &= \sum_{t=1}^{n} \lambda_t |\alpha_t|^2 = \sum_{t=r}^{n} \lambda_t |\alpha_t|^2 \\ &\geqq \lambda_r \sum_{t=r}^{n} |\alpha_t|^2 = \lambda_r y_r^* y_r = \lambda_r. \end{aligned}$$

† If $r=1$ this condition can be omitted.

2. Deduction of Theorem α from Theorem β. The following lemma is required.

LEMMA. *Let* $\delta \geqq 0$ *and* $\alpha_1 \leqq \alpha_1' \leqq \alpha_2 \leqq \alpha_2' \leqq \cdots \leqq \alpha_{n-1}' \leqq \alpha_n$ *then the zeros* $\beta_1, \beta_2, \cdots, \beta_n$ *of the polynomial*

$$F(x) \equiv \prod_{r=1}^{n} (x - \alpha_r) - \delta \prod_{r=1}^{n-1} (x - \alpha_r')$$

are all real; and if numbered in order of magnitude $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n$ *then* $\alpha_r \leqq \beta_r$ $(1 \leqq r \leqq n)$.

Proof. Without loss of generality it may be assumed that $\delta > 0$ and also that

$$\alpha_1 < \alpha_1' < \alpha_2 < \alpha_2' < \cdots < \alpha_{n-1}' < \alpha_n$$

for the general case follows from this by continuity. Then

$$F(\alpha_r) = \operatorname{sgn}(-1)^{n+1-r} \quad (1 \leqq r \leqq n), \qquad F(\infty) = \infty;$$

hence each of the open intervals $(\alpha_1, \alpha_2), \cdots, (\alpha_{n-1}, \alpha_n), (\alpha_n, \infty)$ contains at least one, and so exactly one, zero of $F(x)$.

Proof of Theorem α. Let the characteristic roots of B be δ_r $(1 \leqq r \leqq n)$ where $\delta_r \geqq 0$. There is a unitary matrix T such that

$$T^*BT = \operatorname{diag}(\delta_1, \delta_2, \cdots, \delta_n);$$

T^*AT and T^*CT will have the same characteristic roots as A and C respectively. Thus

$$T^*CT = T^*AT + \sum_{r=1}^{n} I(\delta_r),$$

where $I(\delta_r)$ has δ_r in the rth diagonal place and zero elsewhere.

Thus if the result can be demonstrated when B is an $I(\delta)$ then the result will follow in general; suppose then B is $I(\delta_n)$ where $\delta_n > 0$. Then (A' is the same notation as in Theorem β)

$$\begin{aligned} \det(\lambda I_n - C) &= \det(\lambda I_n - A - I(\delta_n)) \\ &= \det(\lambda I_n - A) - \delta_n \det(\lambda I_{n-1} - A') \end{aligned}$$

and the required result now follows from an application of the above Lemma and use of Theorem β on the characteristic roots of A'.

3. Deduction of Theorem β from Theorem α. Using the notation of Section 2 write

$$A = \begin{bmatrix} A' & 0 \\ 0 & \Delta \end{bmatrix} + \begin{bmatrix} 0 & a \\ a^* & \alpha - \Delta \end{bmatrix} = A_1 + A_2 \text{ (say)}$$

where Δ is a real number to be chosen later.

Clearly the characteristic roots of A_1 are λ_r' $(1 \leqq r \leqq n-1)$ and Δ. A simple calculation gives

$$\det(\lambda I - A_2) = \lambda^{n-2}\left[\lambda^2 - (\alpha - \Delta)\lambda - \sum_{r=1}^{n-1} |a_{rn}|^2\right]$$
$$= \lambda^{n-2}[\lambda^2 - (\alpha - \Delta)\lambda - d^2] \text{ (say)}$$

where $d^2 \geqq 0$; if $d=0$ then Theorem β follows at once, so consider $d^2>0$. It then follows that the characteristic roots of A_2 are 0 repeated $(n-2)$ times and $\lambda^+(\Delta)$, $\lambda^-(\Delta)$ where

$$\lambda^{\pm}(\Delta) = \lambda^{\pm} = \tfrac{1}{2}\{\alpha - \Delta \pm [(\alpha - \Delta)^2 + d^2]^{1/2}\}$$

the positive square root being taken.

Now take $\Delta \leqq \min\{0, \lambda_1'\}$; then λ^+ is positive while λ^- is negative. Write

$$A - \lambda^- I = A_1 + A_2 - \lambda^- I;$$

the characteristic roots of $A - \lambda^- I$ are $\lambda_r - \lambda^-$ $(1 \leqq r \leqq n)$ while those of $A_2 - \lambda^- I$ are $-\lambda^-$ repeated $(n-2)$ times, 0 and $\lambda^+ - \lambda^-$; thus $A_2 - \lambda^- I$ is p.s-d. and, from Theorem α,

$$\lambda_{r+1} - \lambda^- \geqq \lambda_r' \quad (1 \leqq r \leqq n-1), \qquad \lambda_1 - \lambda^- \geqq \Delta;$$

since λ^- can be made negative and as small as is required by choosing Δ to be large and negative it follows that $\lambda_{r+1} \geqq \lambda_r'$ for $1 \leqq r \leqq n-1$.

A similar argument based on choosing Δ large and positive indicates that $\lambda_r \leqq \lambda_r'$ for $1 \leqq r \leqq n-1$ and this completes the proof.

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THE THEORY OF LOANS

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1. Introduction. The small loan business, as it has developed according to legal requirements and necessary collateral, has given rise to a theory somewhat different from that of long term mortgages. Indeed, the usual problem of the consumer with respect to the small loan is to find the interest rate, whereas the problem for the mortgagee is to find the term, the cost or the balance due after a given time.

Approximate formulas which are satisfactory for small loans may give poor results for mortgages due to the increase in size of the term. However, in actual usage this is partly compensated for by use of a smaller interest rate. Also, if the mortgage is paid monthly, the interest rate is usually a monthly interest rate.

Several formulas (see references) have been given for small loans. Approximate formulas, satisfactory for most mortgages, can be obtained by using a higher convergent of the continued fraction. Formulas for mortgages are somewhat extensive for small loans but it seems desirable to establish formulas satisfactory for both fields.

Graduated interest rates are now in extensive use. The formulas for a single rate may be extended to a graduated rate in many cases. The following typical examples are presented as the formulas are developed, one in small loans, and one in mortgages, so that a comparison can be made.

Example I. (*Small loan*). Mr. A. borrows \$100. He repays the loan by making equal payments at the end of each month for the next twelve months. If 1% per month is the interest rate charged, find the payment. Here the payment is $100/a_{\overline{12}|}(1\%) = \$8.884879$. The total cost of the loan is $12(8.884879) - 100 = \$6.618548$.

Example II. (*Mortgage*). Mr. A. borrows \$10,000. He repays the loan by making payments at the end of each of the next 100 months. If $\frac{1}{2}\%$ per month is the interest rate charged, find the payment. Here the payment is $10{,}000/a_{\overline{100}|}(\frac{1}{2}\%) = \$127.3194$. The total cost is \$2731.94.

2. The Cost of a Loan. The total cost of a loan may be expressed as a function of the original loan, the number of payments and the interest rate.

Let B = the original debt
n = the number of monthly payments
R = the monthly payment
r = the monthly interest rate
I = the total cost of the loan
i.e. $I = Rn - B$.

For an ordinary annuity, [1]

$$B = Ra_{\overline{n}|} = R\frac{1-(1+r)^{-n}}{r} = \frac{I}{nr/\{1-(1+r)^{-n}\}-1}.$$

Expanding the right member in a series and solving for I gives

$$(1)\quad I = \frac{B(n+1)r}{2}\left[\frac{1}{1-\frac{(n-1)r}{6}+\frac{(n-1)(n+2)r^2}{36}-\frac{(n-1)(n+2)(n+8)r^3}{540}+\cdots}\right]$$

We now express (1) as a continued fraction

$$I = \frac{B(n+1)r}{2}\left[\cfrac{1}{1-\cfrac{(n-1)r}{6}\Big/1+\cfrac{(n+2)r}{6}\Big/1+\cfrac{(n-2)r}{10}\Big/\cdots}\right].$$

By taking convergents, we obtain approximations to the cost, I.

$$(2)\quad I^{(1)} = \frac{B(n+1)r}{2-\frac{1}{3}(n-1)r}, \quad \text{where} \quad \epsilon_1 < -\frac{(n-1)(n+2)r^2}{36}+\cdots$$

$$(3)\quad I^{(2)} = \frac{B(n+1)r}{2+r}\left(1+\frac{(n-2)r}{6}\right), \quad \text{where } \epsilon_2 < -\frac{(n-1)(n^2-4)r^3}{360}+\cdots$$

$$(4)\quad I^{(3)} = \frac{B(n+1)r}{2}\left|\frac{1+\frac{(n+8)r}{15}}{1-\frac{(n-7)r}{10}+\frac{(n-1)(n-2)r^2}{60}}\right|, \quad \text{where } \epsilon_3 < f(n^4r^4).$$

Thus, the approximate cost of a loan is directly proportional to the principal B, the rate r, and the average time $(n+1)/2$.

Applying these formulas to the examples, we find

Example I.

$I^{(1)} = 6.62139$ $\quad \epsilon_1 \approx -.00284$

$I^{(2)} = 6.61857$ $\quad \epsilon_2 \approx -.00002$

$I^{(3)} = 6.618546$ $\quad \epsilon_3 \approx .000002$

Example II. $I^{(1)} = 2752.04$ $\epsilon_1 \approx -20.10$

$I^{(2)} = 2732.79$ $\epsilon_2 \approx -.85$

$I^{(3)} = 2731.89$ $\epsilon_3 \approx .05.$

It should be noted that the errors for $I^{(2)}$ and $I^{(3)}$ are opposite in sign.

3. The Cost per Period. The cost per period or average cost per period is given by I/n.

$$\frac{I^{(1)}}{n} = \mathrm{Br}\,\frac{(n+1)}{2n - \frac{1}{3}(n^2 - n)r}. \tag{5}$$

For a given loan at a fixed rate r, the cost per period has a small variation. In fact it decreases for a time as n increases and then starts to increase [2]. The minimum value of the cost per period is obtained by differentiating (5) with respect to n, putting this equal to zero and solving for n. This gives

$$n_1 = \sqrt{\left(2 + \frac{6}{r}\right)} - 1. \tag{6}$$

Again,

$$\frac{I^{(2)}}{n} = \frac{Br}{2 + r}\left[1 + \frac{1}{n} + \frac{(n^2 + 3n + 2)r}{6n}\right].$$

The minimum for $I^{(2)}/n$ is

$$n_2 = \sqrt{\left(\frac{6}{r} + 2\right)}. \tag{7}$$

A more accurate value [2] is given by

$$\bar{n} = \sqrt{\left(\frac{6}{r} + 3.8 - .32r\right)}.$$

4. How the Cost is Applied to the Loan. Suppose s payments are still to be paid. That is, $n - s = m$ payments have been made.

Let I_s be the part of the total cost I, which is to be paid as part of the remaining payments. Then $I_s = Rs - B_m$, where B_m is the balance due after m periods. I_s is the proper rebate if the borrower repays the loan at this time. By formula (2)

$$I_s^{(1)} = \frac{B_m(s+1)r}{2 - \frac{1}{3}(s-1)r}. \tag{8}$$

Hence

$$\theta \equiv \frac{I_s^{(1)}}{I} = \frac{B_m(s+1)(2-\frac{1}{3}(n-1)r)}{B(n+1)(2-\frac{1}{3}(s-1)r)}, \tag{9}$$

which is known as the rebate factor. It is that part of the total cost not yet used.

Remembering that $B_m = Rs - I_s$ and $B = Rn - I$, we can change (9) into the form

$$\bar{\theta} = \frac{s(s+1)}{n(n+1)}\left(\frac{1+\frac{1}{3}(n+2)r}{1+\frac{1}{3}(s+2)r}\right), \tag{10}$$

which is presented in [4].

A more accurate value of the rebate factor is found as follows:

$$\theta = \frac{s - a_{\overline{s}|}}{n - a_{\overline{n}|}} = \frac{s(s+1)}{n(n+1)}\left[\frac{1-\frac{1}{3}(s+2)r+\frac{1}{12}(s+2)(s+3)r^2-\cdots}{1-\frac{1}{3}(n+2)r+\frac{1}{12}(n+2)(n+3)r^2-\cdots}\right].$$

Expanding in a continued fraction, we have

$$\theta = \frac{s(s+1)}{n(n+1)}\left[\cfrac{1}{1-\cfrac{(n-s)r}{3}\Bigg/1+\cfrac{(3n-s+7)r}{12}\Bigg/\cdots}\right],$$

which has the convergent

$$\theta^{(1)} = \frac{s(s+1)}{n(n+1)}\left[\frac{1+\frac{(3n-s+7)r}{12}}{1-\frac{(n-3s-7)r}{12}}\right]. \tag{11}$$

Example I (*Small Loans*). Let us find the rebate factor after six payments.

By (10) $\bar{\theta} = 27.44755$ $\epsilon_1 = .008120$

By (11) $\theta^{(1)} = 27.455768$ $\epsilon_2 = -.000093.$

Example II (*Mortgage*). Let us find the rebate after fifty payments.

By (10) $\bar{\theta} = 27.184$ $\epsilon_1 = .112$

By (11) $\theta^{(1)} = 27.3027$ $\epsilon_2 = -.00665.$

It should be noted that although one half of the payments have been made, almost three fourths of the cost has been used.

If we let $s=pn$, $(0<p<1)$, then by (11)

$$\lim_{n\to\infty}\theta^{(1)} = p^2\left[\frac{12+(3-p)r}{12+(3p-1)r}\right].$$

Hence for large values of n, the rebate is approximately proportional to the square of that part of the time remaining.

5. The Time Required to Pay a Loan. Solving the formula $B=Ra_{\overline{n}|}$ at (r) for n gives

$$n = -\frac{\ln\left(1-\frac{Br}{R}\right)}{\ln(1+r)}.$$

Expressing the value of n as a continued fraction,

$$(12)\quad n=\phi\left[\cfrac{1}{1-\cfrac{(\phi+1)r}{1-\frac{r}{6}(\phi-4)-\frac{r^2}{18}(\phi^2+\phi+1)}\quad\cdots}\right],\quad \text{where}\quad \phi=\frac{B}{R}.$$

The corresponding convergent is

$$(13)\quad n=\phi\frac{\left[1-\frac{r}{6}(\phi-4)-\frac{r^2}{18}(\phi^2+\phi+1)\right]}{1-\frac{r}{6}(4\phi-1)-\frac{r^2}{18}(\phi^2+\phi+1)},\quad \epsilon<\frac{r^3}{18}(3\phi-1)(\phi-1)\cdots$$

The value of n as given by (12) is useful in finding the term of the loan, given the amount of the loan, the payment and the rate. It answers the question, "How long will it take to pay off a mortgage?" This formula will give the time to pay off any given balance after m payments have been made. It should be noted that the cost is obtained by (13) without knowing the term.

Example I (Small Loan). $B=100$, $R=8.884879$, $r=.01$ then $\phi=11.25508$. By using (13), we find $n=11.999964$ with $\epsilon_1=.000036$. Hence $I=Rn-B=106.61822$ $\epsilon=.00032$.

Example II (Mortgage). $B=10{,}000$, $R=127.3194$, $\theta=78.54263$. By using (13), we find $n=99.928$ so that $\epsilon_1=.072$. Hence $I=Rn-B=2722.77$, $\epsilon=9.17$.

6. The Cost for a Loan with Graduated Rates. Suppose we have a loan with a graduated rate of interest. A loan may be graduated in two, three, or more brackets. A two bracket loan might be at the rate of $r\%$ on that part of the loan or loan balance which is L or less, at the rate of $s\%$ on that part of the loan or loan balance which is in excess of L but less than B. This is written as $r\%/s\%/L/B$ [5]. Then

$$I = Rn - B = R\left[-\frac{\ln\left\{1 - \frac{Lr}{R}\right\}}{\ln(1+r)} - \frac{\ln\left\{1 - \frac{(B-L)s}{R-Lr}\right\}}{\ln(1+s)} \right] - B.$$

By comparison with Section 5, we have

$$(14) \qquad I = \frac{Lr}{2}\left[\frac{\phi_1 + 1}{1 - \frac{1}{6}r(4\phi_1 - 1)}\right] + Lr\phi_2 + \frac{R\phi_2 s}{2}\,\frac{(\phi_2 + 1)}{1 - \frac{1}{6}s(4\phi_2 - 1)},$$

where $\phi_1 = (L/R)$ and $\phi_2 = (B-L)/(R-Lr)$. In formula (14) the number of payments is not required.

Example. $3\%/2\%/\$100/\$250$, $R = 35.01$. Then $\phi_1 = (L/R) = 2.8563$, $\phi_2 = (B-L)/(R-Lr) = 4.6860$, so that $I = 30.08$ which is correct to the nearest cent.

For the three bracket case, $r\%/s\%/t\%/L/M/B$.

$$(15) \quad I = \frac{Lr}{2}\,\frac{(\phi_1 + 1)}{1 - \frac{1}{6}r(4\phi_1 - 1)} + \frac{R\phi_2 s}{2}\,\frac{(\phi_2 + 1)}{1 - \frac{1}{6}s(4\phi_2 - 1)} + \frac{R\phi_3 t}{2}\,\frac{(\phi_3 + 1)}{1 - \frac{1}{6}t(4\phi_3 - 1)},$$

where

$$\phi_1 = \frac{L}{R}, \qquad \phi_2 = \frac{M - L}{R - Lr}, \qquad \phi_3 = \frac{B - M}{R - Lr - (M - L)s}.$$

Example. (Three bracket rate). Given $2\frac{1}{2}\%/2\%/1\%/\$100/\$200$, $B = 270$, $R = 18.32$. Then $I = \$79.76$.

Formulas (14) and (15) give the cost of the loan without the necessity of determining the times when the breakpoints occur. This is a distinct advantage in computation. For values of B and R, tables could be compiled by use of an electronic calculator giving the cost and term of the loan.

7. Cost for a Loan with Continuous Interest. In this case [3]

$$B = R(1 - e^{-n\delta})/s,$$

so that

$$\bar{I} = \frac{Bx}{2}\left[\frac{1}{1 - \frac{x}{6} + \frac{x^2}{36} - \frac{x^3}{540} - \frac{x^4}{6480}} \cdots\right], \quad \text{where} \quad x = n\delta$$

$$= \frac{Bx}{2}\left[\cfrac{1}{1-\cfrac{\frac{x}{6}}{1+\cfrac{\frac{x}{6}}{1-\cfrac{\frac{x}{10}}{\cdots}}}}\right].$$

Then

$$\bar{I}^{(1)} = \frac{Bx}{2-(x/3)} \qquad \epsilon_1 < \frac{x^2}{36} + \cdots$$

$$\bar{I}^{(2)} = \frac{Bx}{2}\left(1+\frac{x}{6}\right) \qquad \epsilon_2 < \frac{x^3}{360} + \cdots$$

$$\bar{I}^{(3)} = \frac{Bx}{2}\,\frac{(1+(x/15))}{1-\frac{x}{10}+\frac{x^2}{60}} \qquad \epsilon_2 < \frac{x^4}{3240} + \cdots.$$

In a similar manner as in Section 5, we find

$$\bar{I}^{(3)} = \frac{B\phi}{2}\,\frac{(1-\frac{1}{3}\bar{\phi})}{1-\bar{\phi}+\frac{1}{6}\bar{\phi}^2}, \quad \text{where} \quad \bar{\phi} = \frac{B\delta}{R},$$

and

$$n = \frac{B}{R}\,\frac{1-\frac{1}{2}\bar{\phi}}{1-\bar{\phi}+\frac{1}{6}\bar{\phi}^2}.$$

Example [3]. (Continuous interest loan).

$$B = 1{,}906{,}083.04 \qquad R = 100{,}000 \qquad \delta = .02, \qquad n = 24,$$

then

$$\bar{I}^{(2)} = 494{,}056.72 \qquad \epsilon_2 = -139.78$$

$$\bar{I}^{(3)} = 493{,}909.70 \qquad \epsilon_3 = \quad 7.24$$

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SILHOUETTES AND DEGENERATE LOCI OF QUADRATIC FORMS

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1. In a recently published monograph [2], R. S. Underwood presents some convincing demonstrations of the ease with which various problems in diophantine analysis and higher dimensional geometry may be solved by the methods of extended analytic geometry.

Special types of linear substitutions, termed "linear plotting rules," are used to relate a system of two equations in n variables to a readily derived system involving only two variables. Should one of the latter equations possess a "degenerate locus" the complete solution of the original system can be obtained at once, while equally interesting conclusions concerning the nonexistence of a common real solution may be drawn in cases where the derived system has a pair of "silhouettes."

In the first part of this paper we derive necessary and sufficient conditions for the existence of a silhouette for the general homogeneous quadratic and for certain nonhomogeneous forms. In the second part the question of the existence of a degenerate locus is completely resolved for the nonhomogeneous quadratic; a constructive method is provided for the determination of both this locus and one of the linear plotting rules by which it may be obtained.

2. The following definitions are needed:

(2.1) *A linear plotting rule* (L. P. R.) for $f(x_0, x_1, \cdots, x_n)=0$ is a pair of equations of the form

$$y_i = \sum_{j=0}^{n} c_{ij}x_j, \qquad i = 1, 2,$$

where $(c_{11}, c_{12}, \cdots, c_{1n})$ and $(c_{21}, c_{22}, \cdots, c_{2n})$ are linearly independent, $y_0=x_0=1$, and for each positive subscript j, c_{1j} and c_{2j} are not both zero.

Under these conditions one may eliminate two of the x_j, say x_1 and x_2, from $f(x_0, x_1, \cdots, x_n)$ to obtain $g(y_0, y_1, y_2, x_3, \cdots, x_n)$ which is then regarded as a function of y_1 and y_2 alone.

(2.2) *The locus of* $g(y_0, y_1, y_2, x_3, \cdots, x_n)=0$ is the totality of points (y_1, y_2) obtained by the L. P. R. from all sets $\{x_j\}$ which satisfy $f(x_0, x_1, \cdots, x_n)=0$.

(2.3) A locus is *a silhouette* provided any region of the y_1, y_2-plane (aside from discrete lines, points, and line segments) remains uncovered.

(2.4) A locus is *degenerate* or *a curve-locus* under an L. P. R. if all x_j $(j>0)$ occur vacuously in g; that is, if $g(y_0, y_1, y_2, x_3, \cdots, x_n)=g(y_0, y_1, y_2)$.

Some homogeneous quadratic functions, $Q(x_1, x_2, \cdots, x_n)$, including diagonal forms, may be written in the form

$$\sum_{j=1}^{n} A_j\left(x_j + \sum_{k=1}^{j-1} b_{jk}x_k\right)^2$$

by completing the square on successive x_j. In such cases we say that

(2.5) the jth term is *dominated by* x_j,

(2.6) any term with positive (negative) A_j is *a positive (negative) term*, and

(2.7) the sum of all positive (negative) terms is *the positive (negative) component* of $Q(x_1, x_2, \cdots, x_n)$.

To illustrate some of these concepts let us consider the system

$$(2.8)\qquad \begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, \\ \dfrac{(2x_1 - 3)^2}{4} + \dfrac{(x_2 - 1)^2}{3} + \dfrac{(2x_3 - 3)^2}{4} + \dfrac{(x_4 + 1)^2}{5} = 1. \end{cases}$$

If we choose the L. P. R.,

$$y_1 = x_1 + x_3, \qquad y_2 = x_2 + x_4,$$

the first equation in (2.8) becomes

$$(y_1 - x_3)^2 + (y_2 - x_4)^2 + x_3^2 + x_4^2 = 1$$

whence, completing the square on x_3 and on x_4,

$$(2x_3 - y_1)^2 + (2x_4 - y_2)^2 = 2 - y_1^2 - y_2^2.$$

The left-hand member being nonnegative then implies $y_1^2+y_2^2 \leqq 2$. The second equation of (2.8) is similarly transformed yielding

$$(2.9)\qquad \begin{cases} y_1^2 + y_2^2 \leqq 2, \\ \dfrac{(y_1 - 3)^2}{2} + \dfrac{y_2^2}{8} \leqq 1. \end{cases}$$

While it is probably not obvious whether (2.8) possesses real solutions, it is apparent that the silhouettes of (2.9)—closed elliptic regions—do not intersect. Since no solution (y_1, y_2) of the latter exists we infer that the former has no solution (x_1, x_2, x_3, x_4).

A basic weakness of this method lies in the fact that even when two y-loci overlap the sets of x-coordinates satisfying one equation of the original system need not satisfy the other. This is illustrated by the inconsistent system (2.8) under the L. P. R.,

$$y_1 = x_1, \qquad y_2 = x_2 + x_3 + x_4,$$

which results in

$$(2.10)\qquad \begin{cases} 3y_1^2 + y_2^2 \leqq 3, \\ \dfrac{(2y_1 - 3)^2}{4} + \dfrac{(2y_2 - 3)^2}{36} \leqq 1. \end{cases}$$

The latter has overlapping silhouettes and is satisfied, for example, by (1, 0).

However, all x-coordinates of a degenerate y-locus satisfy the generating x-equation, and some x-coordinates of the other y-locus will satisfy its generic equation. Thus, when one y-locus is degenerate the x-system will have a solution if and only if the y-system does.

3. The question of existence of silhouettes. Fundamentally we shall consider the possibility of finding an L. P. R. whereby either the positive component or the negative component of a homogeneous quadratic form will be dominated by y_1 and y_2, the remaining terms composing a definite form with a zero extremum which, in turn, will yield the boundary of a silhouette. Three preliminary results will be useful:

LEMMA 1. *There exist linear substitutions*

$$y_1 = x_1 + \sum_{j=2}^{n} c_j x_j$$

with $n \geqq 2$ *and no* $c_j = 0$, *which take* $Q = (x_1/a_1)^2 - \sum_{j=2}^{n} (x_j/a_j)^2$ *into* $Ay_1^2 + Q_1$ *where* Q_1 *is of rank* $n-1$ *and independent of* y_1. *Furthermore,* Q_1 *may be made*

(1) *negative definite for any* n;

(2) *positive definite for* $n = 2$.

Proof. With the c_j unspecified initially, we define $S = \sum_{j=2}^{n} (a_j c_j)^2 \neq a_1^2$. Direct substitution of

$$x_1 = y_1 - \sum_{j=2}^{n} c_j x_j, \qquad x_j = u_j - \frac{y_1 a_j c_j}{a_1^2 - S}, \qquad (j = 2, 3, \cdots, n)$$

transforms Q into

$$\frac{y_1^2}{a_1^2 - S} + \frac{1}{a_1^2}\left(\sum_{j=2}^{n} c_j u_j\right)^2 - \sum_{j=2}^{n} (u_j/a_j)^2.$$

With the x_j $(j = 1, 2, \cdots, n)$ independent, the u_j $(j = 2, 3, \cdots, n)$ are independent of y_1. (Note that $x_1, x_2, \cdots, x_n, y_1$ are *not* linearly independent). The nonsingular transformation preserves index and rank so that

(1) by choosing $c_j = a_1/(a_j\sqrt{n})$, $A = 1/(a_1^2 - S)$ is positive whence the index of the form in u_j is zero and the rank $n-1$; and

(2) when $n = 2$, if we take $c_j = a_1\sqrt{2}/a_j$, A will be negative and the form in u_j will have index and rank 1.

Note that for $n > 2$ a choice of c_j making $a_1^2 < S$ would produce an indefinite form in the u_j, independent of y_1.

Reversing signs and inequalities enables us to obtain a dual for each of our results. In this manner we have

LEMMA 2. *There exist linear substitutions* $y_1=\sum_{j=1}^{n-1} c_jx_j+x_n$ *with* $n\geqq 2$ *and no* $c_j=0$, *which take* $Q=\sum_{j=1}^{n-1}(x_j/a_j)^2-(x_n/a_n)^2$ *into* $Ay_1^2+Q_1$, *where* Q_1 *is of rank* $n-1$ *and independent of* y_1. *Furthermore,* Q_1 *may be made*

(1) *positive definite for any* n,

(2) *negative definite for* $n=2$.

Here the result is established by defining $S=\sum_{j=1}^{n-1}(a_jc_j)^2\neq a_n^2$ and substituting $x_n=y_1-\sum_{j=1}^{n-1}c_jx_j$,

$$x_j = u_j + \frac{y_1 a_j^2 c_j}{S - a_n^2}, \qquad (j = 1, 2, \cdots, n-1),$$

to find $A=1/(S-a_n^2)$. Case (1) follows when the c_j are chosen so that $A<0$; Case (2) results by choice of c_j making $A>0$. For completeness we should mention that if the c_j were chosen so that $S=a_1^2$ in Lemma 1, subsequent choice of $x_j=a_j^2c_jk$ $(j=2, 3, \cdots, n)$ and substitution of $x_1=y_1-\sum_{j=2}^{n}c_jx_j$ would reduce Q to $(y_1^2-2y_1Sk)/S$ which is seen to represent any number N for $y_1\neq 0$ by proper choice of k. In other words, such a choice of c_j would produce an indefinite form in the u_j for all but one y_1 and would therefore fail to produce a silhouette. The same statement applies to the form in Lemma 2.

LEMMA 3. *For* $y_1=\sum_{j=1}^{n}c_jx_j$ *and* $S=\sum_{j=1}^{n}(a_jc_j)^2\neq 0$, $Q=\sum_{j=1}^{n}(x_j/a_j)^2$ *has a minimum of* y_1^2/S. *The negative definite form* $-Q$ *has a maximum of* $-y_1^2/S$.

Proof.

$$Q \equiv \sum_{j=1}^{n}\left(\frac{Sx_j - a_j^2c_jy_1}{a_jS}\right)^2 + y_1^2/S \geqq y_1^2/S,$$

equality being possible for $x_j=a_j^2c_jy_1/S$.

4. Some results are now easily obtained. Throughout this section we consider equations $Q(x_1, \cdots, x_n)=C$ with C constant and Q of the form $\sum_{j=1}^{m}(x_j/a_j)^2-\sum_{j=m+1}^{n}(x_j/a_j)^2$, $0\leqq m\leqq n$, $0<n$.

THEOREM 1. *If* $m=0$, *there is an L. P. R. for which* $Q=C$ *has a silhouette.*

Proof. If $n=1$, $-(x_1/a_1)^2=C$ represents a pair of lines which are parallel, coincident, or imaginary as C is negative, zero, or positive.

For $n>1$ let $y_1=x_1$, $y_2=\sum_{j=2}^{n}x_j$. By Lemma 3, $Q\leqq -(y_1/a_1)^2-y_2^2/\sum_{j=2}^{n}a_j^2$ whence $Q=C$ implies that $(y_1/a_1)^2+y_2^2/\sum_{j=2}^{n}a_j^2\leqq -C$. This silhouette consists of the interior and circumference of an ellipse which may be real, a point, or imaginary.

THEOREM 1′. *If* $n=m$, *there is an L. P. R. for which* $Q=C$ *has a silhouette*

Proof. This is merely the dual of Theorem 1.

THEOREM 2. *If $m=1$, there is an L. P. R. for which $Q=C$ has a silhouette.*

Proof. Since $n=1$ is covered by Theorem 1′, we consider $n>1$. Let $y_1=x_1+\sum_{j=2}^{n-1} c_jx_j$, $y_2=x_n$ with the c_j chosen so that $S=\sum_{j=2}^{n-1}(a_jc_j)^2<a_1^2$. By Lemma 1 (1),

$$C = Q \leqq \frac{y_1^2}{a_1^2 - S} - \frac{y_2^2}{a_n^2},$$

which represents a region with a hyperbola (or when $C=0$ a pair of intersecting lines) as boundary. The dual result is:

THEOREM 2′. *If $n-m=1$, there is an L. P. R. for which $Q=C$ has a silhouette.*

THEOREM 3. *If $m=2$ and $n=4$, there is an L. P. R. for which $Q=C\neq 0$ has a silhouette.*

Proof. For $C>0$ we set $y_1=x_1+c_3x_3$, $y_2=x_2+c_4x_4$ with $(a_3c_3)^2<a_1^2$ and $(a_4c_4)^2<a_2^2$ to have, by Lemma 1 (1),

$$C = Q \leqq \frac{y_1^2}{a_1^2 - (a_3c_3)^2} + \frac{y_2^2}{a_2^2 - (a_4c_4)^2}$$

which represents the exterior and circumference of a real, nondegenerate ellipse, thereby providing a silhouette.

For $C<0$, setting $y_1=c_1x_1+x_3$, $y_2=c_2x_2+x_4$ with $(a_1c_1)^2<a_3^2$ and $(a_2c_2)^2<a_4^2$ results in

$$C = Q \geqq \frac{y_1^2}{(a_1c_1)^2 - a_3^2} + \frac{y_2^2}{(a_2c_2)^2 - a_4^2}$$

which by Lemma 2 (1) is a "surrounded ellipse" as above.

If C were zero, both of the boundaries above would be point ellipses so that the (y_1, y_2) values would cover the entire plane; this would not constitute a silhouette. Theorem 3 is self-dual.

THEOREM 4. *If $m=2$ and $n>4$, there is an L. P. R. for which $Q=C>0$ has a silhouette.*

Proof. We take $y_1 = x_1 + c_3x_3$, $y_2 = x_2+\sum_{j=4}^{n} c_j x_j$ with $(a_3c_3)^2< a_1^2$ and $S=\sum_{j=4}^{n}(a_jc_j)^2<a_2^2$. Lemma 1 again gives us a surrounded ellipse,

$$\frac{y_1^2}{a_1^2 - (a_3c_3)_2^2} + \frac{y_2^2}{a^2 - S} \geqq C.$$

The dual result reads,

THEOREM 4′. *If $n-m=2$ and $n>4$, there is an L. P. R. for which $Q=C<0$ has a silhouette.*

5. Now we consider the nonexistence of a silhouette under an L. P. R. The constant terms, c_{10} and c_{20}, serve only to translate the axes. With a nonzero c_{1j}, x_j may be isolated by dividing by c_{1j} and scaling the y_1-axis by setting $y_1' = y_1/c_{1j}$; similarly for y_2. Thus the type of substitution used in the preceding lemmas and theorems is perfectly representative of the general L. P. R.

If m and $n-m$ both exceed 2, it is apparent that no L. P. R. will enable the *pair* of y_1 and y_2 to dominate either the positive component or the negative component of Q since each contains at least *three* independent terms. An indefinite form involving $n-2$ of the original x_j and independent of y_1 and y_2 will thus result under any L. P. R., permitting $Q=C$ to be solved in reals for all (y_1, y_2). The entire y_1, y_2-plane is covered and no silhouette can occur.

For $m=2$, $n=4$, and $C=0$, should y_1 and y_2 dominate the positive component, one would have $f(y_1, y_2)=g(x_3, x_4)$ with f and g both positive definite, and the entire plane would be covered. Similarly, if y_1 and y_2 dominate the negative component, an indefinite form results and the whole-plane locus still occurs.

The cases $m=2$, $n>4$ and its dual are also seen to yield coverings of the entire plane except under the conditions of Theorems 4 and 4'.

6. We know that any homogeneous quadratic may be put into the diagonal form of Section 4 rationally by the method of completing the square on successive x_j [1]. The existence or nonexistence of a silhouette may be determined from this diagonal form and interpreted in terms of the original variables.

Our results may be generalized and summarized in the following

THEOREM 5. *If $Q(x_1, x_2, \cdots, x_n)$ is a homogeneous quadratic form and if C is constant, then there is an L. P. R. for which $Q=C$ has a silhouette if and only if at least one of the following holds:*

(i) *The index of Q is less than 2,*
(ii) *The rank of Q exceeds its index by less than 2,*
(iii) *The index of Q is 2, the rank 4, C is nonzero,*
(iv) *The index of Q is 2 and C is positive,*
(v) *The rank of Q exceeds its index by 2 and C is negative.*

7. Use of the foregoing methods will not yield a complete treatment of the nonhomogeneous quadratic. Although such forms may be made homogeneous by introducing a pseudo-variable, t, we would have no latitude with the term dominated by t. However, if one has the form

$$F = \sum_{j=1}^{m} (x_j/a_j)^2 - \sum_{j=m+1}^{n} (x_j/a_j)^2 + \sum_{j=n+1}^{p} a_j x_j, \qquad 0 \leqq m \leqq n < p,$$

the direct sum of a homogeneous quadratic and a linear expression, several results may be obtained.

THEOREM 6. *If $m=0$, there is an L. P. R. for which $F=C$ has a silhouette.*

Proof. Choose y_1 by Lemma 3 and let $y_2=\sum_{j=n+1}^{p} a_j x_j$. The resulting silhouette consists of the circumference and interior of a parabola. Dually,

THEOREM 6′. *If $n=m$, there is an L. P. R. for which $F=C$ has a silhouette.*

Similarly we let y_2 replace the linear component and y_1 dominate the positive component of the quadratic to obtain:

THEOREM 7. *If $m=1$, there is an L. P. R. for which $F=C$ has a silhouette.*

Dually,

THEOREM 7′. *If $n-m=1$, there is an L. P. R. for which $F=C$ has a silhouette.*

Remembering that the linear terms may be given arbitrary sign, it is not difficult to realize that for all other values of m and $n-m$ an indefinite form must remain under any L. P. R. so that again we summarize in

THEOREM 8. *If F is the direct sum of a homogeneous quadratic Q and a linear form L, then there is an L. P. R. for which $F=C$ has a silhouette if and only if not both:*

(i) *the index of Q exceeds* 1,

(ii) *the rank of Q exceeds its index by* 1.

8. The question of existence of degenerate loci. We now consider the quadratic form $Q(x_0, x_1, \cdots, x_n) = x'Ax$ with $A = (a_{ij}) = A'$ and $x' = (x_0, x_1, \cdots, x_n)$. Let A_0 be the $n+1 \times n$ submatrix obtained by deleting the first column of A. Then,

THEOREM 9. *$Q=0$ has a degenerate locus if and only if the rank of A_0 does not exceed* 2.

Proof of necessity. Let $y=Cx$ be an L. P. R. which takes Q into degenerate g with $y'=(y_0, y_1, y_2)$ and

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_{10} & c_{11} & c_{12} & \cdots & c_{1n} \\ c_{20} & c_{21} & c_{22} & \cdots & c_{2n} \end{pmatrix}$$

let $g=y'By$ with $B=(b_{ij})=B'$. Then $A=C'BC$ may also be written as

$$\left(\begin{array}{c|c} C' & \begin{matrix} 0 \\ \hline I \end{matrix} \end{array}\right)\left(\begin{array}{cc} B & 0 \\ 0 & 0 \end{array}\right)\left(\begin{array}{c|c} \multicolumn{2}{c}{C} \\ \hline 0 & I \end{array}\right)$$

with I the $n-2$ identity and the 0's denoting null matrices. We rewrite this in block form as

$$\begin{pmatrix}1 & c_0' & 0\\ 0 & c_1' & 0\\ 0 & c_2' & I\end{pmatrix}\begin{pmatrix}b_0 & b_1 & 0\\ b_1' & b_2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ c_0 & c_1 & c_2\\ 0 & 0 & I\end{pmatrix}$$

$$= \begin{pmatrix} b_0 + c_0' b_1' + b_1 c_0 + c_0' b_2 c_0 & (b_1 + c_0' b_2)(c_1, c_2) \\ (c_1, c_2)'(b_1' + b_2 c_0) & (c_1, c_2)' b_2 (c_1, c_2) \end{pmatrix}.$$

If A_{00} represents the matrix obtained by deleting the first row and first column of A, then $A_{00} = (c_1, c_2)' b_2 (c_1, c_2)$ and the first row of A_0 is $(a_{01}, a_{02}, \cdots, a_{0n}) = (b_1 + c_0' b_2)(c_1, c_2)$. Obviously the rank of A_{00} cannot exceed 2, and if its rank is 2 then $(a_{01}, a_{02}, \cdots, a_{0n})$ is linearly dependent on the row vectors of A_{00}. Thus the rank of $A_0 \leqq 2$.

Proof of sufficiency. Should A_{00} be of rank zero, Q would be a linear form and $Q=0$ can be partitioned arbitrarily into $y_1+y_2=0$. If A_{00} is of rank 1, $Q=(L_1)^2+L_2+C$ with L_1, L_2 homogeneous linear forms and L_2 possibly proportional to L_1. An appropriate L. P. R. is evident in either case.

We shall consider A_{00} to be of rank 2 and without loss of generality assume that rows a_{1j}, a_{2j} are linearly independent. Then under the assumption that the rank of $A_0 \leqq 2$, there exists $T=(t_{ij})$ of the form

$$\begin{pmatrix}1 & 0 & 0 & 0 & \cdots & 0\\ t_{10} & 1 & 0 & t_{13} & \cdots & t_{1n}\\ t_{20} & 0 & 1 & t_{23} & \cdots & t_{2n}\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}$$

such that $T'AT = B \dot{+} N$ with N the $n-2$ square null matrix and B diagonal. Setting $U=T^I$ and using block forms,

$$A = \begin{pmatrix}1 & u_0' & 0\\ 0 & I & 0\\ 0 & u_2' & I\end{pmatrix}\begin{pmatrix}b_0 & 0 & 0\\ 0 & b_2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ u_0 & I & u_2\\ 0 & 0 & I\end{pmatrix}$$

$$= \begin{pmatrix} b_0 + u_0' b_2 u_0 & u_0' b_2 & u_0' b_2 u_2 \\ b_2 u_0 & b_2 & b_2 u_2 \\ u_2' b_2 u_0 & u_2' b_2 & u_2' b_2 u_2 \end{pmatrix}$$

so that $b_2 = (b_{ij}) = (a_{ij})$ for $i, j = 1, 2$ and

$$b_0 \equiv b_{00} = a_{00} - u_0' b_2 u_0 = a_{00} - (u_0' b_2) b_2^I (b_2 u_0).$$

The desired L. P. R. is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_{10} & c_{11} \cdots c_{1n} \\ c_{20} & c_{21} \cdots c_{2n} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = (u_0, I, u_2) \begin{pmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

or,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = b_2^I \begin{pmatrix} a_{10} & a_{11} \cdots a_{1n} \\ a_{20} & a_{21} \cdots a_{2n} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

9. We shall illustrate the preceding with an example. Let

$$Q = 6 + 2x_1^2 + 6x_1 + x_2^2 - 2x_1x_2 - 4x_2 + 4x_3^2 + 4x_3x_1 + 4x_3 + 13x_4^2 + 8x_4x_3 - 6x_4x_2 + 10x_4x_1 + 16x_4.$$

Q has a degenerate locus under some L. P. R. since

$$(a_{ij}) = \begin{pmatrix} 6 & 3 & -2 & 2 & 8 \\ 3 & 2 & -1 & 2 & 5 \\ -2 & -1 & 1 & 0 & -3 \\ 2 & 2 & 0 & 4 & 4 \\ 8 & 5 & -3 & 4 & 13 \end{pmatrix}$$

meets the rank requirements. We then evaluate in order,

$$(b_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad b_2^I = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad b_{00} = 6 - (3, -2)\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 1,$$

$g(y_1, y_2) = 2y_1^2 - 2y_1y_2 + y_2^2 + 1$, and

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 & -1 & 2 & 5 \\ -2 & -1 & 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + x_1 + 2x_3 + 2x_4 \\ -1 + x_2 + 2x_3 - x_4 \end{pmatrix}$$

is the linear plotting rule in question. As a final check we may verify that

$$Q \equiv 2(1 + x_1 + 2x_3 + 2x_4)^2 - 2(1 + x_1 + 2x_3 + 2x_4)(-1 + x_2 + 2x_3 - x_4) + (-1 + x_2 + 2x_3 - x_4)^2 + 1.$$

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A COMBINATORIAL PROBLEM

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1. Introduction. The following combinatorial problem in plane geometry [1, p. 102, problems 7, 8, and 9] is one of the many intriguing problems due to George Polya. *A convex polygon with n sides is dissected into $n-2$ triangles by $n-3$ diagonals. If D_n denotes the number of different dissections for given n, find a general formula for D_n.*

Now it is easy to see that $D_3=1$, $D_4=2$, $D_5=5$. And it is only slightly more difficult to see that $D_6=14$. For larger values of n, however, the situation becomes much more involved and, at least at first glance, the possibility of finding a general formula must seem remote.

Polya's ingenious solution to this problem makes use of generating functions and depends on the identity

$$D_n = D_2D_{n-1} + D_3D_{n-2} + \cdots + D_{n-1}D_2$$

which is true for all $n \geqq 3$ if we define $D_2=1$. The following solution, discovered by inductive reasoning in the best tradition of Polya, is also of interest because of the remarkable identities involved. We present an expository account of both the inductive and deductive phases of the solution and close by mentioning two additional interesting identities which follow from the fundamental recursive relation on which the solution rests.

2. The inductive approach. We begin by fixing our attention on any one of the vertices of a given convex n-gon which we label V_1. For $3 \leqq r \leqq n$, let $D_{n,n-r}$ denote the number of dissections of the n-gon with precisely $n-r$ diagonals emanating from V_1. Then it is clear that

$$D_n = \sum_{r=3}^{n} D_{n,n-r}. \tag{1}$$

Evaluating this sum for the first few values of n we obtain the following table which exhibits some very interesting properties.

$$
\begin{aligned}
D_3 &= 1 \\
D_4 &= 1 + 1 \\
D_5 &= 1 + 2 + 2 \\
D_6 &= 1 + 3 + 5 + 5 \\
D_7 &= 1 + 4 + 9 + 14 + 14 \\
D_8 &= 1 + 5 + 14 + 28 + 42 + 42 \\
D_9 &= 1 + 6 + 20 + 48 + 90 + 132 + 132 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{2}
$$

It is easy to see that the entries on the right side of each equation are just the partial sums of the entries on the right side of the preceding equation. Thus, we are led to guess that

$$D_{n,n-r} = \sum_{j=3}^{r} D_{n-1,n-1-j} \tag{3}$$

for $n>3$, $3\leqq r\leqq n$ where we set $D_{n,-1}=0$ for $n\geqq 3$. Or equivalently, we guess that

$$D_{n,n-r} = D_{n,n-(r-1)} + D_{n-1,n-1-r} \tag{3'}$$

for $n>3$, $3\leqq r\leqq n$ where we take $D_{n,n-2}=0$ for $n>3$ as well as $D_{n,-1}=0$ for $n\geqq 3$ as above.

Presuming this recursive relation to be true, we now apply the Gregory interpolation formula [2] to find formulas for $D_{n,n-r}$ for the various values of r and hence find the desired formula for D_n. For example, from (2) we have that the successive values for $D_{n,n-5}$ for $n\geqq 5$ are 2, 5, 9, 14, 20, $\cdots$. These give rise to the difference table

$$
\begin{array}{ccccccc}
2 & & 5 & & 9 & 14 & 20 \\
& 3 & & 4 & 5 & & 6 \\
& & 1 & & 1 & 1 &
\end{array}
$$

and hence to the formula

$$
\begin{aligned}
D_{n,n-5} &= 2\binom{n-5}{0} + 3\binom{n-5}{1} + \binom{n-5}{2} \\
&= \frac{(n-1)(n-4)}{2!}.
\end{aligned}
$$

In a similar way we obtain

$$D_{n,n-6} = \frac{n(n-1)(n-5)}{3!}$$

for $n \geqq 6$ and

$$D_{n,n-7} = \frac{(n+1)n(n-1)(n-6)}{4!}$$

for $n \geqq 7$. In general we are led to guess that

$$D_{n,n-r} = \frac{(n+r-6)!(n-r+1)}{(n-2)!(r-3)!} \tag{4}$$

for $n \geqq 3$, $3 \leqq r \leqq n$. Note that this formula is also true for $r = n+1$ in view of our previous definition of $D_{n,-1}$.

Presuming that (4) is true, we have from (1) that

$$D_n = \sum_{r=3}^{n} \frac{(n+r-6)!(n-r+1)}{(n-2)!(r-3)!} \tag{5}$$

for $n \geqq 3$. To evaluate this sum we set

$$S_m = \sum_{r=3}^{m} \frac{(n+r-6)!(n-r+1)}{(n-2)!(r-3)!}$$

and investigate the values of S_m for $3 \leqq m \leqq n$ inductively. Simple calculations show that $S_3 = 1$, $S_4 = n-2$, $S_5 = n(n-3)/2!$, and $S_6 = (n+1)n(n-4)/3!$. Thus, we are led to guess, and can easily prove by induction on m, that

$$S_m = \frac{(n+m-5)!(n-m+2)}{(n-1)!(m-3)!} \tag{6}$$

Since $D_n = S_n$, we see from (6) that

$$D_n = \frac{2(2n-5)!}{(n-1)!(n-3)!} = \frac{1}{n-1}\binom{2n-4}{n-2}. \tag{7}$$

Thus, though we still lack proof, we have obtained the desired formula for D_n.

3. The deductive proof. The preceding work shows that (7) follows from (4) and we now show that (4) can be deduced from (3′).

Clearly all $n-3$ diagonals can emanate from V_1 in only one way. Thus, $D_{n,n-3} = 1$ and (4) is true for $r = 3$ and all $n \geqq 3$. For $r > 3$ the proof proceeds by induction on $n+r$. With $r > 3$ the smallest value of $n+r$ is 8 obtained when $r = n = 4$ and (4) is clearly true in this case. Now assume that (4) is true for all values of n and r such that $n+r = k$ where k is any positive integer not less than 8, $n \geqq 4$, and $3 < r \leqq n$. Let $m+s = k+1$ with $m \geqq 4$, $3 < s \leqq m$. Then by (3′)

$$D_{m,m-s} = D_{m,m-(s-1)} + D_{m-1,m-1-s}.$$

By the inductive assumption

$$D_{m,m-(s-1)} = \frac{(m+s-7)!(m-s+2)}{(m-2)!(s-4)!}$$

and

$$D_{m-1,m-1-s} = \frac{(m+s-7)!(m-s)}{(m-3)!(s-3)!}.$$

Therefore

$$D_{m,m-s} = \frac{(m+s-7)!(m-s+2)}{(m-2)!(s-4)!} + \frac{(m+s-7)!(m-s)}{(m-3)!(s-3)!}$$

$$= \frac{(m+s-6)!(m-s+1)}{(m-2)!(s-3)!}$$

as required and the proof is complete.

Now the proof of (7) will be complete if we can prove the recursive relation in either form (3) or (3′). The proof depends on the following lemma which is interesting in itself.

LEMMA. *For* $n>3$, $3\leqq r\leqq n$

$$D_{n,n-r} = \sum_{s=3}^{r} D_{s-1}D_{n-s+2,n-r-1}, \tag{8}$$

where we set $D_{2,-1}=1$.

Proof. In case $r=n$, the sum on the right reduces to $D_{n-1}D_{2,-1}=D_{n-1}$ and this equals $D_{n,0}$ since all possible dissections of the given n-gon with no diagonals emanating from V_1 can be obtained by joining the two vertices adjacent to V_1 with a diagonal and then dissecting the remaining $(n-1)$-gon in all D_{n-1} ways.

For $3\leqq r<n$, we begin by numbering the vertices of the given n-gon consecutively from 1 to n starting with V_1. Now for $3\leqq s\leqq r$ let $D_{n,n-r,s}$ denote the number of dissections of the n-gon with precisely $n-r$ diagonals emanating from V_1 and where V_s is the smallest numbered vertex to which any of the diagonals is drawn. Then it follows that

$$D_{n,n-r} = \sum_{s=3}^{r} D_{n,n-r,s}.$$

Now the diagonal drawn to V_s divides the given n-gon into an s-gon and an $(n-s+2)$-gon. A dissection of the desired sort will result from dissecting the s-gon in such a way that no diagonal emanates from V_1 and dissecting the $(n-s+2)$-gon in such a way that precisely $n-r-1$ diagonals emanate from V_1.

The first of these operations can be carried out in precisely D_{s-1} ways since in each case a diagonal must join V_2 and V_s and there is no other restriction. And the second operation can clearly be carried out in precisely $D_{n-s+2,n-r-1}$ ways. Thus, $D_{n,n-r,s}=D_{s-1}D_{n-s+2,n-r-1}$ and $D_{n,n-r}=\sum_{s=3}^{r} D_{s-1}D_{n-s+2,n-r-1}$ as claimed.

COROLLARY. *For* $n>3$, $D_{n,1}=D_{n-1}$.

Proof. For $n=4$ the result is immediate from (2). The following argument holds for $n\geqq 5$. We have noted previously that $D_{n,0}=D_{n-1}$. Therefore it follows from the lemma that

$$\begin{aligned} D_{n,1} &= D_{n,n-(n-1)} \\ &= \sum_{s=3}^{n-1} D_{s-1}D_{n-s+2,0} \\ &= \sum_{s=3}^{n-1} D_{s-1}D_{n-s+1} \\ &= \sum_{s=3}^{n-2} D_{s-1}D_{n-s+1} + D_{n-2}D_2 \\ &= \sum_{s=3}^{n-2} D_{s-1} \sum_{r=3}^{n-s+1} D_{n-s+1,n-s+1-r} + D_{n-2}D_2. \end{aligned}$$

Now the last double sum is over all lattice points (s, r) in the triangle bounded by the lines $r=n-s+1$, $r=3$, $s=3$. These same lattice points can be described as those with $s\geqq 3$, $r\geqq 3$ which lie on the lines $s+r=k+3$ with $3\leqq k\leqq n-2$. For fixed k, s ranges between 3 and k and so the above sum can be rewritten to give

$$\begin{aligned} D_{n,1} &= \sum_{k=3}^{n-2} \sum_{s=3}^{k} D_{s-1}D_{n-s+1,n-k-2} + D_{n-2}D_2 \\ &= \sum_{k=3}^{n-2} D_{n-1,n-1-k} + D_{n-1,0} \\ &= D_{n-1} \end{aligned}$$

by (8) and (1).

Proof of (3′). Since $D_{n,n-2}=0$ for $n>3$, the assertion of (3′) for $r=3$ is simply that $D_{n,n-3}=D_{n-1,n-4}$ for all $n>3$. This is trivially true since, as noted above, $D_{n,n-3}=1$ for $n\geqq 3$.

Since $D_{n-1,-1}=0$ for $n>3$, the assertion of (3′) for $r=n$ is simply that $D_{n,0}=D_{n,1}$ for all $n>3$. This follows immediately from the corollary since, as noted earlier, $D_{n,0}=D_{n-1}$.

Finally, we prove (3′) for $n\geqq 5$, $3<r<n$ by induction on $n+r$. Under these restrictions the smallest value for $n+r$ is 9, obtained when $r=4$ and $n=5$. That

(3′) is true in this case is easily seen in (2). Now assume that (3′) is true for all admissible values of n and r such that $n+r=k$, where $k\geqq 9$ is a positive integer. Let $m+t=k+1$ with $m\geqq 5$, $3<t<m$. Then by the lemma

$$\begin{aligned} &D_{m,m-(t-1)} + D_{m-1,m-1-t} \\ &\quad = \sum_{s=3}^{t-1} D_{s-1}D_{m-s+2,m-t} + \sum_{s=3}^{t} D_{s-1}D_{m-s+1,m-t-2} \\ &\quad = \sum_{s=3}^{t-1} D_{s-1}\{D_{m-s+2,m-t} + D_{m-s+1,m-t-2}\} + D_{t-1}D_{m-t+1,m-t-2}. \end{aligned}$$

But

$$\begin{aligned} D_{m-s+2,m-t} + D_{m-s+1,m-t-2} &= D_{m-s+2,m-s+2-(t-s+2)} + D_{m-s+1,m-s+1-(t-s+3)} \\ &= D_{m-s+2,m-s+2-(t-s+3)} \\ &= D_{m-s+2,m-t-1}, \end{aligned}$$

by the induction assumption and

$$D_{m-t+1,m-t-2} = 1 = D_{m-t+2,m-t-1}.$$

Therefore we have that

$$D_{m,m-(t-1)} + D_{m-1,m-1-t} = \sum_{s=3} D_{s-1}D_{m-s+2,m-t-1} = D_{m,m-t},$$

again by the lemma, and the proof is complete.

4. More interesting identities. It follows from the recursive property (3′) that equations (2) can be written in the form

$$\begin{aligned} D_3 &= D_2 \\ D_4 &= D_3 + D_3 \\ D_5 &= (D_4 - D_3) + D_4 + D_4 \\ D_6 &= (D_5 - 2D_4) + (D_5 - D_4) + D_5 + D_5 \\ D_7 &= (D_6 - 3D_5 + D_4) + (D_6 - 2D_5) + (D_6 - D_5) + D_6 + D_6 \\ D_8 &= (D_7 - 4D_6 + 3D_5) + (D_7 - 3D_6 + D_5) + (D_7 - 2D_6) + (D_7 - D_6) \\ &\quad + D_7 + D_7 \\ D_9 &= (D_8 - 5D_7 + 6D_6 - D_5) + (D_8 - 4D_7 + 3D_6) + (D_8 - 3D_7 + D_6) \\ &\quad + (D_8 - 2D_7) + (D_8 - D_7) + D_8 + D_8 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{9}$$

where, for example, $D_{7,3}=D_6-2D_5$ and $D_{7,4}=D_6-3D_5+D_4$. Equations (9) can be simplified to

$$
\begin{aligned}
D_3 &= D_2 \\
D_4 &= 2D_3 \\
D_5 &= 3D_4 - D_3 \\
D_6 &= 4D_5 - 3D_4 \\
D_7 &= 5D_6 - 6D_5 + D_4 \\
D_8 &= 6D_7 - 10D_6 + 4D_5 \\
D_9 &= 7D_8 - 15D_7 + 10D_6 - D_5 \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{10}
$$

The coefficients on the right of equations (10) apparently follow a regular pattern and the Gregory interpolation formula can again be used to obtain appropriate expressions. For example, the coefficients of the terms in the second column yield the difference table

$$
\begin{array}{ccccc}
1 & 3 & 6 & 10 & 15 \\
 & 2 & 3 & 4 & 5 \\
 & & 1 & 1 & 1
\end{array}
$$

and hence the formula

$$\binom{n-5}{0} + 2\binom{n-5}{1} + \binom{n-5}{2} = \binom{n-3}{2}.$$

Computing all of the coefficients in this way one can obtain, for example,

$$D_8 = \binom{6}{1} D_7 - \binom{5}{2} D_6 + \binom{4}{3} D_5.$$

After considering a number of similar examples, it is reasonable to guess that,

$$D_n = \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} \binom{n-1-k}{k} D_{n-k} \qquad \text{for } n \geqq 3, \tag{11}$$

or, if all the terms are put on the left,

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-1-k}{k} D_{n-k} = 0, \tag{12}$$

where the square brackets are used to denote the greatest integer not exceeding $\frac{1}{2}(n-1)$.

Now a look at (10) and (9) reveals that the coefficients of the terms in (10) are the same as the coefficients of the terms in the parentheses on the right of the equations in (9). Thus, it is not difficult to guess from (12) that

$$D_{n,s} = \sum_{j=0}^{[s/2]} (-1)^j \binom{s-j}{j} D_{n-1-j}, \qquad \text{for } n \geqq 3,\ 0 \leqq s \leqq n-3. \tag{13}$$

Now (13) can be proved from (3′) by mathematical induction and (11) can be obtained from (13) as follows. By (1)

$$D_n = \sum_{r=3}^{n} D_{n,n-r} = \sum_{s=0}^{n-3} D_{n,s}.$$

Therefore, using (13),

$$\begin{aligned} D_n &= \sum_{s=0}^{n-3} \sum_{j=0}^{[s/2]} (-1)^j \binom{s-j}{j} D_{n-1-j} \\ &= \sum_{s=0}^{n-3} \sum_{k=1}^{[(s+2)/2]} (-1)^{k+1} \binom{s+1-k}{k-1} D_{n-k} \\ &= \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} D_{n-k} \sum_{s=2k-2}^{n-3} \binom{s+1-k}{k-1} \\ &= \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} D_{n-k} \sum_{i=0}^{n-2k-1} \binom{k-1+i}{k-1} \\ &= \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} \binom{n-1-k}{k} D_{n-k}, \end{aligned}$$

since

$$\sum_{i=0}^{n-2k-1} \binom{k-1+i}{k-1} = \binom{n-1-k}{k}$$

by (7) page 803 of [2].

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A NOTE ON A METHOD OF BRADSHAW FOR TRANSFORMING SLOWLY CONVERGENT SERIES AND CONTINUED FRACTIONS

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1. The purpose of this note is to place in the perspective of a more general inquiry certain methods for transforming slowly convergent series and continued fractions proposed by Bradshaw [1], [2], [11].

2. The series which he treats are of two forms, viz.

(1) $$S = \sum_{n=1}^{\infty} a_n,$$

and

$$S' = \sum_{n=1}^{\infty} (-1)^{n-1} a_n, \tag{2}$$

where a_n is the function

$$a_n = \left(\sum_{s=0}^{\rho} t_s n^s \right)^{-1}. \tag{3}$$

He modifies the series (1) and (2) by the term by term addition of the series

$$-b_0 = \sum_{n=1}^{\infty} (b_n - b_{n-1}) \tag{4}$$

and determines the quantities b_n by imposing the condition that if the magnitude of the successive terms in (1) and (2) behave like $n^{-\beta}$, the terms in the transformed series should behave like $n^{-\beta-\alpha}$, or more precisely

$$1 + \frac{b_n - b_{n-1}}{a_n} = O(n^{-\alpha}). \tag{5}$$

The functions b_n in the case of the series (1) and $(-1)^n b_n$ in the case of (2) are identified as a sequence of rational functions for $\alpha = 0, 1, 2, \cdots$ whose coefficients may be derived from a system of linear equations from those in (3) and which may be established as successive convergents of a continued fraction.

Using his method he derived the expansions

$$\sum_{r=1}^{\infty} r^{-2} = \sum_{r=1}^{n} r^{-2} + \frac{2}{2n+1+} \frac{1}{3(2n+1)+} \frac{16}{5(2n+1)+} \cdots \frac{s^4}{(2s+1)(2n+1)+} \cdots \tag{6}$$

and

$$\sum_{r=1}^{\infty} (-1)^{r-1} r^{-1} = \sum_{r=1}^{n} (-1)^{r-1} r^{-1} + (-1)^n \frac{1}{2n+1+} \frac{1}{2n+1+} \frac{4}{2n+1+} \cdots \frac{s^2}{2n+1+} \cdots . \tag{7}$$

3. Continuing the work of Stieltjes [3], [4] and Airey [5], Bickley and Miller [6], [7] have devised a method for transforming slowly convergent series which is applicable to a larger class of series than those given by (1) and (2) (as, in the event, is the method of Bradshaw).

Defining the converging factor C_n by the relation

$$R_n = u_n C_n, \tag{8}$$

where R_n is the remainder term of the slowly convergent series

$$S = \sum_{r=1}^{\infty} u_r \tag{9}$$

and is given by

$$S = \sum_{r=1}^{r-1} u_r + R_n, \tag{10}$$

they show that if it is possible to determine constants ρ_s, $s=0, 1, \cdots$, such that

$$\frac{u_{n+1}}{u_n} = \rho_0 + \rho_1 n^{-1} + \cdots + \rho_{s-1} n^{-s+1} + O(n^{-s}), \tag{11}$$

then a series

$$C_n \sim \sum_{s=-1}^{+\infty} \alpha_s n^{-s} \tag{12}$$

may be established. Again the coefficients α_s, $s=-1, 0, 1, \cdots$, are derived from a set of linear equations based upon the condition

$$C_n = \sum_{s=-1}^{n-1} \alpha_s n^{-s} + O(n^{-h}). \tag{13}$$

(Actually the formalism differs in the two cases $\rho_0=0$, $\rho_0 \neq 0$.)

Using their method they obtained the expansion

$$\sum_{r=1}^{\infty} r^{-2} = \sum_{r=1}^{n-1} r^{-2} + n^{-2}\left\{n - \frac{1}{2} + \sum_{r=1}^{\infty} (-1)^r B_r n^{-2r+1}\right\} \tag{14}$$

and

$$\sum_{r=1}^{\infty} (-1)^{r-1} r^{-1} = \sum_{r=1}^{n-1} (-1)^{r-1} r^{-1} + (-1)^n \sum_{s=0} (-1)^{s-1} T_s (2n)^{-2s-1}. \tag{15}$$

4. It is possible, by using a variety of methods [8], the most efficient of which is provided by the $q-d$ algorithm [9], uniquely to determine the coefficients of the Stieltjes type continued fraction

$$\frac{c_0}{z-}\,\frac{q_1^{(0)}}{1-}\,\frac{e_1^{(0)}}{z-} - \cdots \frac{q_r^{(0)}}{1-}\,\frac{e_r^{(0)}}{z-} \cdots \tag{16}$$

from those of certain formal series

$$\sum_{s=0}^{\infty} c_s z^{-s-1} \tag{17}$$

by imposing the conditions that

$$\sum_{s=0}^{\infty} c_s z^{-s-1} - \cfrac{c_0}{z - \cfrac{q_1^{(0)}}{1 - \cfrac{e_1^{(0)}}{z - \cdots \cfrac{q_r^{(0)}}{1}}}} = O(z^{-2r-1}) \tag{18}$$

and

$$\sum_{s=0}^{\infty} c_s z^{-s-1} - \cfrac{c_0}{z - \cfrac{q_1^{(0)}}{1 - \cfrac{e_1^{(0)}}{z - \cdots \cfrac{e_r^{(0)}}{z}}}} = O(z^{-2r-2}). \tag{19}$$

The coefficients in the continued fraction

$$\cfrac{c_0}{z - \alpha_0^{(0)} - \cfrac{\beta_0^{(0)}}{z - \alpha_1^{(0)} - \cdots \cfrac{\beta_{r-1}^{(0)}}{z - \alpha_r^{(0)}} \cdots}} \tag{20}$$

which is the even part of (16) may be determined from the relation

$$\sum_{s=0}^{\infty} c_s z^{-s-1} - \cfrac{c_0}{z - \alpha_0^{(0)} -} - \cfrac{\beta_0^{(0)}}{z - \alpha_1^{(0)} -} - \cdots - \cfrac{\beta_{r-1}^{(0)}}{z - \alpha_r^{(0)} -} = O(z^{-2r-1}). \tag{21}$$

The $q-d$ algorithm relationships are

$$q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)}, \qquad q_{r+1}^{(m)} e_r^{(m)} = q_r^{(m+1)} e_r^{(m+1)}, \tag{22}$$

and the starting values are

$$e_0^{(m)} = 0, \qquad q_1^{(m)} = c_{m+1}/c_m, \qquad m = 0, 1, \cdots. \tag{23}$$

The coefficients in (20) are related to those in (16) by

$$\alpha_{r+1}^{(0)} = q_{r+1}^{(0)} + e_r^{(0)}, \qquad \beta_{r-1}^{(0)} = q_r^{(0)} e_r^{(0)}. \tag{24}$$

5. From the preceding remarks it can be seen that Bradshaw's continued fractions may be obtained from the Bickley-Miller expansions by expanding $u_n C_n$ as a series in inverse powers of n and applying the $q-d$ algorithm relationship to the coefficients of this series. In general this procedure is more efficient than that proposed by Bradshaw, for the derivation of each of his rational functions necessitates the solution of a completely independent set of linear equations; the Bickley-Miller method and the $q-d$ algorithm are however recursive procedures in which the coefficients in the approximation of one degree assist in the computation of those in the next.

In particular it can be seen that the expansions (14) and (15) may be derived by expanding the integrals

$$\int_0^{\infty} e^{-nt} \operatorname{sech} t\, dt \tag{25}$$

and

$$\int_0^\infty te^{-(2n+1)t}(\sinh t)^{-1}dt \tag{26}$$

in inverse powers of n, while (6) and (7) may be derived from the Stieltjes [10] expansions

$$\int_0^\infty e^{-zt}\operatorname{sech} t\,dt = \frac{1}{z+}\,\frac{1^2}{z+}\,\frac{2^2}{z+}\cdots\frac{(r-1)^2}{z+}\cdots \tag{27}$$

and

$$\int_0^\infty te^{-zt}(\sinh t)^{-1}dt = \frac{1}{z+}\,\frac{1^4}{3z+}\,\frac{2^4}{5z+}\cdots. \tag{28}$$

6. Bradshaw has also applied his method to the transformation of slowly convergent continued fractions [9]. He modifies the slowly convergent continued fraction

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots \tag{29}$$

by transforming the ρth convergent

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_\rho}{b_\rho} \tag{30}$$

of (29) by the inclusion of a term d_ρ, thus

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_\rho}{d_\rho}, \tag{31}$$

where successive modifying terms d_ρ are determined by imposing the condition that

$$d_{\rho-1} - b_{\rho-1} - \frac{a_\rho}{d_\rho} = O(\rho^{-\alpha}), \qquad \alpha = 1, 2, \cdots. \tag{32}$$

Using this method he transforms the slowly convergent continued fraction

$$1 + \frac{2}{8n-4+}\,\frac{1.3}{8n-4+}\,\frac{3.5}{8n-4+}\cdots\frac{(2\rho-1)(2\rho+1)}{8n-4+}\cdots \tag{33}$$

by means of the modifying continued fraction

$$\mathrm{d}_\rho = 2\left\{\frac{2\rho+4n-1}{2+}\,\frac{(2n-1)2n}{2\rho+1+}\cdots\frac{(2n-2+\sigma)(2n-1+\sigma)}{2\rho+1+}\cdots\right\}. \tag{34}$$

7. Wynn [12] has also proposed a method for the numerical transformation of slowly convergent continued fractions of the form

$$\frac{a_0}{b_0+}\ \frac{c_0}{d_0+}\ \cdots\ \frac{y_0}{z_0+}\ \frac{a_1}{b_1+}\ \cdots\ \frac{y_1}{z_1+}\ \frac{a_2}{b_2+}\ \cdots, \tag{35}$$

where $a_n, b_n, \cdots, z_n$ are polynomials in n. He determines the coefficients in the formal expansion

$$u_n = \sum_{s=-k}^{\infty} \alpha_s n^{-s}, \tag{36}$$

where

$$u_n = \frac{a_n}{b_n+}\ \frac{c_n}{d_n+}\ \cdots\ \frac{y_n}{z_n+}\ \frac{a_{n+1}}{b_{n+1}+}\ \frac{c_{n+1}}{d_{n+1}+}\ \cdots \tag{37}$$

by imposing the condition that

$$u_n - \sum_{s=-k}^{h} \alpha_s n^{-s} = O(n^{-h-1}). \tag{38}$$

He develops a formal recursive procedure based upon use of the difference equation

$$u_n = \frac{a_n}{b_n+}\ \frac{c_n}{d_n+}\ \cdots\ \frac{y_n}{z_n + u_{n+1}}, \tag{39}$$

which is equivalent to

$$p(n)u_n + q(n)u_{n+1} + r(n)u_nu_{n+1} = s(n), \tag{40}$$

where $p(n)$, $q(n)$, $r(n)$, $s(n)$ are again polynomials in n.

8. Again it may be seen that Bradshaw's continued fractions may be derived more efficiently by applying the $q-d$ algorithm to the coefficients of the Wynn expansion.

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MATHEMATICAL NOTES

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A THEOREM ON METRIZABILITY OF MOORE SPACES

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It is a well-established procedure in topology to consider spaces in which certain general properties are assumed to hold only on the boundaries of domains in the space. For instance, J. N. Younglove recently has made use of this procedure to establish metrizability of certain normal spaces [4]. This paper is devoted to proving that each Moore space is metrizable provided the boundary of each domain is completely separable.

The statement that the space S is a Moore space [3] means that S is a regular Hausdorff space with a sequence $G_1, G_2, \cdots$ of collections of open sets, called regions, such that for each positive integer i, G_i covers S, G_{i+1} is a subcollection of G_i, and the star of each point with respect to the G_i's is a basis of neighborhoods for the point.

The statement that a collection of point sets is discrete means that the closures of these point sets are mutually exclusive and any subcollection of these closures has a closed sum.

The statement that the space S is strongly screenable means that if G is an open covering of the space S, then there is a sequence $H_1, H_2, \cdots$ of discrete collections of mutually exclusive open sets such that $\sum H_i$ is a refinement of G which covers S.

The statement that a space S is completely separable means that S has a countable basis.

The statement that the space S is peripherally completely separable means that if B is the boundary of a domain in S and G is a collection of domains covering B, then some countable subcollection of G covers B.

THEOREM. *If S is a peripherally completely separable Moore space then S is strongly screenable, and thus metrizable.*

Proof. Denote by H an open covering of the space S, by K the set to which x belongs if and only if x is the only element of a degenerate region, by L the set to which x belongs if and only if x is a limit point of K, and by M the set to which x belongs if and only if x is a point of $S-(K+L)$. Then no point of M is a limit point of L, else that point is a limit point of K and not even in M. Furthermore, K is a domain and L is the boundary of K. Since S is peripherally completely separable, some countable subcollection H_1 of H covers L.

Now suppose there is an uncountable subset T of M such that T has no limit point. Each point of T is a limit point of M since T contains no degenerate region and no point of M is a limit point of either K or L. So $M-T$ is a domain and T is a subset of the boundary B of $M-T$. But B is completely separable and T is an uncountable subset of B. Hence, T has a limit point. But Jones has proved [2] that if each uncountable subset of a Moore space has a limit point, then that space is a completely separable metric space. Thus M is completely separable. Denote by H' a countable subcollection of H such that H' covers M.

Denote by H_2 the collection to which the region R belongs if and only if the only element of R is a point of $(K-H_1^*\cdot K)$. Clearly, H_2 is a discrete collection of mutually exclusive domains. Denote by F_1, F_2, F_3, $\cdots$ a sequence such that each F_i is a collection whose only element is some domain of H_1+H' and each domain of H_1+H' is the only element of some F_i. This sequence is at most countable. Then the sequence H_2, F_1, F_2, $\cdots$ is a countable sequence such that each of H_2 and each F_i is a discrete collection of domains, each is a refinement of H, and $H_2^*+F_1^*+F_2^*+\cdots$ is S. Thus S is strongly screenable. By a result due to Bing [1], S is metrizable.

COROLLARY 1. *If S is a peripherally completely separable Moore space then $S=S_1+S_2+S_3$ where S_1 is open and completely separable, S_2 is closed and completely separable with each point a limit point of S_3, S_3 contains only degenerate regions, and $S_i\cdot S_j$ does not exist for i different from j.*

Proof. Take S_1 to be M, S_2 to be L, and S_3 to be K of the above theorem.

COROLLARY 2. *If S is a separable, peripherally completely separable Moore space, then S is completely separable.*

Proof. By the theorem, S is metrizable. Since S is a separable, metrizable space, S is completely separable.

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SOME INTERESTING SPECIAL CASES OF THE HYPERGEOMETRIC SERIES

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1. Introduction and Main Results. The hypergeometric series is defined by

$$F(a, b; c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \cdots \tag{1}$$

For our present purposes we shall assume that a, b, c and z are real numbers and shall exclude negative integral values of a, b and c, although it will be evident from later results that some of these restrictions may be removed.

The following properties are known ([1], pages 239, 240, 280).

A. The series (1) converges absolutely for $|z| < 1$ and diverges for $|z| > 1$.

B. $$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} du$$

if $c > b > 0$.

C. $$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}, \qquad 0 < x < 1.$$

D. $$2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2x), \qquad x > 0.$$

In this article we shall obtain some special properties of the hypergeometric series. The main results, valid for $p > 0$ are as follows.

I. $$F(\tfrac{1}{2}, p; 3p; \tfrac{3}{4}) = \left(\frac{16}{27}\right)^p \frac{\Gamma(p)\Gamma(3p)}{\{\Gamma(2p)\}^2},$$

II. $$F(1-p, p; 3p; \tfrac{1}{2}) = \left(\frac{1}{2}\right)^p \frac{\Gamma(p)\Gamma(3p)}{\{\Gamma(2p)\}^2},$$

III. $$F(1-p, \tfrac{1}{2}; 2p+\tfrac{1}{2}; \tfrac{1}{4}) = \frac{2\Gamma(p)\Gamma(2p+\tfrac{1}{2})}{3\Gamma(p+\tfrac{1}{2})\Gamma(2p)},$$

From these it is easy to deduce other results, for example

IV. $$\frac{F(\tfrac{1}{2}, p; 3p; \tfrac{3}{4})}{F(1-p, p; 3p; \tfrac{1}{2})} = \left(\frac{32}{27}\right)^p$$

as well as simple expressions for $F(\tfrac{1}{2}, p; 3p; \tfrac{3}{4})F(\tfrac{1}{2}, 1-p; 3-3p; \tfrac{3}{4})$ and $F(1-p, p; 3p; \tfrac{1}{2})F(p, 1-p; 3-3p; \tfrac{1}{2})$ where $0 < p < 1$, using Property C.

An investigation of the standard literature on the subject, such as [1] and [2] together with references therein, failed to reveal the results I and III. Result II was found, however, to be a special case of equation (51), page 104 in [2]. A study of the classical transformation theory of Goursat ([2] pages 64–68,

92–93, 110–114, and [3]) failed to indicate any obvious intimate connection with results I and III.

2. Proofs of Main Results. To prove I we make use of the identity $\sin 3\theta = \sin\theta(3-4\sin^2\theta)$ to show that

$$\int_0^{\pi} \sin^{2p-1}3\theta d\theta = \frac{1}{3}\int_0^{\pi} \sin^{2p-1}\phi d\phi$$

$$= \frac{1}{3}\frac{\Gamma(\frac{1}{2})\Gamma(p)}{\Gamma(p+\frac{1}{2})} = \int_0^{\pi/3} \sin^{2p-1}\theta(3-4\sin^2\theta)^{2p-1}d\theta$$

Then letting $4\sin^2\theta = 3u$ in this last integral we obtain

$$\int_0^1 u^{p-1}(1-u)^{2p-1}(1-\tfrac{3}{4}u)^{-1/2}du = 2^{2p+1}\sqrt{\pi}\,\Gamma(p). \tag{2}$$

Combining this with Property B gives

$$F(\tfrac{1}{2}, p; 3p; \tfrac{3}{4}) = \frac{\Gamma(3p)}{\Gamma(p)\Gamma(2p)}\int_0^1 u^{p-1}(1-u)^{2p-1}(1-\tfrac{3}{4}u)^{-1/2}du \tag{3}$$

from which I follows after using Property D.

To prove II we proceed similarly using the fact that

$$\int_0^{\pi/4} \sin^{2p-1}4\theta d\theta = \frac{1}{4}\int_0^{\pi} \sin^{2p-1}\phi d\phi$$

$$= \frac{1}{4}\frac{\Gamma(\frac{1}{2})\Gamma(p)}{\Gamma(p+\frac{1}{2})}$$

$$= 2^{4p-2}\int_0^{\pi/4} \sin^{2p-1}\theta\cos^{2p-1}\theta(1-2\sin^2\theta)^{2p-1}d\theta$$

and then letting $2\sin^2\theta = u$.

A proof of III follows in like manner on using the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ in the integral

$$\int_0^{\pi/6} \cos^{2p-1}3\theta d\theta. \tag{4}$$

3. Special Results. As special cases of the above results we mention the following:

$$1 + \frac{1}{8} + \frac{1\cdot 3}{2\cdot 4}\frac{p(p+1)}{3p(3p+1)}\left(\frac{3}{4}\right)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{p(p+1)(p+2)}{3p(3p+1)(3p+2)}\left(\frac{3}{4}\right)^3 + \cdots$$

$$= \left(\frac{16}{27}\right)^p \frac{\Gamma(p)\Gamma(3p)}{\{\Gamma(2p)\}^2}, \tag{5}$$

$$(6)\quad 1-\frac{(n^2-1)}{3n+3}\frac{1}{2^2}+\frac{(n^2-1)(n^2-9)}{(3n+3)(3n+5)}\frac{1}{2!2^4}-\frac{(n^2-1)(n^2-9)(n^2-25)}{(3n+3)(3n+5)(3n+7)}\frac{1}{3!2^6}+\cdots=\frac{\sqrt{\pi}}{2^{(3n+1)/2}}\frac{\Gamma\left(\frac{3n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)\Gamma(n+1)},$$

$$(7)\quad \sum_{k=0}^{\infty}\frac{(3k)!}{k!k!k!}\left(\frac{1}{54}\right)^k=\frac{3\sqrt[3]{4}}{8\pi^2}\{\Gamma(\tfrac{1}{3})\}^3,$$

$$(8)\quad 1+\frac{1}{2^3}+\frac{1\cdot 5}{2!}\frac{1}{2^6}+\frac{1\cdot 5\cdot 9}{3!}\frac{1}{2^9}+\cdots=\sqrt[4]{2}.$$

Result (5) follows from I while (6) follows from II and D on letting $p=(n+1)/2$. Results (7) and (8) are obtained from (6) on letting $n=-\frac{1}{3}$, and $n=-\frac{1}{2}$ respectively. In a similar manner other cases of interest can be obtained.

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LOCALLY RECURRENT FUNCTIONS

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In a recent paper [1] K. A. Bush has exhibited a nonconstant continuous function which is everywhere locally recurrent and which, furthermore, is locally recurrent (l.r.) on each side save on an exceptional set of measure zero. It is the object of this note to show that the exceptional set, while it must have the cardinality of the continuum, may be of zero dimension.

A function $f(x)$ is said to be *locally recurrent (l.r.) on the left* [*right*] at the point x if, for all positive ϵ, there exists $x'\in(x-\epsilon, x)$ $[x'\in(x, x+\epsilon)]$ such that $f(x')=f(x)$. It is *locally recurrent (l.r.)* at x if it is either l.r. on the left or l.r. on the right at x.

We express any real number x in the form

$$(1)\qquad x=m+\sum_{r=1}^{\infty}\frac{x_r}{(r+1)!},$$

where m is any integer and x_r is one of the integers $0, 1, 2, \cdots, r$. This expression is unique except that any rational x has two representations, one with $x_r=r$ and the other with $x_r=0$ for all sufficiently large r.

We now define

$$f(x) = m + \sum_{r=1}^{\infty} \frac{u_r}{2^r}, \tag{2}$$

where each $u_i = 0$ or 1 and the sequence $\{u_i\}$ is defined by the sequence $\{x_i\}$ in the following manner:

$$\begin{aligned} u_k \neq u_{k-1} \quad &\text{if} \quad x_{k-1} = 0, && x_k = 1, \\ &\text{or} \quad x_{k-1} > 0, && x_k = 0, \\ &\text{or} \quad x_{k-1} < k-1, && x_k = k, \\ &\text{or} \quad x_{k-1} = k-1, && x_k = k-1, \quad u_1 = x_1. \end{aligned} \tag{3}$$

In all other cases $u_k = u_{k-1}$. Thus the table corresponding to [1] p. 202, is the following:

TABLE 1

(4)

x_{k-1} \ x_k	0	1	2	$\cdots$	$k-2$	$k-1$	k
0		×					×
1	×						×
2	×						×
⋮	⋮						⋮
$k-2$	×						×
$k-1$	×					×	

The significance of Table 1 is that $u_k \neq u_{k-1}$ if and only if X appears as the entry for the pair (x_{k-1}, x_k). The first and last columns of Table 1 ensure that the two alternative representations of a rational x yield the same value for $f(x)$. Hence $|f(x') - f(x)| < 1/2^{r-3}$ whenever $|x' - x| < 1/r!$, so that $f(x)$ is continuous.

Apart from the specification of the first and last columns, the most important property of Table 1 is that, for large k, each row contains at least two blanks and at least two crosses, so that x_k can be altered without changing u_k. The smallest table satisfying these conditions may be regarded as yielding the simplest continuous l.r. function $f_0(x)$. Thus for the definition of $f_0(x)$ we replace (1), (2) by

$$x = m + \sum 4^{-r} x_r, \qquad f_0(x) = \sum 2^{-r} u_r$$

with $u_1 = 1$, and then we use a table like Table 1. It will be found, however,

that the exceptional set for $f_0(x)$ is of dimension $\frac{1}{2}$, and we now return to the study of $f(x)$.

Since $f(m)=m$ it follows that $f(x)$ is nonconstant, and indeed takes every finite value. If $x=p/q$ is rational and $r\geqq q+2$, we find that each of $f(x\pm 2\epsilon)$ is equal to $f(x)$, where $\epsilon=\sum_{j>r}(j!)^{-1}<r^{-1}$. Hence $f(x)$ is certainly both left-l.r. and right-l.r. at all rational points, which are henceforth omitted from consideration.

Next suppose that x is such that there is an infinity of values of r for which $x_r\geqq 2$. Choose such a value of $r\geqq 3$ and define x' (see (1)) by taking $m'=m$, $x_i'=x_i$ for $i<r$ and, for all $j\geqq r$, either $x_j'=u_j$ or $x_j'=1-u_j$, whichever makes $u_r'=u_r$ (that is, we take the former alternative if $x_{r-1}=u_{r-1}=0$ or $x_{r-1}u_{r-1}>0$, and the latter otherwise). Then a glance at the entries for $x_i=0$ or 1 shows that $f(x')=f(x)$. On the other hand, since $x_r'\leqq 1<x_r$ and $x_{r+1}'\leqq 1<r+1$, we have $x-(r!)^{-1}\leqq x'<x$ and, since r can be taken arbitrarily large, it follows that the function $f(x)$ is left-l.r. at the point x.

Now define $E_n=\{\text{irrational } x: x_i\leqq 1 \text{ for } i\geqq n\}$, $E'=\bigcup_1^\infty E_n$. Then we have shown that $f(x)$ is left-l.r. except possibly on E'. Similarly, since $f(x)$ is an odd function of x, $f(x)$ is right-l.r. except possibly on the set $E''=\bigcup_1^\infty \{\text{irrational } x: x_i\geqq i-1 \text{ for } i\geqq n\}$. The sets E' and E'' being disjoint, it follows that $f(x)$ is l.r. everywhere and l.r. on each side save on the exceptional set $E=E'\cup E''$ whose dimension is that of E', where we may confine attention to the interval $0\leqq x\leqq 1$.

Let k and ϵ be arbitrary positive constants and for each value of n choose $N=N(k,\epsilon,n)$ such that $N^k>8$, $2N>n$ and $2^N>n!/\epsilon$. Fixing the values of $x_1, x_2, \cdots, x_{2N-1}$ restricts x to an interval of length $1/(2N)!<N^{-N}$ and E_n is contained in a set of $2^{2N-n}n!$ such intervals, so that the k-dimensional outer measure $\mu_k(E_n)$ of E_n satisfies the inequality $\mu_k(E_n)\leqq 2^{2N-n}n![(2N)!]^{-k}<2^{-n}n!(4/N^k)^N<2^{-N-n}n!<\epsilon/2^n$. Hence $\mu_k(E')<\epsilon$ for all $\epsilon>0$, and so $\mu_k(E')=0$. This shows that $\dim(E')\leqq k$ for all $k>0$ and completes the proof of the assertion that $\dim(E)=\dim(E')=0$.

On the other hand, let $g(x)$ be any nonconstant continuous function and choose $a<b$ such that $g(a)\neq g(b)$. Define $x(\eta)=\min\{x: a\leqq x\leqq b, g(x)=\eta\}$. Then, by continuity, $x(\eta)$ is well-defined for all real η lying strictly between $g(a)$ and $g(b)$ and, by definition, $g(x)$ cannot be locally recurrent on the left at $x(\eta)$, so that $x(\eta)\in E'(g)\subseteq E(g)$. But clearly different values of $\eta\in(g(a), g(b))$ yield different values of $x(\eta)\in E'(g)$, so that $E'(g)$ (and *a fortiori* $E(g)$) has the cardinality of the continuum.

Since the first two columns of Table 1 are complementary, it follows that, unless $x_k=0$, we may always decrease x_k in such a way as to alter u_k. We therefore have:

$$(5)\quad \liminf_{y\to x-}\left|\frac{f(y)-f(x)}{y-x}\right|=0,\quad x\notin E';\quad \limsup_{y\to x-}\left|\frac{f(y)-f(x)}{y-x}\right|=\infty,\quad \text{for all } x.$$

I do not know whether a continuous function can satisfy (5) for all x, but my guess is that it can. Such a function, if odd or even, could have no unilateral

derivative and, according to [2], an even function with no unilateral derivative exists which satisfies (5) except on a set of arbitrarily small positive measure. Our exceptional set, of course, is much smaller than this but, on the exceptional set, $f(x)$ has an infinite unilateral derivative. Both $f(x)$ and $f_0(x)$ satisfy almost everywhere the conditions $f_-(x)=f_+(x)=-\infty$, $f^-(x)=f^+(x)=+\infty$.

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OVALS AND FINITE BOLYAI-LOBACHEVSKY PLANES*

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L. M. Graves [2] has proposed a set of postulates for finite Bolyai-Lobachevsky planes. Except for nontriviality conditions a finite B-L plane is a set $\mathfrak{P}$ of points together with certain specified subsets called lines such that: (I) Two points determine one and only one line; (II) Through each point X not on the line p there pass at least two lines not meeting p; (III) If a subset of $\mathfrak{P}$ contains three points not on a line and contains all of the lines through any pair of points, then that subset contains all points of $\mathfrak{P}$.

Graves gives an example and suggests that it would be interesting to know additional constructions for such planes. In this note, we point out that one of the classical models of the Bolyai-Lobachevsky plane satisfies the Graves postulates in the finite case.

A projective plane π is said to be of order n if each line contains exactly $n+1$ points. Let us restrict ourselves to the case where n is odd. A set $\mathfrak{C}$ of $n+1$ points is said to be an *oval* if no three points of $\mathfrak{C}$ are collinear. A line which contains exactly one point of $\mathfrak{C}$ is said to be a *tangent line.* A line which contains two points of $\mathfrak{C}$ is called a *secant line*; a line which contains no points of $\mathfrak{C}$ is called an *exterior line.*

Qvist [4] has shown that the points of π occur in three disjoint classes: (1) The points in $\mathfrak{C}$, all of which lie on exactly one tangent line (2) The *exterior points*, which lie on exactly two tangent lines and (3) The *interior points*, which lie on no tangent lines. It readily follows that: (a) There are $\frac{1}{2}n(n-1)$ interior points; (b) No tangent line contains any interior points; (c) Each secant line contains $\frac{1}{2}(n-1)$ interior points; and (d) Exterior lines contain exactly $\frac{1}{2}(n+1)$ interior points [3].

Now let $\mathfrak{P}$ be the set of interor points of $\mathfrak{C}$. If we take the B-L lines to be the set theoretic intersections of the (exterior and secant) lines of π with $\mathfrak{P}$, it is immediate that conditions I and II are satisfied. We shall now show that III is satisfied if n is odd and greater than 7.

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Suppose that the set $\mathfrak{S} \subset \mathfrak{P}$ contains three noncollinear points P, Q, R of $\mathfrak{P}$ and, therefore, all of the points on the sides of the triangle PQR. Since the line PQ contains at least $\frac{1}{2}(n-1)$ points of $\mathfrak{P}$, $\mathfrak{S}$ includes at least $\frac{1}{2}(n-1)$ lines through R. Since each of these lines contains at least $\frac{1}{2}(n-1)$ points, $\mathfrak{S}$ contains at least $\frac{1}{2}(n-1)[\frac{1}{2}(n-1)-1]+1$ points. If $n>7$, this last quantity is greater than $n+1$. But each point of $\mathfrak{P}$ lies on at most $n+1$ lines. Hence every point of $\mathfrak{P}$ lies on a line which contains two points of $\mathfrak{S}$. This implies that every point of $\mathfrak{P}$ belongs to $\mathfrak{S}$.

Graves calls a B-L plane *homogeneous* if the collineation group is transitive on the points. Now every collineation of π which carries $\mathfrak{C}$ into itself induces a collineation of the B-L plane. Hence we shall have a homogeneous plane if the collineation group of π which fixes $\mathfrak{C}$ is transitive on interior points. We shall show that our example is homogeneous if π is Desarguesian and $n \not\equiv 1 \bmod 8$.

Segre [5] has shown that if π is the Desarguesian projective plane coordinatised by a field, then every oval $\mathfrak{C}$ is a conic—i.e. is the set of points whose coordinates satisfy a nondegenerate equation of the second degree.

The group of projective collineations which fix an oval $\mathfrak{C}$ in a Desarguesian projective plane is thus isomorphic to the orthogonal group and is of order $(n+1)n(n-1)$ [1]. Let us call this group Ω and let us investigate the subgroups of Ω which fix certain elements of π.

Note that if an element of Ω fixes three points on a line l, then it fixes every point on l and is a perspective collineation with axis l whose center is the pole of l. If an element of Ω fixes a secant line l (pointwise or not) then the two points on $l \cap \mathfrak{C}$ must be either fixed or interchanged. If a line is fixed pointwise, consideration of the other fixed lines shows that the collineation is an involution. In particular if an element ω of Ω fixes two points A and B of $\mathfrak{C}$ and one other point of the line AB, then ω is an involution with axis AB or is the identity.

Now let l be a secant line containing the interior point D and the points A and B of $\mathfrak{C}$. Let E be the pole of AB, let m be the polar of D, and let $m \cap l = F$. Let ω be any element of Ω which fixes l and D, interchanges A with B. Then either ω^2 is the identity or ω^2 is an involution which fixes l pointwise.

Now if F is an exterior point, its polar ED is a secant line. Since the line ED and the point D are fixed, ω must fix or interchange the two points in $ED \cap \mathfrak{C}$. In either case ω^2 fixes the four non-collinear points A, B, and the two points in $ED \cap \mathfrak{C}$. That is, ω^2 is the identity. Now every collineation which fixes D and l must also fix E and F. Moreover, for n odd, every projective collineation of order two in a finite plane is a perspectivity. We conclude that if F is exterior, the subgroup of Ω which fixes D and l is of order four, consisting of the identity and the three involutions with respective centers D, E, and F.

Next, suppose that F is interior and that ω is of order four. Then ω must fix F and permute the remaining $\frac{1}{2}(n+1)-1=\frac{1}{2}(n-1)$ interior points of EF in cycles of length 4. [The pole of a secant line is necessarily exterior, so that E is exterior.] It follows that $n-1 \equiv 0 \bmod 8$. If this is not the case, we again conclude that the subgroup of Ω fixing D and l is at most of order 4.

Now every interior point D lies on $\frac{1}{2}(n+1)$ secant lines; the image of a secant line must be a secant line. We conclude that subgroup of Ω fixing D is at most of order $2(n+1)$ unless $n \equiv 1 \bmod 8$. Since Ω is of order $(n+1)n(n-1)$, D must occur in a transitive class at least of length $\frac{1}{2}n(n-1)$. But the image of an interior point is an interior point and there are $\frac{1}{2}n(n-1)$ interior points. Hence all interior points are in the same transitive class under Ω. Hence the corresponding B-L plane is homogeneous unless $n \equiv 1 \bmod 8$.

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ON THE ZEROS OF POLYNOMIALS

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1. Rahman [4, Theorem 1] has proved the following theorem.

Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial with real or complex coefficients such that for* $n \geqq 2$

$$|a_0| + |a_1| + \cdots + |a_{n-1}| \leqq n|a_n|, \tag{1}$$

$$2^{(n+1)/2} r^n (n+1) \left|\frac{a_n}{a_0}\right| < 1, \tag{2}$$

$$r \geqq \tfrac{1}{2}. \tag{3}$$

Then the least number of zeros of $p(z)$ *lying outside the circle* $|z| = r$ *is* $(n+1)/2$ *or* $(n/2)+1$ *according as* n *is an odd or an even positive integer.*

We prove

THEOREM 1. *If*

$$|a_1| + |a_2| + \cdots + |a_n| < (n-k)|a_0|, \qquad k = 0, 1, \cdots, n-2, \tag{4}$$

then the polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ *has at least* $k+1$ *zeros in* $|z| > \frac{1}{2}$.

If $|z| = 1$ then

$$\left|\frac{p(z)}{a_0}\right| \leqq \frac{|a_n| + |a_{n-1}| + \cdots + |a_1|}{|a_0|} + 1 < n - k + 1,$$

by (4).

$$= \alpha_p \left(\sum_{r=1}^{n} \frac{1}{|z|^{q(r+1)}} \right)^{1/q}.$$

Hence for $|z|>1$,

$$|P(z)| > |a_n||z|^{n+1}\left\{1 - \frac{\alpha_p}{[|z|^q(|z|^q-1)]^{1/q}}\right\} > 0,$$

if $|z|^{2q}-|z|^q-\alpha_p^q \geqq 0$, i.e. if $|z| \geqq [\frac{1}{2}\{1+(1+4\alpha_p^q)^{1/2}\}]^{1/q}$. As before we can now prove that all the zeros of $P(z)$ and also those of $p(z)$ lie in

$$|z| < [\tfrac{1}{2}\{1 + (1 + 4\alpha_p^q)^{1/2}\}]^{1/q}.$$

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ON THE nth DERIVATIVE OF THE INVERSE FUNCTION

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Ostrowski ([1, Appendix C], [2]) has given an inductive proof of an explicit formula for the nth derivative of the inverse function. His subsidiary conditions are derived from a certain recurrence formula. We will give a direct derivation of an explicit formula which we will show to be identical with that of Ostrowski. There are fewer subsidiary equations in our formulation, thus simplifying the calculation of these formulas whether done by hand or on a computer. Our proof is based on a theorem of Jabotinsky [3, Theorem 2]. We will indicate how Ostrowski's subsidiary conditions follow immediately from considerations of dimensional analysis. Finally, we will use our results to give a formal expression for a simple zero of a function possessing a Taylor series expansion. See also [4], [5].

Let $y=f(x)$ possess a Taylor series expansion with f' nonzero and let $x=g(y)$ denote the inverse of $f(x)$. Define $f_{m,n}$ by

$$[f(x)]^m = \sum_{n=-\infty}^{n=+\infty} f_{m,n}x^n,$$

where m is a positive or negative integer. Define $g_{m,n}$ analogously. Then Jabotinsky's theorem states that

$$g_{m,n} = \frac{m}{n} f_{-n,-m} \qquad \text{for } n \neq 0. \tag{1}$$

Since $g^{(n)}/n!=g_{1,n}$, we have

$$g^{(n)} = (n-1)!f_{-n,-1}, \tag{2}$$

where $f_{-n,-1}$ denotes the coefficient of $1/x$ in the expansion of y^{-n}. For later convenience we define

$$a_i = f^{(i)}/i!, \qquad A_i = a_i/a_1. \tag{3}$$

Then

$$y = f(x) = \sum_{i=1}^{\infty} a_i x^i, \qquad a_1 \neq 0.$$

Let $v=y/a_1x-1$. Then

$$\begin{aligned} y^{-n} &= (a_1x)^{-n}(1+v)^{-n} \\ &= (a_1x)^{-n}\left[1 - nv + \cdots + \frac{(-1)^r}{r!}n(n+1)\cdots(n+r-1)v^r + \cdots\right]. \end{aligned} \tag{4}$$

From the multinomial theorem we have

$$v^r = \sum r! \prod_{i=2} \frac{(A_i)^{j_i}}{j_i!} x^T, \qquad T = \sum_{i=2} (i-1)j_i, \tag{5}$$

with the sum taken over all terms such that

$$\sum_{i=2} j_i = r, \qquad j_i \geqq 0. \tag{6}$$

From (2), (3), (4), (5), (6), we conclude that

$$g^{(n)} = (f')^{-n} \sum (-1)^r (n+r-1)! \prod_{i=2}^{n} \frac{(A_i)^{j_i}}{j_i!} \tag{7}$$

with the sum taken over all j_i such that

$$\sum_{i=2}^{n} (i-1)j_i = n-1, \qquad j_i \geqq 0, \tag{8}$$

and where $r=\sum_{i=2}^{n} j_i$. For $n=1$, take $j_i=0$ for all i. The constraints (8) stem from the fact that we are interested in the coefficient of $1/x$.

To show that (7) and (8) are equivalent to Ostrowski's formulas, we define $\beta_i=j_i$, $i=2, 3, \cdots, n$, and $\beta_1=n+r$. Then it is easy to show that

$$g^{(n)} = \sum \frac{(-1)^{\beta_1+n}(\beta_1-1)!}{\prod_{i=2}^{n} (\beta_i)!(i!)^{\beta_i}} (f')^{-\beta_1} \prod_{i=2}^{n} (f^{(i)})^{\beta_i} \tag{9}$$

with the β_i subject to the constraints

$$(10)\qquad \beta_1 \leqq 2n-1, \qquad \beta_i \geqq 0, \qquad \sum_{i=2}^{n} \beta_i = \beta_1 - n, \qquad \sum_{i=2}^{n} i\beta_i = \beta_1 - 1.$$

This is the formula given by Ostrowski [2, Eqs. (C5–C7), pg. 142]. For example, to show that $\beta_1 \leqq 2n-1$, we note that

$$r = \sum_{i=2}^{n} j_i \leqq \sum_{i=2}^{n} (i-1)j_i = n-1.$$

$$\therefore\ \beta_1 = r + n \leqq 2n - 1.$$

We can get some feeling for the constraints (10) from considerations of dimensional analysis. With x we associate the dimension P while with y we associate the dimension R. We write this as $\dim(x) = P$, $\dim(y) = R$. Clearly, $\dim(g(y)) = P$, $\dim(g^{(j)}(y)) = PR^{-j}$, $\dim(f^{(j)}(x)) = RP^{-j}$.

We note that

$$g^{(j+1)}(y) = \frac{1}{f'(x)} \frac{d}{dx} (g^{(j)}(y))$$

with $g'(y) = 1/f'(x)$. It seems clear therefore that $g^{(n)}$ must be of the form

$$g^{(n)} = \sum a_{\beta_1\beta_2\cdots\beta_n} (f')^{-\beta_1} \prod_{i=2}^{n} (f^{(i)})^{\beta_i}$$

with $\beta_i \geqq 0$. We have

$$\begin{aligned} \dim(g^{(n)}) = PR^{-n} &= \dim\left((f')^{-\beta_1} \prod_{i=2}^{n} (f^{(i)})^{\beta_i}\right) \\ &= (RP^{-1})^{-\beta_1} \prod_{i=2}^{n} (RP^{-i})^{\beta_i} \\ &= R^V P^W, \end{aligned}$$

with

$$V = \left(-\beta_1 + \sum_{i=2}^{n} \beta_i\right) \quad \text{and} \quad W = \left(\beta_1 - \sum_{i=2}^{n} i\beta_i\right).$$

Therefore

$$-n = -\beta_1 + \sum_{i=2}^{n} \beta_i, \qquad 1 = \beta_1 - \sum_{i=2}^{n} i\beta_i.$$

As an application of our explicit formula for $g^{(n)}$ we will write down a formal expression for a simple zero of $f(x)$ if $f(x)$ possesses a Taylor series expansion. Letting x be an approximation to α, we have

$$\alpha = g(0) = g(y - y) = x + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} f^k(x) g^{(k)}(y).$$

Substituting (7) yields

$$\alpha = x - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} u^k(x) \sum (-1)^r (k+r-1)! \prod_{i=2}^{k} \frac{(A_i)^{j_i}}{j_i!}, \tag{11}$$

where $u(x)=f(x)/f'(x)$, $r=\sum_{i=2}^{k} j_i$ and where the inner sum is taken over all j_i, such that $\sum_{i=2}^{k}(i-1)j_i=k-1$.

If $f(x)$ possesses only a finite number of continuous derivatives, (11) may be replaced by a finite sum which approximates the zero. Observe that retention of only the first two terms of (11) yields Newton's iteration function $\Phi(x)=x-u(x)$, while retention of the first three terms yields the often rediscovered iteration function $\Phi(x)=x-u(x)-u^2(x)A_2(x)$. For additional details as well as a discussion of the modification of (11) for the case of multiple zeros, the reader is referred to [5].

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AN EXTENSION OF CAUCHY'S GENERALIZED LAW OF THE MEAN

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The purpose of this note is to extend Cauchy's generalized law of the mean to functions which have finite derivates. Theorems of Karamata [2] and Vučković [4] as corrected by Utz [3] are special cases of our result.

DEFINITION. *Let the function f be defined for $\xi \leqq x < \xi + h$. If there exists a sequence $b_1, b_2, b_3, \cdots$ of positive numbers tending to 0 and less than h such that*

$$\lim_{n\to\infty} \frac{f(\xi + b_n) - f(\xi)}{b_n}$$

exists, this limit is called a right derivate of f at ξ and is denoted by $D_r f(\xi)$. Left derivates are analogously defined and denoted.

If the function f is defined on the open interval $a<x<b$ and if ξ is a point such that $a<\xi<b$, we shall say that f has bounded derivates at ξ, provided the set of all derivates of f at ξ is bounded. Furthermore, we shall say that f has

bounded derivates on $a<x<b$, provided it has bounded derivates at each point of the interval.

THEOREM. *If f and g are functions which are continuous at each point of the finite closed interval $a\leqq x\leqq b$ and which have bounded derivates at each point of the open interval $a<x<b$, then there exist numbers ξ, p, q with $a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$, and derivates $D_rf(\xi)$, $D_lf(\xi)$, $D_rg(\xi)$, $D_lg(\xi)$ for which*

$$[pD_rf(\xi)+qD_lf(\xi)][g(b)-g(a)]=[pD_rg(\xi)+qD_lg(\xi)][f(b)-f(a)].$$

Proof. Let $a\leqq x\leqq x'\leqq b$, let I be the closed interval $[x, x']$, let $\phi(I)=f(x')-f(x)$ and let $\sigma(I)=g(x')-g(x)$. Then ϕ and σ are additive continuous functions of an interval on the closed interval $I_0=[a, b]$. Hence [1, p. 2] there exist a point ξ and a descending sequence of intervals $\{I_n\}_{n\in\omega}=\{[a_n, b_n]\}_{n\in\omega}$, where ω is the set of positive integers, with the following properties:

(1) $I_0\supset I_1\supset I_2\supset\cdots\supset I_n\supset\cdots$,

(2) $\lim_{n\to\infty}(b_n-a_n)=0$,

(3) ξ is an interior point of each I_n,

(4) $[f(b_n)-f(a_n)][g(b)-g(a)]=[g(b_n)-g(a_n)][f(b)-f(a)]$ for each n.

Since $b_n-a_n\neq 0$ we may divide each side of (4) by b_n-a_n and rearrange to obtain

$$\left[\frac{f(b_n)-f(\xi)}{b_n-\xi}\,\frac{b_n-\xi}{b_n-a_n}+\frac{f(\xi)-f(a_n)}{\xi-a_n}\,\frac{\xi-a_n}{b_n-a_n}\right][g(b)-g(a)]$$

(A)

$$=\left[\frac{g(b_n)-g(\xi)}{b_n-\xi}\,\frac{b_n-\xi}{b_n-a_n}+\frac{g(\xi)-g(a_n)}{\xi-a_n}\,\frac{\xi-a_n}{b_n-a_n}\right][f(b)-f(a)].$$

In the sequel symbols ω will denote infinite subsets of ω_k. Since

$$0<(b_n-\xi)/(b_n-a_n)<1,$$

there exist a number p, a number q, and a subsequence $\{I_n\}_{n\in\omega_1}$ of the sequence $\{I_n\}_{n\in\omega}$, such that

$$\lim_{(n\in\omega_1)\to\infty}(b_n-\xi)/(b_n-a_n)=p$$

and

$$\lim_{(n\in\omega_1)\to\infty}(\xi-a_n)/(b_n-a_n)=q.$$

We note that p and q are nonnegative and that $p+q=1$.

Since f has bounded derivates at ξ there exist a subsequence $\{I_n\}_{n\in\omega_2}$ of the subsequence $\{I_n\}_{n\in\omega_1}$, and a right derivate $D_rf(\xi)$ of f at ξ, such that

$$\lim_{(n\in\omega_2)\to\infty}[f(b_n)-f(\xi)]/[b_n-\xi]=D_rf(\xi).$$

Continuing in an analogous manner we can show there exist subsequences $\{I_n\}_{n\in\omega_3}$, $\{I_n\}_{n\in\omega_4}$, and $\{I_n\}_{n\in\omega_5}$ of $\{I_n\}_{n\in\omega}$ such that $\omega_2\supset\omega_3\supset\omega_4\supset\omega_5$,

$$\lim_{(n\in\omega_3)\to\infty}[f(\xi)-f(a_n)]/[\xi-a_n]=D_lf(\xi),$$

$$\lim_{(n\in\omega_4)\to\infty}[g(b_n)-g(\xi)]/[b_n-\xi]=D_rg(\xi),$$

$$\lim_{(n\in\omega_5)\to\infty}[g(\xi)-g(a_n)]/[\xi-a_n]=D_lg(\xi).$$

Hence, letting $n\in\omega_5$ and taking the limit of both sides of (A) as $n\to\infty$ gives

$$[pD_rf(\xi)+qD_lf(\xi)][g(b)-g(a)]=[pD_rg(\xi)+qD_lg(\xi)][f(b)-f(a)],$$

with $p\geqq 0$, $q\geqq 0$, and $p+q=1$. This completes the proof.

References

1. S. Kametani, A new formulation of mean value theorem, Nat. Sci. Rep., Ochanomizu Univ., 1 (1951) 1–5.

2. J. Karamata, Sur la formule des accroissements finis, Srpska Akad. Nauka. Zb. Rad., vol. 7, Matematički Inst., Serbo-Croatian, French Summary, 1 (1951) 119–124.

3. W. R. Utz, Mean value theorems for functions with right and left derivatives, this MONTHLY, 62 (1955) 178–179.

4. V. Vučković, Quelques extensions des théorèmes de moyenne, Srpska Akad. Nauka. Zb. Rad., vol. 18, Matematički Inst., Serbo-Croatian, French Summary, 2 (1952) 159–166.

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

THE ZEROS OF A FUNCTION $\sum_{i=0}^{n}a_iz^{\alpha_i}$

W. D. BOUWSMA, Pennsylvania State University

If $0\leq\alpha_0<\cdots<\alpha_n$ are positive integers and $a_n\neq 0$, it is a simple corollary of the fundamental theorem of algebra that the function

$$f(z)=\sum_{i=0}^{n}a_iz^{\alpha_i} \tag{1}$$

has exactly α_n zeros in the complex plane. Now when the α's are not integers, the concept of the number of zeros of such a function may be extended in a very

THE EQUIVALENCE OF TWO MAXIMALITY PRINCIPLES

ALEXANDER R. BEDNAREK, University of Buffalo†

If R is an arbitrary binary relation over the nonempty set X, a subset of X is called *R-coherent* if and only if for every pair of distinct elements x, $y \in X$, x and y are R-comparable; that is, $x\,R\,y$ or $y\,R\,x$. In his paper Wallace [1] showed that the proposition

(W) Any R-coherent subset of X is contained in a maximal R-coherent subset of X.

is equivalent to the axiom of choice. The equivalence of (W) and the proposition

(M) There exists a maximal R-coherent subset of X.

is contained implicitly in Wallace's argument. A recent classroom discussion stimulated a search for a brief set-theoretic demonstration of this equivalence. In particular, we wished to avoid reference to the concept of partial order. Such a demonstration follows.

THEOREM. (M) *is equivalent to* (W).

Proof. Since $\{x\}$, for any $x \in X$, is an R-coherent set, it follows immediately that (W) implies (M).

Let C be any R-coherent subset of X. Define X_C as follows:

$$X_C = \{y \mid y \in X - C, \text{ and } y, x \text{ are R-comparable for every } x \in C\}.$$

If X_C is vacuous, then C is a maximal R-coherent subset of X. If $X_C \neq \emptyset$, given (M), there exists an R-coherent subset $C_1 \subset X_C$ that is maximal in X_C. Consider the set $C^* = C \cup C_1$.

Statement 1. C^* is an R-coherent subset of X. This is an immediate consequent of the definition of X_C and the fact that C and C_1 are R-coherent subsets of X.

Statement 2. C^* is a maximal R-coherent subset of X. Deny! Assume there exists an R-coherent subset C' of X having C^* as a proper subset. The set $C' - C$ is an R-coherent subset of X, and C_1 is a proper subset of $C' - C$. But $C' - C$ is a subset of X_C; since C' is R-coherent and $C \subset C'$, every element of $C' - C$ is R-comparable with every $x \in C$ and is, furthermore, an element of $X - C$. Therefore, we have a contradiction, for C_1 is a maximal R-coherent subset of X_C. Hence, C^* is a maximal R-coherent subset of X, and C^* contains C.

Reference

1. A. D. Wallace, A substitute for the axiom of choice, Bull. Amer. Math. Soc., 50 (1944) 278.

† Present address is University of Akron.

BOUNDS FOR ROOTS OF ALGEBRAIC EQUATIONS

H. GUGGENHEIMER, University of Minnesota

Most texts on the theory of equations teach the method of synthetic division to find bounds for the roots of an algebraic equation. Since it is not worthwhile to spend much time on the determination of such bounds, a faster and more automatic method could be useful. The result presented here refines one of C. Runge.

A classic theorem states that no root of an algebraic equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

may have absolute value greater than $1+\max |a_i|$. The theorem is best possible in a certain sense, for the equation

$$x^N - \sum_{i=0}^{N-1} x^i = 0,$$

has a root in the interval $2-\epsilon<x<2$ for $N \geqq N(\epsilon)$. In special cases, however, much more can be said.

For convenience in formulation of the theorems, we put

$$\sqrt[0]{M} = M + 1.$$

In the algebraic equation (1), *let* a_q *be the first nonzero coefficient, and* a_p *the greatest coefficient in absolute value. All roots are smaller in absolute value than* $\max(2, \sqrt[q-1]{|a_p|})$.

If the algebraic equation (1) *has real coefficients, let* a_q *be the first negative coefficient, and* a_p *the negative coefficient with greatest absolute value. All real roots are smaller than* $\max(2, \sqrt[q-1]{|a_p|})$.

The two statements are proved by the same idea. We prove the second one. For positive x, the sum of the terms with negative coefficients in (1) is not smaller than

$$-a_p(x^{n-q} + x^{n-q-1} + \cdots + x + 1) = -a_p \frac{x^{n-(q-1)} - 1}{x - 1}.$$

Under our conditions, x^n is greater than the sum of the negative terms, since for $x \geqq \max(2, \sqrt[q-1]{-a_p})$

$$x^{q-1}(x - 1) \geqq -a_p,$$

i.e.

$$x^n \geqq -a_p \frac{x^{n-(q-1)}}{x - 1} > -a_p \frac{x^{n-(q-1)} - 1}{x - 1}.$$

Since the left-hand side of (1) is positive, the result follows.

Example. The real roots of

$$x^{15} + 8.6x^7 - 5712.1x^3 + 1250.8x - 233.4 = 0,$$

are in the interval $-2 < x < 2.3$. An upper bound for the absolute values of the roots is 3.7. All computations are rapidly done by slide rule.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

ELEMENTARY SCIENCE

With support from the National Science Foundation, the American Association for the Advancement of Science started work in April on course content improvement in elementary and junior high school science. Through the resources of its membership and 299 affiliated scientific societies, the Association hopes to encourage scientists in all fields to make further contributions in designing elementary courses and identifying useful and stimulating course sequences for the early years of education. A second objective is to assist in correlating the work of a large number of scientific groups working in the various fields and science curriculum groups in the many state departments of education and local school systems. The greatly improved science courses now being taught in secondary schools over the United States have brought out the need for new thinking about what sort of science instruction should precede them. The major science curriculum studies have been concerned with senior high school science. Many of those working on the new senior high school science courses believe that the high school years may be too late to kindle interest in science for many students.

First steps in the new AAAS elementary program were two eight-day conferences in the summer of 1962 for scientists and teachers to make general plans for course content improvement. The conferences were held at Cornell University and the University of Wisconsin. Participants in these conferences included scientists, teachers, and representatives of state and city education committees which have prepared science sequences for elementary and junior high school grades. A major question concerned the value of courses linking the various subjects, as compared with units in the separate sciences. In this connection, the relation of science to mathematics was an important consideration.

AAAS has organized a commission on elementary science which will maintain a continuous review of work being done on elementary courses under its

own and other auspices, develop interest within the scientific community, recruit scientists to work on course materials, arrange for conferences and other communication among the many groups working on the elementary level, serve as an advisory body on this work, and provide interpretation to schools in their selection and use of the science materials.

NEW COOPERATIVE MATHEMATICS TESTS

The Cooperative Test Division of Educational Testing Services has announced publication of new-course-oriented achievement tests in arithmetic (grades 7, 8, and 9), elementary algebra, intermediate algebra, and high school geometry.

The new cooperative mathematics tests have been under development since 1959 (see this MONTHLY, 69 (1962) 223–225). Some of the newer emphases in language and content can be found in the tests but, in the main, important aspects of traditional mathematics continue to be measured. Students in both traditional and new classes will find the tests stimulating and understandable. Unlike tests which measure routine manipulation of symbols or the recall of specific information, these tests emphasize the student's understanding of mathematical ideas and his ability to reason with insight. Where possible, a few of the major shifts in the mathematics curriculum have been represented in the tests. For example, intuitive solid geometry questions, as well as questions involving rectangular coordinates, are included in the geometry test. Inequalities can be found throughout the series of tests.

Materials available with the tests are: answer sheets, scoring stencils, and an interpretative booklet. Tests can be ordered from the Cooperative Test Division, Educational Testing Service, Princeton, New Jersey, and Los Angeles 27, California.

REGIONAL COUNSELORS IN PHYSICS

A Regional Counselor Program in Physics was launched last fall as a joint effort of the American Association of Physics Teachers and the American Institute of Physics. Forty-eight Regional Counselors throughout the United States and Puerto Rico were appointed. The purpose of the program is to promote local cooperation for better physics teaching. Initially, the Counselors are directing their efforts toward the strengthening of high school physics courses.

A recent report from the Counselors indicates that they have attended national and local meetings of physics groups and educational organizations, brought together teachers for luncheon meetings, issued newsletters to teachers concerning new developments in physics teaching, distributed educational literature, acted as consultants in curriculum review and program evaluation, organized public information activities, and assisted in providing information about various national programs. Interviews between the Counselors and state superintendents of education and/or science supervisors have been held in 21 states.

—A report from the *AIP Educational Newsletter*

MICHIGAN ACADEMIC YEAR INSTITUTE

The University of Michigan is conducting a National Science Foundation Academic Year Institute for biology and mathematics teachers during the 1962–63 school year. Fifteen mathematics teachers were selected for a year's work in mathematics on the doctoral level. It was expected that all participants would have the equivalent of a strong master's degree in mathematics and will be prepared, for example, for real and complex variable theory and algebra beyond the Birkhoff and MacLane level. This program is designed especially for teachers who are interested in positions as mathematics supervisors or consultants to state departments of education.

Some applicants were accepted from junior colleges and teachers colleges. In particular, teachers who are currently employed as mathematics supervisors or consultants to state departments of education were encouraged to apply. The program will run through both semesters and the summer of the 1962–63 academic year.

COMMUNICATIONS MEDIA IN IMPROVING SCIENCE AND MATHEMATICS INSTRUCTION

The research staff in the division of educational communications of the New York State Education Department, with assistance from the National Defense Education Act under Title VII, has started a study to determine the effects of carefully selected communications media (films, television, filmstrips, magnetic tapes, etc.) when introduced in science and mathematics courses in a manner intended to increase learning.

A detailed analysis of instruction of ninth graders in algebra is being made during the present school year. The study involves matched groups. The control group receives regular instruction. An experimental group receives instruction treated with audio-visual materials using the learning profile technique as a guide. Classes in the study consist of approximately 30 average or above-average students. Each set of control and experimental groups has the same teacher.

AAAS-ACLS TEACHER EDUCATION CONFERENCE

In late February the American Association for the Advancement of Science joined with the American Council of Learned Societies in a conference in New York where some 65 participants gathered to discuss the complex questions of improved education for teachers of the humanities, social studies, sciences, and mathematics. The keynote speakers were Howard Mumford Jones of Harvard University and Arthur H. Livermore of Reed College. Their speeches emphasized that our future high school teachers must grow and develop in the rich soil of liberal arts and sciences, and that administrators should be persuaded to employ teachers not only as teachers but as teacher-scholars.

The conference heard reports on Yale University's program for the master of arts in teaching, the AAAS-NASDTEC Teacher Preparation-Certification Study, the College Entrance Examination Board program of summer institutes

for English teachers, and the work of the Committee on the Undergraduate Program of the Mathematical Association of America.

Resolutions were produced by some six discussion groups and a final conference report is available from AAAS or ACLS. Presiding at the general sessions were Dr. Jeremiah S. Finch, formerly Dean at Princeton University and now on the staff of the Conant teacher education study, and Dr. Thomas S. Hall, formerly Dean at Washington University in St. Louis.

NEW SCIENCE FILM

The Office of Education has announced the production and availability for distribution of a film report on new patterns of teaching high school science. The film is entitled "A Probing Mind" and gives attention to such topics as understanding the place of science in our life today, growth of skill in methods of scientific inquiry, and development of a probing, questing mind about things of science. The film shows how teachers today are mobilizing modern educational media—films, television, recordings, teaching machines, and the resources of well-equipped science laboratories—to achieve these aims.

It is a 28-minute film produced by the National Educational Television and Radio Center. The film is available for loan from most state departments of public instruction. Work is underway on a somewhat similar film in mathematics.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1541. *Proposed by G. C. Bush, Queen's University, Kingston, Canada*

Find the maximum and minimum number of "Friday the 13th's" that can occur in a year.

E 1542. *Proposed by Azriel Rosenfeld, Yeshiva University*

Find the number of n-tuples $(x_1, \cdots, x_n)$, $0 \leqq x_i < m$, such that the greatest common divisor $(x_1, \cdots, x_n, m) = 1$.

E 1543. *Proposed by S. W. Golomb, Jet Propulsion Laboratory, California Institute of Technology*

Find a necessary and sufficient condition that an $a \times b$ rectangle can be exactly covered (completely, and without overlaps) with "L-tetrominoes," i.e., with shapes composed of four unit squares like that formed by consecutively joining the points (0, 0), (3, 0), (3, 1), (1, 1), (1, 2), (0, 2), (0, 0) of a rectangular cartesian coordinate system.

E 1544. *Proposed by H. W. Hickey, Alexandria, Virginia*

A solitaire player shuffles a deck of cards numbered from 1 to n inclusive, and lays them face up, one at a time. Every time he finds a card whose number is greater than any card turned up previously, he scores one point (the first card automatically gives him a point, whatever it is). What is the expectation of his score, assuming all arrangements of the deck are equiprobable?

E 1545. *Proposed by Y. Matsuoka, University of St. Andrews, Scotland*

Let p be a positive integer. Prove that

$$\sum_{n=1}^{\infty}\left\{(n+p) \Big/ \prod_{k=0}^{2p}(n+k)^2\right\} = 1/4p\{(2p)!\}^2.$$

E 1546. *Proposed by N. V. Glick, North American Aviation, Inc.*

Consider the sequence of functions $f_n(x) = nx - [nx]$ defined on any non-degenerate interval $[a, b]$. Prove that no subsequence converges.

E 1547. *Proposed by Simon Vatriquant, Brussels, Belgium*

Find the locus of a point P such that the tangents issued from it to a given parabola ω form with the axis and a given line l a Zeeman quadrilateral (i.e., a quadrilateral one side of which is parallel to the Euler line of the triangle formed by the remaining three sides).

SOLUTIONS

Interlocking Committees

E 1501 [1962, 165]. *Proposed by Brother U. Alfred, St. Mary's College, California*

There are 15 men on the board of directors of a large company and 20 committees are associated with the board. It is required to set up these committees so that: (1) Each board member shall belong to four committees; (2) Each committee shall have three board members; (3) No two committees shall have more than one board member in common.

I. *Solution by C. W. Trigg, Los Angeles City College.* Arrange letters representing the 15 men in three adjacent columns. Form a second array by displacing the second column cyclically downward by one step, and the third column by

two steps. The process may be repeated until a sixth array duplicates the first. Thus,

A	*B*	*C*	*A*	*Q*	*N*	*A*	*M*	*F*	*A*	*H*	*R*	*A*	*E*	*K*	*A*	*B*	*C*
D	*E*	*F*	*D*	*B*	*R*	*D*	*Q*	*K*	*D*	*M*	*C*	*D*	*H*	*N*	*D*	*E*	*F*
G	*H*	*K*	*G*	*E*	*C*	*G*	*B*	*N*	*G*	*Q*	*F*	*G*	*M*	*R*	*G*	*H*	*K*
L	*M*	*N*	*L*	*H*	*F*	*L*	*E*	*R*	*L*	*B*	*K*	*L*	*Q*	*C*	*L*	*M*	*N*
P	*Q*	*R*	*P*	*M*	*K*	*P*	*H*	*C*	*P*	*E*	*N*	*P*	*B*	*F*	*P*	*Q*	*R*

We now have a situation of 25 committees of 3 with each man on 5 committees and no two committees with more than one man in common. Any four of the five arrays constitutes a solution to the problem proposed here. Many other solutions result from permutations of the letters in the first array.

Indeed, this procedure may be applied to arrange kn men on n^2 committees of k with each man on n committees and no two committees with more than one man in common, $n \geqq k$.

II. *Solution by D. L. Silverman, Beverly Hills, California.* Number the directors from 1 to 15, and form the 15 committees of the type $(k, k+1, k+3)$ and the 5 committees of the type $(k, k+5, k+10)$, where all numbers are reduced modulo 15. Conditions (1), (2), and (3) are readily verified.

III. *Solution by Michael Goldberg, Washington, D. C.* The solution of the famous Kirkman schoolgirl problem (see Ball's *Mathematical Recreations*, 11th ed., 1939, Chap. X, an explicit solution exhibited on p. 298) gives 35 triplets of 15 girls (5 triplets for each of 7 days) in which no two triplets have more than one girl in common. For the present problem, it suffices to eliminate any three days, giving a remainder of 20 triplets in which each member appears exactly four times and no two triplets have more than a member in common.

IV. *Solution by H. K. Hilton, Chicago, Illinois.* The finite geometry in 3-space, with 3 points to a line, has 35 lines, 7 through each point. Regarding the directors as points, and the committees as lines, the problem may be solved by removing 3 sets of 5 nonintersecting lines from the geometry.

V. *Solution by Jack Evernden and Robert Spira, Berkeley, California.* Take the directors as the vertices, center, and centers of the faces of a cube. 16 of the committees are the diagonals of the cube and its faces, giving 4 committees per director except for the face centers, which have only 2. But these face centers form an octahedron, and we can form the 4 additional committees from alternate triangles on the octahedron, and all conditions of the problem are easily verified.

VI. *Solution by the proposer.* If we take 6 points in general position in space and construct planes through them in all possible ways, we obtain 20 planes and

15 lines, such that 4 planes pass through each line and 3 lines lie in each plane. The cross-section of this configuration by another plane consists of 20 lines and 15 points, such that 4 lines pass through each point and 3 points lie on each line. This solves the problem.

Also solved by A. N. Aheart, E. J. Andrews, Merrill Barnebey, H. F. Bennett, W. Bluger, Robert Bowen, Brother L. F. Zirkel, W. E. Buker, R. J. Bumcrot, P. J. Burke, C. N. Burt, G. I. Chisholm, Jr., D. I. A. Cohen, J. F. Darling, Janet L. Dell'Omo, Monte Dernham, Jane Di Paola, Underwood Dudley, E. S. Eby and W. A. Ferguson (jointly), Jane Evans, L. D. Goldstone, Ned Harrell, A. R. Hyde, Arthur Itkin, A. J. Keeping, Frank Kocher, Betty Levine, C. A. McComas, D. C. B. Marsh, Robert Matulis, Otto Mond, D. A. Moran, P. N. Muller, J. C. Nichols, R. J. Oberg, C. S. Ogilvy, Robert Page, W. E. Patten, C. F. Pinzka, Peter Rasmussen, C. A. Ritcher, David Rothman, S. J. Ryan, N. P. Salz, Mario Tan, Patrick Twomey, J. E. Yeager, S. D. Yoelin, and David Zeitlin.

Burke, Di Paola, and Itkin pointed out that this problem is known to statisticians experienced in the field of experimental design, and is an example of a partially balanced incomplete block design with two associate classes. See Bose, Clatworthy, and Shrikhande, "Tables of partially balanced design with two associate classes," *North Carolina Agricultural Experiment Station Technical Bulletin No. 107* (1954), p. 236. Burke proved that there are at least 20!15! different solutions to the problem, which justifies a remark by Pinzka that a complete compilation of solutions would be dangerous in the hands of a dean (cf. *Sat. Rev.*, Mar. 24, 1962, p. 31). Patten gave consideration to the allied problem of assigning 15 board members to 20 committees, where the board members are of different rank and the committees of different importance, so that the more important committees will have the higher ranking members.

The Fibonacci Sequence Again

E 1502 [1962, 165]. *Proposed by H. S. Shapiro, New York University.*

Let

$$\frac{1 + a_1x + a_2x^2 + \cdots}{1 + b_1x + b_2x^2 + \cdots} = 1 + c_1x + c_2x^2 + \cdots,$$

where $0 \leqq a_i \leqq 1$, $0 \leqq b_i \leqq 1$. Prove that $|c_n|$ does not exceed the nth Fibonacci number.

I. *Solution by Hironori Onishi, Massachusetts Institute of Technology.* The recursive solution for c_n is given by

$$(1) \qquad c_1 = a_1 - b_1, \qquad c_n = a_n - b_n - \sum_{k=1}^{n-1} c_k b_{n-k} \qquad \text{for } n \geqq 2.$$

Let $n \geqq 2$. Substitute

$$c_{n-1} = a_{n-1} - b_{n-1} - \sum_{k=1}^{n-2} c_k b_{n-1-k}$$

(or $c_1 = a_1 - b_1$ when $n = 2$) into (1) to obtain

$$c_n = (a_n - b_n) - (a_{n-1} - b_{n-1})b_1 + \sum_{k=1}^{n-2} c_k(b_{n-1-k}b_1 - b_{n-k})$$

(when $n=2$ the last sum is vacuous). Since $0 \leqq a_i \leqq 1$, $0 \leqq b_i \leqq 1$, we find, by taking the absolute value,

$$|c_n| \leqq 1 + 1 + \sum_{k=1}^{n-2} |c_k|, \qquad |c_1| \leqq 1.$$

But the nth Fibonacci number $F_n = 1 + \sum_{k=0}^{n-2} F_k$. Hence, by induction $|c_n| \leqq F_n$.

(1) This proof tells how to obtain the extreme case. It is unique and given by $a_n = 0$, $b_n = 1$, if n is odd, and $a_n = 1$, $b_n = 0$, if n is even.

(2) The result has the following corollary: Any power series about the origin whose constant term is 1 and all other coefficients lie between 0 and 1 inclusive has no zero within the circle about the origin of radius $(\sqrt{5}-1)/2$.

(3) Dr. J. E. Potter, of M.I.T. Instrumentation Lab., pointed out that c_n is given by

$$c_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ b_1 & 1 & 0 & \cdots & 0 & a_1 \\ b_2 & b_1 & 1 & \cdots & 0 & a_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & 1 & a_{n-1} \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_1 & a_n \end{vmatrix}, \qquad \begin{matrix} 0 \leqq a_i \leqq 1, \\ 0 \leqq b_i \leqq 1, \end{matrix}$$

so that the result is a statement about such determinants.

II. *Solution by the proposer.* Denote the ratio by $f(x)$, and multiply numerator and denominator by $1 - b_1 x$. We get

$$f(x) = \frac{1 + (a_1 - b_1)x + (a_2 - b_1 a_1)x^2 + \cdots}{1 + (b_2 - b_1^2)x^2 + (b_3 - b_1 b_2)x^3 + \cdots}$$

$$\ll \frac{1 + x + x^2 + \cdots}{1 - x^2 - x^3 - \cdots} = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n,$$

where F_n is the nth Fibonacci number.

Also solved by J. L. Brown, Jr., Leonard Carlitz, C. I. A. Cohen, Underwood Dudley, Jane Evans, A. J. Keeping, R. J. Oberg, Frederick Rickey, and C. C. Wong.

Euler's Constant

E 1503 [1962, 165]. *Proposed by Richard Sinkhorn, University of Wisconsin*

Evaluate $\lim_{x \to 0} [(\pi/2) \csc (\pi x/2) - \Gamma(x)]$, where $\Gamma(x)$ is the gamma function.

Solution by M. S. Klamkin, AVCO Research. Since $(\pi/2)\csc(\pi x/2) = 1/x + O(x)$ and $\Gamma(x) = [\Gamma(1+x)]/x = 1/x - \gamma + O(x)$, where γ is Euler's constant $(0.577 \cdots)$, it follows that the desired limit is γ.

Also solved by C. M. Becker, Robert Bowen, C. H. Burt, M. S. Demos, Serge Dubuc, Underwood Dudley, Ragnar Dybvik, Jane Evans, J. A. Faucher, H. A. Heckart, Erwin Just, P. G. Kirmser, P. B. Manchester, D. C. B. Marsh, R. J. Oberg, W. A. Peterson, C. F. Pinzka, David Rothman, F. C. Smith, Evelyn A. Strawbridge, Sherwood Washburn, David Zeitlin, and the proposer.

Most solvers applied l'Hospital's rule twice to the equivalent quotient

$$[\pi x/2 - \sin(\pi x/2)\Gamma(x+1)]/x \sin(\pi x/2)$$

to obtain $-\Gamma'(1)=\gamma$.

Special Normal Trigrade

E 1504 [1962, 165]. *Proposed by R. C. Lyness, Great Singleton, England*

Find positive integers x, y, z, a, b, c, d such that: (1) $x^n+y^n+z^n=a^n+b^n+c^n+d^n$ for $n=1, 2, 3$, (2) $z>y>x$, and (3) $z\neq y+x$.

Solution by Michael Goldberg, Washington, D. C. The solution of

$$x^n + y^n + z^n + w^n = a^n + b^n + c^n + d^n \tag{1}$$

for $n=1, 2, 3$ is given in Dickson's *Introduction to the Theory of Numbers*, 1929, pp. 55–58, and is further amplified in Gloden's *Mehrgradige Gleichungen*, 1944, pp. 35–39. Each solution yields other solutions by adding (or subtracting) the same integer from each element. One of the smallest solutions given by Gloden is (1, 39, 47, 73; 3, 29, 61, 67). By subtracting one unit, and then dividing the elements by 2, we obtain (0, 19, 23, 36; 1, 14, 30, 33). Hence x, y, $z=19$, 23, 36; and a, b, c, $d=1$, 14, 30, 33. Also, this satisfies condition (3). Another solution, (7, 9, 16; 1, 4, 12, 15), does not satisfy condition (3). Some other satisfactory solutions are: (9, 11, 22; 2, 4, 15, 21), (11, 13, 22; 1, 7, 18, 20), (18, 19, 41; 5, 6, 28, 39), (22, 30, 47; 2, 15, 40, 42), (31, 48, 109; 4, 21, 55, 108), (367, 486, 619; 3, 343, 528, 598), (595, 731, 1176; 45, 406, 1001, 1050), (1351, 2319, 2968; 42, 1197, 2662, 2737), (1701, 2305, 3556; 105, 1276, 2856, 3325).

Also solved by Robert Bowen, D. C. B. Marsh, C. F. Pinzka, G. C. Thompson, and the proposer.

Editorial Note. The diophantine system (1) above, where the exponent n takes successively the values 1, 2, 3, is called the *normal trigrade*, and is represented by the abridged notation x, y, z, $w\overset{3}{=}a$, b, c, d. In connection with the general solution of the normal trigrade the reader might like to consult J. Chernick, "Ideal solutions of the Tarry-Escott problem," this MONTHLY, Dec. 1937, pp. 626–633, and A. Gloden, "Normal trigrade and cyclic quadrilaterals with integral sides and diagonals," this MONTHLY, April 1951, pp. 244–247.

Construction of a Triangle

E 1505 [1962, 166]. *Proposed by R. L. Caskey, Oklahoma State University*

Construct a triangle given, in position, a vertex, the midpoint of the side opposite that vertex, and the nine-point center. Describe the region in which the nine-point center must be located so that the problem will have a solution when the other two points are arbitrarily placed.

Solution by C. F. Pinzka, University of Cincinnati. Denote the given points by A, M, and N, respectively. The centroid G is on AM and is determined by $AG=2GM$. The circumcenter O is on the Euler line through G and N and is determined by $OG=2GN$, G lying between N and O. The side opposite A is that chord of the circumcircle which passes through M and is perpendicular to MO.

The construction is possible if and only if $OA>OM$. Thus N must lie in the open half-plane containing A and bounded by the line perpendicular to AM and intersecting it at a point one fourth the way from M to A.

In case $G=N$, then $O=N$. In case N is the midpoint of AM, $O=M$ and any diameter not meeting A serves as the side opposite A.

Also solved by N. A. Court, Rufus Crane, J. F. Darling, R. G. Dean, R. H. F. Denniston, David Elberman, Jane Evans, Michael Goldberg, L. D. Goldstone, Ned Harrell, H. K. Hilton, Erwin Just, D. C. B. Marsh, Yung-liang Wang, Charles Wexler, and the proposer.

Editorial Note. Strictly speaking, one should omit from the permissible region for N the two points which form equilateral triangles with A and M, for if N is at one of these points the triangle will degenerate into a line segment. If A, M, N are collinear, the triangle will be isosceles, and if N coincides with G, the triangle will be equilateral. The proposer showed that the triangle will be a right triangle if N lies on the circle whose center is $\frac{3}{8}$ of the distance from M to A and whose radius is equal to $AM/8$. All points N inside this circle yield an acute triangle; all permissible points outside the circle yield an obtuse triangle. Finally, although it is tacitly assumed that A, M, N are distinct, a solution does exist if $N=A$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5051. *Proposed by Olga Taussky, California Institute of Technology*

Let $A=(a_{ik})$ be an $n\times n$ matrix with $a_{ik}=a_{n+1-i,\,n+1-k}$. Show that corresponding to any characteristic root there exists a vector $x=(x_1, \cdots, x_n)$ with either $x_i=x_{n+1-i}$, $i=1, \cdots, n$ or with $x_i=-x_{n+1-i}$, $i=1, \cdots, n$.

5052. *Proposed by Seth Warner, Duke University.*

A multiplicative seminorm on the algebra E (over the complex numbers) of all entire functions is, by definition, a function p from E into the nonnegative

real numbers with the following properties: $p(0)=0$, $p(\lambda f)=|\lambda|p(f)$, $p(f+g) \leqq p(f)+p(g)$, and $p(fg) \leqq p(f)p(g)$ for all entire functions f, g and all complex numbers λ. If E is furnished with the topology of uniform convergence on compact sets (the compact-open topology), is every multiplicative seminorm on E necessarily continuous?

5053. *Proposed by H. S. Shapiro, New York University*

Let $P(x, y, z)$ be a polynomial such that

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = 0, \quad \text{where} \quad r = (x^2 + y^2 + z^2)^{1/2}.$$

Prove that P is a multiple of $x^2+y^2+z^2$.

5054. *Proposed by Richard Sinkhorn, The Boeing Company, Wichita, Kansas*

Evaluate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

5055. *Proposed by Peter Yff, American University, Beirut, Lebanon*

If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal.

5056. *Proposed by C. W. Kohls and M. E. Mahowald, Syracuse University*

In the book *Topology* by J. G. Hocking and G. S. Young, a space is defined to be locally compact if each point belongs to an open set whose closure is compact. On p. 72, it is asserted that local compactness is invariant under open mappings. Give an example to show that this need not be the case if the spaces involved are not Hausdorff spaces.

5057. *Proposed by F. M. Hornyak, Virginia Polytechnic Institute*

An analytic expression $f(m, n)$ is sought for the sum of the m elements in the nth row of a rectangular array of numbers formed as follows: The elements of the first row are all unity. Each of the m elements in the second row is obtained by adding all elements in the first row directly above and to the left; hence, these are the integers 1 through m. Each element of the third row is the sum of all element in the second row directly above and to the right of the element; hence we have the elements $\frac{1}{2}m(m+1)$ through m. The array is completed by continuing similar additions alternately to the left and to the right.

SOLUTIONS

Representations with Nonintegral Radix

4938 [1960, 1933; 1962, 65]. *Proposed by R. W. McChesney, Albion College*

It is known that every real number $x \in [0, 1)$ has at most two representations of the form $x = \sum_{m=1}^{\infty} a_m N^{-m}$, where N is any integer greater than 1 and

each a_i is one of the set 0, 1, $\cdots$, $N-1$. Now consider a representation of the form $x=\sum a_m b^{-m}$, where $x\in[0, 1)$, and a_i is one of 1, 2, $\cdots$, $n-1$; n an integer greater than 1 and $n-1\leqq b<n$.

(a) Show that an x can exist having more than two such representations.

(b) Prove that at most two representations of x exist if

$$\tfrac{1}{2}[n-1+\sqrt{(n^2+2n-3)}]\leqq b<n.$$

Comment by G. M. Bergman, University of California, Berkeley. The result given in (b) is false. In fact, for any b and $n(>1)$ there are values of x having infinitely many representations.

It is easy to show that any real number $x\in[0, 1)$ can be written $x=\sum a_m b^{-m}$. Since $\sum(n-1)b^{-m}=(n-1)/(b-1)$, we have $(n-1)/(b-1)-x=\sum(n-1-a_m)b^{-m}$. It follows that any number in $((n-1)/(b-1)-1, (n-1)/(b-1)]$ can be so written, and therefore any number in $[0, (n-1)/(b-1)]$ can be similarly represented.

We use the notation $.a_1a_2a_3\cdots$, and put 0^k to designate a sequence of k noughts. Now $(n-1)/(b-1)=.dddd\cdots$, where d is the digit $n-1$. Since $(n-1)/(b-1)>1$, we can choose k such that $M=1.0^k\,1\,0^k\,1\,0^k\cdots\in[0, (n-1)/(b-1)]$. As $M<.dddd\cdots$, we can write $M(=1.0^k\,1\,0^k\,1\,0^k\cdots)=0.X$ where X is some infinite sequence of digits. Then we will also have

$$M=1.0^k\,1\,0^k\,1\,0^k\cdots=1.0^{k+1}X=1.0^k\,1\,0^{k+1}X=1.0^k\,1\,0^k\,1\,0^{k+1}X,$$

and so on. (To make this fit the conditions of the problem exactly we must displace it one or more places to the right of the "decimal" point so as to make it less than 1.)

An Inequality in the Complex Plane

4975 [1961, 577]. *Proposed by Robert Spira, Berkeley, California*

Prove or disprove: If $\mathrm{Re}(z)\geqq 1$, then for any positive integer n,

$$|z^{n+1}-1|>|z^n|\cdot|z-1|.$$

Solution by Marlow Sholander, Western Reserve University. The inequality holds for $\mathrm{Re}(z)>1$ [or $\mathrm{Re}(z)=1$, $z\neq 1$]. We may assume $\mathrm{Re}(z)=r\cos\theta\geqq 1$, where $r>1$ and $0<\theta<\pi/2$. From the cosine law, we have

$$|z^{n+1}-z^n|^2=r^{2n+2}+r^{2n}-2r^{2n+1}\cos\theta,$$

$$|z^{n+1}-1|^2=r^{2n+2}+1-2r^{n+1}\cos(n+1)\theta.$$

We are to show that, for $n=1, 2, \cdots$,

$$r^{2n}-1<2r^{2n+1}\cos\theta-2r^{n+1}\cos(n+1)\theta. \tag{1}$$

We first consider $r=1+d$ where $d\geqq 2/n(n-2)$, $n\geqq 3$. Then $r^2-1=2d+d^2\leqq(n-1)^2d^2=(1+nd-r)^2<(r^n-r)^2$. Hence

$$2r^{2n+1}\cos\theta - 2r^{n+1}\cos(n+1)\theta \geqq 2r^{2n+1}\cos\theta - 2r^{n+1}$$
$$\geqq 2r^{2n} - 2r^{n+1} > r^{2n} - 1.$$

We next consider $d<2/n(n-2)$. Since $r\cos\theta\geqq 1$, $1-2/n(n-2)<\cos\theta<1-\theta^2/2+\theta^4/24$. Hence $\theta^4-12\theta^2+48/n(n-2)>0$, and since $\theta^2<6$, $\theta^2\leqq 6-6\sqrt{D}$ where $D=1-4/3n(n-2)$. When $n\geqq 4$, we find $8n^4+10n^3-87n^2-50n-16>0$ or $1-6/(n+2)^2<\sqrt{D}$. Hence $\theta^2\leqq 6-6\sqrt{D}<36/(n+2)^2<4\pi^2/(n+2)^2$. Since $(n+2)\theta<2\pi$, $\cos(n+1)\theta<\cos\theta$. The last inequality also holds when $n=3$ and, independently of r, when $n=1$, 2. Thus in all remaining cases the right hand side of (1) is greater than $2r^{2n+1}\cos\theta-2r^{n+1}\cos\theta\geqq 2r^{2n}-2r^n>r^{2n}-1$, and the proof is complete.

Also solved by I. N. Baker, H. E. Bray, Robert Breusch, W. G. Brown, A. E. Currier, G. Di Antonio, N. G. de Bruijn and J. H. van Lint, D. Ž. Djoković, B. R. Henry, Don Passman, Gilbert Stengle, and Gabor Szegö.

Unique Solution

4977 [1961, 672]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let S be a system having a finite number of elements with two operations, addition and multiplication, satisfying the following axioms:

1) Under addition the elements of S form an abelian group.

2) For a, b, c in S, $a\neq b$, the equation $xa=xb+c$ has a unique solution.

Prove that also the equation $ay=by+c$ has a unique solution.

Solution by Barbara L. Osofsky, Douglass College, Rutgers—The State University. Let $T=\{ay-by \mid y\in S\}$ for $a\neq b$, a, $b\in S$. Assume $ay-by=az-bz$ for $y\neq z$, y, $z\in S$. Then $ay=az+(by-bz)$. But $by=bz+(by-bz)$. A contradiction follows by axiom (2). Therefore T and S are of the same order, and every element c in S must be expressible in the form $ay-by=c$ for some (unique) y in S.

Also solved by R. G. Albert, W. G. Brown, H. S. Cash, R. A. Cunningham-Green, J. C. Ferrar, N. V. Glick, G. A. Heuer, A. P. Hillman and P. B. Manchester, Wilbur Jonsson, Stanley Kertzner, G. A. Kozlowski, Jr., W. S. Lawton, Shen Lin, Jiang Luh, C. R. MacCluer, W. S. Martindale, 3rd, D. C. B. Marsh, D. W. Miller, Henryk Minc, R. T. Shannon, Guy Torchinelli, B. R. Toskey, W. F. Trench, C. J. Vanderlin, Jr., J. H. van Lint, Seth Warner, W. C. Waterhouse, and the proposer. Late solution by C. I. Lubin

Finite Series

4978 [1961, 673]. *Proposed by R. Nathan, University of Idaho*

Let $n>0$ and $k\geqq 2$ be integers. Show that

$$\sum_{i=0}^{n}\frac{(-1)^i}{i![(i+1)k-1](n-i)!}=\frac{k^n}{(k-1)(2k-1)\cdots[(n+1)k-1]}.$$

I. *Solution by R. C. Read, University of London, England.* Expanding the right member into partial fractions in the usual way, we have

$$\frac{k^n}{(k-1)(2k-1)\cdots[(n+1)k-1]} = \sum_{\mu=0}^{n} \frac{A_\mu}{(\mu+1)k-1},$$

where

$$A_\mu = \lim_{k\to 1/(\mu+1)} \left\{ \frac{k^n}{(k-1)\cdots(\mu k-1)[(\mu+2)k-1]\cdots[(n+1)k-1]} \right\}$$

$$= \frac{1}{(-\mu)(-\mu+1)\cdots(-1)\cdot 1\cdot 2\cdots(n-\mu)} = \frac{(-1)^\mu}{\mu!(n-\mu)!},$$

from which the required result follows at once.

II. *Solution by David Zeitlin, Remington Rand Univac. St. Paul, Minn.* Let

$$n!g(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{x^{ki+k-1}}{ki+k-1}.$$

Then $g(0)=0$, $g(1)$ is the given sum, and $n!g'(x)=x^{k-2}(1-x^k)^n$, and

$$n!g(1) = \int_0^1 x^{k-2}(1-x^k)^n dx = \frac{1}{k} \frac{\Gamma(n+1)\Gamma(1-1/k)}{\Gamma(n+2-1/k)}.$$

By known properties of the gamma function, the desired result is obtained immediately.

III. *Solution by H. W. Gould, West Virginia University.* The result is not new. For instance, it follows from problem 14, page 313, Widder, *Advanced Calculus*. It also follows from a much more general problem 4458 [1953, 53] upon setting $f(x)=1$.

Also solved by J. J. Conde Aldemira, E. B. Andersen, M. T. L. Bizley, W. G. Brown, R. G. Buschman, L. Carlitz, Robert Frucht, S. H. Greene, Carl Hammer, V. C. Harris, J. M. Horner, A. G. Konheim, W. S. Lawton, J. B. Linder, Jiang Luh, D. C. B. Marsh, Immanuel Marx, B. J. M. Morselt, R. G. Nath, Jim Nulton, G. B. Parrish, Stanton Philipp, B. E. Rhoades, R. E. Shafer, Arnold Singer, Michael Skalsky, Leon Steinberg, Guy Torchinelli, W. F. Trench, R. L. Van de Wetering, J. H. van Lint, and W. C. Waterhouse.

A Vanishing Determinant

4979 [1961, 673]. *Proposed by H. S. Shapiro, New York University*

If z_1, z_2, z_3 are distinct numbers of modulus 1, and

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1^m & z_2^m & z_3^m \\ z_1^n & z_2^n & z_3^n \end{vmatrix} = 0,$$

then either two rows or two columns are identical.

Solution by Leonard Carlitz, Duke University. The example $m=1$, $n=3$; $z_1=-1$, $z_2=-\omega$, $z_3=-\omega^2$, $\omega^2+\omega+1=0$ shows that the conclusion is not correct as stated. It is correct, however, if *identical* is replaced by *proportional.*

The vanishing of the determinant implies the existence of three numbers α, β, γ that are not all zero such that

$$(1) \qquad \begin{cases} \alpha+\beta+\gamma=0 \\ \alpha z_1^m+\beta z_2^m+\gamma z_3^m=0 \\ \alpha z_1^n+\beta z_2^n+\gamma z_3^n=0. \end{cases}$$

If $\alpha=0$ then $\beta=-\gamma$ and the second and third columns are identical. Similarly if $\beta=0$ or $\gamma=0$. We may accordingly suppose that $\alpha=-1$, $\beta+\gamma=1$, $\beta\gamma\neq 0$. Then (1) reduces to

$$z_1^m=\beta z_2^m+(1-\beta)z_3^m, \qquad z_1^n=\beta z_2^n+(1-\beta)z_3^n,$$

so that

$$(3) \qquad z_1^m-z_3^m=\beta(z_2^m-z_3^m), \qquad z_1^n-z_3^n=\beta(z_2^n-z_3^n).$$

Remembering that $|z_i|=1$ and taking conjugates, we get

$$(4) \qquad z_2^m(z_1^m-z_3^m)=\bar{\beta}z_1^m(z_2^m-z_3^m), \qquad z_2^n(z_1^n-z_3^n)=\bar{\beta}z_1^n(z_2^n-z_3^n).$$

If $z_2^m-z_3^m=0$ then $z_1^m-z_3^m=0$ and the second row is proportional to the first. Similarly if $z_2^n-z_3^n=0$. Thus comparison of (3) and (4) yields $z_2^m\beta=\bar{\beta}z_1^m$, $z_2^n\beta=\bar{\beta}z_1^n$, so that $z_1^{m-n}=z_2^{m-n}$.

In exactly the same way, if we had solved (2) for $1-\beta$ we would have obtained $z_1^{m-n}=z_3^{m-n}$. Hence if we put $\lambda=z_1^{m-n}=z_2^{m-n}=z_3^{m-n}$ it is evident that the second and third rows are proportional.

Also solved by A. C. Aitken, E. B. Andersen, R. H. C. Newton, G. B. Parrish, K. A. Post, and the proposer.

Editorial Note. The proposer draws the interesting COROLLARY: *If the trinomial equation* $z^n+az^m+b=0$ $(1\leqq m<n$ *are integers,* $ab\neq 0)$ *has three distinct roots on* $|z|=1$, *then* m *and* n *have a common factor.*

Nonassociative Products

4983 [1961, 807]. *Proposed by M. S. Klamkin, University of Buffalo, and L. A. Shepp, University of California*

Determine the number of different products, $P_n(r)$, if the factors are to be taken $r+1$ at a time, in $a_1a_2a_3\cdots a_n$ by inserting parentheses and keeping the order of the elements a_i unchanged. The different products which arise will be due entirely to the nonassociative character of the multiplication. The explicit products for $n=4$, $r=1$ are given by $((a_1a_2)(a_3a_4))$, $(a_1(a_2(a_3a_4)))$, $(((a_1a_2)a_3)a_4)$, $(a_1((a_2a_3)a_4))$, $((a_1(a_2a_3))a_4)$; whence $P_4(1)=5$. This problem generalizes the case for $r=1$ (Bateman Project, *Higher Transcendental Functions*, III, 1955, p. 230).

Solution by John B. Kelly, Michigan State University. We follow the method of generating functions given by N. Jacobson (*Lectures in Abstract Algebra*, vol. I, pp. 18–19) for the case $r=1$. Let

$$y = \sum_{n=1}^{\infty} P_n(r)x^n.$$

One easily observes the recursion formula

$$P_n(r) = \sum P_{n_1}(r)P_{n_2}(r) \cdots P_{n_{r+1}}(r), \tag{1}$$

the summation being extended over all solutions of $n_1+n_2+\cdots+n_{r+1}=n$ in nonnegative integers. From (1) it follows that $y^{r+1}-y+x=0$, whence using a method given by J. S. Frame for inverting trinomials (this MONTHLY, April 1957) we find that

$$P_n(r) = 0, \qquad n \not\equiv 1 \pmod{r}$$

$$P_n(r) = \frac{1}{n}\binom{(r+1)k}{k}, \quad \text{where} \quad n = kr+1.$$

(See also Pólya und Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1954, Bd. I, Aufgabe 211.)

Also solved by W. G. Brown, L. Carlitz, Henryk Minc, B. J. M. Morselt, G. N. Raney, R. C. Read and Michael Skalsky.

Editorial note. The present result is included in *Some problems of nonassociative combinations*, [1] I. M. H. Etherington, Edinburgh Math. Notes, no. 32 (1940), 1–6, and [2] I. M. H. Etherington and A. Erdélyi, ibid., 7–12. It is observed in [1] that $P_n(r)$ is the number of ways in which a convex polygon with $n+1$ sides can be divided into $(r+2)$-gons by nonintersecting diagonals.

For a related problem which includes the present result as a special case, see G. N. Raney, *Functional composition patterns and power series reversion*, Trans. Amer. Math. Soc., 94 (1960) 441–451.

Probability of a Nilpotent Matrix

4985 [1961, 808]. *Proposed by P. T. Bateman, University of Illinois*

Fine and Herstein (*Illinois Journ. Math.*, 2 (1958), 499–504) proved that if K is a finite field with q elements, then the probability that an $n\times n$ matrix over K be nilpotent is q^{-n}. Show that their result implies the following more general fact: If R is a commutative ring with a finite number of elements, the probability that an $n\times n$ matrix with elements in R be nilpotent is r^{-n}, where r is the number of elements in the residue class ring of R modulo its radical.

Solution by L. Carlitz and S. L. Warner, Duke University. Let M denote a matrix of order n with elements in the finite commutative ring R. Let N denote the radical of R and let $R'=R/N$. The probability that M be nilpotent is equal to the probability that a matrix with elements in R' be nilpotent because the radical is itself nilpotent.

By Wedderburn's theorem R' is a direct sum of finite fields, $GF(q_1), \cdots, GF(q_m)$. Therefore, by the theorem of Fine and Herstein, the probability that M be nilpotent is equal to

$$(q_1q_2 \cdots q_m)^{-n} = r^{-n},$$

where r is the order of R'.

Also solved by Harley Flanders, W. S. Martindale, 3rd, Barbara L. Osofsky, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Die innere Geometrie der metrischen Räume. By Willi Rinow. Springer, Berlin-Göttingen-Heidelberg, 1961. xv+520 pp. DM83.

Chapter headings (and numbers of pages devoted to each) are as follows: I Metrische Geometrie und Topologie (57); II Stetige Abbildungen (41); III Die innere Metrik (41); IV Theorie der Kürzesten (46); V Fundamentalgruppe und Überlagungsräume (50); VI Existenzsätze für geodätische Kurven (53); VII Theorie der Krümmung (65); VIII Das Clifford-Kleinsche Raumformenproblem (59); IX Räume der Krümmung $\leqq 0$ (77); X Sphäroide und Räume vom elliptischen Typ (19).

This is the latest of several distinguished contributions to "differential geometry without differentials," initiated by Gauss's discovery of isometric invariance of the curvature which now bears his name, stimulated by Menger's metric investigations, and continued in the treatises of A. D. Aleksandrov (1948; Math. Rev. 10, 619), L. M. Blumenthal (1953; Math. Rev. 14, 1009), and H. Busemann (1955; Math. Rev. 17, 779). These works study such important notions as arc-length, geodesics, and curvature in a purely *metric* way, avoiding the classical regularity assumptions wherever possible. The metric approach differs from the classical analytical approach in its details and also in its spirit. Its spirit (and its advantages) are conveyed by the following quotations from Busemann's reviews of books by Aleksandrov (*loc. cit. supra*) and Pogorelov (1952; Math. Rev. 16, 162), and from the preface to his own book: "Nothing can make a geometer happier than seeing geometry prevail in geometry." " . . . can be used to solve important problems in the realm of classical differential geometry . . . hitherto inaccessible to the classical methods. This is highly

inconvenient for those who justify their disregard for the new methods by their exclusive interest in smooth surfaces." "The direct approach has produced many new results and has materially generalized many known phenomena, thereby revealing two startling facts: much of Riemannian geometry is not truly Riemannian and much of differential geometry requires no derivatives. . . . The present methods also offer the esthetical and methodological advantage of a geometric approach to geometric problems. . . . Nor is there any reason why reduction of hypotheses (and their exact formulation!) should be undesirable in differential geometry but meritorious in all other parts of mathematics." All these comments apply to the present work by Rinow.

For a detailed description of the relationship of the various books on metric geometry to each other and to the classical approach, the reader may consult the reviews cited above and the prefaces of the books in question. Though Rinow's book makes little contact with that of Blumenthal, it does depend heavily on Aleksandrov's theory of curvature and on much of Busemann's work. Rinow's principal departure is his greater use of topological background and methods. Though homology theory is (almost) avoided, he develops and employs some elementary homotopy theory, Borsuk's absolute retracts, the Lusternik-Schnirelmann category, and Sperner's lemma. In fact, his first six chapters form a good base for a one-year course in metric topology and the whole book (though not entirely self-contained) could be used for a two-year course on the topology and geometry of metric spaces. Being wider in scope than its predecessors, Rinow's book may have wider influence.

Though generally good, the index could be enlarged (e.g., it omits *Krümmung*). A few of the authors listed in the bibliography are unmentioned in the index and apparently also in the text. The book mentions a few unsolved problems, but might well include more (see, e.g., Busemann's book cited above). The volume's physical makeup is excellent.

VICTOR KLEE
University of Washington

Elements of functional analysis. By L. A. Lusternik and V. J. Sobolev. (Translated from the Russian by A. E. Labarre, Jr.; H. Izbicki; and H. W. Crowley.) Frederick Ungar, New York, 1961. ix+227 pp. $8.50.

Elements of functional analysis. By L. A. Lusternik and V. J. Sobolev. (Russian Monographs and Texts on Advanced Mathematics and Physics, Volume 5.) Hindustan Publ. Corp. Distributed in Western Hemisphere by Gordon and Breach, New York, 1961. ix+411 pp. $12.50.

These books are two different English translations of a book first published in 1951 presenting in elementary terms the primary ideas and techniques of functional analysis. The subjects treated are: metric spaces, linear spaces, linear operators, linear functionals and weak convergence, completely continuous

(=compact) operators in a Banach space with a basis, the spectral theorem for bounded self adjoint operator in Hilbert space, and some topics on nonlinear functions in Banach spaces.

A German translation was published in 1955, but no English translation was made until now. This is unfortunate, since most of the material found here has appeared, at approximately the same level, in the textbooks of A. C. Zaanen, *Linear Analysis*, North Holland, Amsterdam, 1953, and A. E. Taylor, *Introduction to functional analysis*, John Wiley, New York, 1958. Only the nonlinear theory is not to be found in these latter books. Throughout the books under review, the authors are careful to present "concrete" applications and examples of the abstract theory. For this reason the reviewer feels that the book should not be overlooked by instructors or students.

Of the two translations, the first is clearly superior. Although the Ungar volume has its slips, it accurately renders the Russian into clear and pleasant English, is satisfactorily printed on paper of standard quality, and shows evidence of having been proof-read. Unfortunately, the Gordon-Breach volume has none of these virtues and would begin to be competitive at approximately one-third its listed price.

ROBERT G. BARTLE
University of Illinois

Guide to Tables in Mathematical Statistics. By J. A. Greenwood and H. O. Hartley. Princeton University Press, Princeton, N. J., 1962. lxii+1014 pp. $8.50.

A reviewer of this book is in danger of being so enthusiastic as to appear ridiculous. A careful examination of the Table of Contents and index, accompanied by a random perusal of the "text" will, I believe, leave one agog at the enormous amount of careful work there reflected. The reader is provided with a convenient and informative access to nearly all tables of mathematical statistics published prior to 1961.

The book is meticulously structured, the print pleasant, the discussion terse and clear. The subject index has been done with a fine grid; for example, there are approximately 140 sub-entries under the main entry "approximation." A singular feature is the cataloging of tables of combinatorial patterns usable as experimental designs.

The authors are most circumspect in their use of words, so much so that one is reminded of Humpty Dumpty when he remarked somewhat testily, "If I'd meant that, I'd have said it."

I believe this work will and should find its way onto the shelf of all practicing statisticians as well as that of many mathematicians. It has to be used to be appreciated, and at $8.50 for 62+1014 pages, one can afford to do so.

ROGER S. PINKHAM
Rutgers—The State University

Fourier transforms and convolutions for the experimentalist. By R. C. Jennison. Pergamon Press, New York, 1961. 120 pp. $5.00.

This is avowedly a tract for experimentalists; its aim is to make available to them the principles and practical uses of the Fourier transform. To quote the author, "newcomers to the field of the Fourier integral are likely to be perfectly at home with the behaviour of optical systems illuminated by point sources or by spectral lines"; perhaps so, though the assumption is somewhat surprising.

The author deliberately refrains from any discussion of the mathematical conditions which legitimize operations on Fourier transforms, and this is eminently reasonable given the aim of the book. But it is by no means clear to the reviewer that the experimentalist will find the book a clear introduction to the practical value of the Fourier transform: many illustrations (mainly from optics) are touched on lightly, and none is pursued in sufficient detail to initiate the reader into the use of Fourier transforms in an actual problem. The mathematical reader who wishes to understand the physicist's approach to this theory is likely to be discouraged by the brevity and technicality of the author's allusions to physical interpretations of the transform.

JOHN MCNAMEE
University of Alberta

Discrete Variable Methods in Ordinary Differential Equations. By Peter Henrici. Wiley, New York, 1962. 407 pp. $11.50.

The coverage of the subject of numerical solutions of differential equations in this relatively short treatise is almost encyclopedic. Most major methods have been covered along with a detailed analysis of the various kinds of errors which can arise in the process of obtaining an approximate solution. In the present development, generalized formulas are analyzed, which can be reduced to various special methods by a proper choice of coefficients, thereby enabling the author to develop a more well-rounded error analysis and opening the way for the discovery of new integration formulas. Both fixed-point, single and partial double precision, and floating-point arithmetic are covered in these analyses; fixed-point arithmetic seems to be emphasized somewhat more than floating-point arithmetic. The majority of computer users probably use floating-point arithmetic; perhaps, in view of some of the results obtained here, this usage should be reconsidered.

Sound theoretical proofs are given for various types of absolute error bounds. A novel feature of the text is the development of a probabilistic theory of round off errors. The resulting bounds evidently give much more realistic appraisals for the errors in most cases. Several interesting numerical examples are given, along with flow charts for some of the methods considered.

I expect that this volume will prove to be of great practical interest to many, as well as an inspiration for further research in the field of numerical analysis.

The book is divided into three major parts: one-step methods for initial value problems, multistep methods for initial value problems, and boundary value problems. I would like to try teaching from this text. After noting the compact coverage in many sections, and the author's experience in using about two and one half chapters of the seven in the book for a one-semester senior level course, I imagine that that would be about as much as one might comfortably hope to cover.

In following carefully through most of the proofs a considerable amount of back-reference to previous theorems, equations, etc., is required. It is unfortunate, therefore, that a numbering system was not used which would connect these items with a particular section, and that these section numbers did not show up at the top of all pages.

Very few misprints were noted, and most of these were of a trivial nature.

A. B. FARNELL
General Dynamics/Astronautics

Advanced Pure Mathematics. By J. D. Hodson, Macmillan, London, 1961. xiii +456 pp. $10.00.

This English text completes the course for Advanced Level Examinations up to Scholarship Standard. It is a sequel to an earlier book of the author's which provides an introduction to the calculus. Building on this foundation it carries analysis through partial differentiation and an introduction to differential equations, and, in the last chapter, develops initial topics in statistics. There is no multiple integration, but in addition to the analysis there is considerable material from college algebra and a large amount of geometry. The analytical geometry of the conics is developed in detail with the equation of the conic in general position emphasized. There are also theorems from college geometry and a substantial development of projective geometry, both synthetic and analytic. Altogether, there is a wealth of mathematics here for the student at this level.

The material is presented in integrated form, topics from algebra, geometry, and calculus being included in each of the six chapters. This is sound pedagogically but does break up the logical development of the different subject areas. Mastery of the sections is furthered by excellent, extensive sets of exercises, and at the end of each chapter except the last there are four "Test Papers," which provide examinations at the level indicated on the work just covered.

In its treatment of mathematics the book is more classical than modern. The word "set" does not appear in the entire book, and differentials are frequently identified with increments. On the other hand, the style is direct, motivation is good, explanations are clear, and the exercises and tests provide a fine opportunity for the student to develop skill and power in doing mathematics of the kind presented.

DONALD H. BALLOU
Middlebury College

Mathematical Statistics. By S. S. Wilks. Wiley, New York, 1962. xvi+644 pp. $15.00.

This book is structured much like the book of the same name by the same author which was first published in 1953. The new book contains vastly more information and is more tightly knit. Although the result of a powerful compression, I believe it will prove to be readable by students.

Kendall and Stuart's *Advanced Theory of Statistics, volumes I, II, and III*, and Cramèr's *Methods of Mathematical Statistics* are the books most likely to invite comparison with Wilks' *Mathematical Statistics*. Cramèr's book contains much less statistical information and much less that is current. The volumes by Kendall and Stuart and that of Wilks, while all encyclopedic in flavor, are poles apart in style. The organization of the former is loose, rather like the beads of a necklace, while the latter exhibits the integrated appearance of a piece of sculpture. The author's deliberations are sensed "almost everywhere."

The powerful compression and meticulous organization of the book constitute both a strength and a weakness. The inexperienced student may think that nothing remains to be done. Then too, since the author has purposely omitted discussion of a data analysis and methodology, it is possible to get a biased view of the field as a whole.

There seem to be relatively few misprints in the text but a much higher fraction in the problems. The only error noted was the result stated in problem 8.42 (this after changing a misprinted u to v).

Some of the new notation will be objected to by nearly everyone, but this is probably inevitable. The reviewer, for example, keenly dislikes the notation $R(\frac{1}{2}, 1)$ for the uniform distribution on $[0, 1]$, but this is a minor point, to be sure.

The reviewer understands that there is a student's edition available at $12.50.

This is a remarkable book which is destined to be a classic in the field. The author is to be praised for such an immense effort so carefully carried out.

ROGER S. PINKHAM
Rutgers—The State University

Non-Euclidean Geometry. By S. Kulczycki. (Translated from the Polish by S. Knapowski.) Pergamon Press, 1961, 208 pp. $10.00.

This introduction to hyperbolic geometry is a worthy successor to the classical textbooks by Bonola 1906, Sommerville 1914, Carslaw 1916, and Liebmann 1923. It combines the historical interest of Bonola with the precision of Carslaw and Liebmann. Concerning the two thousand years of frantic efforts to deduce the Parallel Postulate from the rest of Euclid's assumptions, the author writes: "There is something deeply moving in the epic of those heroic strivings towards ideal scientific perfection—disinterested efforts directed solely by the love of knowledge." Again, concerning the fact that Euclid himself was intelligent enough to place this among his Postulates, "What Saccheri considered a fault in

Euclid's exposition we now take to be one of his chief merits." Concerning Lobatchevsky and Bolyai, "It is truly amazing to what extent the trains of thought of the two scholars were related; they were in essence both based on the properties of the *horosphere*. . . . Both were ignored by the scientific world. Both were bitterly disappointed, but they reacted in different ways. Bolyai, exasperated, closed his mouth and withdrew from scientific activity. . . . The extraordinary steadfastness of spirit shown by Lobatchevsky during his twenty-five year struggle in utter isolation has very few equals in the history of science." These quotations reveal something of the author's pleasant style and also of translator Stanislaw Knapowski's excellent command of English.

The mathematical development begins with Hjelmslev's theorem (see, e.g., Coxeter, *Introduction to Geometry*, Wiley, New York, pp. 47, 269), which enables the author (following Hjelmslev himself) to prove that a particular transformation of hyperbolic space ("mapping j") is a collineation. O being a fixed point, each point A yields a corresponding point A' as follows. In any plane through OA, take a ray a making an acute angle α with OA. Let A_0 be the orthogonal projection of A on a. Rotate A_0 through $-\alpha$ about O to obtain A' on OA. (The analogous transformation of Euclidean space is the dilatation $O(\cos \alpha)$. After hyperbolic trigonometry has been developed, we see that $\tanh OA' = \tanh OA \cos \alpha$.) The whole space is thus transformed into the interior of a sphere whose chords represent whole lines. We thus have the Beltrami-Klein projective model imbedded in the hyperbolic space itself! The notion of a bundle of parallels follows at once, and a horosphere is defined as the locus of images of a fixed point by half-turns about the lines of such a bundle. On page 123 we find a very ingenious proof that the intrinsic geometry of a horosphere is Euclidean (cf. Coxeter, *op. cit.*, p. 304). Another neat application of "mapping j" is the proof on p. 75 that any two divergent (or ultraparallel) lines have a common perpendicular. The famous formula

$$\cot \tfrac{1}{2}\Pi(q) = e^q,$$

for the angle of parallelism $\Pi(q)$, is proved in the manner of Lobatchevsky himself, and this is followed by the classical formulae of hyperbolic trigonometry, including spherical trigonometry as a by-product. "Cartesian" coordinates x, y are defined as the orthogonal projections of the radius vector r on two rectangular axes. It is observed that the equation for a straight line is linear in Beltrami's coordinates $X = \tanh x$, $Y = \tanh y$, in terms of which

$$\tanh^2 r = X^2 + Y^2.$$

There are misprints on pages 37 ("axiom be" for "axiom to be") and 42 ("when Legendre" for "then Legendre"). The definition of symmetry (meaning reflection) on page 61 is not very happily expressed, and the reader might at first imagine that the dilatation $O(m)$ is another transformation belonging to absolute geometry. Incidentally, a nice new absolute theorem is proved on page 69.

One appendix uses a central inversion to prove that two sections of a dihedral angle are congruent. Another explains the term "hyperbolic functions" (cf. Coxeter, *The Pentagon*, 14, 1954, p. 39). Finally, there is a table of these functions and an index.

This lively and well printed book will surely be enjoyed by every teacher of geometry.

H. S. M. COXETER
University of Toronto

Tables of Weber Functions, Volume 1. By I. Ye. Kireyeva and K. A. Karpov. Translated from the Russian by Prasenjit Basu. Pergamon Press, New York, 1961. xxiv+340 pp. $20.00.

The Weber (parabolic cylinder) functions are solutions of $D_p''(z) + (p+\frac{1}{2}-\frac{1}{4}z^2)D_p(z)=0$; their importance is well known. With x and p real and with values assigned to $D_p(0)$ and $D_p'(0)$, the real component $u_p(x)$ and imaginary component $v_p(x)$ of $D_p(x+xi)$ are tabulated for

$$\pm x = 0\ (0.01)\ 5.00, \qquad p = 0\ (0.1)\ 2;$$

$$\pm x = 5.00\ (0.01)\ 10.00, \qquad p = 0\ (0.05)\ 2.$$

An introduction gives pertinent formulas, references, and several examples of the use of the tables.

Two misprints in the table of contents give the impression that the book contains about ten per cent of what it actually contains. The headings on pages 124–141 are incorrect.

Assuming accuracy of the tabulated values, this large table of Weber functions should prove to be of great value.

EARL D. RAINVILLE
University of Michigan

Topics in the Theory of Surfaces in Elliptic Space. By A. V. Pogorelov. Translated from the Russian by Royer and Roger, Inc. Edited with a preface by Richard Sacksteder. Gordon and Breach, New York, 1962. xiv+131 pp. $5.95.

This is an intriguing little book in which the author's purpose is to prove some theorems on uniqueness, on infinitesimal deformations, and on regularity of convex surfaces in elliptic space, the latter being defined as a complete three-dimensional Riemannian manifold with constant curvature, homeomorphic to projective space. It is the method of proof which is of interest. By means of certain transformations the author is able to transfer the situation from elliptic space to Euclidean space, there apply known theorems, the results of which are then transferred back to elliptic space.

Thus in Chapters 1 and 2, the author discusses the concepts of elliptic space and convex surfaces therein, develops the fundamental formulas for curves and

surfaces in elliptic space, and considers briefly manifolds of curvature not less than K. In Chapters 3 and 4, he proves the theorems by which congruent figures, isometric surfaces and locally convex isometric surfaces in elliptic space are transformed to the corresponding figures in Euclidean space and vice versa. Chapters 5, 6 and 7 contain the proofs of the afore-mentioned theorems, those for the regularity theorems in the last chapter depending on the results of A. D. Alexandrov for surfaces of bounded specific curvature in Euclidean space and on his theorems on the realization of manifolds of curvature not less than K in space of curvature K.

There are many misprints in the book, there is no index, and the bibliography consists of only six items, three each by Pogorelov and A. D. Alexandrov, even though reference is made in the body of the book to results of others such as Vekua, Efimov and Sauer.

Since the material is related in large part to that in the book "Convex Surfaces" by H. Busemann (Interscience 1958), and since the terminology is consistent with that established in the latter (e.g. swerve of a curve), it seems that the book is best suited for a seminar-type course on convex surfaces, using both works, following a course in classical differential geometry.

GEORGE F. FEEMAN
Williams College

Differential Equations. By L. E. El'sgol'ts. Authorized English translation. Hindustan Publ. Corp., Delhi, 1961. Gordon and Breach, New York, distributors for Western Hemisphere. 360 pp. $14.50.

The author not only presents the topics usually treated in elementary textbooks on the subject, but covers a number of more advanced topics as well. One finds chapter headings such as: Systems of Differential Equations, Stability Theory, Differential Equations with Diverging Arguments, and Partial Differential Equations of the First Order. In addition, careful attention is given to existence theorems, including dependence on initial conditions and parameters. Besides Picard's iteration method and the Cauchy-Peano method there is a proof of the fundamental existence theorem using functional analysis. In the chapter on stability the author discusses, among other things, autonomous systems and Lyapunov's theorems on stability.

In general the book is carefully prepared and concisely written, though in some parts the translation is cumbersome and awkward. Though there are the usual number of small typographical errors, it is a little disturbing to find two vectors described as "linearly dependent" when "linearly independent" is intended as on page 211. The book suffers from the lack of an index.

The author has intended the book for a year's course and clearly there is ample material for the student with a good background in elementary real variable theory. Those looking for "applications" will definitely be disappointed.

R. L. VAN DE WETERING
San Diego State College

Theories of Economic Growth and Development. By Irma Adelman. Stanford University Press, Stanford, California, 1961. ix+164 pp. $5.00.

The main contribution of this volume is an extremely competent and lucid description of the earlier standard models of economic development. Professor Adelman is a master of exposition and she handles her subject with obvious competence. She has mastery of both the literature and her tools and this shows up in the quality of her product.

Unfortunately, the subject itself—the formal theory of economic growth, is in a rather unsatisfactory state and this is what prevents Mrs. Adelman from producing an exciting book. It is symptomatic that, as in many previous writings on the subject, most of the volume is devoted to discussion of the work of Adam Smith (1776), David Ricardo (approx. 1817), Karl Marx (circa 1867), and Joseph Schumpeter (1911). The obvious implication is, that while much impressive work is currently in progress in other areas of economics, recent contributions to the theory of economic development are not extremely exciting. There has, indeed, been a fair amount of recent model building activity here, and contributions have been made by many eminent economists, but none of their models has really been able to produce any degree of enthusiasm. Some of them do not yield very interesting results, others have been too abstract for the purposes at hand—the explanation of differences in rates of expansion of production in different economies, and the things which can be done to influence the rate of development. Still other such models have simply failed to carry conviction as reasonable representations of the economic facts. Indeed, Mrs. Adelman does conclude the volume with a model of her own, but this does not come off perceptibly better than other recent contributions.

Of particular interest is the author's attempt to translate the older analyses into mathematical terms. The effort is not entirely wasted since it does provide a handy framework for comparison and contrast of the various models. But by and large the results of this translation are not impressive. Very little is done by way of mathematical manipulation, and practically no theorems are derived. It has somewhere been said that little insight into the process is gained by talking of "the man, M, who went walking with his dog, D." I'm afraid that in this case, while symbols are employed somewhat more fruitfully, the difference is not very substantial. For example, it seems to me that little help is obtained from the translation of the Marxian view that institutional arrangements are determined by productive technology into the relationship

$$U = g\left(S, \frac{dS}{dt}, \frac{d^2S}{dt^2}, \cdots, t\right),$$

where U is some (unspecified) index "which represents the socio-cultural milieu," S symbolizes "the society's fund of applied knowledge" and t represents time. Of course, this is a particularly unfortunate example, but there are others like it.

The fact of the matter is that, so far, mathematical analysis has just not helped very much in this sub-area of economics. This to me, clearly, is just a reflection of the underdeveloped state of development theory.

WILLIAM J. BAUMOL
Princeton University

Theory of Markov Processes. By E. B. Dynkin. Translated from the Russian by D. E. Brown. Prentice-Hall, Englewood Cliffs, N. J., 1961. x+210 pp. $8.95.

A treatise on the foundations of Markov processes, with emphasis on greatest possible generality. Recommended for connoisseurs of Markov processes only.

JOHN G. KEMENY
Dartmouth College

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

PERSONAL ITEMS

Professor E. J. McShane, University of Virginia, has been reappointed to a six-year term on the Board of the National Science Foundation.

Professor E. J. Schlesinger, Connecticut College, represented the Association at the inauguration of Dr. H. D. Babbidge, Jr. as President of the University of Connecticut on October 20, 1962.

Massachusetts Institute of Technology: Professor Bertram Kostant, University of California, Berkeley, has been appointed Professor; Associate Professor P. E. Conner, Jr., University of Virginia, has been appointed Associate Professor; Dr. W. G. Strang, Mathematical Institute, Oxford, England, has been appointed Assistant Professor; Assistant Professor N. L. Alling, Purdue University, and Drs. G. M. Kelly and A. J. Weir have been appointed Lecturers; Drs. J. L. Alperin, Mathematical Institute, Oxford, England, E. B. Curtis, Harvard University, J. S. Lew, Kirtland Air Force Base, New Mexico, Urlich Hirzebruch and W. G. H. Schaal have been appointed C. L. E. Moore Instructors; Drs. H. B. Enderton and A. B. Ramsay, Harvard University, D. R. Anderson, and A. D. MacGillivray have been appointed Instructors; Drs. H. F. G. Schade and Hirowo Yosinobu have been appointed Research Associates; Assistant Professors K. M. Hoffman and Gian-Carlo Rota have been promoted to Associate Professors; Professor Lodewijk Woltjer, University of Leiden, Netherlands, has been appointed Visiting Professor; Associate Professor M. O. Rabin, Hebrew University, Israel, has been appointed Visiting Associate Professor; Assistant Professor W. F. Stinespring, University of Chicago, has been appointed Visiting Assistant Professor; Professor Eric Reissner will spend the academic year on leave on a Guggenheim Fellowship.

University of Delaware: Dr. R. J. Libera, Rutgers, the State University, and Mr. R. J. Weinacht, University of Maryland, have been appointed Assistant Professors; Assistant Professor R. J. Weinacht will be on leave of absence with a National Science Foundation Post-Doctoral Fellowship at the Courant Institute of Mathematical Sci-

ences, New York University; Mr. Howard Wilson, University of Maryland, has been appointed Instructor; Associate Professor Russell Remage, Jr. has been promoted to Professor; Assistant Professor W. E. Baxter has been promoted to Associate Professor; Professor G. C. Webber has been named H. Fletcher Brown Professor of Mathematics.

University of Washington: Associate Professor I. L. Glicksberg, University of Notre Dame, has been appointed Professor; Assistant Professor H. G. Ellis, University of Utah, Drs. D. G. Cantor, Princeton University, R. E. DeMarr, Moscow University, Russia, R. A. Gangolli, Massachusetts Institute of Technology, R. R. Phelps, University of California, Berkeley, R. W. Ritchie, Dartmouth College, and Alan Troy, University of Illinois, have been appointed Assistant Professors; Professor Jerzy Los, Polish Academy of Sciences, Poland, has been appointed Visiting Professor; Dr. C. M. Bessage, University of Warsaw, Poland, has been appointed Visiting Assistant Professor; Associate Professor J. R. Isbell has been promoted to Professor; Assistant Professors S. P. Avann, H. H. Corson and Ronald Pyke have been promoted to Associate Professors; Dr. D. S. Newman has been promoted to Assistant Professor.

Wesleyan University: Dr. E. K. Blum, Space Technology Laboratories, Los Angeles, California, has been appointed Professor and Director of the Computer Laboratory; Dr. E. F. Assmus, Jr., Columbia University, has been appointed Lecturer.

Mr. Timon Anderson, Ohio State University, has been appointed Instructor at the University of British Columbia.

Dr. J. F. Andrus, Lockheed Aircraft, Marietta, Georgia, has accepted a position as Consulting Mathematician with General Electric, Huntsville, Alabama.

Mr. C. L. Bannister, Army Ballistic Missile Agency, Redstone Arsenal, Alabama, has accepted a position with the G. C. Marshall Space Flight Center, Huntsville, Alabama.

Dr. R. D. Bengston, University of Nebraska, has accepted a position as Aerospace Technologist with the National Aeronautics and Space Administration, Hampton, Virginia.

Dr. D. G. Beverage, University of Florida, has been appointed Assistant Professor at the University of Texas.

Dr. R. C. Bollinger, Westinghouse, Pittsburgh, Pennsylvania, has been appointed Assistant Professor at Pennsylvania State University.

Miss Deborah A. Boyd, Bryn Mawr College, has been appointed Instructor at Wittenberg University.

Dr. D. G. Brennan, Massachusetts Institute of Technology, has accepted a position as President of the Hudson Institute, Harmon-on-Hudson, New York.

Mr. E. W. Brown, Mountain States Telephone and Telegraph, Phoenix, Arizona, has accepted a position as Staff Representative with American Telephone and Telegraph, New York, New York.

Associate Professor G. C. Caldwell, North Carolina State College, has been appointed Assistant Director of the School of Mathematics at Georgia Institute of Technology.

Mr. A. J. Carlan, Syntron Company, Homer City, Pennsylvania, has accepted a position as President of Power Components, Scottdale, Pennsylvania.

Mr. S. C. Choi, Boeing Airplane, Renton, Washington, has accepted a position as Application Analyst with Control Data, Palo Alto, California.

Mr. R. L. Cordle, University of Kentucky, has accepted a position as Research and Development Engineer with Celanese, Charlotte, North Carolina.

Dr. D. W. Crowe, University College, Ibadan, Nigeria, has been appointed Assistant Professor at the University of Wisconsin.

Mr. L. M. Damewood, Bowdoin College, has been appointed Assistant Professor at Eastern Oregon College.

Mr. J. R. Downing, University of Cincinnati, has been appointed Assistant Professor at Illinois State Normal University.

Professor Philip Dwinger, Purdue University, has been appointed Professor at the Mathematical Institute, Technological University, Julianalaan, Netherlands.

Mr. R. L. Edwards, Jr., Johns Hopkins University, has accepted a position as Senior Design Engineer with Aircraft Armaments, Cockeysville, Maryland.

Associate Professor C. C. Faith, Pennsylvania State University, has been appointed Professor at Rutgers, the State University.

Dr. E. A. Fay, University of Florida, has accepted a position as Research Mathematician Statistician with the U. S. Naval Ordnance Test Station, China Lake, California.

Rev. J. C. Friedell, Catholic University of America, has been appointed Instructor at Loras College.

Mr. E. A. Garbaty, University of Nebraska, has been appointed Assistant Professor at the State University of New York, Geneseo.

Mr. K. W. Garrison, Convair Astronautics, San Diego, California, has accepted a position as Systems Analyst with Aeronutronic, Newport Beach, California.

Mrs. Constance J. Golden, Harvard University, has accepted a position as Associate Scientist with Lockheed, Palo Alto, California.

Assistant Professor Dorothy M. Gollmar, State University of New York, Cortland, has been appointed Assistant Professor at Wisconsin State College, Whitewater.

Miss Joann M. Grayless, Occidental College, has accepted a position as Research Engineer with North American Aviation, Downey, California.

Dr. E. H. Hanson, United Electro-Dynamics, Pasadena, California, has been appointed Professor at the U. S. Naval Postgraduate School.

Mr. D. E. Harris, Jr., C.E.I.R., Houston, Texas, has accepted a position as Mathematical-programmer with General Atomics, San Diego, California.

Mr. J. M. Hartman, Dartmouth College, has accepted a position as Methods Analyst with U. S. Steel, Gary, Indiana.

Assistant Professor D. W. Hight, Arkansas State College, has been appointed Assistant Professor at Kansas State College at Pittsburg.

Miss Julia G. Hill, University of California, Davis, has accepted a position as Associate Scientist with Lockheed, Palo Alto, California.

Mr. G. S. Jerpseth, State University of Iowa, has been appointed Instructor at Gustavus Adolphus College.

Mrs. Helen Jick, Bronx Community College, has been promoted to Assistant Professor.

Professor F. I. John, Purdue University, has accepted a position as Senior Statistician with C.E.I.R., Dugway, Utah.

Dr. B. F. Jones, Jr., Rice University, has been appointed Assistant Professor.

Dr. T. F. Kimes, Carnegie Institute of Technology, has been appointed Assistant Professor and Chairman of the Department of Mathematics at Austin College.

Dr. J. P. King, University of Kentucky, has been appointed Assistant Professor at Lehigh University.

Mr. M. S. Klamkin, Avco, Wilmington, Massachusetts, has been appointed Professor of Engineering at the University of Buffalo.

Mr. J. E. Koch, Idaho State College, has accepted a position as Mathematician with Phillips Petroleum, Idaho Falls, Idaho.

Mr. Andrew Kraus, Philco, Palo Alto, California, has accepted a position as Design Engineer with Martin, Denver, Colorado.

Mr. H. E. Krehbiel, Niles High School, Niles, Michigan, has been appointed Assistant Professor at Iowa Wesleyan College.

Dr. Solomon Kullback has retired from the Department of Defense and has been appointed Professor at George Washington University.

Mr. J. B. Lane, University of Vermont, has been appointed Instructor at the University of South Carolina.

Mr. J. F. Latimer, U. S. Naval Ordnance Test Station, China Lake, California, has been appointed Assistant Professor at the University of Alaska.

Mr. Chin-Yeang Lim, Dartmouth College, has been appointed Instructor at Western Reserve University.

Dr. J. A. Lindberg, Jr., Yale University, has been appointed Assistant Professor at Syracuse University.

Dr. H. G. Loomis, Amherst College, has accepted a position as Mathematician with the Syracuse University Research Corporation, Syracuse, New York.

Mr. F. L. McMains, Jr., University of Oregon, has accepted a position with Picatinny Arsenal, Dover, New Jersey.

Mr. R. A. Marking, University of Wisconsin, has been appointed Instructor at the U. S. Naval Nuclear Power School.

Mr. P. L. Mattson, Curtis Publishing Company, has accepted a position as Associate Programming Engineer with Martin, Orlando, Florida.

Assistant Professor Gus Mavrigian, Youngstown University, has been awarded the Watson Foundation Distinguished Professor Award. Professor Mavrigian will be on leave for the academic year 1962–63 at the Carnegie Institute of Technology as a NSF Faculty Fellow.

Mr. M. W. Milligan, Oklahoma State University, has been appointed Associate Professor at Albion College.

Assistant Professor L. J. Montzingo, Jr., University of Buffalo, has been appointed Associate Professor and Head of the Mathematics Department at Seattle Pacific College.

Mr. Clay Nichols, Tulsa Public Schools, Tulsa, Oklahoma, has accepted a position as Programmer with Shell Oil, Houston, Texas.

Mr. D. L. Outcalt, Ohio State University, has been appointed Instructor at Claremont Men's College.

Mr. W. A. Peterson, St. Louis University, has been appointed Assistant Professor at Wheeling College.

Dr. E. A. Pflumm, Republic Aviation, East Farmingdale, Long Island, has accepted a position as Technical Specialist with International Telephone and Telegraph Laboratories, Nutley, New Jersey.

Mr. T. G. Price, West Virginia University, has accepted a position as Aerospace Engineer with NASA Manned Spacecraft Center, Houston, Texas.

Dr. N. J. Pullman, Syracuse University, has been appointed Lecturer at McGill University.

Dr. L. B. Rall, Virginia Polytechnic Institute, has been appointed a member of the U. S. Army Mathematics Research Center, University of Wisconsin.

Assistant Professor F. A. Raymond, University of Wisconsin, has been appointed Visiting Associate Professor at the University of California, Berkeley.

Dr. W. E. Ritter, Massachusetts Institute of Technology, has been appointed J. W. Young Research Instructor at Dartmouth College.

Mr. A. B. Robinson, Huntingdon College, has accepted a position as Computer Programmer with Systems Development, Santa Monica, California.

Mr. F. D. Roscoe, Lawrence Radiation Laboratory, Livermore, California, has accepted a position as Senior Programmer with Federal Electric, Lompoc, California.

Professor C. E. Rusch, State University of New York, Oneonta, has been appointed Chairman of the Department of Mathematics at Parsons College.

Mr. J. T. Ryan, OMEGA Technical Operations, Washington, D. C., has accepted a position as Systems Programmer with Computer Sciences, Palos Verdes, California.

Mr. Bobby Sanders, Florida State University, has been appointed Assistant Professor at Texas Christian University.

Dr. D. E. Sands, Lawrence Radiation Laboratory, Livermore, California, has been appointed Assistant Professor at the University of Kentucky.

Dr. J. E. Scroggs, University of Arkansas, has been appointed Assistant Professor at the University of Texas.

Mr. J. A. Shneer, George Washington University, has accepted a position as Engineer with North American Aviation, Downey, California.

Mr. B. T. Sims, Iowa State University, has been appointed Assistant Professor at San Jose State College.

Mr. F. W. Sinn, Jr., International Electric, Paramus, New Jersey, has accepted a position with the technical staff of Computronics, Fort Lee, New Jersey.

Assistant Professor L. B. Sklar, Trenton State College, has accepted a position with the technical staff of Hughes Aircraft, Culver City, California.

Mr. T. P. Sodano, Magnavox, Fort Wayne, Indiana, has accepted a position as Development Engineer with Goodyear Aircraft, Akron, Ohio.

Mr. F. W. Spraktes, University of Idaho, has accepted a position as Mathematician Statistician with Phillips Petroleum, Idaho Falls, Idaho.

Miss Sharon A. Steimel, Michigan College of Mining and Technology, has accepted a position as Programmer with General Electric, Schenectady, New York.

Mr. H. E. Stern, Lockheed Aircraft, Marietta, Georgia, has accepted a position as Physicist with NASA Marshall Space Flight Center, Huntsville, Alabama.

Mr. C. J. Swauger, Jr., University of Kentucky, has been appointed Instructor.

Mr. J. J. Swift, Lockheed Aircraft, Burbank, California, has accepted a position as Supervisor with General Electric, Valley Forge, Pennsylvania.

Mr. T. J. Thomas, University of Pittsburgh, has accepted a position as Associate Engineer with Douglas Aircraft, Santa Monica, California.

Dr. D. W. Vanderjagt, Florida State University, has been appointed Instructor at Central College, Pella, Iowa.

Mr. R. G. Welling, Washburn University, has been appointed Assistant Professor at Olivet Nazarene College.

Mr. T. C. Weston, University of North Dakota, has accepted a position with Boeing, Seattle, Washington.

Dr. L. H. Williams, Duke University, has been appointed Assistant Professor and Consultant to the Computing Center of Florida State University.

Mr. D. G. Wilson, Iowa State University, has accepted a position as Associate Mathematician with International Business Machines, Bethesda, Maryland.

Mr. L. B. Winrich, University of Wyoming, has accepted a position as Mathematician with Minneapolis-Honeywell, Minneapolis, Minnesota.

Mr. J. E. Wood, Hampden-Sydney College, has been appointed Instructor at the Patrick Henry Branch of the University of Virginia.

Mr. N. R. Wright, Las Vegas High School, Las Vegas, Nevada, has accepted a position with Harcourt, Brace and World, New York, New York.

Professor Guy Stevenson, University of Louisville, died on May 8, 1962. He was a member of the Association for 35 years.

LECTURES ON MATHEMATICAL RESEARCH

The Department of Mathematics of the George Washington University, Washington, D. C., announces a series of thirty lectures on the current situation in fifteen areas of mathematical research. The lectures are sponsored by the Office of Naval Research and will be given at the University during the academic years 1962–63 and 1963–64. The lectures will be published under the editorship of Dr. Thomas Saaty.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS ANNOUNCED

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of the General Mathematics Examination of the 1962 Preliminary Actuarial Examinations are as follows:

FIRST PRIZE OF $200

Mansbach, Peter	Harvard University

ADDITIONAL PRIZES OF $100 EACH

Breznay, George B., Jr.	Harvard University
Dornhoff, Larry L.	University of Nebraska
Lipsman, Ronald L.	City College of New York
Margolin, Barry H.	City College of New York
Snell, Roy C.	Queen's University
Turner, James E.	University of Toronto
Walsh, Timothy R.	University of Toronto
Weinstein, Alan D.	Massachusetts Institute of Technology

The Society of Actuaries has authorized a similar set of nine prizes for the May 1963 Examination on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations: Part 1. Language Aptitude Examination. (Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.) Part 2. General Mathematics Examination. (Algebra, Trigonometry, coordinate geometry, differential and integral calculus.) Part 3. Special Mathematics Examination. (Probability and statistics.)

The 1963 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians and will be administered by the Society of Actuaries at centers throughout the United States and Canada on November 14, 1962 (closing date is October 1, 1962) and May 15, 1963 (closing date is April 1, 1963.)

Further information concerning these Examinations can be obtained from the Society of Actuaries, 208 LaSalle Street, Chicago 4, Illinois.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-THIRD SUMMER MEETING OF THE ASSOCIATION

The Forty-third Summer Meeting of the Mathematical Association of America was held at the University of British Columbia, Vancouver, B. C., Canada, from Monday, August 27 through Wednesday, August 29, 1962, in conjunction with summer meetings of the American Mathematical Society and the Society for Industrial and Applied Mathematics. There were registered 594 persons, including 407 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning and on Wednesday afternoon. All sessions were held in the Auditorium of the University of British Columbia. The eleventh series of Earle Raymond Hedrick Lectures were delivered by Professor A. M. Gleason of Harvard University. Presiding

officers at the three Earle Raymond Hedrick Lectures were President A. W. Tucker and First Vice-President H. S. M. Coxeter, at the remainder of the first session Second Vice-President R. A. Rosenbaum, at the session on Teaching Machines Professor H. M. MacNeille, at the session on Inequalities Professor Leo Moser, at the Report on the International Congress of Mathematicians at Stockholm Professor M. D. Marcus, and at the session on Differential Equations Professor R. A. Beaumont. The Program Committee for the meeting consisted of M. D. Marcus, Chairman; R. A. Beaumont, Leo Moser, Ivan Niven, Patrick Suppes, and L. B. Williams.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *The Coordinate Problem*, Lecture I, by Professor A. M. Gleason, Harvard University.

In the study of mathematical structures, especially when computations are to be made, it is important to have a system for naming all of the elements. Moreover, it is essential that the names be so chosen that the structural relations between the various elements can be expressed by relations between their names. When the structure has cardinal $\aleph_0$ it is natural to take integers or finite sequences of integers as names. When the cardinal is c, it is appropriate to take real numbers or sequences of real numbers as names.

Most mathematical systems are described initially in terms of purely synthetic ideas with no reference to the real number system. Theorems concerning the existence of analytic representations for different types of structures were discussed.

Survey of Mathematics Programs in Institutions Granting Bachelor's and Higher Degrees, by Dr. C. B. Lindquist, Office of Education, U. S. Department of Health, Education, and Welfare.

In 1960–61, 1,014 U. S. colleges and universities offered bachelor degrees in mathematics. Participating in the U. S. Office of Education survey were 877 institutions, enrolling 380,164 students in freshman mathematics courses in Fall 1960; 94,847 in sophomore courses; 104,270 in junior-senior courses. Of entering freshmen 52.1 percent were enrolled in college algebra, trigonometry, or courses of equivalent level; 24.9 percent in courses of lower level; and 23.0 percent in courses of higher level. During the past decade many undergraduate curriculums have undergone considerable change. The number of doctorate-granting institutions reached 97 in 1961; master's programs also increased significantly.

The Problem of Staffing College Departments of Mathematics, by Professor A. W. Tucker, Princeton University.

It was shown that mathematics majors constitute a rapidly increasing percentage of the population of students receiving bachelor's degrees in science and mathematics. Projections to 1970 indicate an acceleration of this trend, thus making even more acute the already severe shortage of qualified college teachers of mathematics.

A discussion following Professor Tucker's lecture was opened by remarks of Professor E. E. Moise of Harvard University who, as Chairman of the Joint Committee on the Doctor of Arts Degree, urged that mathematicians consider the problem realistically, and Professor L. E. Bush of Kent State University and Professor D. W. Robinson of Brigham Young University who spoke on the difficulties in securing competent staff at their own institutions.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Gleason. (See abstract with Lecture I.)

Session on Teaching Machines

Research Problems and Findings in Programmed Instruction, by Professor L. M. Stolurow, Training Research Laboratory, Department of Psychology, University of Illinois, Urbana.

This paper presents basic tutorial concepts, relates them to programmed self-instruction (PSI), and to research findings. PSI developed from (a) Socratic instruction; (b) reinforcement feedback (providing knowledge of results and new information immediately after response); (c) sequential organization of subject matter to build larger concepts and/or skills from smaller units.

Research indicates that students learn as well or better under varying PSI systems than by conventional methods. Two applications of PSI used are: (a) program by itself; (b) program supplementary to conventional teaching; the latter is probably more effective. Results from use in mathematics were discussed, both successes and failures.

The Role of Programmed Material in the Teaching of Mathematics, by Dr. J. E. Forbes, Director of Research in Mathematics, Britannica Center for Studies in Learning and Motivation, Palo Alto, California.

A brief introduction to the "history" and "evolution" of programmed materials was given. The strengths and limitations of this mode of instruction were examined. A report was given of the experience of teachers with programs, stressing the variety of "teacher roles" which have evolved. Finally, a survey of existing types of program utilization and some conjectures concerning future utilization were made.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Gleason. (See abstract with Lecture I.)
Business Meeting of the Association.

Session on Inequalities

To Introduce Inequalities: Two Principles, by Professor George Polya, Stanford University.

The speaker discussed two principles which may be suitable for winning interest in the study of inequalities. One of these is a heuristic principle, the "Principle of Non-Sufficient Reason." Here is one of its many possible formulations: "If, for several unknowns, the conditions of the problem are the same, we may expect to find, for all those unknowns, the same value." Many examples were shown in which this principle does suggest the right result and a few in which it does not. The other principle can be intuitively expressed as a statement about the center of gravity of a system of masses contained in a convex region. It is originally due to Jensen; some applications going beyond Jensen's formulation were sketched.

Some Inequalities in the Differential Geometry of Surfaces, by Professor E. F. Beckenbach, University of California, Los Angeles.

J. L. W. V. Jensen based the theory of inequalities on the systematic application of convexity theorems. It was indicated that Jensen's complex-variable formula, relating the geometric-mean modulus of a meromorphic function to the location of its zeros and poles, itself implies a convexity theorem and thus yields inequality results. The relationship of Jensen's formula to theorems of Hadamard and Hardy, and to F. Riesz's subsequent development of subharmonic-function theory was traced, as was its role in the Nevanlinna theory of meromorphic functions. Finally, a generalization of Jensen's formula and of Nevanlinna's theory to meromorphic minimal surfaces was presented.

Some Methods for Proving Matrix Inequalities, by Professor M. D. Marcus, University of California, Santa Barbara.

Let M be a set of n-square complex matrices and let g be a real valued function defined on M. Inequalities that involve the values of g can take several forms: those that relate $g(A)$ to the characteristic roots and entries of A or to the values of g on other matrices. Also of interest is a comparison of the values of g with those of some of other function h. Examples are the Hadamard and Minkowski determinant inequalities and the Weyl inequalities relating singular values and characteristic roots. A unified treatment that includes these results was described together with substantial generalizations made by many authors over the last few years. Some unanswered questions involving particular choices of M and g were listed.

FOURTH SESSION OF THE ASSOCIATION

A Report on the International Congress of Mathematicians at Stockholm, by Professor V. L. Klee, Jr., University of Washington.

The Fields medals (awarded to young mathematicians for distinguished achievement in mathematical research) went to Lars Hörmander, University of Stockholm, and John Milnor, Princeton University. Hörmander's work on differential operators was discussed by L. Gårding, and the work of Milnor in algebraic topology (and especially in founding the subject of differential topology) was discussed by H. Whitney.

The Congress has been invited to meet in Russia in 1966 at a location to be determined later.

In addition to reporting the above facts, and to discussing briefly the work of Hörmander and Milnor, the speaker touched on a few of the invited addresses and contributed papers given in Stockholm. Since there were more than 70 of the former, and more than 800 of the latter, there was only brief mention of some of them.

Session on Differential Equations

The Sturm-Liouville Problem, by Professor Hans Sagan, University of Idaho.

A selfadjoint homogeneous differential equation of second order is transformed into a system of two differential equations of first order and polar coordinates are introduced. Based on a fundamental theorem by *H. Prüfer* which compares the polar representations of the solutions of two selfadjoint differential equations of second order with each other, the location of the zeros of the solutions and the distribution of the eigenvalues of selfadjoint boundary value problems were discussed. The introduction of polar coordinates permits a very simple and compelling geometric demonstration of the Sturm-Liouville theorems which makes this method especially appealing and suitable for instructional purposes.

Symmetric Differential Operators, by Professor R. W. McKelvey, University of Colorado.

This was a report on aspects of the theory of symmetric operators in Hilbert space, as this theory bears upon the study of ordinary differential boundary value problems. Of particular interest were the eigenfunction expansions associated with a symmetric differential operator T and generated by the selfadjoint extensions of T into possibly larger Hilbert spaces. The characterization of these extensions directly in terms of boundary conditions was described in relation to A. V. Straus' theory of the generalized resolvent. The expansions corresponding to various boundary conditions were then examined and compared.

Stability and Asymptotic Behavior, by Professor Fred Brauer, University of Wisconsin.

In elementary courses, the ordinary differential equations arising from physical problems are usually treated by making approximations to give linear equations with constant coefficients, which are easily solved.

It is possible to estimate the errors introduced by these approximations using reasonably elementary methods. These methods serve two useful purposes. They introduce the student to the important concepts of stability and asymptotic behavior, and they should do much to dispel the popular misconception that if a differential equation cannot be solved explicitly then it cannot be studied at all.

SPECIAL SESSIONS OF THE ASSOCIATION

The four films produced by the Committee on Production of Films were shown in the Auditorium as follows: On Monday at 4:55 P.M. and also on Tuesday at 8:10 P.M. the latest film produced by the Committee "The Kakeya Problem" (in color and with animation) with Professor A. S. Besicovitch as lecturer; on Monday at 7:30 P.M. "What is an Integral?" (a kinescope) with Professor Edwin Hewitt as lecturer; on Tuesday at 3:15 P.M. "Theory of Limits" (in black and white) with Professor E. J. McShane as lecturer; and on Tuesday at 7:00 P.M. "Mathematical Induction" (in color) with Professor Leon Henkin as lecturer.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Sunday at 10:00 A.M. in the Study Room of the International House at the University of British Columbia with twenty-nine members present.

The Board elected Professor M. Gweneth Humphreys as Governor from the Maryland-D. C.-Virginia Section to fill the unexpired part of the term ending June 30, 1965, of Professor C. F. Stephens who had resigned.

The Board elected Professor H. M. Gehman as an Associate Editor of the MONTHLY to fill the unexpired part of the term ending December 31, 1966 of Professor L. J. Montzingo, Jr., who had resigned. Professor Gehman will be the official in the publication office at Buffalo responsible for the MONTHLY. Professor F. A. Ficken, as Editor of the MONTHLY, announced that Professor M. H. Protter had resigned as Associate Editor in charge of MATHEMATICAL NOTES and that Professor J. H. Curtiss, already an Associate Editor, has been appointed as his successor.

The Board voted to invite Professor Hans Rademacher of the University of Pennsylvania to deliver the twelfth series of Earle Raymond Hedrick Lectures at the 1963 Summer Meeting.

The Board approved the following schedule of future meetings: University of California, Berkeley, January 26–28, 1963; University of Colorado, August 26–28, 1963; Miami, Florida, January, 1964; University of Michigan, August, 1964; Denver, Colorado, January, 1965; Cornell University, August, 1965; Rutgers-The State University, New Brunswick, August, 1966; University of Toronto, August, 1967.

The Board authorized President Tucker to appoint a Committee on Membership charged with the responsibility to find ways and means of having persons, who for the good of mathematics should become members of the Association, join and also find possible ways of approaching those who lose membership because of delinquency in the payment of dues, in the hope of being able to retain such members in the Association.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Tucker presiding. The Secretary reported that the membership of the Association was 12,105, an increase of 9% since the corresponding date last year.

The Secretary then reported on some of the actions taken by the Board of Governors on Sunday.

He introduced Professor M. G. Humphreys, as the newly elected member of the Board of Governors from the Maryland-D. C.-Virginia Section, and Professor Bernard Jacobson of Franklin and Marshall College, as the newly appointed Associate Director of CUPM.

On behalf of the members attending the meeting, he expressed deep appreciation to the members of the local Committee on Arrangements for the excellent facilities and services provided for the meeting and introduced Professors B. N. Moyls and R. D. James as those primarily responsible for the success of the meeting.

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in room 3205 of the Buchanan Building. Second Vice-President R. A. Rosenbaum presided. Approximately fifty persons were present representing 26 of the 27 Sections of the Association. Professor C. T. Salkind gave a report as Chairman of the Committee on High School Contests, Professor Harvey Gelder for the Committee on Secondary School Lecturers, Professor R. E. Gaskell for the Committee on Visiting Lecturers, and Professor Bernard Jacobson for CUPM.

There was a lengthy discussion of various ways and means of stimulating the preparation of papers for presentation at sectional meetings.

A discussion on what sections are doing to promote undergraduate research was highlighted by a report of Professor D. W. Western in which he announced that the Philadelphia Section is considering the possibility of making time available at their next sectional meeting for reports by undergraduate students or high school students on results obtained in connection with undergraduate research participation. Such results may be the outcome of activity of a formal nature, such as by a participant in an NSF sponsored project in Undergraduate Science Education, or of informal mathematical creativity, possibly in connection with a course of study.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. Invited addresses were given by Professor C. C. Lin of the Massachusetts Institute of Technology on "Hydrodynamic Stability—A Study in Applied Mathematics" at 9:00 A.M. on Wednesday; by Professor Charles Loewner of Stanford University on "Semigroups in Analysis and Geometry" at 2:00 P.M. on Thursday; by Professor Albert Nijenhuis of the University of Washington on "Derivations and Structure" at 2:00 P.M. on Tuesday; by Professor R. S. Pierce of the University of Washington on "Representations of Lattices" at 9:00 A.M. on Friday; and by Professor R. L. Vaught of the University of California, Berkeley, on "Models of Complete Theories" at 9:00 A.M. on Thursday.

The Society for Industrial and Applied Mathematics met Wednesday morning and Thursday and Friday. Invited addresses were given by Professor S. P. Diliberto of the University of California, Berkeley, on "Applications of Transformation Theorems to Solution Expansion Problems" at 10:15 A.M. on Wednesday; by Professor H. B. Mann of Ohio State University on "The Addition Theorems of Group Theory and Their Applications" at 10:15 A.M. on Thursday; and by Dr. Frank Proschan of Boeing Scientific Research Laboratories on "Probabilistic Models in Reliability Theory" at 3:15 P.M. on Friday.

The Pi Mu Epsilon Fraternity held a Dutch-treat breakfast meeting on Tuesday at 8:00 A.M. in the "Common Block."

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of B. N. Moyls, Chairman; H. L. Alder, N. J. Divinsky, R. D. James, R. S. Pierce, R. A. Restrepo, W. H. Simons, G. L. Walker.

Registration headquarters were located in the main floor corridor of the Buchanan Building. Dormitory housing, residence accommodations, and cafeteria facilities were provided by the University of British Columbia. The Mathematical Sciences Employment Register was maintained in room 1221 of the Buchanan Building.

The Department of Mathematics was host at a Tea on Tuesday from 4:00 P.M. to 6:00 P.M. in the courtyard of the Buchanan Building. A tandem chess exhibition was held on Tuesday at 8:00 P.M. in International House with all visitors challenged to play against Professor E. Macskasy (Canadian Chess Champion) and Professor N. J. Divinsky (Manitoba Chess Champion and Editor of Chess Chat), both of the Department of Mathematics at the University of British Columbia, and Professor Leo Moser (Alberta Chess Champion) of the University of Alberta. About 30 players participated.

An Indian-style barbecue was held on Wednesday at 6:00 P.M. on the Common Block lawn.

HENRY L. ALDER, *Secretary*

ACADEMIC AND CORPORATE MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendments adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in Vancouver on August 26, 1962, elected to membership in the Association the second set of applicants for academic and corporate memberships (for election of first applicants see pages 337–338 of the April issue of this MONTHLY). Approval for election to membership was given to the following 37 applicants for academic and 1 applicant for corporate memberships:

Academic Membership

University of Alabama
Brooklyn College
University of California at Berkeley
University of California at Davis
University of California at Riverside
Canisius College
Collier-Blocker Junior College
East Carolina College
Emory University
Florida A & M University
Harvard University
Hiram College
Howard Payne University
Howard University
Indiana University
Iowa State University
Kent State University
University of Kentucky
Louisiana Polytechnic Institute
North Carolina State College
Ohio State University
Ohio University
Purdue University
Queen's University
Reed College
University of Rhode Island
St. Francis College
San Jose State College
University of Texas
Trinity College
Tulane University
United States Military Academy
University of Utah
Wabash College
University of Washington
Washington State University
Wesleyan University

Corporate Membership

Westinghouse Electric Corporation

HENRY L. ALDER, *Secretary*

FREE AND INEXPENSIVE ITEMS

Mathematics and Your Career. A ten-page pamphlet, revised in 1962, designed to assist teachers and guidance counselors in advising students in their occupational choice and future education. Free. Bureau of Labor Statistics, 341 Ninth Avenue, Room 1025, New York 1, N. Y.

Employment Outlook for Mathematicians, Statisticians and Actuaries. Reprinted from 1961 Occupational Outlook Handbook, pp. 148–157. Included is information on employment, nature of work, training and other qualifications, employment outlook, earnings and working conditions. 10 cents. Superintendent of Documents, Government Printing Office, Washington 25, D. C.

Fellowship and Research Opportunities in Mathematics 1962–1963. Free. Division of Mathematics, NAS-NRC, 2101 Constitution Avenue, Washington 25, D. C.

New Thinking in School Mathematics. Reviewed in this MONTHLY, February 1962, p. 180. Free. MAA office.

The William Lowell Putnam Mathematical Competitions. Reprinted from this MONTHLY, January 1961, pp. 18–33. Problems and solutions from the 18th, 19th, and 20th competitions. 25 cents. MAA office.

A Conference on Mathematics Curricula in Institutes. Reprinted from this MONTHLY, January 1961, pp. 33–38. Free. MAA office.

Note: "MAA office" is an abbreviation for: Mathematical Association of America, c/o University of Buffalo, Buffalo 14, N. Y.

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Annual Meeting at the University of California at Berkeley, on January 25, 26, and 27, 1963. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registration, either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $3.00 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms or for position description forms. These forms must be completed and returned to Providence not later than January 4, 1963, in order to be included free of charge in the listings at the Annual Meeting at Berkeley. The printed listings will be available for distribution both during and after the meeting. Forms which arrive after this closing date, but before January 14, will be included in the register at the meeting for a late registration fee of $3.00.

CALENDAR OF FUTURE MEETINGS

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

Forty-fourth Summer Meeting, University of Colorado, Boulder, August 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Executive Director.

ALLEGHENY MOUNTAIN, Pennsylvania State University, University Park, May 4, 1963.

ILLINOIS, Northern Illinois University, De Kalb, May 10–11, 1963.

INDIANA

IOWA, Iowa State University, Ames, April 19–20, 1963

KANSAS, Sterling College, Sterling, April 27, 1963.

KENTUCKY, University of Louisville, April 27, 1963.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1963.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 8, 1962.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University, East Lansing, March 23, 1963.

MINNESOTA

MISSOURI, Southwest Missouri State College, Springfield, April, 1963.

NEBRASKA, University of Nebraska, Lincoln, May 3–4, 1963.

NEW JERSEY

NORTHEASTERN, Connecticut General Life Insurance Company, Bloomfield, Connecticut, November 24, 1962.

NORTHERN CALIFORNIA, University of California, Berkeley, January 1963.

OHIO, Ohio State University, Columbus, May 4, 1963.

OKLAHOMA

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Franklin and Marshall College, Lancaster, Pennsylvania, November 24, 1962.

ROCKY MOUNTAIN, Brigham Young University, Provo, Utah, Spring, 1963.

SOUTHEASTERN, University of Chattanooga, Chattanooga, Tennessee, March 29–30, 1963.

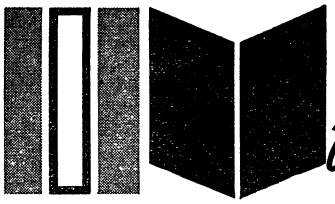
SOUTHERN CALIFORNIA, University of California, Riverside, March 9, 1963.

SOUTHWESTERN, Arizona State College, Flagstaff, April 1963.

TEXAS, North Texas State University, Denton, April 19–20, 1963.

UPPER NEW YORK STATE, University of Buffalo, April 27, 1963.

WISCONSIN, Carroll College, Waukesha, May 4, 1963.


from ADDISON-WESLEY

COLLEGE ALGEBRA: A Modern Approach

By Burnett R. Toskey, *Seattle University*

This new book bridges the gap between trigonometry and calculus. Features include: (1) Notations and concepts of symbolic logic and set theory used throughout: (2) Complete survey of structure of number system, from natural numbers to complex numbers: (3) Detailed discussion of functions applicable to a follow-up course in calculus.

220 pp., 77 illus., 1962—$5.75

FUNDAMENTAL CONCEPTS OF ELEMENTARY MATHEMATICS

By Charles Brumfiel and Robert E. Eicholz,
Ball State Teachers College
and Merrill E. Shanks, *Purdue University*

Presents the structure of arithmetic, and relates it to algebra and geometry. For the student with inadequate high school background, it will give the mathematical maturity necessary for later subjects. Because it emphasizes the *structure* of the subject, it will be valuable to prospective teachers in either content or methods courses.

340 pp., 336 illus., 1962—$6.50

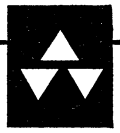
ORDINARY DIFFERENTIAL EQUATIONS

By L. S. Pontryagin, *Moscow State University*

This modern text for an introductory course emphasizes the applications of differential equations to the theory of oscillations and the theory of automatic control. By means of engineering problems, it offers a thorough grounding in the methods and techniques of differential equations. Applied mathematicians and engineers will be particularly interested in this text.

298 pp., 70 illus., 1962—$8.75

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NEW AND NOTABLE . . .

BASIC PROGRAMMING CONCEPTS AND THE IBM 1620 COMPUTER

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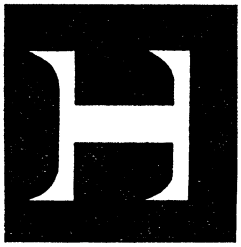
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INVERSION OF FOURIER AND LAPLACE TRANSFORMS

R. P. BOAS, JR., Northwestern University

1. Introduction. I present here a simple proof of the basic Fourier transform formulas under hypotheses that suffice for many applications. The calculations are as simple as those of some of the well-known heuristic derivations of the formulas, but have the advantage that they can easily be made rigorous. I shall first present the formal calculations, which should appear plausible to anyone with a rudimentary knowledge of the formal aspects of Fourier series, and then justify them under the assumption that the function being transformed is absolutely integrable and of bounded variation on $(-\infty, \infty)$, with the usual normalization at points of discontinuity (i.e., the value of the function is the midpoint of the jump). The complex inversion formula for Laplace transforms, under similar hypotheses, is an immediate corollary.

This proof of the Fourier transform formulas is essentially an adaptation to a simpler setting of A. Weil's proof for functions of L^2 on locally compact Abelian groups [2].

2. Basic formulas. Let G be a given function on $(-\infty, \infty)$. We take its Fourier transform in the form

$$g(x) = \int_{-\infty}^{\infty} e^{2\pi ixu} G(u)\,du. \tag{2.1}$$

We want to show that

$$G(u) = \int_{-\infty}^{\infty} g(x) e^{-2\pi ixu}\,dx, \qquad -\infty < u < \infty, \tag{2.2}$$

(Fourier inversion) and that

$$\int_{-\infty}^{\infty} |\, G(u)\,|^2 du = \int_{-\infty}^{\infty} |\, g(x)\,|^2 dx, \tag{2.3}$$

(Parseval formula). Once we have (2.3) we can easily get the convolution formula for the transform of a product,

$$\int_{-\infty}^{\infty} f(x) g(t - x)\,dx = \int_{-\infty}^{\infty} e^{2\pi itu} F(u) G(u)\,du. \tag{2.4}$$

3. Formal calculations. With g defined by (2.1) we have, when k is an integer,

$$\begin{aligned} g(x - k) &= \sum_{n=-\infty}^{\infty} \int_{n}^{n+1} e^{2\pi iu(x-k)} G(u)\,du \\ &= \sum_{n=-\infty}^{\infty} \int_{0}^{1} e^{2\pi ixt} e^{2n\pi ix} e^{-2\pi ikt} G(n + t)\,dt \end{aligned} \tag{3.1}$$

$$= \int_0^1 e^{-2\pi ikt} e^{2\pi ixt} \sum_{n=-\infty}^{\infty} e^{2n\pi ix} G(n+t)dt$$

$$= \int_0^1 e^{-2\pi ikt} \psi(x, t)dt,$$

where

$$\psi(x, t) = e^{2\pi ixt} \sum_{n=-\infty}^{\infty} e^{2n\pi ix} G(n+1). \tag{3.2}$$

That is, $g(x-k)$ is the kth Fourier coefficient (on (0, 1)) of $\psi(x, t)$ as a function of t.

We now compute the Fourier coefficients of $\psi(x, t)e^{-2\pi ixt}$, as a function of x, in two ways: one by direct inspection of (3.2); the other by using the Fourier series of $\psi(x, t)$, whose coefficients are $g(x-k)$ (by (3.1)), and multiplying it by $e^{-2\pi ixt}e^{-2\pi inx}$ and integrating to get the nth Fourier coefficient of $\psi(x, t)e^{-2\pi ixt}$. We obtain then,

$$\begin{aligned} G(n+t) &= \int_0^1 e^{-2\pi ixt} e^{-2\pi inx} \psi(x, t)dx \\ &= \int_0^1 e^{-2\pi ixt} e^{-2\pi inx} \sum_{k=-\infty}^{\infty} g(x-k) e^{2\pi ikt} dx \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 g(x-k) e^{2\pi ikt} e^{-2\pi ix(n+t)} dx \\ &= \sum_{k=-\infty}^{\infty} \int_{-k}^{-k+1} g(u) e^{-2\pi iu(n+t)} du \\ &= \int_{-\infty}^{\infty} g(u) e^{-2\pi iu(n+t)} du, \qquad 0 < t < 1. \end{aligned} \tag{3.3}$$

In other words,

$$G(t) = \int_{-\infty}^{\infty} g(u) e^{-2\pi iut}\, du, \qquad -\infty < t < \infty,$$

and this is the inversion formula (2.2). A change of variable gives us an equivalent but probably more familiar pair: if

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} G(u)du \tag{3.4}$$

then

$$G(u) = \int_{-\infty}^{\infty} g(x) e^{iux} dx. \tag{3.5}$$

By comparing the two expansions of $\psi(x, t)$, namely (3.2) and the Fourier series in t with coefficients $g(x-k)$, we obtain incidentally the Poisson summation formula,

$$e^{-i\pi xt}\sum_{k=-\infty}^{\infty} e^{2\pi ikt}g(x-k) = e^{i\pi xt}\sum_{n=-\infty}^{\infty} e^{2n\pi ix}G(t+n), \tag{3.6}$$

where g and G are connected by (2.1), (2.2). For the alternative pair g, G connected by (3.4), (3.5), we have

$$A^{\frac{1}{2}}e^{\frac{1}{2}ixt}\sum_{n=-\infty}^{\infty} e^{-inAt}g(x-nA) = A^{-\frac{1}{2}}e^{-\frac{1}{2}ixt}\sum_{n=-\infty}^{\infty} e^{-inBx}G(t+nB), \quad AB = 2\pi, \tag{3.7}$$

from which the forms usually quoted are obtained by specializing the parameters. The very symmetric form (3.7) used to be given by G. H. Hardy in his lectures; an equivalent form appears in ([1], p. 217).

4. The Parseval formula. Parseval's formula for Fourier series states that if $f(x)$ has Fourier coefficients c_k with respect to $e^{-2\pi inx}$ over (0, 1), then

$$\int_0^1 |f(x)|^2dx = \sum_{-\infty}^{\infty} |c_k|^2.$$

Apply this to $\psi(x, t)$, as a function of t, and its Fourier coefficients $g(x-k)$ as given by (3.1):

$$\int_0^1 |\psi(x,t)|^2dt = \sum_{k=-\infty}^{\infty} |g(x-k)|^2. \tag{4.1}$$

Now apply Parseval's formula to $\psi(x, t)e^{-2\pi ixt}$ as a function of x, with Fourier coefficients $G(n+t)$ as given by (3.2):

$$\int_0^1 |\psi(x,t)|^2dx = \sum_{n=-\infty}^{\infty} |G(n+t)|^2. \tag{4.2}$$

Integrating (4.1) and (4.2) with respect to x and t, respectively, and equating the results, we find

$$\begin{aligned}\int_{-\infty}^{\infty} |g(x)|^2dx &= \int_0^1 \sum_{k=-\infty}^{\infty} |g(x-k)|^2dx = \int_0^1 \sum_{n=-\infty}^{\infty} |G(n+t)|^2dt \\ &= \int_{-\infty}^{\infty} |G(t)|^2dt,\end{aligned} \tag{4.3}$$

and this is the Parseval formula (2.3). With the transforms in the form (3.4), (3.5), it reads

$$\int_{-\infty}^{\infty} |g(x)|^2dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(u)|^2du,$$

and the convolution formula (2.4) becomes

$$\int_{-\infty}^{\infty} f(x)g(t-x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu}F(u)G(u)du.$$

5. Justification of the calculations. We need three lemmas on functions of bounded variation.

LEMMA 1. *If G is of bounded variation on $(0, \infty)$ then the total variation $V(N)$ of G over $[N, \infty)$ tends to 0 as $N \to \infty$.*

Here $V(N)$ means $\lim_{M\to\infty} \mathrm{var}_{[N,M)} G$. This is $\lim_{M\to\infty} \mathrm{var}_{[0,M)} G - \mathrm{var}_{[0,N)} G$, and this difference tends to 0 as $N \to \infty$.

LEMMA 2. *Let G be absolutely integrable on $(-\infty, \infty)$ and of bounded variation on $(-\infty, \infty)$. Then the series*

$$\sum_{n=-\infty}^{\infty} |G(n+t)| \tag{5.1}$$

converges uniformly in t on $[0, 1]$.

Since $|G|$ is of bounded variation if G is, we may suppose that $G(x) \geqq 0$. Then we have

$$\begin{aligned}\left|\int_N^{M+1} G(u+t)du - \sum_{n=N}^{M} G(n+t)\right| &= \left|\sum_{n=N}^{M}\int_0^1 G(n+u+t)du - \sum_{n=N}^{M} G(n+t)\right| \\ &= \left|\sum_{n=N}^{M}\int_0^1 \{G(n+u+t) - G(n+t)\}du\right| \\ &\leqq \int_0^1 \sum_{n=N}^{M} |G(n+u+t) - G(n+t)|\, du,\end{aligned}$$

and the right hand side does not exceed the total variation of G on $[N, \infty)$ if $0 \leqq t \leqq 1$. This total variation approaches 0 as $N \to \infty$ by Lemma 1. Hence the sums $\sum_{n=N}^{M} G(n+t)$ are small with the integrals

$$\int_N^{M+1} G(u+t)du = \int_{N+t}^{M+t+1} G(u)du \leqq \int_N^{M+2} G(u)du, \qquad 0 \leqq t \leqq 1,$$

and the last integral is arbitrarily small if N and M are large. This establishes the uniform convergence of (5.1).

LEMMA 3. *If G is of bounded variation on $[0, 1]$, the partial sums of its Fourier series are bounded by a number that depends only on the maximum absolute value and the total variation of G.*

This lemma seems to be too trivial to appear in books on Fourier series.

The partial sums are of the form

$$\int_0^1 D_n(t)G(x+t)dt,$$

where D_n is the Dirichlet kernel. We may suppose G positive and decreasing and then, by the second law of the mean, the integral has the form $G(x)\int_0^\xi D_n(t)dt$, which has a bound independent of n and ξ ([3], vol. I, p. 57, Lemma 8.2).

Lemma 2 shows that the series in (3.2) converges uniformly; we may therefore carry out the calculations of (3.1). It is clear that $\psi(x, t)$ has period 1 in t. The total variation of the series in (3.2) does not exceed

$$\sum_{n=-\infty}^{\infty} \operatorname*{var}_{0\leqq t\leqq 1} G(n+t) \leqq \operatorname*{var}_{-\infty<t<\infty} G(t).$$

Hence the Fourier series of $\psi(x, t)$, as a function of t, represents $\psi(x, t)$ for each x (assuming that G and hence ψ is normalized, i.e. $G(t)=\frac{1}{2}[G(t+0)+G(t-0)]$):

$$\psi(x, t) = \sum_{k=-\infty}^{\infty} g(x-k)e^{2\pi ikt}. \tag{5.2}$$

This series converges boundedly in x (uniformly in t) by Lemma 3. Hence we can multipy (3.2) by $e^{-2\pi ixt}e^{-2\pi inx}$ and integrate term by term, and we have then established (2.2) under our hypotheses.

We now justify the calculations of Section 4 leading to the Parseval formula. In the first place we applied the Parseval formula for Fourier series to $\psi(x, t)$, both as a function of x and as a function of t. This is justified since $\psi(x, t)$ is bounded in each variable: it is of bounded variation in t, and it is bounded in x for each t since (5.2) converges boundedly in x.

Next we have to integrate the series on the right hand sides of (4.1) and (4.2) term by term. Since $G(t)\to 0$ as $|t|\to\infty$, $|G(t)|^2$ is integrable because $|G(t)|$ is integrable. By Lemma 2, the last two terms on the right of (4.3) are equal. Therefore $\int_0^1 dt\int_0^1 |\psi(x, t)|^2 dx$ is finite (from (4.2)) and since the integrand is positive, the integral in the other order is also finite and has the same value (here we are appealing to Fubini's theorem from Lebesgue integration, or to an appropriate substitute). Hence from (4.1) the second term of (4.3) is equal to the term that follows it.

6. Laplace transforms. It is worth remarking that by changing our formulas slightly we can prove the complex inversion formula for Laplace transforms, a formula that is too often supposed to be of forbidding depth.

Let $f(t)e^{-ct}$ be absolutely integrable and of bounded variation on $[0, \infty)$ for some real c (and normalized by $f(t)=\frac{1}{2}[f(t+0)+f(t-0)]$, $t>0$; $f(0)=\frac{1}{2}f(0+)$). For $\Re(s)>c$, the Laplace transform

$$F(s) = \int_0^\infty e^{-su}f(u)du$$

converges, and if we put $s=\sigma+it$, we have

$$F(\sigma + it) = \int_0^\infty e^{-ist}e^{-\sigma u}f(u)du, \qquad -\infty < t < \infty.$$

Thus $(2\pi)^{-1}F(\sigma+it)$, as a function of t, is the Fourier transform (in the form (3.5)) of the function that is $f(u)$ for $u>0$ and 0 for $u<0$. Since this function satisfies our hypotheses, we have, by (3.4),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}F(\sigma + it)dt = \begin{cases} f(u), & u > 0, \\ 0, & u < 0, \end{cases} \qquad \sigma > c.$$

In a different and more customary notation,

$$f(u) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{uz}F(z)dz, \qquad \sigma > c,\ u > 0.$$

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A CRUCIAL CORRECTION

W. F. EBERLEIN, University of Rochester

In my paper "The Spin Model of Euclidean 3-Space" [this MONTHLY, 69 (1962) 587–598] vectors are represented as operators and it is important to distinguish between the inner product $A\cdot B$ and the operator product AB. The second equation on page 597 should read, and did read in manuscript and galley proof, as follows:

$$i\hbar I\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{e\mathfrak{A}}{c}\right)^2 + e\Phi I\right]\psi$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}(\nabla\mathfrak{A} + \mathfrak{A}\nabla) + \left(\frac{e^2\mathfrak{A}^2}{2mc^2} + e\Phi I\right)\right]\psi.$$

Unfortunately, in the passage from galley proof to publication, $\nabla\mathfrak{A}+\mathfrak{A}\nabla$ got changed into $\nabla\cdot\mathfrak{A}+\mathfrak{A}\cdot\nabla$. Since the whole point of the argument is to show that these two expressions differ by just the spin term, the effect of the misprint is to stop the spin of the electron.

THE PRODUCT OF TWO MEASURES

S. K. BERBERIAN, State University of Iowa

1. Introduction. It has seemed to me, as an occasional teacher of courses in measure and integration, that it is all too easy to exaggerate the role of infinity, that is, the extended real number ∞. Set functions, for instance measures, with possibly infinite values seem natural enough; but for the point functions that arise in the basic theory, namely measurable and integrable functions, infinite values are often an artificial complication, for it is generally possible, and easy, to replace the function in question by a real valued function to which it is equal almost everywhere. Particularly in connection with Fubini's theorem, the main outlines of the theory can easily be obscured, and I have done it, by purely technical difficulties with infinity. My attempts to minimize these difficulties have led to the present article, in which the context of Fubini's theorem, namely product measure, is given an exposition which clings to the (finite) real numbers as long as possible.

In the treatise [1], an elegant theory of product measures is developed for σ-finite measure spaces (measure spaces in which every measurable set is a countable union of sets of finite measure). The aim of the present note is twofold: (1) to show how this theory may be extended, in a simple and natural way, to obtain the product of any two measure spaces, and (2) to give an exposition of (1) which exhibits the essential dependence of the general case on the case of finite measure spaces (measure spaces in which every measurable set has finite measure). The results of (1) are possibly new; in any case they do not appear in the standard textbooks on measure theory which I examined.

In the way of prerequisites, I assume that the reader is familiar with the basic Lebesgue theory, as covered say in the first five chapters of [1], as well as Sections 33 and 34 in Chapter VII of [1]. My terminology is that of [1], with the following simplification: the terms "measurable" and "integrable" will be applied only to real valued functions. We recall now a few of the basic definitions of [1]. A class of sets is *monotone* if it is closed under the formation of increasing countable unions and decreasing countable intersections. A *ring* [resp. *σ-ring*] is a nonempty class of sets which is closed under the formation of set theoretic differences and finite [resp. countable] unions. I deviate from the notation of [1] by writing $\mathfrak{S}(\mathfrak{A})$ for the σ-ring generated by a class $\mathfrak{A}$ of sets. A *measurable space* is a pair $(X, \mathcal{S})$, where $\mathcal{S}$ is a σ-ring of subsets of X; the sets in $\mathcal{S}$ are called *measurable* (it will not be necessary to assume that each point of X belongs to some measurable set). A *measure space* is a triple $(X, \mathcal{S}, \mu)$, where $(X, \mathcal{S})$ is a measurable space and μ is a measure on $\mathcal{S}$. Aside from routine techniques, key use will be made of the following three results:

THE LEMMA ON MONOTONE CLASSES [1, 6B]. *If a monotone class contains a ring $\mathfrak{R}$, then it also contains $\mathfrak{S}(\mathfrak{R})$.*

THE UNIQUE EXTENSION THEOREM [1, 13A]. *If $\mathfrak{R}$ is a ring, and μ_1, μ_2 are measures on $\mathfrak{S}(\mathfrak{R})$ whose restrictions to $\mathfrak{R}$ are equal and σ-finite, then $\mu_1=\mu_2$ on $\mathfrak{S}(\mathfrak{R})$.*

THE MONOTONE CONVERGENCE THEOREM [1, 27B]. *If f_n is an increasing sequence of integrable functions, converging pointwise to the integrable function f, then the integral of f_n converges to the integral of f.*

Throughout the paper, $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ denote measurable spaces, and $(X\times Y, \mathcal{S}\times\mathcal{T})$ is the product measurable space [1, 33F]. Generic elements of $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{S}\times\mathcal{T}$ are denoted by the letters E, F, and M, respectively, and the use of these letters in any context is meant to indicate the σ-ring from which the sets are drawn. A set of the form $E\times F$ will be called a *measurable rectangle*; thus, $\mathcal{S}\times\mathcal{T}$ is the σ-ring generated by the measurable rectangles. The letters f and g are reserved for functions defined on X and Y, respectively.

2. Preliminaries. If M belongs to $\mathcal{S}\times\mathcal{T}$ and x is a point of X, the *x-section* of M is defined to be the set $M_x=\{y\in Y: (x, y)\in M\}$; M_x belongs to $\mathcal{T}$ by [1, 34A]. Similarly if y is a point of Y, the y-section of M is $M^y=\{x\in X: (x, y)\in M\}$, and M^y belongs to $\mathcal{S}$. We refrain from taking sections of sets not in $\mathcal{S}\times\mathcal{T}$, hence all sections encountered will be measurable. If $g: Y\to X\times Y$ is defined by $g(y)=(x, y)$, evidently $M_x=g^{-1}(M)$; since g^{-1} preserves the usual set theoretic operations, the following lemma is immediate (the symbol $M_n\uparrow M$ means that M_n is an increasing sequence of sets whose union is M):

LEMMA 1. *If M, N, and M_n $(n=1, 2, 3, \cdots)$ are sets in $\mathcal{S}\times\mathcal{T}$, and x is a point of X, then:*

(i) $(\bigcup M_n)_x=\bigcup(M_n)_x$

(ii) $(\bigcap M_n)_x=\bigcap(M_n)_x$

(iii) *If $M\subset N$, then $M_x\subset N_x$.*

(iv) *If $M_n\uparrow M$, then $(M_n)_x\uparrow M_x$.*

(v) *If $M_n\downarrow M$, then $(M_n)_x\downarrow M_x$.*

We denote by R the set of all real numbers, while R_e denotes the set of all extended real numbers. Whenever convenient, R_e is regarded as a complete lattice in the natural way. If α and α_n $(n=1, 2, 3, \cdots)$ are extended real numbers, the symbol $\alpha_n\uparrow\alpha$ means that $\alpha_m\leqq\alpha_n$ whenever $m\leqq n$, and $\alpha=$l.u.b. α_n. If $f_n: X\to R_e$ is a sequence of functions, $f_n\uparrow f$ means that $f_n(x)\uparrow f(x)$ for each x.

For the rest of the section, $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ denote arbitrary measure spaces.

DEFINITION 1. *For each M in $\mathcal{S}\times\mathcal{T}$, the functions $f_M: X\to R_e$ and $g^M: Y\to R_e$ are defined by the formulas*

$$f_M(x) = \nu(M_x)$$

$$g^M(y) = \mu(M^y).$$

If $M=E\times F$, then $M_x=F$ when x is in E, and $M_x=\varnothing$ otherwise. Similarly for y-sections. In terms of characteristic functions, this implies at once:

LEMMA 2. *For every measurable rectangle* $E\times F$,

$$f_{E\times F} = \nu(F)\chi_E$$
$$g^{E\times F} = \mu(E)\chi_F.$$

If moreover $\mu(E)<\infty$ *and* $\nu(F)<\infty$, *these functions are integrable, and*

$$\int f_{E\times F}d\mu = \int g^{E\times F}d\nu = \mu(E)\nu(F).$$

Suppose M and N are disjoint sets in $\mathcal{S}\times\mathcal{T}$. Since M_x and N_x are disjoint by Lemma 1, the additivity of ν implies $\nu[(M\cup N)_x]=\nu(M_x\cup N_x)=\nu(M_x)+\nu(N_x)$. Similarly for y-sections. Summarizing:

LEMMA 3. *If* M *and* N *are disjoint sets in* $\mathcal{S}\times\mathcal{T}$, *then* $f_{M\cup N}=f_M+f_N$, *and* $g^{M\cup N}=g^M+g^N$.

Suppose M_n is a sequence of sets in $\mathcal{S}\times\mathcal{T}$, and $M_n\uparrow M$. Since $(M_n)_x\uparrow M_x$ and ν is continuous from below [1, 9D], $\nu[(M_n)_x]\uparrow\nu(M_x)$. Evidently:

LEMMA 4. *If* $M_n\uparrow M$, *then* $f_{M_n}\uparrow f_M$ *and* $g^{M_n}\uparrow g^M$.

3. The product of two finite measure spaces. Throughout this section, $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are finite measure spaces. It is easy to see that a finite measure is bounded [1, 9D]; we write $\|\mu\|_\infty=$ l.u.b. $\{\mu(E): E\in\mathcal{S}\}$, and similarly for $\|\nu\|_\infty$. Evidently:

LEMMA 5. *For every* M *in* $\mathcal{S}\times\mathcal{T}$, $0\leqq f_M\leqq\|\nu\|_\infty$ *and* $0\leqq g^M\leqq\|\mu\|_\infty$.

Quoting [1, 9E], we have:

LEMMA 6. *If* $M_n\downarrow M$, *then* $f_{M_n}\downarrow f_M$ *and* $g^{M_n}\downarrow g^M$.

LEMMA 7. *For every* M *in* $\mathcal{S}\times\mathcal{T}$, *the functions* f_M *and* g^M *are measurable.*

Proof. Let us denote by $\mathcal{R}$ the ring generated by the measurable rectangles $E\times F$; then, $\mathcal{S}(\mathcal{R})=\mathcal{S}\times\mathcal{T}$. Suppose first that M belongs to $\mathcal{R}$; by [1, 33E], M may be written as a finite disjoint union of rectangles $E\times F$, hence the measurability of f_M and g^M follow at once from Lemmas 2 and 3.

Denote by $\mathfrak{M}$ the class of all sets M such that both f_M and g^M are measurable. Since $\mathfrak{M}$ contains $\mathcal{R}$, and is clearly a montone class by Lemmas 4 and 6, $\mathfrak{M}$ contains $\mathcal{S}\times\mathcal{T}$ by the lemma on monotone classes.

For the sake of completeness, we include a proof of the following variant of [1, 27E]:

LEMMA 8. *On a finite measure space, every bounded measurable function is integrable.*

Proof. Say $f: X \to R$ is measurable with respect to $\mathcal{S}$, and $|f(x)| \leqq K < \infty$ for all x. Let $E = \{x : f(x) \neq 0\}$; then E is measurable, and $|f| \leqq K\chi_E$. Since $\mu(E) < \infty$, $K\chi_E$ is integrable, hence so is f by [1, 27A].

LEMMA 9. *For every M in $\mathcal{S} \times \mathcal{T}$, the functions f_M and g^M are integrable, and*

$$(*) \qquad \int f_M d\mu = \int g^M d\nu.$$

Proof. The functions f_M and g^M are bounded by Lemma 5, measurable by Lemma 7, hence integrable by Lemma 8.

If $M = E \times F$, then M satisfies (*) by Lemma 2. Suppose next that M belongs to $\mathcal{R}$, with notation as in Lemma 7; writing M as a finite disjoint union of rectangles $E \times F$, it is clear from Lemma 3 that M also satisfies (*).

Let us define set functions ϕ and ψ on $\mathcal{S} \times \mathcal{T}$ by the formulas

$$\phi(M) = \int f_M d\mu$$

$$\psi(M) = \int g^M d\nu.$$

We shall show that ϕ and ψ are (finite) measures on $\mathcal{S} \times \mathcal{T}$. Clearly $\phi(\varnothing) = 0$, and $M \subset N$ implies $\phi(M) \leqq \phi(N)$ by part (iii) of Lemma 1; moreover, ϕ is finitely additive by Lemma 3. If $M_n \uparrow M$, then $f_{M_n} \uparrow f_M$ by Lemma 4, hence $\phi(M_n) \uparrow \phi(M)$ by the monotone convergence theorem. It follows at once that ϕ is a finite measure on $\mathcal{S} \times \mathcal{T}$. Similarly for ψ.

Since $\phi = \psi$ on $\mathcal{R}$, it follows from the unique extension theorem that $\phi = \psi$ on $\mathcal{S} \times \mathcal{T}$; in other words, (*) holds for every M in $\mathcal{S} \times \mathcal{T}$.

THEOREM 1. *If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are finite measure spaces, there exists a unique measure π on $\mathcal{S} \times \mathcal{T}$ such that*

$$\pi(E \times F) = \mu(E)\nu(F)$$

for every measurable rectangle $E \times F$. Moreover, π is finite, and for every M in $\mathcal{S} \times \mathcal{T}$,

$$\pi(M) = \int f_M d\mu = \int g^M d\nu.$$

Proof. Define $\pi: \mathcal{S} \times \mathcal{T} \to R$ by $\pi(M) = \int f_M d\mu$; as shown in the proof of Lemma 9, π is a finite measure, and $\pi(M) = \int g^M d\nu$. By Lemma 2, $\pi(E \times F) = \mu(E)\nu(F)$.

Suppose ρ is any measure on $\mathcal{S} \times \mathcal{T}$ such that $\rho(E \times F) = \mu(E)\nu(F)$ for every measurable rectangle $E \times F$. Evidently ρ is a finite measure on the ring $\mathcal{R}$ generated by the $E \times F$, and $\rho = \pi$ on $\mathcal{R}$, hence $\rho = \pi$ on $\mathcal{S} \times \mathcal{T}$ by the unique extension theorem.

DEFINITION 2. *With notation as in Theorem* 1, *we write* $\pi=\mu\times\nu$ *for the unique (finite) measure on* $\mathcal{S}\times\mathcal{T}$ *such that* $(\mu\times\nu)(E\times F)=\mu(E)\nu(F)$ *for every measurable rectangle* $E\times F$.

It should be observed that in the foregoing (and ensuing) theory of product measures, one can get along with x-sections and the functions f_M. Nevertheless, a symmetric development is desirable preparation for the main application, Fubini's theorem, and costs little in the way of extra exertion.

4. The l.u.b. of a directed family of measures. Suppose now we are given arbitrary measure spaces $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$; we wish to define the "product" measure of a set M in $\mathcal{S}\times\mathcal{T}$. If $P\times Q$ is a rectangle with sides of finite measure, it is plausible that $(P\times Q)\cap M$ should be susceptible of being measured by the finite techniques of the preceding section, yielding, so to speak, the $P\times Q$-th measure of M; the "true" measure of M can then be taken to be the least upper bound of the $P\times Q$-th measures, as $P\times Q$ varies over all finite rectangles. In speaking of the "true" measure of M, we do not mean to imply that M is in any sense exhausted by finite rectangles, for it is quite possible that there exist points of M not covered by any finite rectangle; however, such pathology cannot arise if the factor spaces are σ-finite, thus the construct of product measure to be given in the next section will yield, for σ-finite measure spaces, the usual satisfactory product measure, well-behaved enough to yield a Fubini theorem.

In the present section, we explain the necessary technique for taking least upper bounds. Let us denote by I a fixed set of indices, partially ordered, and directed to the right: for each pair of indices i and j, there exists an index k such that $i\leqq k$ and $j\leqq k$.

A family $\{\alpha_i: i\in I\}$ of extended real numbers, satisfying $0\leqq\alpha_i\leqq\infty$, is said to be *increasingly directed* in case $i\leqq j$ implies $\alpha_i\leqq\alpha_j$; if $\alpha=$l.u.b. α_i, we shall then write $\alpha_i\uparrow\alpha$. Keeping in mind that the extended real numbers involved are $\geqq 0$, and operating under the arithmetic conventions made in [1, §0], the following is an elementary calculation with least upper bounds:

LEMMA 10. *If* $\alpha_i\uparrow\alpha$ *and* $\beta_i\uparrow\beta$, *then* $\alpha_i+\beta_i\uparrow\alpha+\beta$ *and* $\alpha_i\beta_i\uparrow\alpha\beta$.

DEFINITION 3. *Let* $\{\mu_i: i\in I\}$ *be a family of measures defined on a ring* $\mathcal{R}$. *The family is said to be increasingly directed in case* $i\leqq j$ *implies* $\mu_i\leqq\mu_j$; *that is, for each* M *in* $\mathcal{R}$, $\{\mu_i(M): i\in I\}$ *is an increasingly directed family of extended real numbers.*

LEMMA 11. *If* $\{\mu_i: i\in I\}$ *is an increasingly directed family of measures on a ring* $\mathcal{R}$, *and if* μ *is the set function on* $\mathcal{R}$ *defined by the formula*

$$\mu(M) = \text{l.u.b.}_i\, \mu_i(M),$$

then μ *is a measure on* $\mathcal{R}$. *Notation*: $\mu=$l.u.b. μ_i.

Proof. It is clear from the definition of μ that $\mu(\varnothing)=0$, and $M\subset N$ implies $\mu(M)\leqq\mu(N)$. If M and N are disjoint, then $\mu_i(M\cup N)=\mu_i(M)+\mu_i(N)\uparrow\mu(M)+\mu(N)$ by Lemma 10, hence μ is finitely additive.

If M_n is an increasing sequence of sets in $\mathfrak{R}$ whose union M is in $\mathfrak{R}$, thus $M_n \uparrow M$, then $\mu(M_n) \uparrow \mu(M)$ results from the associativity of least upper bounds: $\mu(M) = \text{l.u.b.}_i\ \mu_i(M) = \text{l.u.b.}_i\ [\text{l.u.b.}_n\ \mu_i(M_n)] = \text{l.u.b.}_n\ [\text{l.u.b.}_i\ \mu_i(M_n)] = \text{l.u.b.}_n\ \mu(M_n)$. Combining this result with finite additivity, it is easy to see that μ is countably additive, hence is a measure. For future use, we note the following:

LEMMA 12. *If μ_1, μ_2 are finite measures on $\mathfrak{S}$, and ν_1, ν_2 are finite measures on $\mathfrak{T}$, such that $\mu_1 \leqq \mu_2$ and $\nu_1 \leqq \nu_2$, then $\mu_1 \times \nu_1 \leqq \mu_2 \times \nu_2$ on $\mathfrak{S} \times \mathfrak{T}$.*

Proof. The class $\mathfrak{M}$ of all M in $\mathfrak{S} \times \mathfrak{T}$ such that $\mu_1 \times \nu_1(M) \leqq \mu_2 \times \nu_2(M)$ is evidently closed under finite disjoint unions, and contains every measurable rectangle $E \times F$, hence it contains the ring $\mathfrak{R}$ they generate. If M_n is a sequence of sets in $\mathfrak{M}$, and $M_n \uparrow M$, then M belongs to $\mathfrak{M}$ by a general property of measures; if $M_n \downarrow M$, then M belongs to $\mathfrak{M}$ by a property of finite measures. Thus, $\mathfrak{M}$ is a monotone class containing $\mathfrak{R}$, hence it contains $\mathfrak{S}(\mathfrak{R}) = \mathfrak{S} \times \mathfrak{T}$.

5. The product of any two measure spaces. Throughout this section $(X, \mathfrak{S}, \mu)$ and $(Y, \mathfrak{T}, \nu)$ are fixed measure spaces, not necessarily finite. For each E in $\mathfrak{S}$, we denote by μ_E the *contracted* set function defined on $\mathfrak{S}$ by the formula $\mu_E(E_1) = \mu(E \cap E_1)$; clearly μ_E is a measure on $\mathfrak{S}$, and $\mu_E \leqq \mu$.

We denote by $\mathfrak{S}_\phi$ the set of all P in $\mathfrak{S}$ such that $\mu(P) < \infty$; evidently $\mathfrak{S}_\phi$ is a ring, and is moreover an "ideal" in $\mathfrak{S}$, that is, $P \cap E$ belongs to $\mathfrak{S}_\phi$ for every P in $\mathfrak{S}_\phi$ and E in $\mathfrak{S}$. Similarly, $\mathfrak{T}_\phi$ is the set of all Q in $\mathfrak{T}$ such that $\nu(Q) < \infty$. The use of the letters P and Q is meant to indicate the rings from which the sets are drawn. A set of the form $P \times Q$ will be called a *finite rectangle.*

LEMMA 13. *For every finite rectangle $P \times Q$, there exists a unique (finite) measure $\tau^{P \times Q}$ on $\mathfrak{S} \times \mathfrak{T}$ such that*

$$\tau^{P \times Q}(E \times F) = \mu(P \cap E)\nu(Q \cap F)$$

for every measurable rectangle $E \times F$.

Proof. Apply Theorem 1 to the finite measure spaces $(X, \mathfrak{S}, \mu_P)$ and $(Y, \mathfrak{T}, \nu_Q)$; in other words, take $\tau^{P \times Q} = \mu_P \times \nu_Q$.

The class of all finite rectangles $P \times Q$, ordered by inclusion, is evidently directed to the right. We propose to apply Lemma 11 to the family of measures $\tau^{P \times Q}$, hence must check first that the family is increasingly directed:

LEMMA 14. *If $P_1 \times Q_1 \subset P_2 \times Q_2$, then $\tau^{P_1 \times Q_1} \leqq \tau^{P_2 \times Q_2}$.*

Proof. If $P_1 \times Q_1$ is empty, the conclusion is clear; otherwise, $P_1 \subset P_2$ and $Q_1 \subset Q_2$. For any E in $\mathfrak{S}$, $\mu_{P_1}(E) = \mu(P_1 \cap E) \leqq \mu(P_2 \cap E) = \mu_{P_2}(E)$, thus $\mu_{P_1} \leqq \mu_{P_2}$; similarly, $\nu_{Q_1} \leqq \nu_{Q_2}$, hence our assertion follows from Lemma 12.

DEFINITION 4. *According to Lemma* 11, *the formula*

$$\pi(M) = \text{l.u.b.}\ \{\tau^{P \times Q}(M) : P \in \mathfrak{S}_\phi,\ Q \in \mathfrak{T}_\phi\}$$

defines a measure π on $\mathfrak{S} \times \mathfrak{T}$. Briefly, $\pi = \text{l.u.b.}\ \tau^{P \times Q}$.

LEMMA 15. *If* $P_1\times Q_1\subset P_2\times Q_2$, *then the contraction* $(\tau^{P_2\times Q_2})_{P_1\times Q_1}$ *is identical with* $\tau^{P_1\times Q_1}$.

Proof. Let us write $\tau_k=\tau^{P_k\times Q_k}$ $(k=1, 2)$. If $P_1\times Q_1$ is empty, the conclusion is clear; otherwise, $P_1\subset P_2$ and $Q_1\subset Q_2$. For any measurable rectangle $E\times F$, $(\tau_2)_{P_1\times Q_1}(E\times F)=\tau_2[(P_1\times Q_1)\cap(E\times F)]=\tau_2[(P_1\cap E)\times(Q_1\cap F)]$ $=\mu_{P_2}(P_1\cap E)\nu_{Q_2}(Q_1\cap F)=\mu(P_2\cap P_1\cap E)\nu(Q_2\cap Q_1\cap F)=\mu(P_1\cap E)\nu(Q_1\cap F)$ $=\tau_1(E\times F)$; it follows that the finite measures $(\tau_2)_{P_1\times Q_1}$ and τ_1 agree on the ring $\mathfrak{R}$ generated by the $E\times F$, hence they agree on $\mathfrak{S}(\mathfrak{R})=\mathcal{S}\times\mathcal{T}$.

LEMMA 16. *Let* π *be the measure described in Definition* 4. *For every finite rectangle* $P\times Q$, *the contraction* $\pi_{P\times Q}$ *is identical with* $\tau^{P\times Q}$.

Proof. Given M in $\mathcal{S}\times\mathcal{T}$. If $P_1\times Q_1\supset P\times Q$, then $\tau^{P_1\times Q_1}[(P\times Q)\cap M]$ $=(\tau^{P_1\times Q_1})_{P\times Q}(M)=\tau^{P\times Q}(M)$ by Lemma 15; passing to the l.u.b. over $P_1\times Q_1$, clearly $\pi[(P\times Q)\cap M]=\tau^{P\times Q}(M)$.

THEOREM 2. *If* $(X, \mathcal{S}, \mu)$ *and* $(Y, \mathcal{T}, \nu)$ *are arbitrary measure spaces, there exists a unique measure* π *on* $\mathcal{S}\times\mathcal{T}$ *such that*
(1) $\pi(P\times Q)=\mu(P)\nu(Q)$ *for every finite rectangle* $P\times Q$,
(2) *for each* M *in* $\mathcal{S}\times\mathcal{T}$,

$$\pi(M)=\text{l.u.b.}\{\pi[(P\times Q)\cap M]: P\in\mathcal{S}_\phi,\ Q\in\mathcal{T}_\phi\}.$$

Proof. Let π be the measure described in Definition 4. If $P\times Q$ is any finite rectangle, then by Lemma 16, $\pi(P\times Q)=\pi[(P\times Q)\cap(P\times Q)]=\pi_{P\times Q}(P\times Q)$ $=\tau^{P\times Q}(P\times Q)=\mu(P\cap P)\nu(Q\cap Q)=\mu(P)\nu(Q)$; this establishes (1).

If M belongs to $\mathcal{S}\times\mathcal{T}$, then quoting Lemma 16 and the definition of π, we have (all l.u.b.'s are taken over the family of $P\times Q$'s) l.u.b. $\pi[(P\times Q)\cap M]$ $=$l.u.b. $\pi_{P\times Q}(M)=$l.u.b. $\tau^{P\times Q}(M)=\pi(M)$.

Suppose ρ is another measure on $\mathcal{S}\times\mathcal{T}$ satisfying (1) and (2). If $P\times Q$ is a finite rectangle, let us show first that the contraction $\rho_{P\times Q}$ is identical with $\pi_{P\times Q}$ on the ring $\mathfrak{R}$ generated by the measurable rectangles $E\times F$. For any such rectangle, $\rho_{P\times Q}(E\times F)=\rho[(P\times Q)\cap(E\times F)]=\rho[(P\cap E)\times(Q\cap F)]$ $=\mu(P\cap E)\nu(Q\cap F)=\tau^{P\times Q}(E\times F)=\pi_{P\times Q}(E\times F)$. Thus, $\rho_{P\times Q}$ and $\pi_{P\times Q}$ have restrictions to $\mathfrak{R}$ which are equal and finite; by the familiar application of the unique extension theorem, $\rho_{P\times Q}=\pi_{P\times Q}$ on $\mathcal{S}\times\mathcal{T}$. Then $\rho=\pi$ results at once from the conditions (2).

DEFINITION 5. *If* $(X, \mathcal{S}, \mu)$ *and* $(Y, \mathcal{T}, \nu)$ *are arbitrary measure spaces, we denote by* $\mu\times\nu$ *the unique measure on* $\mathcal{S}\times\mathcal{T}$ *such that*
(1) $(\mu\times\nu)(P\times Q)=\mu(P)\nu(Q)$ *for every finite rectangle,*
(2) $(\mu\times\nu)(M)=$l.u.b. $\{(\mu\times\nu)[(P\times Q)\cap M]: P\in\mathcal{S}_\phi, Q\in\mathcal{T}_\phi\}$ *for each* M *in* $\mathcal{S}\times\mathcal{T}$.

Thus, the result of Lemma 16 may be written as follows: $(\mu\times\nu)_{P\times Q}=\mu_P\times\nu_Q$.

LEMMA 17. *If* $E\times F$ *is a measurable rectangle with* σ*-finite sides, then* $(\mu\times\nu)(E\times F)=\mu(E)\nu(F)$.

Proof. Let P_n be a sequence in $\mathcal{S}_\phi$ such that $P_n \uparrow E$, and similarly let $Q_n \uparrow F$. Evidently $P_n \times Q_n \uparrow E \times F$, hence $(\mu \times \nu)(E \times F) = \text{l.u.b. } (\mu \times \nu)(P_n \times Q_n) = \text{l.u.b. } \mu(P_n)\nu(Q_n) = \mu(E)\nu(F)$ by Lemma 10.

THEOREM 3. *If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces, there exists a unique measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{T}$ such that*

$$(\mu \times \nu)(P \times Q) = \mu(P)\nu(Q) \tag{1}$$

for every finite rectangle $P \times Q$. Moreover, $\mu \times \nu$ is σ-finite, and

$$(\mu \times \nu)(E \times F) = \mu(E)\nu(F) \tag{1'}$$

for every measurable rectangle $E \times F$.

Proof. Let us denote by $\mathcal{R}_\phi$ the ring generated by the finite rectangles $P \times Q$; in view of the assumed σ-finiteness, it is easy to see that $\mathcal{S}(\mathcal{R}_\phi) = \mathcal{S} \times \mathcal{T}$. The measure $\mu \times \nu$ given by Definition 5 satisfies (1') by Lemma 17; since every set in $\mathcal{S} \times \mathcal{T}$ can be covered by a countable union of finite rectangles [1, Exercise 5.7], it is clear that $\mu \times \nu$ is σ-finite.

If ρ is any measure on $\mathcal{S} \times \mathcal{T}$ satisfying (1), its restriction to $\mathcal{R}_\phi$ is clearly a finite measure coinciding with $\mu \times \nu$, hence $\rho = \mu \times \nu$ on $\mathcal{S}(\mathcal{R}_\phi) = \mathcal{S} \times \mathcal{T}$ by the unique extension theorem.

We close with a few remarks on the application to Fubini's theorem. A function $h: X \times Y \to R_e$ is said to possess an *iterated integral* $\int\int h d\nu d\mu$ in case: there exists a set E in $\mathcal{S}$ of measure zero, and a function f integrable with respect to μ, such that for each x in $X - E$, the function $h_x(y) = h(x, y)$ is integrable with respect to ν, and $\int h_x d\nu = f(x)$. The iterated integral is then defined to be $\int\int h d\nu d\mu = \int f d\mu$. The formulation and proof of Fubini's theorem now follow as in [1]. The critical role played by σ-finiteness in this theory can be condensed to the following argument. If M is any measurable set in $X \times Y$, then $M \subset E \times F$ for a suitable measurable rectangle. Let $P_n \uparrow E$ and $Q_n \uparrow F$; then $P_n \times Q_n \uparrow E \times F$, and setting $M_n = M \cap (P_n \times Q_n)$, we have $M_n \uparrow M$. Thus, properties over M can be discussed as limits of properties over the M_n, where M_n is confined to a finite rectangle $P_n \times Q_n$, and can be treated in the context of the finite measure space $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu_{P_n} \times \nu_{Q_n})$.

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AN EXTENSION OF THE NONHOMOGENEOUS FARKAS THEOREM

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1. Introduction. In 1902, J. Farkas [6] proved a theorem concerning finite systems of homogeneous linear inequalities. Using the common dot product notation for vectors this theorem may be stated as follows.

THEOREM 1.1. *Let* $A = \{x_1, x_2, \cdots, x_n\}$ *be a nonempty finite subset of Euclidean m dimensional space E. Consider the system*

$$x \cdot x_i \geqq 0 \qquad (1 \leqq i \leqq n) \tag{1}$$

of homogeneous linear inequalities. The inequality

$$x \cdot x_0 \geqq 0 \tag{2}$$

is satisfied by every vector x which satisfies system (1) *if and only if there exist nonnegative numbers* λ_i $(1 \leqq i \leqq n)$ *such that*

$$x_0 = \sum_{i=1}^{n} \lambda_i x_i.$$

Recently, in [2], the following extension of Theorem 1.1 was proved.

THEOREM 1.2. *Let A be any nonempty subset of a real locally convex linear topological space E, let* A^* *denote the set of all continuous linear functionals* x^* *on E such that* $x^*(x) \geqq 0$ *for all x in A, and let B be the set of all elements x in E such that* $x^*(x) \geqq 0$ *for all* x^* *in* A^*. *Then B is the closed conical hull of A.*

The analogue of Theorem 1.1 for systems of nonhomogeneous linear inequalities is

THEOREM 1.3. *Let* $A = \{x_1, x_2, \cdots, x_n\}$ *be a nonempty finite subset of Euclidean m dimensional space E. Consider the system*

$$x \cdot x_i \geqq \alpha_i \qquad (1 \leqq i \leqq n), \tag{1}$$

where the x, x_i *are in E and the* α_i *are real numbers. The inequality*

$$x \cdot x_0 \geqq \alpha_0, \qquad x_0 \text{ in } E,\ \alpha_0 \text{ real}, \tag{2}$$

is satisfied by every vector x satisfying system (1) *if and only if there exist nonnegative numbers* λ_i $(1 \leqq i \leqq n)$ *such that*

$$x_0 = \sum_{i=1}^{n} \lambda_i x_i \quad \text{and} \quad \alpha_0 \leqq \sum_{i=1}^{n} \lambda_i \alpha_i.$$

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It is the purpose of this paper to state and prove a theorem which generalizes Theorem 1.3 in a manner analogous to the way Theorem 1.2 extends Theorem 1.1. The statement of this "generalized nonhomogeneous Farkas theorem" will be given in Section 2 (Theorem 2.1) after certain definitions have been listed and the notation which will be used has been explained. The proof of this theorem will be given in Section 4, Section 3 being devoted to the several preliminary lemmas needed.

Recently, R. J. Duffin, after reading an abstract of this paper, kindly communicated to the author that "points of contact" exist between Theorem 2.1 and his duality theorem for infinite programs (Theorem 1, [3]). These points of contact are strong indeed. In fact, Duffin's theorem contains the generalized nonhomogeneous Farkas theorem as a special case. Section 5 of this paper has been added to demonstrate this fact.

2. Definitions. Notation. Statement of theorem. The notation introduced here will be used throughout the remainder of this paper unless otherwise specifically mentioned. For terminology not explicitly defined in this section the reader may refer to standard reference sources such as [1], [4] and [7].

Let E denote a real locally convex linear topological space. Recall that a linear topological space is *locally convex* if it has a neighborhood base at the origin θ consisting of convex sets. R and R^+ will be used to denote the real number system (with its usual locally convex topology) and the set of nonnegative real numbers, respectively. The space $E \times R$ with its locally convex product topology will be denoted by $\mathcal{E}$. E^* and $\mathcal{E}^*$ denote the conjugate spaces of all continuous linear functionals on E and $\mathcal{E}$, respectively.

Let A be a nonempty subset of E. To each element x in A let α_x be a corresponding real number and let $\mathfrak{A}$ be the set of all elements (x, α_x) for x in A. If x^* is any element of E^* it is easily seen that f, defined by $f(x, \alpha) = x^*(x) - \alpha$, is a continuous linear functional on $\mathcal{E}$. $\mathfrak{A}^*$ denotes the set of all such elements f of $\mathcal{E}^*$ which have the property that $f(x, \alpha) = x^*(x) - \alpha$ and $f(x, \alpha_x) \geqq 0$ for all (x, α_x) in $\mathfrak{A}$. Let $\mathfrak{R}^+ = \{\theta\} \times R^+$, that is, $\mathfrak{R}^+$ is the set of all elements (θ, α) where θ is the origin of E and α is a nonnegative real number. Let $\mathfrak{B}$ signify the set of all (x, α) in $\mathcal{E}$ such that $f(x, \alpha) \geqq 0$ for all f in $\mathfrak{A}^*$.

The *conical hull* of $\mathfrak{A}$, denoted by $C(\mathfrak{A})$, is the set of all finite linear combinations with nonnegative coefficients of elements of $\mathfrak{A}$.

THEOREM 2.1. *If $\mathfrak{A}^*$ is nonempty then $\mathfrak{B} = \overline{C(\mathfrak{A}) - \mathfrak{R}^+}$, (the closure in the topology of $\mathcal{E}$ of $C(\mathfrak{A}) - \mathfrak{R}^+$).*

3. Preliminary discussion. Lemmas. Before giving the lemmas needed for the proof of Theorem 2.1 it is desirable to convince the reader that Theorem 2.1 does indeed extend Theorem 1.3. For this purpose suppose that, in Theorem 2.1, the space E is m dimensional Euclidean space with its usual locally convex topology, that the set A is the finite subset $A = \{x_1, x_2, \cdots, x_n\}$ of E, and that $\alpha_{x_i} = \alpha_i$ $(1 \leqq i \leqq n)$. Then $\mathfrak{A} = \{(x_i, \alpha_i) : 1 \leqq i \leqq n\}$ is a finite subset of $\mathcal{E}$. Since, in

this case, each element x^* of E^* is determined by an element x in E in the sense that x^* is in E^* if and only if there exists an x in E such that $x^*(y) = x \cdot y$ for all y in E, f is in $\mathfrak{A}^*$ if and only if $f(y, \beta) = x \cdot y - \beta$ for some x in E such that $x \cdot x_i - \alpha_i \geqq 0$ $(1 \leqq i \leqq n)$. Thus, (x_0, α_0) is in $\mathfrak{B}$ if and only if $x \cdot x_0 \geqq \alpha_0$ for all x such that $x \cdot x_i \geqq \alpha_i$ $(1 \leqq i \leqq n)$. Also (x_0, α_0) is in $C(\mathfrak{A}) - \mathfrak{R}^+$ if and only if $(x_0, \alpha_0) = (x_0, \beta_1) - (\theta, \beta_2)$ where (x_0, β_1) is in $C(\mathfrak{A})$ and (θ, β_2) is in $\mathfrak{R}^+$. Since (x_0, β_1) is in $C(\mathfrak{A})$ if and only if there exist nonnegative numbers λ_i $(1 \leqq i \leqq n)$ such that $x_0 = \sum_{i=1}^n \lambda_i x_i$ and $\beta_1 = \sum_{i=1}^n \lambda_i \alpha_i$, and (θ, β_2) is in $\mathfrak{R}^+$ if and only if $\beta_2 \geqq 0$, therefore (x_0, α_0) is in $C(\mathfrak{A}) - \mathfrak{R}^+$ if and only if $x_0 = \sum_{i=1}^n \lambda_i x_i$ and $\alpha_0 = \sum_{i=1}^n \lambda_i \alpha_i - \beta_2$. Thus, using the notation of Theorem 2.1, the conclusion of Theorem 1.2 reads: $\mathfrak{B} = C(\mathfrak{A}) - \mathfrak{R}^+$. To complete the argument that Theorem 2.1 is an extension of Theorem 1.3 it is only needed to show that if E is Euclidean m space and A is finite then

$$\mathfrak{C}(\mathfrak{A}) - \mathfrak{R}^+ = \overline{\mathfrak{C}(\mathfrak{A}) - \mathfrak{R}^+}.$$

This fact is an immediate consequence of Theorem 1.3. The following lemmas are included here, however, because they may be of interest in themselves and because, taken in order, they constitute a proof of this fact, independent of Theorem 1.3.

LEMMA 3.1 (H. E. CLARK). *Let X be any real linear space and S any nonempty subset of X. If $\mathfrak{F}$ denotes the family of all finite linearly independent subsets of S then $C(S) = \cup\{C(F): F \in \mathfrak{F}\}$.*

Proof. For each $F \in \mathfrak{F}$, $F \subset S$ and, hence, $C(F) \subset C(S)$. Thus $\cup\{C(F): F \in \mathfrak{F}\} \subset C(S)$. To prove the opposite inclusion assume the contrary. That is, assume there exists an element x_0 in $C(S)$ such that for every linearly independent subset $\{y_1, y_2, \cdots, y_t\}$ of S for which $x_0 = \sum_{i=1}^t \beta_i y_i$ it is true that $\beta_i < 0$ for at least one of the coefficients β_i. Clearly we may take $x_0 \neq \theta$. Let r be the smallest number of elements of S such that

$$x_0 = \sum_{i=1}^r \alpha_i x_i, \quad \alpha_i > 0, \quad x_i \in S, \quad x_i \neq \theta \qquad (1 \leqq i \leqq r). \tag{1}$$

That x_0 has such a representation follows from the definition of $C(S)$ and the fact that $x_0 \neq \theta$. Let F_r denote the set $\{x_1, x_2, \cdots, x_r\}$ of elements used in the representation (1) of x_0. Let T_s be a maximal linearly independent subset of F_r. By a re-ordering of F_r, if necessary, we can write $T_s = \{x_1, x_2, \cdots, x_s\}$, $s \leqq r$. T_s spans the same subspace as does F_r and, in particular, x_0 can be expressed as a linear combination of the elements of T_s,

$$x_0 = \sum_{i=1}^s \beta_i x_i. \tag{2}$$

By assumption at least one β_i is negative. Let $c = \max\{-\beta_i/\alpha_i: 1 \leqq i \leqq s\}$. For some k, $1 \leqq k \leqq s$, $c = -\beta_k/\alpha_k$. By reordering F_r and T_s if necessary, $c = -\beta_1/\alpha_1$.

Multiplying (1) by c, adding this to (2) and defining $\beta_i=0$ for $s+1\leqq i\leqq r$, we get

$$x_0 = \sum_{i=1}^{r} \frac{(c\alpha_i + \beta_i)}{c+1} x_i. \tag{3}$$

For each i, $c\alpha_i+\beta_i\geqq 0$ and $c\alpha_1+\beta_1=-\beta_1+\beta_1=0$. Thus,

$$x_0 = \sum_{i=2}^{r} \gamma_i x_i, \tag{4}$$

where $\gamma_i=(c\alpha_i+\beta_i)/(c+1)\geqq 0$, $(2\leqq i\leqq r)$. Since (4) is a representation of type (1) of x_0 using fewer than r elements the desired contradiction has been achieved.

LEMMA 3.2. *Let X be any real linear topological T_1-space. If F is a finite, linearly independent subset of X then $C(F)$ is closed.*

Proof. Let Y be the linear subspace of X spanned by F. Y is closed in X and is itself an n dimensional real linear topological T_1-space with basis $F=\{x_1, x_2, \cdots, x_n\}$. The mapping h defined by $h(\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n)=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a linear homeomorphism of Y onto n dimensional Euclidean space E^n. The image of $C(F)$ under h is the closed first orthant $\{(\alpha_1, \alpha_2, \cdots, \alpha_n): \alpha_i\geqq 0, 1\leqq i\leqq n\}$ in E^n and $C(F)$ is therefore, closed in Y. But since Y is closed in X, also $C(F)$ is closed in X.

LEMMA 3.3. *Let X be any real linear topological T_1-space. If S is any finite subset of X then $C(S)$ is closed.*

Proof. By Lemma 3.1, $C(S)=\cup\{C(F): F\in\mathfrak{F}\}$. By Lemma 3.2, for each $F\in\mathfrak{F}$, $C(F)$ is a closed set. Since S is finite, the family $\mathfrak{F}$ is finite and, being the union of a finite number of closed sets, $C(S)$ is closed.

LEMMA 3.4. *If E is a real linear topological T_1-space and $\mathfrak{A}$ is a finite subset of $\mathcal{E}=E\times R$ then $C(\mathfrak{A})-\mathfrak{R}^+$ is closed in $\mathcal{E}$.*

Proof. It is easily verified that $C(\mathfrak{A})-\mathfrak{R}^+=C(\mathfrak{A}\cup\{(\theta, -1)\})$. Being the topological product of T_1-spaces, $\mathcal{E}$ is a T_1-space. Since $\mathfrak{A}\cup\{(\theta, -1)\}$ is a finite subset of $\mathcal{E}$, $C(\mathfrak{A}\cup\{(\theta, -1)\})=C(\mathfrak{A})-\mathfrak{R}^+$ is closed in $\mathcal{E}$ by Lemma 3.3.

The remainder of this section is devoted to those lemmas leading directly to the proof of Theorem 2.1.

LEMMA 3.5. *If $(x_0, \alpha_0)\notin\overline{C(\mathfrak{A})-\mathfrak{R}^+}$ then there exists an f_0 in $\mathcal{E}^*$ such that $f_0(x_0, \alpha_0)<0$ and $f_0(x, \alpha)\geqq 0$ for all (x, α) in $\overline{C(\mathfrak{A})-\mathfrak{R}^+}$.*

Proof. Since $\mathcal{E}$ is a locally convex space, $\overline{C(\mathfrak{A})-\mathfrak{R}^+}$ is convex, and $\{(x_0, \alpha_0)\}$ is compact, an application of Proposition 4, page 73 of [1] insures the existence of an element f_0 of $\mathcal{E}^*$ such that $f_0(x_0, \alpha_0)<\beta$ and $\overline{C(\mathfrak{A})-\mathfrak{R}^+}\subset\{(x, \alpha): f_0(x, \alpha)>\beta\}$. That $\beta<0$ and that $\overline{C(\mathfrak{A})-\mathfrak{R}^+}\subset\{(x, \alpha): f_0(x, \alpha)\geqq 0\}$ follows easily from the fact that $\overline{C(\mathfrak{A})-\mathfrak{R}^+}$ is a convex cone with vertex $(\theta, 0)$ in $\mathcal{E}$. (See Lemma 2.1 in [2].)

The proof of the following lemma is a straightforward application of basic definitions and, to save space, is omitted.

LEMMA 3.6. *$f\in\mathcal{E}^*$ if and only if there exists an x_f^* in E^* and a t_f in R such that for all (x, α) in $\mathcal{E}$, $f(x, \alpha)=x_f^*(x)-\alpha t_f$.*

4. Proof of Theorem 2.1. Since

$$\mathcal{B}=\bigcap_{f\in\mathcal{A}^*}\{(x, \alpha): f(x, \alpha)\geqq 0\}$$

is the intersection of closed half spaces in $\mathcal{E}$, $\mathcal{B}$ is a closed convex cone with vertex $(\theta, 0)$ in $\mathcal{E}$. If $(x, \alpha_x)\in\mathcal{A}$, then $f(x, \alpha_x)\geqq 0$ for all f in $\mathcal{A}^*$ so $\mathcal{A}\subset\mathcal{B}$ and since $\mathcal{B}$ is a convex cone, $C(\mathcal{A})\subset\mathcal{B}$. If (y, β) is any element of $C(\mathcal{A})-\mathcal{R}^+$ then $(y, \beta)=(y, \beta_1)-(\theta, \beta_2)$ where $(y, \beta_1)\in C(\mathcal{A})$ and $\beta_2\geqq 0$. For any f in $\mathcal{A}^*$, $f(y, \beta_1)\geqq 0$ since $C(\mathcal{A})\subset\mathcal{B}$ and, because $f(x, \alpha)=x^*(x)-\alpha$ for some x^* in E^* and all (x, α) in $\mathcal{E}$, $f(\theta, \beta_2)=x^*(\theta)-\beta_2=-\beta_2$. Thus, f in $\mathcal{A}^*$ and (y, β) in $C(\mathcal{A})-\mathcal{R}^+$ imply $f(y, \beta)=f(y, \beta_1)+\beta_2\geqq 0$ and, consequently, $C(\mathcal{A})-\mathcal{R}^+\subset\mathcal{B}$. Since $\mathcal{B}$ is closed, $\overline{C(\mathcal{A})-\mathcal{R}^+}\subset\mathcal{B}$.

Assume that there exists an element (x_0, α_0) in $\mathcal{B}$ with (x_0, α_0) not in $\overline{C(\mathcal{A})-\mathcal{R}^+}$. We shall arrive at the desired contradiction by exhibiting an element h in $\mathcal{A}^*$ such that $h(x_0, \alpha_0)<0$ (which is impossible if $(x_0, \alpha_0)\in\mathcal{B}$). By Lemma 3.5 there exists an f_0 in $\mathcal{E}^*$ such that $f_0(x_0, \alpha_0)<0$ and $f_0(x, \alpha)\geqq 0$ for all (x, α) in $\overline{C(\mathcal{A})-\mathcal{R}^+}$. By Lemma 3.6 there exists an x_0^* in E^* and a t_0 in R such that, for all (x, α) in $\mathcal{E}$, $f_0(x, \alpha)=x_0^*(x)-\alpha t_0$. Since $(\theta, -1)\in\overline{C(\mathcal{A})-\mathcal{R}^+}$, $f_0(\theta, -1)=x_0^*(\theta)-(-1)t_0=t_0\geqq 0$. There are now two possibilities to consider: $t_0>0$ or $t_0=0$.

If $t_0>0$ define $h=(1/t_0)f_0$. For each (x, α_x) in $\mathcal{A}$, $h(x, \alpha_x)=(1/t_0)f_0(x, \alpha_x)\geqq 0$ since $\mathcal{A}\subset\overline{C(\mathcal{A})-\mathcal{R}^+}$ and f_0 is nonnegative on $\overline{C(\mathcal{A})-\mathcal{R}^+}$. For each (x, α) in $\mathcal{E}$, $h(x, \alpha)=(1/t_0)f_0(x, \alpha)=(1/t_0)[x_0^*(x)-\alpha t_0]=(1/t_0)x_0^*(x)-\alpha=x_1^*(x)-\alpha$ where $x_1^*=(1/t_0)x_0^*\in E^*$. Thus, $h\in\mathcal{A}^*$. But $h(x_0, \alpha_0)=(1/t_0)f_0(x_0, \alpha_0)<0$ since $f_0(x_0, \alpha_0)<0$ and $t_0>0$.

If $t_0=0$ then $f_0(x, \alpha)=x_0^*(x)$ for all (x, α) in $\mathcal{E}$. Let f_1 be any element in $\mathcal{A}^*$ (which, by hypothesis, is nonempty). Then $f_1(x, \alpha)=x_1^*(x)-\alpha$ for each (x, α) in $\mathcal{E}$ and some x_1^* in E^*. Since $f_1\in\mathcal{A}^*$, $f_1(x, \alpha)\geqq 0$ for all (x, α) in $\overline{C(\mathcal{A})-\mathcal{R}^+}$, and since $(x_0, \alpha_0)\in\mathcal{B}$, $f_1(x_0, \alpha_0)\geqq 0$. Let $k=-(f_1(x_0, \alpha_0)+1)/f_0(x_0, \alpha_0)$. Since $f_0(x_0, \alpha_0)<0$, $k>0$. Define $h=f_1+kf_0$. Certainly $h\in\mathcal{E}^*$, and $h(x, \alpha_x)\geqq 0$ for all (x, α_x) in $\mathcal{A}$. For each (x, α) in $\mathcal{E}$, $h(x, \alpha)=f_1(x, \alpha)+kf_0(x, \alpha)=[x_1^*(x)-\alpha]+kx_0^*(x)=[x_1^*(x)+kx_0^*(x)]-\alpha=x_2^*(x)-\alpha$ where $x_2^*=x_1^*+kx_0^*\in E^*$. Thus, $h\in\mathcal{A}^*$. Finally, $h(x_0, \alpha_0)=f_1(x_0, \alpha_0)+[-(f_1(x_0, \alpha_0)+1)/f_0(x_0, \alpha_0)]f_0(x_0, \alpha_0)=f_1(x_0, \alpha_0)-f_1(x_0, \alpha_0)-1=-1<0$. This completes the proof.

Comment. If $f\in\mathcal{E}^*$ and $f(x, \alpha)\geqq 0$ for all (x, α) in $\overline{C(\mathcal{A})-\mathcal{R}^+}$ then $\{(x, \alpha): f(x, \alpha)=0\}$ is called a *support* for $\overline{C(\mathcal{A})-\mathcal{R}^+}$. If $f(x, \alpha)=x^*(x)-\alpha t$ and $t\neq 0$, $\{(x, \alpha): f(x, \alpha)=0\}$ is said to be *nonvertical* and if $t=0$ it is said to be *vertical*. The second half of the above proof shows that not only is $\overline{C(\mathcal{A})-\mathcal{R}^+}$ the intersection of all the closed half spaces determined by its supports but, in fact,

$\overline{C(\mathfrak{A})-\mathfrak{R}^+}$ is the intersection of all closed half spaces determined by its non-vertical supports.

5. Duffin's duality theorem for infinite programs implies Theorem 2.1. The notation and definitions used before will continue to have the same meaning in this section. Before stating Duffin's duality theorem (denoted by D.D.T.) some additional notation and definitions will be needed.

Let V be any real linear space with a locally convex topology and let V^* denote its conjugate space. Let P be a closed convex cone with vertex at the origin in E and let Q be a closed convex cone with vertex at the origin in V. Let $P^*=\{u^*\in E^*: u^*(u)\geqq 0,\ u\in P\}$ and $Q^*=\{v^*\in V^*: v^*(v)\geqq 0,\ v\in Q\}$. A *program* is a set (T, v_0, u_0) where T is a linear transformation mapping E^* into V, $v_0\in V$ and $u_0\in E$. Its *dual program* is the set $(-S, -u_0, -v_0)$ where S is the adjoint to T, mapping V^* into E, such that, for all $u^*\in E^*$ and all $v^*\in V^*$, $u^*(Sv^*)=v^*(Tu^*)$. The program (T, v_0, u_0) is *consistent* if there exist $u^*\in P^*$ such that $Tu^*-v_0\in Q$. Each such u^* is said to be *feasible*. The *value* of a consistent program is: $M=\inf\{u^*(u_0): u^* \text{ is feasible}\}$. The dual program is *sub-consistent* if there exist sequences (nets if the spaces are not metric) $\{v_n^*\}$ in Q^* and $\{p_n\}$ in P such that $\lim (Sv_n^*+p_n)=u_0$. Such sequences are called *feasible*. The *sub-value* of this sub-consistent dual program is: $m'=-\sup\{\limsup v_n^*(v_0): \{v_n^*\} \text{ is feasible}\}$.

D.D.T. (THEOREM 1, [3]). *A program is consistent and has a finite value M if and only if its dual program is sub-consistent and has a finite sub-value m'. Moreover, $M+m'=0$.*

Now, to show that D.D.T. implies Theorem 2.1, let $V=\prod_{x\in A} R_x$, the topological product where, for each $x\in A$, $R_x=R$. Then, as is well known, V is the set of all real valued functions v with domain A, and V^* is the direct sum $\oplus_{x\in A} R_x$. Furthermore, $v^*(v)=\sum_{x\in A} v^*(x)v(x)$, this being a finite sum since $v^*(x)\neq 0$ for at most a finite number of x in A. Let $Q=\{v\in V: v(x)\geqq 0,\ x\in A\}$. Clearly Q is a closed convex cone with vertex at the origin in V and $Q^*=\{v^*\in V^*: v^*(x)\geqq 0,\ x\in A\}$. Let P be the one point set, the origin in E. Then $P^*=E^*$. Select T to be the linear transformation from E^* into V defined as follows: for each $u^*\in E^*$ let v be that element of V such that, for each $x\in A$, $v(x)=u^*(x)$, and define $Tu^*=v$. Finally, take $v_0\in V$ so that $v_0(x)=\alpha_x$ for each $x\in A$.

With the generality of D.D.T. restricted in the above manner we note that the program (T, v_0, u_0) is consistent if and only if there exists a $u^*\in E^*$ such that $u^*(x)-\alpha_x\geqq 0$ for each $x\in A$. Thus (T, v_0, u_0) is consistent if and only if $\mathfrak{A}^*\neq\varnothing$. Note also that if (T, v_0, u_0) is finite valued with value M then $u^*(u_0)\geqq M$ for each feasible u^* and, consequently, $(u_0, M)\in\mathfrak{B}$. Conversely, if $(u_0, \beta_0)\in\mathfrak{B}$ for any β_0 then (T, v_0, u_0) is finite valued with value $M\geqq\beta_0$.

Assume, as in Theorem 2.1, that $\mathfrak{A}^*\neq\varnothing$. Let (u_0, β_0) be any element of $\mathfrak{B}$. Then (T, v_0, u_0) has finite value $M\geqq\beta_0$. By D.D.T. the dual program

$(-S, -u_0, -v_0)$ is sub-consistent with finite sub-value m' and $M+m'=0$. If $v^* \in Q^*$ then, for any $u^* \in E^*$,

$$u^*(Sv^*) = v^*(Tu^*) = \sum_{x \in A} v^*(x)(Tu^*)(x) = \sum_{x \in A} v^*(x)u^*(x) = u^*\left(\sum_{x \in A} v^*(x)x \right)$$

Thus, $Sv^* = \sum_{x \in A} v^*(x)x$. Also $v^*(v_0) = \sum_{x \in A} v^*(x)\alpha_x$. Thus, $v^* \in Q^*$ implies $(Sv^*, v^*(v_0)) \in C(\mathfrak{A})$. Choose $\{v_n^*\}$ in Q^* such that $\{v_n^*\}$ is feasible and $\lim v_n^*(v_0) = -m' = M$. For each n, define $\beta_n = v_n^*(v_0) - (M - \beta_0)$. Then $(Sv_n^*, \beta_n) = (Sv_n^*, v_n^*(v_0)) - (\theta, M-\beta_0) \in C(\mathfrak{A}) - \mathfrak{R}^+$ and, $\lim (Sv_n^*, \beta_n) = \lim (Sv_n^*, v_n^*(v_0)) - (\theta, M-\beta_0) = (u_0, M) - (\theta, M-\beta_0) = (u_0, \beta_0)$. Thus, $(u_0, \beta_0) \in \overline{C(\mathfrak{A}) - \mathfrak{R}^+}$ and, because (u_0, β_0) was any element of $\mathfrak{B}$, $\mathfrak{B} \subset \overline{C(\mathfrak{A}) - \mathfrak{R}^+}$.

To get the reverse inclusion suppose now that (u_0, β_0) is any element of $\overline{C(\mathfrak{A}) - \mathfrak{R}^+}$. Let $\{(u_n, \beta_n)\}$ be in $C(\mathfrak{A}) - \mathfrak{R}^+$ such that $\lim u_n = u_0$ and $\lim \beta_n = \beta_0$. For each n, $\beta_n = \beta_n^{(1)} - \beta_n^{(2)}$, where $\beta_n^{(2)} \geqq 0$ and $(u_n, \beta_n^{(1)}) \in C(\mathfrak{A})$, i.e.,

$$u_n = \sum_{i=1}^{p_n} \lambda_{ni} x_{ni}, \qquad \beta_n^{(1)} = \sum_{i=1}^{p_n} \lambda_{ni} \alpha_{x_{ni}},$$

where $\lambda_{ni} \geqq 0$, $x_{ni} \in A$ for $i = 1, 2, \cdots, p_n$. For each n let v_n^* be that element of V^* defined by $v_n^*(x_{ni}) = \lambda_{ni}$, $i = 1, 2, \cdots, p_n$ and $v_n^*(x) = 0$ if $x \neq x_{ni}$, $i = 1, 2, \cdots, p_n$. Then $v_n^* \in Q^*$, $Sv_n^* = u_n$ and $v_n^*(v_0) = \beta_n^{(1)}$. Note that $\lim Sv_n^* = u_0$, $\{v_n^*\}$ is feasible and $(-S, -u_0, -v_0)$ is sub-consistent. Because $\mathfrak{A}^* \neq \varnothing$ there exist feasible u^*. For any feasible u^*, $u^*(Sv_n^*) = v_n^*(Tu^*) = v_n^*(Tu^* - v_0) + v_n^*(v_0) \geqq v_n^*(v_0)$ since $Tu^* - v_0 \in Q$ and $v_n^* \in Q^*$. Thus, for feasible u^*, $u^*(u_0) = \lim u^*(Sv_n^*) \geqq \lim v_n^*(v_0) = \lim \beta_n^{(1)} = \lim (\beta_n + \beta_n^{(2)}) \geqq \lim \beta_n = \beta_0$ and, since $u^*(u_0) \geqq \beta_0$, $(u_0, \beta_0) \in \mathfrak{B}$. Since (u_0, β_0) was arbitrary in $\overline{C(\mathfrak{A}) - \mathfrak{R}^+}$ we have $\overline{C(\mathfrak{A}) - \mathfrak{R}^+} \subset \mathfrak{B}$. This completes the demonstration that D.D.T. implies Theorem 2.1.

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PRELIMINARY RECOMMENDATIONS FOR AN HONORS PROGRAM

The Committee on the Undergraduate Program in Mathematics, supported by the National Science Foundation and the Ford Foundation, is a committee of the Mathematical Association of America, charged with making recommendations for the overall improvement of college and university mathematics curricula at all levels and in all educational areas. Moreover, the Committee devotes its energies to reasonable efforts in realizing these recommendations.

The Membership of the Committee is as follows: R. Creighton Buck, Chairman; E. G. Begle, Roy Dubisch, David Gale, W. T. Guy, Jr., Leon A. Henkin, John G. Kemeny, Edwin E. Moise, John C. Moore, Frederick Mosteller, Henry O. Pollak, G. Baley Price, I. M. Singer, Patrick Suppes, R. J. Walker, A. D. Wallace, Robert J. Wisner, Executive Director, Ex Officio; A. W. Tucker, Ex Officio.

To reflect one of its major concerns, the Committee has established a Panel on Pre-Graduate Training with additional mathematicians to aid with deliberations. This Panel is comprised as follows: John C. Moore, Chairman; Richard D. Anderson, Earl A. Coddington, Leon W. Cohen, Eldon Dyer, Samuel Karlin, Peter D. Lax, Kenneth O. May, Ivan Niven, I. M. Singer, A. D. Wallace, Gerard Washnitzer, R. C. Buck, ex officio; R. J. Wisner, ex officio.

The Panel began in 1960 by constructing an idealized program in terms of what might be suitable for honors students in mathematics, and the purpose of this paper is to report the resultant recommendations of this Panel, which have also received the approval of the full Committee.

Introduction. This preliminary report outlines an honors program in mathematics for undergraduate students. Although few institutions will wish to follow the program herein set forth in every detail, the proposal will nevertheless serve the following purposes:

1. It indicates the level of mathematical maturity desirable for the beginning graduate student who expects to create, teach, or apply the central disciplines of mathematics: algebra, analysis, and geometry.
2. It indicates changes that are now feasible in some colleges and universities, and which can be achieved in the near future at others.
3. It helps to locate the areas in which new textual material and new courses are needed.
4. It serves as a basis for designing curricula suitable for majors whose interests lie largely in applications, as well as for those outside the honors group.

Although this report is concerned primarily with mathematical content, the Panel is aware that there are other essential aspects of an effective program. A mere collection of material, however rich and full it is mathematically, cannot

guarantee understanding on the part of the student. We return to this important question at the end of the report.

In developing the program, we have taken for granted the need for closing the gap between undergraduate instruction and contemporary mathematics. The gap alluded to is not simply a matter of level or advancement. More than that, we are concerned with a gulf between the spirit and outlook of the undergraduate program and the current literature in mathematics. The trend that we urge, which is, of course, already observable at many colleges and universities, is facilitated by the new programs in secondary schools. Indeed, our proposal presupposes that the student has the background of the School Mathematics Study Group program in secondary school, or the equivalent. Thus, it is presumed that the student entering college with the intention of majoring in mathematics is prepared to begin a serious course in calculus—a body of theorems, not simply a collection of devices and procedures for solving a certain class of problems.

The Program. *Introductory Undergraduate Mathematics.* Whereas this report as a whole pertains to the programs of students majoring in mathematics, the introductory undergraduate courses we propose are suitable for a much wider class of honors students, namely all those seeking a solid foundation of basic mathematics. We recommend that for such students the beginning college program be devoted to a study of calculus, analytic geometry, linear algebra, and differential equations, with the main emphasis on calculus as in the traditional program. This comprises five semesters at three hours per week. Since the topics are integrated, it is not possible to make a clear separation into semester units. However, the proportions are roughly three semesters of calculus and analytic geometry, one semester of linear algebra, and one semester of differential equations. Since it totals five semesters, the work might well extend into the junior year in many institutions. (For schools not on the semester plan, our proposal should be subjected to the obvious transformation.) The three semesters of calculus encompass the traditional year course in the calculus of one variable and also topics in "advanced calculus," so that the calculus of functions in n-dimensional space is available in the early college years.

There are variations possible in the arrangement of all this material, partly because there is some flexibility in the order of presentation, and partly because of the ways the material can be integrated. The linear algebra, for example, can be merged with the calculus to form a four-semester sequence, or the differential equations can be merged with calculus. Thus, if an institution cannot include all this work in the first two years, it can postpone to the upperclass years part of the development of the calculus of several variables, or the linear algebra, or the differential equations.

However, in any arrangement a modicum of the algebra of vector spaces should be incorporated into a calculus course both to gain a richer formulation of the basic analysis and to provide an economical simplifying notation. Similarly,

we recommend the introduction of appropriate geometric and topological concepts (such as intervals, open and closed sets, neighborhoods, compactness) to illustrate the power of the mathematics under study, to add to its understanding, and to develop the student's intuition.

Higher Undergraduate Mathematics. This part of the program will be discussed from two viewpoints: that of the college or university department of mathematics, and that of the student. Every college department offering a major in mathematics should endeavor to provide the following courses at the upper division level:

Complex analysis
Abstract algebra
Geometry-topology
Probability or mathematical physics
Real analysis.

Not every student will take all of these courses; choices will depend on the student's intent.

In addition, to achieve a richer and more comprehensive program, a department should offer, as far as its resources will permit, a balanced selection of courses with topics selected from:

Algebra
Analysis
Applied mathematics (in both the natural and social sciences)
Foundations and logic
Geometry (algebraic, differential, or projective)
Mathematical statistics
Number theory
Topology.

Secondly, from the viewpoint of the student, the following principles should be observed:

(a) that in his upperclass years, at least three of the following four categories should be represented in his course program: (1) algebra; (2) analysis; (3) applied mathematics; (4) geometry-topology,
(b) that included in his program there should be, in order to achieve depth, at least two full year courses—that is, courses in which the first semester is an essential prerequisite to the second,
(c) that a student majoring in mathematics should have at least seven semester courses beyond our suggested introductory college mathematics.

Initiative and Independence. A description of the mathematical content of a sound contemporary undergraduate program in mathematics is only part of the story: a large part, to be sure, but not the whole story. For mathematics is not just so many definitions, postulates, and theorems to be learned by the student in a passive fashion. There is also the active side, with a student gradually becoming independent of his professors until he finally becomes a colleague.

Thus, the student must increasingly take the initiative, not only to construct proofs by himself, but also to develop his imaginative powers so that he can make conjectures for proof or disproof, even perhaps going on to contribute more fully by creating new concepts and theories that he adds to the world of mathematics.

This process has its beginning in a very small way when the student solves textbook problems. Another component is added when the student learns to read the textual material by himself, later making the passage to the reading of papers in the journals, which are more compactly written and therefore more difficult to read. There are other ways in which the undergraduate student can develop his initiative. Some schools pay a lot of attention to various ways of doing this, others very little. We urge that every school face this question, developing some appropriate scheme. Without attempting an exhaustive list, we mention a few common patterns.

There are seminars and colloquia of one kind or another wherein the student makes reports. There is the undergraduate thesis in which a student makes a contribution, original for him but not usually original in the larger sense. There is the developmental course, a version of the Socratic method, in which the student is led to develop a body of mathematical material under the skillful guidance of the professor. The number of teachers who employ the developmental method in its full-blown form is not large, but they are an enthusiastic and dedicated band of people in their devotion to this procedure. A partial, or modified, use of the developmental method is employed widely. It would be desirable for all schools to give attention to the problem of enrolling the student actively in the study of mathematics beyond a mere passive attendance at lectures.

Present Action and Future Plans. Free copies of this document are available now and may be obtained by writing to the CUPM Central Office. In addition, a more complete report including detailed outlines of some of the suggested courses will soon be available to assist colleges in undertaking to institute honors curricula.

It is hoped that the program herein outlined will be discussed at mathematics departmental meetings, at section meetings of the Mathematical Association of America, and at meetings of other interested groups. Regional conferences will be held in order to explain this program in greater detail and on a more personal basis.

For the future, CUPM hopes to modify this program in order that the "normal" undergraduate mathematics major might be better prepared in his consideration of a career in mathematics. In the meantime, we hope that the reader will feel free to add his remarks and suggestions. Such correspondence, as well as requests for additional free copies of this preliminary report, for a more complete final report, or for outlines of suggested courses, should be addressed to the CUPM Central Office: Committee on the Undergraduate Program in Mathematics, Post Office Box 1000, Pontiac, Michigan.

AN ANALYST'S BOOKSHELF

PETER HAGIS, JR., Temple University

At the recent joint meeting of the M.A.A., A.M.A., and A.A.M. held at Bourbaki A. & M. I was most delighted to encounter my old friend and teacher, Professor P. E. B. Umbugio. On the evening of the final day, after the last paper had been presented, the professor and I, along with several other mathematicians, dined at the local inn before going our several ways. The after-dinner conversation, general at first, eventually turned to books and in particular to a discussion of great works in mathematics. Professor Umbugio, as one of the greatest of living analysts, was asked to name the 20 books which had most influenced his career. After several minutes of deep and considered thought he slowly ticked off the list which appears below. To the surprise of all present he mentioned none of the epoch making works of men such as Euler, Gauss, or Lagrange and none of the modern treatises which are ordinarily considered as the standard works in their fields. When questioned about these omissions the professor replied that while he was quite aware of the importance of these books in the development and expansion of mathematics in general he personally, having had neither the time nor the patience to wade through them, had been influenced by them not at all. Such a highly personalized reading list, drawn up by you or me, would, of course, be of little interest or significance to others. Such a list drawn up by an eminent analyst such as Professor Umbugio is another matter altogether. It should certainly be made available to as wide a circle of mathematicians as possible. It is with this in mind that I have decided to send the list to this journal for publication. Unfortunately, as I took no notes at the time and have not been in contact with Professor Umbugio since then, I have had to rely entirely on my memory. Therefore, some of the titles may not be exactly as stated by the professor. I, of course, take full responsibility for any such inaccuracies. Here then is Professor Umbugio's list.

1. The Jacobians and Their Struggle for Independence
2. A Ten Day Diet to Improve Indeterminate Forms
3. Cheaper by the Googol
4. 1001 Best Loved Double Integrals
5. The Torus and I
6. A Short Table of Even Primes (Abridged)
7. Will Success Spoil Runge-Kutta?
8. Dining Out in Hilbert Space
9. 100 Tasty Fillings for Empty Sets
10. Life Begins at e^{π}
11. How to Keep Condensation Points from Dripping into Open Sets
12. A Child's Garden of Tchebycheff Polynomials
13. Tom Swift and his Electric Cycloid
14. A Treasury of Matrices—Upright and Inverted

15. Improving Lipschitz Conditions in the Slums of New York
16. The Decline and Fall of e^{-x}
17. How to Prevent Rust on Riemann Surfaces
18. The Peano Postulates Transcribed for Violin and Cello
19. First Aid for Dedekind Cuts and Bruises*
20. A Collection of Happy Endings for Incomplete Beta Functions

* An appropriate subtitle, believed to have been originated by P. D. Lax, might be "What to do Until the Mathematician Arrives." (Ed.)

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida.

CONCERNING A THEOREM OF HADAMARD

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1. Introduction. The purpose of this paper is to use a certain property of complex number sequences (Theorem 2.1) to state and prove a well-known result of Hadamard in a form which appears to be quite useful (Theorem 2.2). In fact, a theorem of Agnew [1, Theorem 4] is shown to be an immediate corollary of our restatement of Hadamard's result. We also use Theorem 2.1 to extend to the complex case a theorem of Dini concerning positive term sequences.

2. Hadamard's Theorem. For convenience, let S denote the set of all complex number sequences. The first theorem of this section was proved in [2, 3].

THEOREM 2.1. *If* $\{z_i\}_{i=1}^{\infty} \in S$, *then* $\sum_{i=N}^{\infty} |1 - z_{i+1}/z_i|$ *converges for some* N *if and only if* $z_1 + \sum_{i=1}^{\infty} (z_{i+1} - z_i)$ *converges absolutely and its sum is not* 0.

Hadamard's main result in [4] may be stated as follows:

THEOREM 2.2. *Suppose* $\{p_i\}_{i=1}^{\infty} \in S$. *The following two statements are equivalent:*

1) *if* $\{a_i\}_{i=1}^{\infty} \in S$, *then* $\sum_{i=1}^{\infty} a_i$ *and* $\sum_{i=1}^{\infty} p_i a_i$ *both converge or both diverge, and*
2) *the series* $\sum_{i=N}^{\infty} |1 - p_{i+1}/p_i|$ *converges for some* N.

Proof. Suppose no $p_i = 0$. Let $H(h_{ij})$ and $H'(h'_{ij})$ be triangular matrices defined as follows:

$$h_{ij} = \begin{cases} 0 & \text{if } i < j \\ p_j - p_{j+1} & \text{if } j < i \\ p_i & \text{if } i = j, \end{cases} \qquad h'_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1/p_j - 1/p_{j+1} & \text{if } j < i \\ 1/p_i & \text{if } i = j. \end{cases}$$

Suppose $\sum_{p=N}^{\infty} |z_p/Z_{p-1}|$ converges for some N. If $p>N$, then $|z_p/Z_{p-1}| = |(Z_{p-1}-Z_p)/Z_{p-1}|$. Hence by Theorem 2.1, $Z_p \to u \neq 0$ as $p \to \infty$, contrary to the assumption that $\sum_{p=0}^{\infty} z_p$ converges.

Suppose $\sum_{p=N}^{\infty} |w_p/W_p|$ converges for some N. If $p>N$, then $|w_p/W_p| = |(W_p-W_{p-1})/W_p| = |1-(1/W_p)/(1/W_{p-1})|$. Hence by Theorem 2.1, $1/W_p \to y \neq 0$ as $p \to \infty$, contrary to the assumption that $\sum_{p=0}^{\infty} w_p$ diverges. This completes the proof of the theorem.

References

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SOME REMARKS ON ANTI-COMMUTATIVE ALGEBRAS, CORRECTION AND ADDENDUM

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The statement in [1] that there are no perfect Lie algebras of dimension 5 is false. Consider the perfect Lie algebra of dimension 6, defined by

$$e_1e_2 = 2e_3,\ e_1e_3 = 2e_2,\ e_2e_3 = 2e_1,\ e_1e_4 = e_5,\ e_1e_5 = e_4,\ e_2e_4 = e_5,$$
$$e_5e_2 = e_4,\ e_3e_4 = e_4,\ e_5e_3 = e_5,\ e_4e_5 = e_6.$$

If this algebra is factored by its centre $((e_6))$, then it must yield a perfect Lie algebra R of dimension 5, defined by

$$f_1f_2 = 2f_3,\ f_1f_3 = 2f_2,\ f_2f_3 = 2f_1,\ f_1f_4 = f_5,\ f_1f_5 = f_4,\ f_2f_4 = f_5,$$
$$f_5f_2 = f_4,\ f_3f_4 = f_4,\ f_5f_3 = f_5.$$

Now the radical of R is clearly the ideal $N=((f_4, f_5))$. This is the only ideal of dimension 2, since any other such ideal must be solvable and so contained in N. To show that N is the only proper ideal of R it is sufficient to consider quotients, and use the fact that there are no perfect Lie algebras of dimensions 1, 2, 4. It follows that R is not semi-simple, but $RX=X$ for all ideals X.

We have shown, therefore, that if $\mathfrak{A}$ denotes the class of all perfect Lie algebras, $\mathfrak{B}$ the class of all Lie algebras P such that $PX=X$ for all ideals X, and $\mathfrak{S}$ the class of all semi-simple Lie algebras, then we have *proper* inclusions

$$\mathfrak{A} \supset \mathfrak{B} \supset \mathfrak{S}.$$

Reference

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A CHARACTERIZATION OF POLYNOMIAL DOMAINS OVER A FIELD

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Usually a Euclidean domain (D, φ) is defined to be an integral domain D on which there is a nonnegative integral valued function φ satisfying the following conditions:

ED1. $\varphi(a)=0$ if $a=0$, the zero of D.

ED2. $\varphi(a)\leqq\varphi(b)$ if a divides b.

ED3. For any pair of elements a, $b\neq 0$ in D, there exist elements q and r in D such that

$$a = bq + r \quad \text{and} \quad \varphi(r) < \varphi(b).$$

For example, see Zariski-Samuel [1] p. 23; later we shall make use of results stated there.

In general, the existence of q and r is not assumed to be unique. The purpose of this note is to point out the fact that if their uniqueness is assumed then D is either a field (commutative, always) or a polynomial domain over a field. We shall prove

THEOREM 1. *A Euclidean domain (D, φ) is either a field or a polynomial domain over a field if (D, φ) satisfies the condition*:

(U). *For any pair of elements a, $b\neq 0$ in D, there is a unique pair of elements q, r in D such that*

$$a = bq + r \quad \textit{and} \quad \varphi(r) < \varphi(b).$$

N. Jacobson proved the same result (see [2]), due to the lemma given below. It is based on the slightly different definition of Euclidean domain which re quires conditions ED1, ED3 and

ED2'. $\varphi(ab) = \varphi(a)\cdot\varphi(b)$ for all a, b, in D,

in place of ED2. This note shows, therefore, that Jacobson's result still holds in wider sense.

To prove the theorem, we need a lemma:

LEMMA. *For a Euclidean domain (D, φ), the condition* (U) *is equivalent to condition* (V):

(V). $\varphi(a+b)\leqq\max(\varphi(a), \varphi(b))$ *for all a, b in D.*

Proof of the lemma. (V)$\Rightarrow$(U). Suppose, for a, $b\neq 0$ in D, that there are two pairs of elements q_1, r_1, and q_2, r_2, such that

$$\begin{aligned} a &= bq_1 + r_1 \quad \text{and} \quad \varphi(r_1) < \varphi(b), \\ &= bq_2 + r_2 \quad \text{and} \quad \varphi(r_2) < \varphi(b). \end{aligned}$$

Then it follows that

$$b(q_1 - q_2) = -(r_1 - r_2) \neq 0,$$

i.e., b is a divisor of r_1-r_2, hence

$$\varphi(b) \leqq \varphi(r_1 - r_2).$$

Now by (V), and since $\varphi(-a)=\varphi(a)$ for any a in D, we have

$$\varphi(b) \leqq \text{Max. } (\varphi(r_1), \varphi(r_2)),$$

which is a contradiction.

(U)$\Rightarrow$(V). Suppose that there are elements a and b in D such that

$$\varphi(a + b) > \varphi(a),\ \varphi(b).$$

Then, for the pair of elements a^2-b^2+b and $a+b$, the uniqueness of quotient and remainder fails. In fact

$$\begin{aligned} a^2 - b^2 + b &= (a + b)(a - b) + b, && \varphi(b) < \varphi(a + b), \\ &= (a + b)(a - b + 1) - a, && \varphi(a) < \varphi(a + b). \end{aligned}$$

Proof of Theorem 1. It follows from ED2 that, for any nonzero element a in D, $\varphi(1)\leqq\varphi(a)$. Now define the set F by

$$F = \{a \in D \mid \varphi(a) \leqq \varphi(1)\}.$$

Then there are two cases to be considered:

i) $F=D$. In this case, all nonzero elements in D are units (i.e., every nonzero elements has an inverse); hence D is a field.

ii) $F\neq D$. In this case there is at least one element a in D such that $\varphi(a)>\varphi(1)$, and we can choose an element x in $D\backslash F$ with the property that $\varphi(1)<\varphi(x)\leqq\varphi(a)$, for all a in $D\backslash F$.

It is to be noticed that F is a subfield of D, since F is closed under subtraction because of condition (V).

Now, by assumption, for any a in $D\backslash F$ there exist unique q and r in D such that

$$a = xq + r, \qquad \varphi(r) < \varphi(x).$$

Clearly, r is in F, and it can be verified that $\varphi(q)<\varphi(a)$. If $\varphi(q)\geqq\varphi(x)$, i.e. q is in $D\backslash F$, we can get, in similar manner, a unique pair of elements q_1, r_1 such that

$$q = xq_1 + r_1, \qquad r_1 \in F, \qquad \varphi(q_1) < \varphi(q).$$

We continue this process. After a finite number of steps it has to break off, since

$$\varphi(a) > \varphi(q) > \varphi(q_1) > \cdots$$

is a decreasing sequence of nonnegative integers. Thus any a in D can be uniquely expressed in the form

$$a = x^n q_{n-1} + x^{n-1} r_{n-1} + \cdots + x r_1 + r,$$

where $q_{n-1}, r_{n-1}, \cdots, r_1, r$ are in F.

Hence D is a polynomial domain over F.

Conversely, if a polynomial domain D is endowed with standard norm function φ, i.e. $\varphi(a)=\deg a+1$, $\deg 0=-1$, then (D, φ) is a Euclidean domain satisfying the condition (U). Furthermore we obtain

THEOREM 2. *For a polynomial domain $D=F[x]$ over a field F, if there is a nonnegative integral valued function φ defined on D such that (D, φ) is a Euclidean domain satisfying the condition* (U), *then φ must be a monotone increasing function of the degree.*

Proof. First we shall establish a relation:

$$\text{(A)} \qquad \varphi(a) = \varphi(x^n) \quad \text{if } \deg a = n.$$

If $n=0$ this is true, so we go on with the induction. Suppose (A) is true for every integer $m<n$. Now any a in D of degree n with leading coefficient 1 can be written

$$a = x^n + b, \quad \text{where} \quad \deg b = m < n.$$

Then, by (V) and the inductive hypothesis, we have

$$\varphi(a) \leqq \text{Max. } (\varphi(x^n), \varphi(x^m)) = \varphi(x^n).$$

On the other hand, $x^n=a-b$ gives

$$\varphi(x^n) \leqq \text{Max. } (\varphi(a), \varphi(x^m)) = \varphi(a).$$

Hence $\varphi(x^n)=\varphi(a)$. Because $\varphi(fa)=\varphi(a)$ holds for any nonzero element f in F, this completes the proof of (A).

Let a, b in D have $\deg a=m$, $\deg b=n$ and $m<n$. Then by (A)

$$\varphi(a) = \varphi(x^m) < \varphi(x^n) = \varphi(b)$$

which shows that φ is a monotone increasing function of the degree.

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2. N. Jacobson, A note on non-commutative polynomials, Ann. of Math., 35 (1934) 209–210.

A NOTE ON CONTENTS

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A content μ is a nonnegative, extended real-valued finitely additive set function defined on a field $\mathfrak{F}$ of sets such that $\mu(\varnothing)=0$. A measure is a countably additive content defined on a σ-field Σ. Throughout this note X will denote a fixed set and Σ a σ-field of subsets of X; all measures and contents considered will be defined on Σ and all sets will be elements of Σ.

Yosida and Hewitt in [3] have proved interesting results for finite contents. The purpose of this note is to extend one of their theorems to the infinite case.

The method employed here is different from their approach which used the hypothesis $\mu(X) < \infty$ in a crucial way. We employ a simple proof valid without regard to the value of $\mu(X)$. Perhaps our method may be useful as a general tool in examining infinite contents.

For any content μ define

$$\mu^*(E) = \inf_{\{E_i\}\in D} \left\{ \sum_{i=1}^{\infty} \mu(E_i) : \bigcup_{i=1}^{\infty} E_i \supseteq E, E_i \in \Sigma \text{ for each } i \right\},$$

where D consists of all countable unions of Σ-sets covering E. It is well known [1, p. 91] that μ^* is an outer measure on the class of all subsets of X.

LEMMA. *μ^* is a measure on Σ.*

Proof. Since μ^* is an outer measure, it remains to prove the relation: $\mu^*(\bigcup_{i=1}^{\infty} A_i) \geqq \sum_{i=1}^{\infty} \mu^*(A_i)$ for a disjoint, countable class $\{A_i\}$ of sets. First, if $A \cap B = \varnothing$, then $\mu^*(A \cup B) \geqq \mu^*(A) + \mu^*(B)$. For let $\bigcup_{i=1}^{\infty} E_i \supseteq A \cup B$ such that $\mu^*(A \cup B) \geqq \sum_{i=1}^{\infty} \mu(E_i) - \epsilon$. Now:

$$\sum_{i=1}^{\infty} \mu(E_i) \geqq \sum_{i=1}^{\infty} \mu((E_i \cap A) \cup (E_i \cap B)) = \sum_{i=1}^{\infty} \mu(E_i \cap A) + \sum_{i=1}^{\infty} \mu(E_i \cap B)$$
$$\geqq \mu^*(A) + \mu^*(B).$$

This follows because $\bigcup_{i=1}^{\infty} (E_i \cap A) \supseteq A$, $\bigcup_{i=1}^{\infty} (E_i \cap B) \supseteq B$, and so, since ϵ is arbitrary, the inequality is obtained for two disjoint sets, and by induction, for any finite number of disjoint sets. Now let $\{A_i\}$ be a countable, disjoint class of sets. By the monotonicity of μ and the preceding results, we see that

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geqq \sum_{i=1}^{n} \mu^*(A_i), \qquad \text{for each } n.$$

Therefore, $\mu^*(\bigcup_{i=1}^{\infty} A_i) \geqq \sum_{i=1}^{\infty} \mu^*(A_i)$, proving Lemma 1.

DEFINITION. *Let μ be a content on Σ. μ is called purely finitely additive* (p.f.a.) *if the relation $0 \leqq \alpha \leqq \mu$ for a measure α implies that $\alpha = 0$. (The ordering is the standard ordering of set functions.)*

Remark. μ^* is the largest measure less than μ. For if α is a measure and $\alpha \leqq \mu$, then, taking outer measures on both sides of the inequality, we see that $\alpha^* \leqq \mu^*$. But $\alpha^* = \alpha$ by a known measure-theoretical result [2, p. 42]. If μ is a finite content, then $\mu - \mu^*$ is p.f.a. Setting $\mu - \mu^* = \mu_f$, we have $\mu = \mu^* + \mu_f$, which is the well-known decomposition theorem of a content into a measure and a purely finitely additive part. This decomposition theorem was proved in [3].

The main result of this note is the following

THEOREM. *If μ is* p.f.a. *and m is any finite measure, then for every $\epsilon > 0$, there is a set B with $\mu(B) = 0$ and $m(X \sim B) < \epsilon$.*

Proof. Since $\mu^* \leqq \mu$ and μ is p.f.a., $\mu^* = 0$. Let $\epsilon > 0$ be given and without loss of generality suppose that $m(X) = 1$. Let ϵ_i be any sequence of positive numbers with $\sum_{i=1}^{\infty} \epsilon_i = \epsilon$. For each ϵ_i, there exists a sequence $\{A_{ij}\}$ of sets with $\bigcup_{j=1}^{\infty} A_{ij} = X$ and $\sum_{j=1}^{\infty} \mu(A_{ij}) \leqq \epsilon_i$. Choose $B_{ij} \subseteq A_{ij}$, so that $\bigcup_{j=1}^{n} B_{ij} = \bigcup_{j=1}^{n} A_{ij}$ for each positive integer n, and so that the B_{ij} form a disjoint sequence for each fixed index i. Since $\lim_{n\to\infty} m(\bigcup_{j=1}^{n} A_{ij}) = 1$, there exists an integer N_i such that $m(\bigcup_{j=1}^{N_i} A_{ij}) > 1 - \epsilon_i$. Put $\bigcup_{j=1}^{N_i} A_{ij} = B_i$. Then:

$$\mu(B_i) = \mu\left(\bigcup_{j=1}^{N_i} A_{ij}\right) = \mu\left(\bigcup_{j=1}^{N_i} B_{ij}\right) = \sum_{j=1}^{N_i} \mu(B_{ij}) \leqq \sum_{j=1}^{N_i} \mu(A_{ij}) \leqq \epsilon_i$$

and $m(B_i) > 1 - \epsilon_i$. Set $B = \bigcap_{i=1}^{\infty} B_i$. Then $\mu(B) \leqq \mu(B_i) \leqq \epsilon_i$ for each i, and so $\mu(B) = 0$. Moreover, $m(X \sim B) = m(\bigcup_{i=1}^{\infty} (X \sim B_i)) \leqq \sum_{i=1}^{\infty} m(X \sim B_i) < \sum_{i=1}^{\infty} \epsilon_i = \epsilon$.

COROLLARY. *If m is a σ-finite measure and μ is a content such that $m(E) > 0$ implies $\mu(E) > 0$ for every set E, then μ is not* p.f.a.

Proof. This is an immediate consequence of the theorem and the fact that, given a σ-finite measure, a finite measure with the same sets of measure zero can always be constructed from it.

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ON AN IDENTITY FOR MULTIPLICATIVE FUNCTIONS

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An arithmetic function is said to be multiplicative if $(m, n) = 1$ implies $f(mn) = f(m)f(n)$ and $f(1) = 1$. If $f(n)$ and $g(n)$ are multiplicative functions, the composition of f by g, denoted $f \odot g$, is defined by $(f \odot g)(n) = \sum f(d)g(n/d)$, where the summation ranges over all divisors d of n. It is well known that composition has the identity $E_0(n)$ ($E_0(1) = 1$, and $E_0(n) = 0$ if $n > 1$), and that for any given multiplicative function $f(n)$, the inverse function f^{-1}, defined by

$$f(n) \odot f^{-1}(n) = E_0(n), \tag{1}$$

exists, is unique, and is also multiplicative.

Vaidyanathaswamy demonstrated ([1] p. 645, in different notation) that any multiplicative function f satisfies the identity

$$f(mn) = \sum f(m/d_1)f(n/d_2)f^{-1}(d_1d_2)C(d_1, d_2), \tag{2}$$

where the summation extends over all divisors d_1 of m and all divisors d_2 of n, f^{-1} is the function satisfying (1), and $C(M, N)$ is a multiplicative function of two variables, defined to be

$$(3)\quad C(M, N) = \begin{cases}(-1)^k \text{ if } M \text{ and } N \text{ have the same } k \text{ distinct prime divisors,}\\ 0 \quad \text{otherwise.}\end{cases}$$

The proof of (2) given in [1] was by the method of generating series. More recently, McCarthy [2] considered a specialization of (2) for quadratic functions. His proof was by induction. In this paper, using the elementary property of f^{-1} given in Lemma 1, we give a direct proof of equation (2) in the case when m and n are powers of the same prime. Then we extend the result to the general case by showing that the right member of (2) actually defines a multiplicative function.

LEMMA 1. *If $f(n)$ is any multiplicative function, and p is prime, then*

$$\sum_{i=0}^{N} f(p^{N-i})f^{-1}(p^i) = \begin{cases}1 & \text{if } N = 0\\ 0 & \text{if } N > 0.\end{cases}$$

This is immediate from (1).

LEMMA 2. *Suppose that $f(n)$ is any multiplicative function. For a fixed prime p, and positive integers a and b, define a function F of two variables by*

$$F(a, b) = -\sum_{\substack{0\le A\le a\\ 0\le B\le b}} f(p^{a-A})f(p^{b-B})f^{-1}(p^{A+B}).$$

Then $F(a, b) = f(p^{a+b})$.

Proof. Changing notation slightly, we have

$$\begin{aligned}F(a, b) &= -\sum_{A=0}^{a} f(p^{a-A}) \sum_{B=0}^{b} f(p^{b-B})f^{-1}(p^{A+B})\\ &= -f(p^a)\sum_{B=0}^{b} f(p^{b-B})f^{-1}(p^B) - \sum_{A=1}^{a} f(p^{a-A}) \sum_{B=0}^{b} f(p^{b-B})f^{-1}(p^{A+B}).\end{aligned}$$

But by Lemma 1, $\sum_B f(p^{b-B})f^{-1}(p^B) = 0$. Thus

$$\begin{aligned}F(a, b) &= -\sum_{A=1}^{a} f(p^{a-A}) \sum_{B=0}^{b} f(p^{b-B})f^{-1}(p^{A+B})\\ &= -\sum_{A=1}^{a} f(p^{a-A}) \sum_{B=0}^{b} f(p^{b+A-(B+A)})f^{-1}(p^{B+A})\\ &= -\sum_{A=1}^{a} f(p^{a-A}) \sum_{B+A=A}^{b+A} f(p^{b+A-(B+A)})f^{-1}(p^{B+A}\\ &= -\sum_{A=1}^{a} f(p^{a-A}) \Bigg[\sum_{B+A=0}^{b+A} f(p^{b+A-(B+A)})f^{-1}(p^{B+A})\\ &\qquad - \sum_{B+A=0}^{A-1} f(p^{b+A-(B+A)})f^{-1}(p^{B+A})\Bigg].\end{aligned}$$

Because of Lemma 1, the first summation in brackets vanishes, and

$$F(a, b) = \sum_{A=1}^{a} f(p^{a-A}) \sum_{B+A=0}^{A-1} f(p^{b+A-(B+A)}) f^{-1}(p^{B+A})$$

$$= \sum_{A=1}^{a} f(p^{a-A}) \sum_{B=-A}^{-1} f(p^{b-B}) f^{-1}(p^{B+A}).$$

Now it is clear that each term in the expansion of $F(a, b)$ contains a factor of the form $f(p^{b+t})$ for positive t. For a fixed t, $1 \leqq t \leqq a$, consider those terms which contain $f(p^{b+t})$. These are the terms for which $b-B=b+t$, or $B=-t$. Then $-A \leqq -t \leqq -1$, or $A \geqq t$. Selecting such terms, we have

$$f(p^{b+t}) \sum_{A=t}^{a} f(p^{a-A}) f^{-1}(p^{A-t}),$$

which is equal to $f(p^{b+t}) \sum_{A-t=0}^{a-t} f(p^{a-t-(A-t)}) f^{-1}(p^{A-t})$. Since there is one such summation for each t, $1 \leqq t \leqq a$, $F(a, b)$ can be written

$$F(a, b) = \sum_{t=1}^{a} f(p^{b+t}) \sum_{A-t=0}^{a-t} f(p^{a-t-(A-t)}) f^{-1}(p^{A-t}).$$

From Lemma 1 we see that if $a-t>0$, the second summation vanishes, so $F(a, b)$ contains no terms involving $f(p^{b+t})$ for $t<a$. If $a-t=0$, or $a=t$, the second summation is identically 1, by Lemma 1. The proof is now complete.

LEMMA 3. *If $f(n)$ is multiplicative, and a function $g(m, n)$ is defined by $g(m, n) = \sum f(m/d_1) f(n/d_2) f^{-1}(d_1 d_2) C(d_1, d_2)$, where the summation is over divisors d_1 of m and divisors d_2 of n, then $g(m, n)$ is multiplicative in both variables.*

Proof. We wish to show that if $(mn, m'n')=1$, then $g(m, n)g(m', n') = g(mm', nn')$. We have

$$g(m, n)g(m', n')$$

$$= \sum_{\substack{d_1|m \\ d_2|n}} f\left(\frac{m}{d_1}\right) f\left(\frac{n}{d_2}\right) f^{-1}(d_1 d_2) C(d_1, d_2) \sum_{\substack{d_1'|m' \\ d_2'|n'}} f\left(\frac{m'}{d_1'}\right) f\left(\frac{n'}{d_2'}\right) f^{-1}(d_1' d_2') C(d_1', d_2')$$

$$= \sum_{\substack{d_1, d_2, \\ d_1', d_2'}} f\left(\frac{m}{d_1}\right) f\left(\frac{m'}{d_1'}\right) f\left(\frac{n}{d_2}\right) f\left(\frac{n'}{d_2'}\right) f^{-1}(d_1 d_2) f^{-1}(d_1' d_2') C(d_1, d_2) C(d_1', d_2')$$

$$= \sum_{\substack{d_1 d_1' | mm' \\ d_2 d_2' | nn'}} f\left(\frac{m}{d_1} \frac{m'}{d_1'}\right) f\left(\frac{n}{d_2} \frac{n'}{d_2'}\right) f^{-1}(d_1 d_1' d_2 d_2') C(d_1 d_1', d_2 d_2')$$

$$= g(mm', nn').$$

THEOREM. *Every multiplicative function $f(n)$ satisfies* (2).

Proof. By Lemma 3, it is sufficient to consider the case when the argument is a power of a prime. Suppose that $f(mn)$ is a multiplicative function, p is prime, and $m=p^a$, $n=p^b$, for a and b nonnegative.

If at least one of a, b is zero, it is clear from (3) that the result is trivial. If both a and b are positive, then by the nature of the function C we see that the right member of (2) is precisely $F(a, b)$, and the theorem follows by Lemma 2.

My thanks to Prof. M. V. Subbarao for his help in the preparation of this paper.

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NOTE ON A CLASS OF ARITHMETICAL FUNCTIONS

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Let $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ be the representation of n as a product of powers of distinct primes and define $\Omega_k(n)=\alpha_1^k+\cdots+\alpha_r^k$ where k is an arbitrary nonnegative integer. Then, as is shown in a previous note [1],

$$\sum_{n\leq x}\Omega_k(n) = x\log\log x + B_k x + O\left(\frac{x}{\log x}\right),$$

where

$$B_k = \gamma + \sum_p \{(1-p^{-1})G_k(p^{-1}) + \log(1-p^{-1})\},$$

and

$$G_k(x) = \sum_{n=1}^{\infty} n^k x^n. \quad \text{Also,} \quad G_k(x) = \theta^k\left(\frac{x}{1-x}\right),$$

where θ is the differential operator $x(d/dx)$. Thus, it is easy to obtain an exact expression for B_k as a sum over the primes for small values of k. This procedure becomes very cumbersome, however, when k is large. The object of the present note is to determine the asymptotic behavior of B_k for large k.

THEOREM.

$$B_k \sim \frac{k!}{2\log^{k+1}2}.$$

Proof.

$$B_k = \gamma + \sum_p \{(1-p^{-1})G_k(p^{-1}) + \log(1-p^{-1})\}$$

$$= \gamma - \log 2 + \frac{1}{2}\sum_{n=1}^{\infty} n^k 2^{-n} + \sum_{p>2} \{(1-p^{-1})G_k(p^{-1}) + \log(1-p^{-1})\}.$$

It is known [2] that

$$\sum_{n=1}^{\infty} n^k 2^{-n} \sim \int_0^{\infty} x^k 2^{-x} dx = \frac{\Gamma(k+1)}{\log^{k+1} 2} = \frac{k!}{\log^{k+1} 2}.$$

Also, since all the terms in the following series are positive, we have

$$\sum_{p>2} \{(1-p^{-1})G_k(p^{-1}) + \log(1-p^{-1})\} \leqq \sum_{p>2} \{G_k(p^{-1}) - p^{-1}\}$$

$$= \sum_{p>2} \sum_{n=2}^{\infty} n^k p^{-n} = \sum_{p>2} p^{-2} \sum_{n=0}^{\infty} (n+2)^k p^{-n}$$

$$\leqq \left(\sum_{p>2} p^{-2}\right)\left(\sum_{n=0}^{\infty} (n+2)^k 3^{-n}\right) = 9\left(\sum_{p>2} p^{-2}\right)\left(\sum_{n=2}^{\infty} n^k 3^{-n}\right)$$

$$= O\left(\int_0^{\infty} x^k 3^{-x} dx\right) = O\left(\frac{k!}{\log^{k+1} 3}\right).$$

We conclude with the remark that the asymptotic expression for B_k is precisely the same as that for the total number of preferential arrangements with k objects [2].

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CHARACTERISTIC ROOTS WITH ALL THEIR REAL PARTS THE SAME

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In a recent note R. D. Larsson [1], raised the problem of determining when all the roots of the equation

$$z + e^{-z}(az + b) = 0 \tag{1}$$

(a, b real) have their real parts the same. As he explains, the result is of interest in the theory of difference-differential equations. He found (apart from a special case) necessary and sufficient conditions for there to be an infinity of roots with their real parts the same. Here I complete the solution by determining when these are all the roots and when there are others. The existence or nonexistence of roots with different real part is of importance in the applications.

The equation (1) may be trivially transformed to one or other of the equations

$$Ze^{Z} = A, \qquad (Z+B)e^{Z+A} = \pm (Z-B) \tag{2}$$

and I have recently "solved" these equations in a series of papers [2, 3, 4]. The particular results of the present note can be deduced from this fairly elaborate general theory, (see, in particular, §2 and Fig. 1 of [4]). But here we find them

independently and much more simply. The first equation of (2) has a long history; full references are given in [2].

If $a=b=0$ in (1), there is just one root. If at least one of a or b does not vanish, a simple application of Rouché's theorem shows that (1) has an infinity of roots, whose modulus tends to infinity. We take this for granted henceforth. We denote the line $\Re(z)=c$ by L.

THEOREM 1. *The necessary and sufficient conditions for* (1) *to have an infinity of roots on* L *are that* $a=\pm e^c$ *and that* b *is either* 0 *or* $-2ca$.

THEOREM 2. *The necessary and sufficient conditions for all the roots of* (1) *to lie on* L *are that* $b=-2ca$ *and that either* (i) $a=e^c$, $c\geqq -2$ *or* (ii) $a=-e^c$, $c\geqq 0$.

THEOREM 3. *If* $c\neq 0$, $a=\pm e^c$, $b=0$, *then all the roots of* (1) *lie on* L *except the root* 0.

If $b=-2ca$ *and either* (iii) $a=e^c$, $c<-2$ *or* (iv) $a=-e^c$, $c<0$, *then all the roots of* (1) *lie on* L *except two real roots* $c\pm\sigma$.

(1) may be written $e^z=-a-b/z$. If this is satisfied by a sequence of values of z for which $\Re(z)=c$ and $|z|\to\infty$, we must have $|a|=e^c$, $a=\pm e^c$. If we write $b=-2aB$, (1) becomes

$$ze^{z-c}=\mp(z-2B). \tag{3}$$

Putting $z=c+iy$ and taking the modulus of each side, we have $c^2+y^2=(c-2B)^2+y^2$, so that $B=0$ or c. This proves necessity in Theorem 1.

Now for sufficiency. If we suppose that $a=\pm e^c$ and $b=-2Ba$, (1) becomes (3). If $B=0$, (3) reduces to $z(e^{z-c}\pm 1)=0$, whose roots are 0, $c+k\pi i$, where k takes all odd or even values according to which sign is taken in the ambiguity. If $B=c$ and we write $z=Z+c$, (3) becomes

$$(Z+c)e^Z=\mp(Z-c). \tag{4}$$

We next prove

LEMMA 1. *All the roots of* (4) *lie on the real or on the imaginary axes.*

Put $Z=X+iY$ (X, Y real) and $T=\mp e^X$ in (4), so that

$$e^{iY}(X+c+iY)=T^{-1}(X-c+iY) \tag{5}$$

or, what is the same thing,

$$T(X+c+iY)=e^{-iY}(X-c+iY). \tag{6}$$

Equating imaginary parts in (5) and (6), we have $(X+c)\sin Y+Y\cos Y=YT^{-1}$ and $TY=Y\cos Y-(X-c)\sin Y$. Hence

$$\mp 2Y\sinh X=Y(T-T^{-1})=-2X\sin Y. \tag{7}$$

But, if $XY\neq 0$, we have $|X|<|\sinh X|$, $|\sin Y|<|Y|$, $|X\sin Y|<|Y\sinh X|$, which contradicts (7). Hence $XY=0$ and this is Lemma 1.

If Z_0 is a root of (4), so is $-Z_0$. We know that (4) has an infinity of roots, and the imaginary axis in the Z-plane is the transform of L. Hence we can complete the proof of the three theorems by examining the real roots of (4). If we write $f(X)=(X+c)e^X+X-c$, $g(X)=(X+c)e^X-X+c$, it is enough to prove

LEMMA 2. *The equation $f(X)=0$ has just one positive real root or none according as $c<-2$ or not.*

LEMMA 3. *The equation $g(X)=0$ has just one positive real root or none according as $c<0$ or not.*

The proofs are routine and I omit them.

[3] gives a series expansion for all but a few roots of (1). In our present case, with $a=\pm e^c$ and $b=-2ce^c$, we write

$$z_k = c + i\left\{k\pi + \sum_{m=0}^{\infty} P_m(c)(k\pi)^{-2m-1}\right\},$$

where $P_m(c)$ is a polynomial in c defined by

$$P_0(c) = 2c, \qquad P_m(c) = -2m(2m-1)\int_0^c (c-t)(t+2)t^{-1}P_{m-1}(t)\,dt$$

for $m \geqq 1$. Provided that k is greater than a certain function of c, the series z_k is convergent. Then the z_k with odd or even k are the roots of (1), according as a is positive or negative.

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Proof. I. If R is assumed to be a rational number such that $R^2>2$ and $R>0$, then it can be verified that:

(a) $$\frac{R^2+2}{2R} < R$$

and

(b) $$\left[\frac{R^2+2}{2R}\right]^2 > 2.$$

Thus, no rational number R can be the least rational with the desired property. The choice of $(R^2+2)/2R$ can be motivated by applying Newton's method for approximating roots at the point $(R, f(R))$ of the function $f(x)=x^2-2$. (A sketch of the graph is, of course, most helpful.)

II. If R is assumed to be a rational number such that $R^2<2$ and $R>0$, then it can be verified that

(a) $$\frac{6R-R^3}{4} > R$$

and

(b) $$\left[\frac{6R-R^3}{4}\right]^2 < 2.$$

Part (b) may be proved as follows: If $[(6R-R^3)/4]^2<2$ then $R^6-12R^4+36R^2-32=(R^2-2)^2(R^2-8)<0$. Now $R^2<8$, since $R^2<2$, and we obtain part (b) by reversing our steps. Thus, no rational number R can be the greatest rational with the desired property. The choice of $(6R-R^3)/4$ can be motivated by applying Newton's method at the point $(R, f(R))$ to the function $f(x)=(x^2-2)/x^2$.

The question of priority involved in the proposed use of the calculus to motivate the presentation of the real number system can lead to important classroom discussion.

EQUALITY IN $\left|\int_a^b f(x)dx\right| \leqq \int_a^b |f(x)|\,dx$

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We establish necessary and sufficient conditions for equality to hold in the well-known inequality $\left|\int_a^b f(x)dx\right| \leqq \int_a^b |f(x)|\,dx$.

LEMMA 1. *Let $f(x)$ be an integrable real or complex valued function on the real interval $[a, b]$, and $g(x)$ be a real or complex valued measurable function such that $|g(x)|=1$ a.e. on $[a, b]$. Then $\left|\int_a^b g(x)\cdot|f(x)|\,dx\right| = \int_a^b |f(x)|\,dx$ if and only if $g(x)=c$ (constant) a.e. on $[a, b]-S$, where $|c|=1$ and $S=\{x: x\in[a, b], f(x)=0\}$.*

Proof. If $\left|\int_a^b g(x)\cdot|f(x)|\,dx\right| = \int_a^b |f(x)|\,dx$ then there exists a constant E such that $|E| = 1$ and

$$\int_a^b g(x)\cdot|f(x)|\,dx = E\int_a^b |f(x)|\,dx.$$

Thus $\int_a^b [E-g(x)]\cdot|f(x)|\,dx = 0$, and consequently, if $\overline{E}$ is the complex conjugate of E,

$$\int_a^b [1-\overline{E}\cdot g(x)]\cdot|f(x)|\,dx = \overline{E}\int_a^b [E-g(x)]\cdot|f(x)|\,dx = 0.$$

Therefore, if $\Re$ and $\Im$ denote real and imaginary parts,

$$\int_a^b \Re[1-\overline{E}g(x)]\cdot|f(x)|\,dx + i\int_a^b \Im[1-\overline{E}\cdot g(x)]\cdot|f(x)|\,dx$$
$$= \int_a^b [1-\overline{E}\cdot g(x)]\cdot|f(x)|\,dx = 0,$$

so that $\int_a^b \Re[1-\overline{E}\cdot g(x)]\cdot|f(x)|\,dx = \int_a^b \Im[1-\overline{E}g(x)]\cdot|f(x)|\,dx = 0$. Since $|\overline{E}\cdot g(x)| = 1$ a.e. in $[a, b]$ it follows that $\Re[1-\overline{E}\cdot g(x)] = 1-\Re[\overline{E}\cdot g(x)] \geqq 0$ a.e. in $[a, b]$; hence $\int_a^b \Re[1-\overline{E}g(x)]\cdot|f(x)|\,dx = 0$ implies that $\Re[1-\overline{E}fg(x)]\cdot|f(x)| = 0$ a.e. in $[a, b]$ (see [1]). If $\Re[1-\overline{E}\cdot g(x)] = 0$ then $\Re[\overline{E}\cdot g(x)] = 1$ which with $|\overline{E}\cdot g(x)| = 1$ implies $\Im[\overline{E}\cdot g(x)] = 0$. Thus we also have $\Im[1-\overline{E}g(x)]\cdot|f(x)| = 0$, and consequently $[1-\overline{E}g(x)]\cdot|f(x)| = 0$ a.e. in $[a, b]$. From this it now follows that $g(x) = (\overline{E})^{-1}$ a.e. in $[a, b]-S$.

The converse is obvious.

THEOREM. *Let $f(x)$ be an integrable real or complex valued function on the real interval $[a, b]$. Then $\left|\int_a^b f(a)dx\right| = \int_a^b |f(x)|\,dx$ if and only if* arg $[f(x)]$ *is constant a.e. on $[a, b]-S$, where $S = \{x: x\in[a, b], f(x) = 0\}$.*

Proof. If arg $[f(x)] = \theta$(constant) a.e. in $[a, b]-S$ then $f(x) = \Re[f(x)] + i\Im[f(x)] = \Re[f(x)](1+ic)$ a.e. in $[a, b]-S$, where $c = \tan\theta$ (the trivial case $\theta = \pm\pi/2$ must be treated separately). Furthermore, arg $[f(x)] = \theta$ a.e. in $[a, b]-S$ implies that either $\Re[f(x)] \geqq 0$ or $\Re[f(x)] \leqq 0$ a.e. in $[a, b]-S$; to be definite, assume the former. Then we have

$$|f(x)| = \sqrt{(1+c^2)}\cdot\Re[f(x)] = \frac{\sqrt{(1+c^2)}}{1+ci}f(x)$$

a.e. in $[a, b]-S$ so that

$$\left|\int_a^b f(x)dx\right| = \left|\int_a^b \frac{1+ci}{\sqrt{(1+c^2)}}\cdot|f(x)|\,dx\right| = \left|\frac{1+ci}{\sqrt{(1+c^2)}}\int_a^b |f(x)|\,dx\right|$$
$$= \int_a^b |f(x)|\,dx.$$

On the other hand, if $\left|\int_a^b f(x)dx\right| = \int_a^b |f(x)|\,dx$, let $g(x) = f(x)/|f(x)|$ for x in $[a, b]-S$ and $g(x)=1$ for x in S. Then $g(x)\cdot|f(x)| = f(x)$ for every x in $[a, b]$, and $\left|\int_a^b f(x)dx\right| = \left|\int_a^b g(x)\cdot|f(x)|\,dx\right|$. Thus $\left|\int_a^b g(x)\cdot|f(x)|\,dx\right| = \int_a^b |f(x)|\,dx$, and $|g(x)| = 1$ on $[a, b]$. Hence it now follows from Lemma 1 that $g(x) = C$ (constant) a.e. on $[a, b]-S$.

If $\theta(x) = \arg [f(x)]$, then $\cos(\theta(x)) + i \sin(\theta(x)) = \Re[f(x)]/|f(x)| + i\Im[f(x)]/|f(x)| = f(x)/|f(x)| = C = C_1 + iC_2$ a.e. on $[a, b]-S$, where C_1, C_2 are real constants. Therefore $\cos(\theta(x)) = C_1$, $\sin(\theta(x)) = C_2$, and consequently $\arg [f(x)] = \theta(x)$ is constant a.e. in $[a, b]-S$.

The proofs given here remain valid if the real interval $[a, b]$ is replaced by a measurable real set.

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SOME REMARKS ON TOPOLOGICAL SPACES WITH AN INFINITE NUMBER OF COMPONENTS

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In this note we shall give several characterizations of topological spaces which have an infinite number of components.

THEOREM 1. *A topological space X has an infinite number of components if and only if there exists in X an infinite sequence of nonempty, disjoint sets which are both open and closed.*

Proof. Sufficiency. Let $\{O_i\}_{i=1}^{\infty}$ have the following properties: (1) $O_i \neq \varnothing$, (2) $O_i \cap O_j = \varnothing$ for $i \neq j$ and (3) O_i is both open and closed for all i. Let C_i be any component of X such that $O_i \cap C_i \neq \varnothing$. Then since O_i and $\mathcal{C}O_i$ constitute a separation for X it follows that $C_i \subset O_i$. Thus $C_i \neq C_j$ for $i \neq j$ and thus X has an infinite number of components.

Necessity. Suppose X has an infinite number of components. Then X is certainly disconnected. Let O_1 and U_1 constitute a separation for X, the notation being chosen so that an infinite number of components of X lie in U_1. Then O_1 is nonempty and both open and closed and U_1 is disconnected. Now let O_2 and U_2 constitute a separation for U_1. Since U_1 is both open and closed it follows that O_2 is open and closed in X and disjoint from O_1. We assume further that the notation has been chosen so that an infinite number of components of X lie in U_2. Suppose now that $O_1, \cdots, O_k, U_k$ have been chosen so that (1) $O_i \neq \varnothing$ and $U_k \neq \varnothing$, (2) O_i and U_k are both open and closed, (3) $O_i \cap O_j = \varnothing$, (4) $O_i \cap U_k = \varnothing$, (5) an infinite number of components of X lie in U_k, and (6) $X = O_1 \cup O_2 \cup \cdots \cup O_k \cup U_k$. Now U_k is disconnected and thus let O_{k+1} and U_{k+1} be a separation for U_k. Again since U_k is open and closed it follows that O_{k+1} and U_{k+1} are open and closed in X. Thus $O_1, O_2, \cdots, O_n, \cdots$ is the required sequence.

COROLLARY 1. *A topological space X has an infinite number of components if and only if for each positive integer n, $X = O_1^* \cup O_2^* \cup \cdots \cup O_n^*$ where O_i^* is nonempty, open and closed, and $O_i^* \cap O_j^* = \varnothing$.*

The proof is left as an exercise.

Let Q be the subspace of the reals consisting of the points $0, 1, \frac{1}{2}, \cdots, 1/n, \cdots$ with the relative topology. Clearly Q has an infinite number of components.

THEOREM 2. *A topological space X has an infinite number of components if and only if Q is a continuous image of X.*

The proof is left as an exercise for the reader.

THEOREM 3. *A topological space X has an infinite number of components if and only if $X \times X$ has an infinite number of components.*

THEOREM 4. *A topological space X has an infinite number of components if and only if X can be expressed as a union of two separated sets in an infinite variety of ways.*

THEOREM 5. *Let X be a topological space with an infinite number of components and X^* a dense subspace. Then X^* has an infinite number of components.*

The proofs of Theorems 3, 4 and 5 are omitted. The converse of Theorem 5 is false as the reader can easily verify.

LEMMA 1. *Let $\{X_\alpha\}_{\alpha \in \Delta}$ be a nonempty family of nonempty topological spaces. Suppose X_{α^*} has an infinite number of components. Then $\Pi_{\alpha \in \Delta} X_\alpha$ has an infinite number of components.*

LEMMA 2. *Let $\{X_\alpha\}_{\alpha \in \Delta}$ be a nonempty family of nonempty topological spaces. Suppose X_{α_i} is disconnected for $i = 1, 2, \cdots$, and the α_i distinct. Then $\Pi_{\alpha \in \Delta} X_\alpha$ has an infinite number of components.*

Corollary 1 may be used to supply proofs for Lemmas 1 and 2.

THEOREM 6. *Let $\{X_\alpha\}_{\alpha \in \Delta}$ be a nonempty family of nonempty topological spaces. Then $\Pi_{\alpha \in \Delta} X_\alpha$ has an infinite number of components if and only if either (1) X_{α^*} has an infinite number of components for some $\alpha^* \in \Delta$ or (2) X_{α_i} is disconnected for distinct $\alpha_1, \alpha_2, \cdots$ in Δ.*

Proof. The sufficiency follows from Lemmas 1 and 2. To show the necessity let $\Pi_{\alpha \in \Delta} X_\alpha$ have an infinite number of components. Suppose neither (1) nor (2) holds. Then X_α is disconnected for at most a finite number of α, say $\alpha_1, \cdots, \alpha_N$. Let $X_{\alpha_j} = C_1^{(j)} \cup \cdots \cup C_{n_j}^{(j)}$ where $C_i^{(j)}$ are the components of X_{α_j} for $j = 1, \cdots, N$ and $i = 1, \cdots, n_j$. Then $\Pi_{\alpha \in \Delta} X_\alpha = \cup \Pi X_\alpha^*$ where

$$
\begin{aligned}
X^*_\alpha &= X_\alpha && \text{for } \alpha \neq \alpha_1, \cdots, \alpha_N, \\
X^*_{\alpha_1} &= C^{(1)}_{i_1} && 1 \leqq i_1 \leqq n_1 \\
&\cdots\cdots\cdots\cdots\cdots && \\
X^*_{\alpha_N} &= C^{(N)}_{i_N} && 1 \leqq i_N \leqq n_N,
\end{aligned}
$$

the union extending over all $i_1, i_2, \cdots, i_N$ such that $1 \leqq i_1, \cdots, i_N \leqq n_1, \cdots, n_N$. Since each term in the above union is connected, it follows that $\Pi_{\alpha\in\Delta} X_\alpha$ has at most $n_1 \cdot n_2 \cdots n_N$ components, a contradiction.

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A STUDY OF MATHEMATICAL ABILITIES

E. G. BEGLE, Stanford University

Recent efforts to improve the mathematics curriculum in our schools have led to a flood of new texts and new programs and have, of course, raised the question of how good these innovations are. Opinions are plentiful and strongly held, but little factual evidence is at hand. In fact, it would be difficult to obtain such evidence, since many of the virtues (or defects) claimed for the new texts are difficult to measure.

Indeed, the purpose of this article is to invite suggestions of appropriate means for ascertaining mathematical development. Any proposal is requested, from specific questions to new types of instruments for measuring whatever it is that a good mathematics program ought to promote. There is no intention, however, of using these suggestions to compare one course with another, and to conclude that one is "better" than the other. Instead, they will be put to a much broader and, hopefully, more productive use—to obtain information of many kinds concerning the effects of the variety of presentations of mathematics now available on the variety of students in our classrooms, information which will be useful to school administrators, teachers, students, parents, and in particular to all those with a continuing interest in providing a better mathematics program for our schools.

As a mechanism for obtaining some of this information, a National Longitudinal Study of Mathematical Abilities has been instituted. The committee charged with the planning and direction of this study consists of Richard Alpert (psychology), Harvard University; Max Beberman, University of Illinois; E. G. Begle, Stanford University; R. P. Dilworth, California Institute of Technology; Jerome Kagan (psychology), Fels Research Institute; M. Albert Linton, Jr., William Penn Charter School, Philadelphia; William G. Lister, State University of New York. Financial support has been supplied by the National Science Foundation. Administrative offices for the study are located in the SMSG headquarters at Stanford University.

In brief, the plan of operation of this study is as follows. At each of the grade levels 4, 7, and 10, approximately 40,000 public school students were selected, representing all parts of the country and a variety of environments. Both new and conventional mathematics curricula are among those which these students will follow.

An extensive battery of tests was administered at the beginning of the school year '61–2 and scoring and analysis of this battery has been started. Additional tests will be administered periodically during the next five years, a period of time which makes possible a study of the cumulative effects of various curricula.

The information collected during the study will include much that has not ordinarily been included in studies of mathematical achievement. The initial battery of tests estimated not only the knowledge and understanding of mathematics which the students brought into the study, but also their attitudes toward various aspects of their mathematical experiences. The battery also measured such things as their fluency in arranging a set of stimuli into new and meaningful combinations and their ability to scan or analyze a problem situation in a disciplined fashion. It is plausible that such traits influence the learning of mathematics, and, in turn, it will be interesting to see how the learning of mathematics influences these traits.

Information will also be collected on the students' use of mathematics outside the mathematics classroom. In particular, records will be kept of elections of science courses by the students and achievement in these courses.

Although many will be lost from the study because of drop-outs, transfers to other schools, etc., there will remain at the end a large pool of students for each of whom we will have detailed information covering a five-year period. From this pool, samples can be drawn to provide answers to many of our questions.

What we are requesting now is assistance with that part of the projected battery, to be used as the study proceeds, which directly involves achievement in mathematics. The questions we wish to ask and the information we need to obtain will depend on our purposes in teaching mathematics. These are many and varied. Among them are: 1) to teach mathematical facts, skills and techniques; 2) to teach understanding of the facts, skills, and techniques; 3) to develop problem-solving ability; 4) to provide a language which can be used in

discussing and explaining the world we live in; 5) to provide a foundation for the further study of mathematics; 6) to provide an example of rigorous, exact thinking; and 7) to pass on to the next generation a part of the culture we have inherited.

Clearly, these purposes are not independent nor is the list exhaustive. Certainly no two individuals will attach the same weights to them, but any teacher of mathematics will attach a nonnegative weight to each of these purposes.

Most teachers feel reasonably confident about measuring success with respect to the first of the above purposes. The usual objective multiple choice type of test seems to be satisfactory. To measure understanding, more comprehensive responses are commonly required, but few teachers are completely happy about the conventional problem formulations. For the remaining purposes above it seems that standard types of tests are inadequate or inapplicable.

New ideas for measuring mathematical achievement and development are therefore needed. Everyone interested in the improvement of instruction in mathematics is urged to think about this problem and to submit suggestions. Not much attention need be paid to the practicality of the suggestions. Clever but impractical ideas suggested by one person can often be put into feasible form by another.

Ideas and suggestions should be sent to: N.L.S.M.A., Cedar Hall, Stanford University, Stanford, California. The more suggestions produced, the greater will be the value of the study.

SCHOOL MATHEMATICS STUDY GROUP OFFERS TO ASSIST TEACHERS OF EXTRAORDINARILY GIFTED STUDENTS

MARIE S. WILCOX, Thomas Carr Howe High School, Indianapolis

The Committee on the Extraordinarily Gifted Student of the School Mathematics Study Group feels that there is a very small but very important group of extraordinarily gifted students who need special attention. Perhaps there are less than a hundred of them in the country.

In general, students in large cities or in communities which include universities are able to secure assistance locally. The School Mathematics Study Group is willing to assist teachers who have extraordinarily gifted students in communities where such local assistance is not available.

Upon writing to the School Mathematics Study Group, School of Education, Stanford University, Stanford, California, these teachers will receive bibliographies which will assist the student with self study, pamphlets concerning professional opportunities for mathematicians, and suggestions about guidance in the choice of a college. When it seems advisable, a conference with a mathematician will be scheduled for one of these gifted students.

CONFERENCE ON FILMS (PRELIMINARY REPORT)

R. A. Good, University of Maryland

A conference on the use of audio-visual aids to mathematical instruction was held in Chicago June 1–3, 1962. The Mathematical Association of America, the National Council of Teachers of Mathematics, and the School Mathematics Study Group jointly sponsored the conference. Approximately 35 persons analyzed the problems from the viewpoints of college and university faculties, the secondary and elementary school teachers, and various administrators. The conference recognized an urgent need for instructional aids in several areas: to assist with the in-service training of teachers, to foster the pre-service training of future teachers, to help those smaller collegiate institutions whose staffs are not prepared to meet the demands for varied courses, and to provide opportunities for those larger institutions whose staffs must cope with heavy enrollments at the freshman-sophomore level.

The conference devoted its attention to the media of films, television, kinescopes, film-strips, and so forth. The filming of several courses—as soon as can be arranged—was suggested. One of these is a beginning course in calculus with analytic geometry. Several others having top priority are for teacher training, both in-service and pre-service. Other proposals included individual lectures on single topics of mathematical importance, preservation on film of talks by outstanding expositors, and documentary productions to foster public relations.

The sponsoring agencies were urged to inaugurate soon the production of the most urgently needed films and to seek the establishment of a Mathematics Film Center which would continue the work on a permanent basis. Extensive experimentation in techniques is hoped for. Considerable stress was placed on the necessity of studies to determine the effectiveness of the various projects in accomplishing their respective goals.

The formal report on the Conference, together with the recommendations adopted, will be released at a later date.

REPORT OF THE COMMITTEE ON STRENGTHENING THE TEACHING OF MATHEMATICS (CSTM) TO THE ILLINOIS SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

Gertrude Hendrix, University of Illinois

The CSTM for the year 1961–62 consisted of seven members. Representing the Illinois Section of the M.A.A. were: John M. H. Olmsted, Southern Illinois University, Carbondale; Charles E. Moulton, MacMurray College, Jacksonville; Gertrude Hendrix, University of Illinois, Urbana. Representing the Illinois Council of Teachers of Mathematics were: Mae Blair, Pekin Community High School; Houston Kirk, Quincy Senior High School; Marian Moore, Southern Illinois University, Carbondale. Rothwell Stephens served on the committee as

a seventh member *ex officio* in his capacity as a member of the Panel on Teacher Training of the Committee on the Undergraduate Program in Mathematics.

In the spring of 1962 the committee conducted a study of opinion in Illinois colleges and universities with regard to the CUPM recommendations for teacher training on Levels I, II, and III—that is, for preparation of elementary, junior high school, and senior high school teachers. A questionnaire was mailed to heads of departments of mathematics in all seventy-seven (77) colleges, universities, and junior colleges certified for training teachers in the state of Illinois. Three days later the same questionnaire was mailed to all members of the Illinois Section of the M.A.A. not included in the first mailing. Answers to the questionnaire are given at the end of this report.

Following an open meeting held in May at the time of the annual meeting of the Illinois section, CSTM submitted the following recommendations:

1. The CSTM is unanimously agreed that the CUPM recommendations for the training of teachers of mathematics for all five levels constitute a reasonable *minimal* goal toward which we should strive for the near future. The CSTM presents the following recommendations in addition to those of the CUPM.
2. Prospective teachers of mathematics should be prepared to teach the solid substance in analytic trigonometry—identities, multiple-angle equations, inverse trigonometric functions, etc.
3. The undergraduate mathematics sequence for teachers should prepare teachers in rigorous mathematical analysis sufficiently to enable them to teach the kind of work in absolute values and inequality relations needed in the first course in calculus.
4. Secondary school teachers need to have studied the new high school courses in algebra and geometry as *content*, rather than just to have learned *about* the new courses in surveys of the high school mathematics curriculum.
5. All school mathematics programs should provide the continuing drill and practice necessary to add skills to previously established understandings.
6. The undergraduate curriculum for elementary school teachers should offer courses *in* arithmetic, rather than courses *about* arithmetic.
7. Because of the current rapid changes in the mathematics curriculum, it is imperative that teachers at all levels become familiar with trends on levels other than their own. For example, (1) teachers at the elementary level should know the high school curricula for which they are preparing their students; (2) secondary school teachers should understand the new elementary school materials to which their students may have been exposed; and (3) teachers in college should be familiar with the content of new elementary and secondary school programs, both to know what their students may have had and to know what prospective teachers may be called upon to teach.

The CSTM recognizes that immediate implementation of the CUPM recommendations presents problems not easy to face. Some of these difficulties are listed in the questionnaire report. The tendency on the part of State Departments of Public Instruction is to grant institutional recognition for teacher certification, and to leave the problem of strengthening the teacher training program to the individual institutions. Greatest skepticism with regard to implementing the CUPM recommendations in Illinois centers on Level I, the twelve(12)- semester-hour requirement for elementary school teachers. Since some institutions not far away have already adopted this requirement, however, and since other subjects taught by elementary school teachers have requirements of sixteen (16) to twenty-four (24) semester hours each, the CSTM recommends that the Chairman of the Illinois Section of the M.A.A. be authorized to transmit to the chairman of the Illinois Commission on Teacher Education and Professional Standards (TEPS) the following resolution:

Be it resolved that the Illinois Section of the Mathematical Association of America urges Illinois representatives of the TEPS Commission to seek the introduction and passage of a bill at the next session of the Illinois State Legislature to require:

1) three (3) years of college preparatory high school mathematics as prerequisite for entrance to the elementary teacher-training program, and
2) twelve (12) semester hours of mathematics, four (4) of which may be obtained by proficiency examination by candidates who present more than the three-year requirement in high school mathematics.

After some discussion the above report was unanimously approved by the Illinois Section of the Mathematical Association of America on Friday, May 11, 1962. The Secretary-Treasurer was authorized by the Chairman to convey the resolution to the appropriate representative of TEPS.

Findings from the CSTM Questionnaire. Fifty-seven replies were received. Two of the replies were from members in industry or government; three were from persons in high school teaching; fifty-two were from mathematics personnel in Illinois institutions of higher learning representing forty-six of the seventy-seven colleges, universities, and junior colleges in Illinois, including thirty-four of the fifty-three institutions of higher learning with four-year teacher training programs approved by the Illinois State Teacher Certification Board.

I. Tabulation of Short Answers

Question	*Partially*	*Yes*	*No*
1. Are you in general agreement with the recommendations of the CUPM for levels I, II, and III? _____. If not, what what differing opinions would you like to see presented to the Illinois Section of the MAA?*	2	53	2
2. Do you feel that your department is now offering the approximate equivalent of these recommendations? _____. If not, how do your offerings differ essentially from the recommendations?*		24	27

3. Are your major and minor *requirements* in mathematics for prospective teachers approximately equivalent to those recommended by the CUPM? _____. If not, how do your requirements differ from those recommended?*	14	34
4. If your department does not now offer, or offers but does not require, the equivalent of the program outlined by the CUPM, do you plan to introduce a program more nearly in accord with the content and spirit of the CUPM recommendations? _____. If so, and if your plans can be announced at this time, please indicate what the anticipated changes are, and when they are going into effect.*	26	7

* See Part II for a summary of answers to the second part of each question.

In summary of the data above, we may say: Fifty-three of the fifty-seven replies favored programs including at least the CUPM recommendations for elementary and secondary school teachers. Of these, six believed that the CUPM recommendations were not strong enough. Twenty-four of fifty-one replying to the second question felt that their departments were *offering* the approximate equivalent of the CUPM recommendations; fourteen of forty-eight replying to Question 3 considered their *requirements* for prospective teachers to be approximately equivalent to those recommended by the CUPM; twenty-six of thirty-three replying to Question 4 indicated that they were planning to introduce a program more nearly in accord with the content and spirit of the CUPM recommendations. The remaining questionnaire items were relevant for college personnel only.

II. Differing Opinions. One-third of those replying to the questionnaire expressed some opinions and made some suggestions differing from the recommendations of the CUPM.

One person thought junior high school teachers should have as much preparation as those on any level in the "structure of the real numbers." He went on to say that senior high school teachers could probably use more analysis and less probability and statistics (than recommended by the CUPM). Two other replies recommended more analysis than the CUPM committee did. Another person thought that a course in logic and sets should be required on all three levels, rather than merely available as an elective. Another wished to see some history of mathematics required for all, and still another wanted "History of Arithmetic" for Level I. One person wished to require three years of college-preparatory high school mathematics for prospective elementary school teachers (Level I). Another wished to see the Level III recommendations applied to Level II also.

There were thirteen expressions of skepticism concerning the CUPM recommendations for teacher training requirements. This skepticism was concentrated on requirements for preparation of elementary school teachers. Some felt that there was not room for twelve semester hours of mathematics in the elementary education curriculum. Some felt that their departments of education could

never be convinced of the need for such training, and they could foresee no source for the pressure needed to move these departments of education in the desired direction.

Other skeptical comments included one that pointed out the difficulty of staffing a college program if the CUPM recommendations were to be put into effect suddenly; another thought such requirements would push the colleges into five-year programs for certification to teach; and still another asserted that college graduates "trained this well will not go into teaching."

The full report also included reference to various meetings of CSTM, and a section of the questionnaire summary on textbook criteria; these items have been deleted by the editors because of lack of space.

DEFERMENT OF ROTC STUDENTS FOR GRADUATE STUDY OF MATHEMATICS

ROBERT J. WISNER, Executive Director, CUPM

The Committee on the Undergraduate Program in Mathematics learned some time ago that U. S. Army policy did not allow ROTC students receiving their bachelor's degree to be deferred from undertaking their active duty commitments in order to continue their studies, *except* for students in "medicine, dentistry, law (in certain cases), osteopathy, veterinary medicine, and theology." A letter prepared by the Committee was sent to appropriate authorities arguing the case that the acute shortage of mathematicians should be realized, and that qualified ROTC students should be deferred for graduate study in mathematics. A separate exchange of letters between Professor John G. Kemeny (of CUPM's Panel on Teacher Training) and certain officials in the Department of Defense has been most illuminating. These letters say that ROTC students interested in and qualified for graduate study in mathematics may be considered on an individual basis for year-to-year deferment subject to continued acceptance in graduate school, with the understanding that an "individual who is granted a delay is still required to fulfill his tour of active military duty prior to the expiration of his military obligation." A critical shortage of military manpower would naturally make deferment less likely. The relevant correspondence has been published in the AMS *Notices* 9 (1962) 275–6.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1548. *Proposed by Taiichi Kitamura, Ibaraki University, Japan*

In any convex quadrilateral $ABCD$, we trisect the sides AB, BC, CD, DA in E, F; G, H; I, J; K, L respectively. Show that the area of the quadrilateral formed by the lines EJ, KH, IF, GL is one-ninth the area of $ABCD$. Generalize the problem.

E 1549. *Proposed by C. R. MacCluer, Ohio State University*

Find all functions $y=f(x)$ with second derivatives such that on each interval $[0, x]$, arc length = area.

E 1550. *Proposed by W. L. Nicholson, General Electric Company, Richland, Washington*

Show that

$$\sum_{(p,q)=1} 1/(2^{p+q} - 1) = 1,$$

where the sum extends over all positive integers p and q such that p and q are relatively prime.

E 1551. *Proposed by Azriel Rosenfeld, Budd Electronics, Inc., N. Y.*

Let $N>K>0$ be integers. How many partitions of K are there such that the parts all divide N? What is the smallest number of parts in such a partition? (This problem is, of course, suggested by the well-known Egyptian restriction to fractions with unit numerators.)

E 1552. *Proposed by David Rearick, University of Colorado*

Let $\{a_k\}$ be a sequence of positive terms approaching zero. For any positive x, let $N(x)$ denote the number of terms a_k which are greater than x. Prove that if $\sum a_k$ converges, then $xN(x)\to 0$ as $x\to 0$. Is the latter condition also sufficient for the convergence of $\sum a_k$?

E 1553. *Proposed by F. D. Parker, University of Alaska*

Find the value of the determinant of order n which is formed in the following way: The elements of the diagonals running in the direction of the main diagonal and beginning with a_{21}, a_{11}, a_{13}, a_{15}, $\cdots$, $a_{1,2k-1}$, $\cdots$ are all unity, and all other elements are zero.

E 1554. *Proposed by Krishna Savati, Ann Arbor, Michigan*

Find all solutions of $f(x) = -f(1/x)$.

E 1555. *Proposed by Alfred Brauer and Aubrey Kempner, University of North Carolina and University of Colorado*

In his paper "On the integer solutions of the equation $x^2+y^2+z^2+2xyz=n$," *Journal of the London Math. Soc.*, 28 (1953) 500–10, L. J. Mordell states: "I do not know anything about the integer solutions of

$$x^3 + y^3 + z^3 = 3 \tag{1}$$

beyond the existence of the four solutions

$$(1, 1, 1), \quad (-5, 4, 4), \quad (4, -5, 4), \quad (4, 4, -5), \tag{2}$$

and it must be very difficult indeed to find out anything about any other solutions."

Prove that the triples (2) are the only solutions of (1) which also satisfy the Diophantine equation

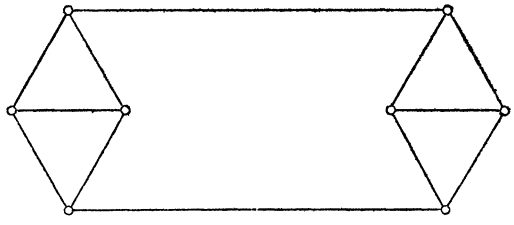
$$x^3 + y^3 + z^3 = x + y + z. \tag{3}$$

SOLUTIONS

A Notched Tetrahedron

E 1506 [1962, 232]. *Proposed by L. R. Ford, Jr., C-E-I-R, Inc., Beverly Hills, Calif.*

The vertices and edges of a certain polyhedron, from which the inside and the faces have been removed, may be imbedded in the plane, allowing stretching. Sketch the original polyhedron if the plane imbedding is as shown.



Solution by Robert Connelly, Carnegie Institute of Technology. It can be a tetrahedron with a notch in the form of a tetrahedral wedge removed from one edge. This is most easily seen by redrawing the graph with one of the triangles outermost, or by viewing the notched tetrahedron through one of the remaining triangular faces.

Also solved by D. I. A. Cohen, Serge Dubuc, E. S. Eby, J. D. Haggard, Ned Harrell, R. T. Hood, R. A. Jacobson, John McGuire, Robert Maas, D. C. B. Marsh, D. A. Moran, Mary Agnes Racki, S. J. Ryan, E. J. Schmahl, and Brian Wichmann.

A number of other "solutions" were received in which some of the distinct vertices of the given figure were incorrectly allowed to coalesce in the final polyhedron.

A Partitioned Square

E 1507 [1962, 232]. *Proposed by D. J. Newman, Yeshiva University*

If a square is partitioned into n convex polygons, show that the total number of edges present is at most $3n+1$.

Solution by the proposer. For each vertex v, let m_v denote the number of edges meeting at v. Except possibly at the four vertices of the square, $m_v \geqq 3$, so that $m_v \leqq 3(m_v-2)$. Hence $\sum_v m_v \leqq 3\sum_v (m_v-2)+2+2+2+2$. Now $\sum_v m_v = 2E$, since all the edges are counted twice, and so the previous inequality gives $2E \leqq 6E-6V+8$, or $E \leqq 3(E-V)+4$. But $E-V=n-1$, by Euler's theorem, and the proof is complete.

To see that $3n+1$ edges are possible, simply divide the square in half by a vertical line and divide the left half into $n-1$ pieces by horizontal lines. We deduce, incidentally, that $V \leqq 2(n+1)$.

Also solved by R. H. Anglin, Walter Bluger, D. I. A. Cohen, Jane Di Paola, Seymour Ditor, Serge Dubuc, Michael Goldberg, David Goodman, J. D. Haggard, S. L. Hunt, A. J. Keeping, J. W. McQuire, D. C. B. Marsh, R. J. Oberg, Nancy Payne, S. J. Ryan, D. L. Silverman, and W. A. Traub.

The solutions submitted by many of the above were incomplete in that they assumed a square partitioned into n convex polygons can always be obtained by drawing a single additional edge in a square partitioned into $n-1$ convex polygons.

Sums of Powers of Integers

E 1508 [1962, 232]. *Proposed by Leonard Carlitz, Duke University*

David Allison has proved (this MONTHLY (1961) 272) that the identity in n

$$\left(\sum_{r=1}^{n} r^k\right)^p = \left(\sum_{r=1}^{n} r^m\right)^q, \qquad p > q,$$

holds only when $p=2$, $q=1$, $k=1$, $m=3$. Show that

$$\left(\sum_{r=1}^{n} (-1)^r r^k\right)^p = \left(\sum_{r=1}^{n} (-1)^r r^m\right)^q, \qquad p > q,$$

is never satisfied for $k \geqq 1$, $m \geqq 1$.

Solution by R. J. Oberg, University of California, Berkeley, Calif. We have

$$\sum_{r=1}^{n} (-1)^r r^k = -\sum_{r=1}^{n} r^k + 2\sum_{j=1}^{[n/2]} (2j)^k = -\sum_{r=1}^{n} r^k + 2^{k+1}\sum_{j=1}^{[n/2]} j^k,$$

whose term of highest degree is $-n^{k+1}/(k+1)$. Hence

$$\left(-\frac{n^{k+1}}{k+1}+\cdots\right)^p = \left(-\frac{n^{m+1}}{m+1}+\cdots\right)^q,$$

which implies p and q have the same parity. But, by Allison's proof, $p=2$, $q=1$. This contradiction establishes the impossibility of the proposed equality.

Also solved by Underwood Dudley, Ralph Greenberg, Erwin Just, A. J. Keeping, Andrzej Makowski, D. C. B. Marsh, David Zeitlin, and the proposer.

A Well-Known Series

E 1509 [1962, 232]. *Proposed by A. G. Konheim, IBM, Yorktown Heights, New York*

With $-1<t<1$, sum the series

$$1+\frac{2}{3}t^2+\frac{2}{3}\frac{4}{5}t^4+\frac{2}{3}\frac{4}{5}\frac{6}{7}t^6+\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}t^8+\cdots.$$

I. *Solution by J. W. Wrench, Jr., David Taylor Model Basin, Washington, D. C.* Let $f(t)$ represent the function defined by the given series over the interval $(-1, 1)$. Then, if $y(t)$ represents $t\cdot f(t)$, we find that the function y satisfies the differential equation $(1-t^2)y'=1+ty$, subject to the condition $y=0$ when $t=0$. The solution to the differential equation, satisfying this condition, is $y=(1-t^2)^{-1/2}\arcsin t$; consequently, the required sum is given by

$$f(t) = (\arcsin t)/t(1-t^2)^{1/2},$$

over the specified interval, except at $t=0$, where the sum is clearly 1, which is the limit value of $f(t)$.

II. *Solution by M. S. Klamkin, University of Buffalo.* The result follows from the known sum

$$(\sin^{-1}t)^2 = t^2+\frac{2}{3}\frac{t^4}{2}+\frac{2}{3}\frac{4}{5}\frac{t^6}{3}+\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{t^8}{4}+\cdots$$

by differentiating both sides and dividing by $2t$.

Also solved by A. D. Anderson and W. R. Faust and A. W. Saenz (jointly), Merrill Barnebey, Lewis Batson, T. L. Beatty, C. R. Berndtson, M. T. L. Bizley, S. E. Bohn, Robert Bowen, Brother Christopher Beaulieu, J. L. Brown, Jr., R. G. Buschman, James Cannon, Flor Cartuyvels, David Croasdale, David Culver and Edward Delahanty (jointly), Gus Di Antonio, W. A. Dolid, Serge Dubuc, Underwood Dudley, Jane Evans, G. P. Farrell, J. A. Faucher, D. P. Giesy, R. M. Gilbert and Alan Weinstein (jointly), Michael Goldberg, L. D. Goldstone, Ralph Greenberg, S. H. Greene, A. H. Halpin, F. W. Herlihy, R. Hines, V. E. Hoggatt, Jr. and I. D. Ruggles (jointly), J. M. Horner, G. J. Janusz, Erwin Just and Norman Schaumberger (jointly), R. G. Kayel, A. J. Keeping, R. P. Kelisky, P. G. Kirmser, D. L. Linfield, D. C. B. Marsh, K. K. Norton, C. S. Ogilvy, G. B. Parrish, Walter Penney, C. F. Pinzka, Gerald Porter, Leo Sauvé, Jim Shilleto, Barry Simon, Philip Smedley, Beverly Smith, F. C. Smith, W. A. Traub, Patrick Twomey, W. C. Waterhouse, J. S. W. Wong, J. E. Yeager, David Zeitlin, and the proposer.

The series was located in *Smithsonian Mathematical Formulae and Tables of Elliptic Function*, 6.42 No. 5, p. 122; Chrystal, *Algebra*, vol. 2, 1906 ed., Ex. xx, No. 7, p. 335 (cites Pfaff as source); Bromwich, *An Introduction to the Theory of Infinite Series*, Prob. 2, p. 197 (claims known to Euler); Knopp, *Theory and Application of Infinite Series*, Ex. 123, Chap. VIII, p. 271; Schuh, *Leerboek der Differentiaal en Integraalrekening*, vol. 2, pp. 154–6; Hobson, *Treatise on Plane Trigonometry*, eqs. 20, 21, 22, pp. 279–80; M. R. Spiegel, this MONTHLY, 60 (1953) 243–7; Taylor, *Advanced Calculus*, p. 632; Edwards, *Differential Calculus for Beginners* (1899), p. 78.

A Set in the Complex Plane

E 1510 [1962, 233]. *Proposed by A. J. Goldman, National Bureau of Standards*

If A and B are sets of complex numbers, let $B-1=\{b-1: b\in B\}$, $a(B-1)=\{a(b-1): b\in B\}$, and $A*B=\bigcap\{a(B-1): a\in A\}$. Suppose that $A=\{z: |z-1|\leqq k\}$, where $k>1$, and that $B=\{z: |z|\geqq 1\}$. Find $A*B$.

Solution by J. E. Vinson, Aerojet-General Corp., Sacramento, Calif. Since $0\in A$, $A*B\subset 0(B-1)=\{0\}$. On the other hand, $0\in a(B-1)$ for all a because $0\in B-1$. Hence $A*B=\{0\}$.

Also solved by Kandiah Balachandran, Donald Bateman, Robert Bowen, G. P. Farrell, J. F. Leetch, D. C. B. Marsh, G. B. Parrish, B. E. Rhoades, James Sui and Elaine Tatham (jointly), and the proposer.

Editorial Note. The above solution assumes that we are working in the finite, or unextended, complex plane. Every solver, with the exception of the proposer, made this assumption. If we are working in the extended complex plane, the set $0(B-1)$ is not well defined, but from continuity considerations would be defined to be the set exterior to the null circle $|z|=0$. It can now be shown that $A*B$ is the point 0 plus the complement of the interior of the limaçon $r=2(k-\cos\theta)$. See, e.g., Cowling, Leighton, Thron, "Twin convergence regions for continued fractions," *Bull. Amer. Math. Soc.*, 50 (1944) 351–7.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5058. *Proposed by Albert Wilansky, Lehigh University*

Must the one point compactification of a locally compact metric space be metrizable?

5059. *Proposed by E. R. Gentile, Universidad del Sur, Argentina*

Let R be an integral ring (possessing unity element and without divisors of zero). R is said to have a left field of quotients if any two nonzero left ideals have a nonzero intersection. Let A be a left R-module. An element a of A is called a (left) R-torsion element if $\mu a=0$ for some $\mu\neq 0$ in R. Let tA be the set of all torsion elements of A. Prove that R has a left field of quotients if and only if for every left R-module A, tA is a submodule.

5060. *Proposed by H. S. Shapiro, New York University*

Let $f(x)$ be continuous for $0 \leqq x \leqq 1$. Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{k}{n}\right) \Big/ 2^n = 0.$$

5061. *Proposed by Eric D. Nix, New York, N. Y.*

Exhibit (or disprove the existence of) a function which is C^∞ on an interval but nowhere analytic.

5062. *Proposed by Jonathan Sondow, University of Wisconsin*

Prove that a topological space X is bicompact if and only if every cover of X has an irreducible subcover.

5063. *Proposed by R. E. Shafer, University of California, Livermore*

Show that

$$(1) \qquad \sum_{n=1}^{\infty} \frac{1}{\sinh^2 n\pi} = \frac{1}{6} - \frac{1}{2\pi}, \qquad (2) \qquad \sum_{n=1}^{\infty} \frac{1}{\cosh^2 (2n-1)\pi/2} = \frac{1}{2\pi}.$$

5064. *Proposed by S. Lajos, L. Eötvös University, Hungary*

A semigroup S is called regular if to each element $a \in S$ corresponds at least one semi-inverse element $x \in S$ such that $axa = a$. Prove that S is regular if and only if $(a)_R(a)_L = (a)_R \cap (a)_L$, for all $a \in S$, where $(a)_L$ denotes the principal left ideal of S generated by a.

SOLUTIONS

A Convergent Integral

4981 [1961, 673]. *Proposed by Lawrence Shepp, University of California, Berkeley*

Show that

$$\int_0^1 \left\{ \sum_{n=1}^{\infty} 2^{-n}(1 - \cos 2^{2n}z) \right\}^{-1} dz < \infty.$$

Solution by N. G. de Bruijn, Technological University, Eindhoven, Netherlands. We observe at the start that all of the terms of the sum are nonnegative. The idea of the proof is to majorize the integrand by strategically replacing the infinite series by one of its terms. It obviously suffices to show that, for $m = 1, 2, \cdots$, the integral from $\pi \cdot 2^{-2^{m+1}}$ to $\pi \cdot 2^{-2^m}$ is less than $C \cdot 2^{-m}$, where C is a constant. Denoting the sum occurring under the integral sign by Σ, we have in that interval

$$(1) \qquad \sum \geqq 2^{-m} b (2^{2m}x)^2,$$

where b is such that $1 - \cos y \geqq by^2 (0 \leqq y \leqq \pi)$). By integrating both sides of (1),

it follows that the integral from $\pi \cdot 2^{-2^{m+1}+2m}$ to $\pi \cdot 2^{-2^m}$ is less than $\pi^{-1}b^{-1}2^{-m}$. So we can focus our attention on the relatively small subinterval $I_m = [\pi \cdot 2^{-2^{m+1}}, \pi \cdot 2^{-2^{m+1}+2m}]$. We shall now use only those terms of the sum for which $m+1 \leqq n \leqq 5m$. Let S_m be the set of those points x of I_m for which $\cos 2^{2^n}x > 0$ holds for all n in that interval. Analyzing the periodic nature of $\cos 2^{2^n}x$ on I_m, it is easy to show that the measure of S_m equals $\mu(S_m) = 2^{-4m}\mu(I_m)$. Put $I_m - S_m = T_m$.

On T_m at least one of the above-mentioned cosines is $\leqq 0$, whence $\Sigma \geqq 2^{-5m}$. So the integral over T_m is at most $2^{5m}\mu(I_m) < \pi \cdot 2^{-m+10}$. On S_m we use $\Sigma \geqq b\pi^2 2^{-2^{m+1}-m}$, which follows from (1). The integral over S_m turns out to be less than

$$b^{-1}\pi^{-2}2^{2^{m+1}+m} \cdot 2^{-4m}\mu(I_m) < b^{-1}\pi^{-1}2^{-m},$$

and the proof is complete.

Also solved by the proposer.

Necessary and Sufficient Condition for Absolute Convergence

4986 [1961, 808]. *Proposed by U. C. Gupta, University of Malaya, Singapore*

A familiar result due to Cauchy gives the following necessary and sufficient condition for the absolute convergence of $\sum a_n$: given $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $\left| \sum_{k=1}^{p} a_{n+k} \right| < \epsilon$, $n \geqq N$, p any positive integer.

Prove the following analogous necessary and sufficient condition for the absolute convergence of $\sum a_n$: given $\epsilon > 0$ and any sequence of positive integers $\{\lambda_n\}$, where $\lambda_1 < \lambda_2 < \cdots$, there exists a positive integer $N = N(\epsilon, \{\lambda_n\})$ such that $\left| \sum_{k=1}^{n} a_{n+\lambda_k} \right| < \epsilon$, $n \geqq N$, p any positive integer.

Solution by R. H. C. Newton, Berkhamsted, Hertfordshire, England. We will put $q(k)$ for λ_k. By the Cauchy condition, if $\sum_n |a_n|$ is convergent, then for $n > N$ and for all $q(p) > 0$, we have

$$\epsilon > \sum_{k=1}^{q(p)} |a_{n+k}| \geqq \sum_{k=1}^{p} |a_{n+q(k)}| \geqq \left| \sum_{k=1}^{p} a_{n+q(k)} \right|.$$

Hence the proposed condition is necessary.

Now suppose that $\sum_n a_n$ is either divergent or conditionally convergent. It follows that not both $\sum_n \mathrm{Rl}(a_n)$ and $\sum_n \mathrm{Im}(a_n)$ are absolutely convergent. Accordingly, a sequence exists, denoted by $p(1), p(2), p(3), \cdots$, such that at least one of the two sums $\sum_k \mathrm{Rl}(a_{p(k)})$ and $\sum_k \mathrm{Im}(a_{p(k)})$ diverges to infinity. Hence $\left| \sum_{k=1}^{n} a_{p(k)} \right|$ tends to infinity with n.

Now write b_k for $a_{p(k)}$, so that, by Cauchy again, a γ exists such that

$$(*) \qquad \left| \sum_{k=1}^{p} b_{n+k} \right| > \gamma > \epsilon,$$

for $n > N$ provided that $p > P$, where P depends upon γ and n.

Now choose a matrix $q(n, k)$ such that $p(n+k)-n=q(n, k)$. It follows from (*) that $\left|\sum_{k=1}^{p} a_{n+q(n,k)}\right|>\gamma$. Hence, choosing $n>N$, a sequence $q(n, 1)$, $q(n, 2), \cdots$ exists such that the proposed condition is violated. Hence that condition is sufficient.

Also solved by N. V. Glick, Erwin Just, Mark Ramras, H. J. Zimmerberg, and the proposer.

A Special Normal Subgroup

4987 [1961, 808]. *Proposed by B. L. T. Dufa Scio, Institute for Defense Analyses*

Let G be a group and H a homomorphism of G into G which commutes with every inner automorphism of G. Let K be the set of elements $x\in G$ such that $H(H(x))=H(x)$. Show that K is a normal subgroup of G and that G/K is Abelian.

Solution by Gordon Keller, University of Buffalo. By hypothesis we have, for any $y,\ z\in G$: $H(yz)=H(y)H(z)$; $H(y^{-1})=(H(y))^{-1}$, $H(e)=e$; $H(y^{-1}zy)=y^{-1}H(z)y$.

K is nonempty since $e\in K$. K is a subgroup of G if $a,\ b\in K$ implies $ab^{-1}\in K$. This is established by $H(H(ab^{-1}))=H(H(a)H(b^{-1}))=H(H(a))H(H(b^{-1}))=H(a)H((H(b))^{-1})=H(a)(H(H(b)))^{-1}=H(a)(H(b))^{-1}=H(a)H(b^{-1})=H(ab^{-1})$.

K is a normal subgroup of G if for any $k\in K$ and $y\in G$, $y^{-1}ky\in K$. This follows from $H(H(y^{-1}ky))=H(y^{-1}H(k)y)=y^{-1}H(H(k))y=y^{-1}H(k)y=H(y^{-1}ky)$.

G/K is Abelian if all commutators of G belong to K. This follows since, for all $y,\ z\in G$, $H(H(y^{-1}z^{-1}yz))=H(H(y^{-1}z^{-1}y)H(z))=H(y^{-1}H(z^{-1})yH(z))=H(y^{-1})H((H(z))^{-1}yH(z))=H(y^{-1})(H(z))^{-1}H(y)H(z)=H(y^{-1}z^{-1}yz)$.

Also solved by Joseph Altinger, D. A. Arterburn, H. F. Bechtell, R. J. Bumcrot, Harry Cash, R. J. Cormier, R. H. Dale, D. M. Danvers, S. E. Dickson, Mary R. Embry, Burton Fein, Arthur Fine, Harley Flanders, T. S. Frank, E. R. Gentile, M. L. Glasser, Richard Goodman, Alfred Gray, C. A. Green, J. M. Hartley, Jr., C. V. Holmes, J. F. Hurley, O. J. Huval, K. D. Johnson, Geoffrey Kandall, John H. Kelly, Mary K. Knox, W. G. Leavitt, Gerald Leibowitz and E. L. Cohen, G. M. Lewis, Jiang Liu, C. R. MacCluer, S. S. Magliveras, W. S. Martindale, M. D. Mavinkurve, Robert Melter, D. E. Myers, M. F. Newman, R. J. Newman, N. F. O'Donohue, Barbara L. Osofsky, Mark Ramras, D. A. Robinson, C. Seguin, J. R. Swenson, Wolmer Vasconcelos, E. W. Wallace, W. C. Waterhouse, N. J. Weiss, Neal Zierler, and the proposer. Late solution by R. A. Cunninghame-Green.

A Set of Special Polynomials

4989 [1961, 933]. *Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan*

Let $f(x)$ be a polynomial of degree n, such that $\int_0^1 x^k f(x)dx=0$ $(k=1, 2, \cdots, n)$. Prove that

$$\int_0^1 \{f(x)\}^2 dx=(n+1)^2\left\{\int_0^1 f(x)dx\right\}^2.$$

Solution by R. L. Van de Wetering, San Diego State College. Set $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$. Since $\int_0^1 x^k f(x)dx=0$, $k=1, 2, \cdots, n$, we have

$$\int_0^1 \{f(x)\}^2 dx = a_0 \int_0^1 f(x)dx. \tag{1}$$

Also

$$\int_0^1 x^k f(x)dx = \frac{a_n}{n+k+1}+\frac{a_{n-1}}{n+k}+\cdots+\frac{a_0}{k+1};$$

$$\frac{a_n}{n+k+1}+\frac{a_{n-1}}{n+k}+\cdots+\frac{a_0}{k+1}=0, \qquad k=1, 2, \cdots, n.$$

Now

$$\frac{a_n}{n+k+1}+\frac{a_{n-1}}{n+k}+\cdots+\frac{a_0}{k+1}=\frac{Q(k)}{(k+1)(k+2)\cdots(n+k+1)},$$

where $Q(k)$ is a polynomial in k of degree at most n. Since $Q(k)=0$, $k=1, 2, \cdots, n$; $Q(k)=C(k-1)(k-2)\cdots(k-n)$ with C a constant. Hence

$$\frac{a_n}{n+k+1}+\frac{a_{n-1}}{n+k}+\cdots+\frac{a_0}{k+1}=\frac{C(k-1)(k-2)\cdots(k-n)}{(k+1)(k+2)\cdots(n+k+1)}. \tag{2}$$

For $k=0$, $\int_0^1 f(x)dx=a_n/(n+1)+\cdots+a_0/1=(-1)^nC/(n+1)$. Multiply (2) by $k+1$ and set $k=-1$ to obtain

$$a_0=(-1)^n(n+1)C \quad \text{or} \quad a_0=(n+1)^2\int_0^1 f(x)dx.$$

Thus, by (1)

$$\int_0^1 \{f(x)\}^2 dx=(n+1)^2\left\{\int_0^1 f(x)dx\right\}^2.$$

Also solved by W. N. Anderson, Jr., Robert Breusch, J. L. Brown, Jr., L. Carlitz, G. Di Antonio, A. R. Di Donato, P. G. Engstrom, A. B. Farnell, J. S. Frame, Emil Grosswald, V. C. Harris, Alvin Hausner, J. C. Hickman, L. M. Kaplan, A. G. Konheim, G. Laman, J. B. Linder, Y. L. Luke, R. G. Nath, E. A. Newberg, J. E. Potter, J. V. Ryff, E. M. Scheuer and J. D. Brooks and A. H. Halpin, Oved Shisha, R. L. Shively, Michael Skalsky, R. H. Urbano, W. C. Waterhouse, J. S. W. Wong, P. W. Zehna, and the proposer. Late solution by G. B. Parrish.

An Integral Identity

4990 [1961, 933]. *Proposed by R. G. Medhurst, Wembley, England*

Prove the identity:

$$\int_{\frac{1}{2}(p+p_0)}^{p_m} dy \int_{p-y+p_0}^{y} f(y)f(x)f(y+x-p)dx = \int_{p_0}^{2p_m-p} d\eta \int_{\frac{1}{2}(\eta+p)}^{p_m} f(\eta)f(\xi)f(\eta-\xi+p)d\xi,$$

and establish under what conditions it holds.

Solution by G. Laman, Technological University, Eindhoven, Netherlands. Denote the region of integration of the first double integral by G and of the second by Γ. Then both G and Γ are triangles defined by

$$G\colon\ y - p_m < 0, \quad x - y < 0, \quad x + y - p - p_0 > 0;$$
$$\Gamma\colon\ \xi - p_m < 0, \quad \eta - p_0 > 0, \quad \eta - 2\xi + p < 0.$$

Now the linear transformation $\xi = y$, $\eta = x + y - p$ does the trick as is easily verified. In both integrals $f(x)$ is defined necessarily on the following intervals:

$$I.\ \tfrac{1}{2}(p + p_0) < x < p_m, \quad \text{II.}\ p + p_0 - p_m < x < p_m, \quad \text{III.}\ p_0 < x < 2p_m - p.$$

As interval I is part of interval II we may state the condition as follows: $f(x)$ must be defined and integrable on the union of the intervals II and III.

Infimum of Boolean Topologies

4991 [1961, 934]. *Proposed by Oscar Varsavsky, Caracas, Venezuela*

Find the infimum of all Boolean topologies defined on a given set T. (A topology is called Boolean if it is Hausdorff, compact, and totally disconnected.)

Solution by W. J. Pervin, Pennsylvania State University. For each $x \in T$ there exists a topology τ_x for which x is the "ideal" point of the one-point-compactification of the discrete space $T - x$; that is, every subset of T which does not contain x is open while the only sets containing x which are open are those whose complements are finite. Each τ_x is clearly a Boolean topology. The infimum, τ, of all Boolean topologies for T can be no larger than $\cap\{\tau_x\colon x \in T\}$ which is the topology made up of $\varnothing$ and all complements of finite sets. This topology is the smallest of all T_1 topologies and so is smaller than any Boolean topology. Thus the infimum τ can be no smaller than that topology. We note that the infimum of all Boolean topologies for T is the same as the infimum of all T_1 topologies.

Also solved by Manuel Berri, Barbara L. Osofsky, W. C. Waterhouse, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Mathematical Discovery, Volume 1. On understanding, learning and teaching problem solving. By George Polya. Wiley, New York, 1962. xv+216 pp. $4.25.

This book is a continuation of *How to Solve It* and *Mathematics and Plausible Reasoning*, and it has the same multiple aims—problem solving, improvement of high school teaching, study of "heuristic". The examples are mostly elementary and the needs of teachers are stressed. However, there is much to interest the general reader. It is intended that this book shall be usable as a text for a seminar in problem solving.

Part One, which makes up the bulk of this volume, is concerned with solving some elementary problems. It is written in a very informal way in an effort to show the actual steps, good and bad, that one might take when attempting to arrive at a solution. The style is quite effective but it may tend to obscure the fact that the final result should be a clearly stated, mathematically correct solution. Presumably this final step is left to the reader, but the author does not always set a good example. In some cases the statement of the problem seems to be too imprecise; the reader may not know what is expected of him. Page 31 contains an apparent division by zero. But these are small points. The book contains an excellent set of problems. There are many of them, mostly elementary, all interesting.

Part Two is incomplete. The rest of it is to be included in a forthcoming second volume.

H. S. ZUCKERMAN
University of Washington

MAA Studies in Mathematics, Volume 1, *Studies in Modern Analysis.* By E. J. McShane, M. H. Stone, E. R. Lorch, C. Goffman. Edited by R. C. Buck. Distributed by Prentice Hall, Englewood Cliffs, N. J., 1962. 182 pp. $4.00.

The series of which this is the initial volume is designed to provide a source of introductory expository essays on a variety of mathematical topics. The need for such material is well recognized, and the mathematical community should be grateful, both to the editors for their initiative, and to the authors for the trouble they have taken to make a presentation of their subjects which is lively and lucid. While no effort has been made to achieve complete homogeneity in the level of presentation, the readers in mind range roughly from the advanced undergraduate to the beginning graduate student. However, many older readers with gaps or rusty spots in their mathematical education (and who doesn't have them?) will undoubtedly find the series of value.

The present volume contains four papers: A Theory of Limits by E. J. McShane, A Generalized Weierstrass Approximation Theorem by M. H. Stone, The Spectral Theorem by Edgar R. Lorch, and Preliminaries to Functional Analysis by Casper Goffman, together with an Introduction by R. C. Buck. As these titles will indicate, the general emphasis is on those sections of analysis which make considerable use of algebra and topology, and, except for McShane's paper, the reader will need some acquaintance with these areas. However, there are enough epsilons and deltas around to convince the "hardest" analyst that the book justifies its title. The general tendency of the exposition is a gentle start with an increase in tempo and also in the demands on the reader's preparation as the author succumbs to the temptation to cover more ground near the end. Thus in some cases one would question whether the student with sufficient sophistication to complete the paper with full understanding would need to be led by the hand so carefully at the beginning. However, if the reader with limited preparation is prepared to forego a complete comprehension of the end of the paper, he can certainly learn much from the beginning, and the general utility of the essays is thereby widened.

A brief discussion of each of the four papers follows:

McShane, using extended-real-valued functions (with a discussion of the extended reals), and a *direction* (filter base) on their domains, presents a general theory of limits which should be understandable in its entirety to any undergraduate who has studied the calculus. Furthermore, such a reader should be delighted to have the various sorts of limits he has encountered so elegantly unified and his knowledge so effortlessly extended. One might wish that he had said a word about the relation of directions to filters, since only a word was needed.

Stone's essay is a reprint of a classic paper on the uniform approximation of functions by functions generated from those in a given class using specified algebraic operations, in turn those of a lattice, a linear lattice and a linear ring. The variety of applications of the theory thus developed gives an admirable example of the beauty and power of general methods, and should convince any budding analyst that he must go into the field with his algebraic tools well sharpened.

Lorch carries the spectral theorem successively through the n-dimensional, completely continuous, bounded and unbounded cases. As one might expect with so much ground to cover, the going gets a little rough toward the end, and the reader for whom this is really an introduction will have to do considerable filling in from other sources. However, he can do this with a firm grasp of the simpler cases and a good idea of the general picture.

As the title implies, Goffman's paper is the least unified of the four. Its prerequisites are somewhat lighter than those for Stone or Lorch, and much of it should be understandable for an advanced undergraduate. The topics considered are: metric spaces, fixed-point theorems, compact spaces, Hilbert and Banach spaces and Banach algebras. Enough applications to classical analysis

are given to show that these generalities are not without their utility in more familiar fields.

For the student for whom this is all more or less new, I should recommend McShane, Goffman, Stone, Lorch as reading order.

W. R. TRANSUE
Kenyon College

Popular Lectures in Mathematics Series, Blaisdell Publishing Company, New York, 1961. $.95 per volume.

Vol. 1 I. S. Sominskii: *The Method of Mathematical Induction*;
Vol. 2 N. N. Vorob'ev: *Fibonacci Numbers*;
Vol. 3 V. A. Uspenskii: *Some Applications of Mechanics to Mathematics*;
Vol. 4 A. N. Kostovskii: *Geometrical Constructions Using Compasses Only*;
Vol. 5 A. S. Smogorzhevskii: *The Ruler in Geometrical Constructions*;
Vol. 6 P. P. Korovkin: *Inequalities*.

These are paper-bound books varying in length from 57 to 86 pages. They are quite elementary (only one, *Inequalities*, uses the notion of limit in an essential way).

The books were intended for bright secondary school students in the Soviet Union. They are written with much care, and almost all statements are proved rigorously. The only one descending from this level of performance is Uspenskii's book on mechanics applied to mathematics (mostly geometry). What might have been an illuminating approach is marred by the author's refusal to say which principles of mechanics are axioms, which are theorems, or which should be accepted because they are "known from every-day experience".

The two on geometry (Kostovskii and Smogorzhevskii) are beautifully and economically written. Smogorzhevskii gives an easy introduction to projective geometry and then settles down to Steiner's theorem on constructions with ruler alone. Kostovskii gives the Mohr-Mascheroni theorem on the sufficiency of the compass alone for all classical constructions, and then investigates constructions using compasses with restricted openings.

The books on induction and inequalities include a multitude of problems, all with solutions. Korovkin's entire book on inequalities is ingeniously based on the theorem that if the product of m positive numbers is 1, their sum is not less than m.

The reproduction (apparently a lithographing process) is just bad enough to keep the reader alert. In places it is illegible.

Aside from being interesting reading, these books could find a use in honors seminars for high school students. The six volumes could make up a fairly well-balanced diet: only algebra is totally absent.

STEPHEN PUCKETTE
The University of the South

Geometry of Complex Numbers. By Hans Schwerdtfeger. University of Toronto Press, Toronto, 1962. xi+186 pp. $4.95.

This is the University of Toronto Mathematical Expositions No. 13. The book's three chapters develop three important related areas of the geometry of complex numbers—the analytic geometry of circles, the Möbius transformations, and the two-dimensional non-Euclidean geometries. The material, now mostly standard, is nicely treated, and largely reflects work of C. Carathéodory and E. Cartan. The book caters to the senior-graduate student and lies (using the author's words) "in the intersection of geometry, analysis, and algebra." Each of the fifteen sections of the book is followed by a set of exercises (called "examples" by the author) designed to furnish the reader further information, rather than to test his mastery of the subject, and indications of proof are frequently supplied. The book seems excellent as a collateral text to be used with a course in the theory of analytic functions, but is probably too restricted in its topics to be used as the basic text for a course in the geometry of complex numbers. At any rate, there are large, attractive, and important areas of the subject not covered by the text, and there are similar lacunae in the bibliography at the end of the book. The work is not without some usual first-edition misprints, of which none noted seems to be serious, though this reviewer was a bit grieved to find his name at one place in the bibliography carelessly given as H. Evans, and elsewhere that of his friend Hoggatt as Hoggat.

HOWARD EVES
University of Maine

Methods of Celestial Mechanics. By D. Brouwer and G. M. Clemence. Academic Press Inc., New York, 1961. xii+598 pp. $15.50.

This book provides a comprehensive background for the practical applications of celestial mechanics. Observational astronomy and theoretical matters are not emphasized. While designed to facilitate numerical work, it does much more. Being developed logically from first principles, it provides motivation for large segments of celestial mechanics. In the exposition few steps are omitted, formulae being displayed in detail. The book is suitable for advanced undergraduate and graduate students, and as a reference work.

The realization of artificial satellites has led to renewed interest in celestial mechanics. Calculating machines have reduced the drudgery of calculations. Much brilliant work has been done in the field of celestial mechanics, especially during the last three centuries. In applications to the motion of bodies in the solar system the accuracies of predictions have been most impressive. It thus is appropriate that the results of the efforts of many be made more fully known in order that maximum benefit be derived therefrom in today's and tomorrow's "space" problems. These considerations have provided the authors, both of whom are greatly skilled in the field, with the motives for writing the book.

Chapters I, II and III lay the foundation for the study of elliptic motion and the theory of gravitational attraction, beginning with Kepler's and Newton's laws. The methods presented suffice for the calculation of the coordinates of a planet in an elliptic orbit at any time. Chapters IV and V treat the calculus of finite differences and the numerical integration of orbits. Chapters VI through IX cover topics from practical and spherical astronomy (aberration, differential correction of orbits). The methods used extensively for numerical calculations in celestial mechanics are developed in detail in Chapters X through XVII (general integrals, lunar and perturbation theories, Hansen's method, canonical variables, Delaunay's method). The treatment deals largely with cases where the principal forces are gravitational. General relativity is not included.

It often is desirable to be able to use the great memoirs of the past for reference. The number of symbols required is large. Thus the selection of notation is difficult. The authors have met this problem well.

The reader should consider the book in the perspective held by astronomers. If he does, matters that mathematicians might prefer to be treated differently become better motivated. Celestial mechanicians have been great computers. Serious students can learn much from their works. The present volume, well illustrated with numerical examples, reflects the results of this deep experience.

The reviewer noted few blemishes in the typography. The book's usefulness would be improved by more extensive use of drawings. Some would welcome a more elementary introduction to the discussion of coordinates and time systems. While a knowledge of calculus will suffice in the early parts of the book, the reader really needs a good background in analytical dynamics, elementary astronomy and the theory of differential equations.

Chapter XVII discusses the motion of an artificial satellite in which the effects of the earth's oblateness and atmospheric drag are considered. When the authors undertake a revision the reviewer would recommend that they include comprehensive sections combining classical celestial mechanics with aeroballistics.

The reviewer recommends this book. If the user masters it he should be in a position to dig deeply into the more advanced phases of the subject.

The views expressed herein are those of the reviewer, and are not necessarily those of the Navy Department.

RICHARD S. BURINGTON
Bureau of Naval Weapons

BRIEF MENTION

Higher Algebra for the Undergraduate (Second Edition). By Marie J. Weiss; revised by Roy Dubisch. Wiley, New York, 1962. ix+171 pp. $4.95.

Substantially revised, this book now contains many more problems and added material on linear algebra.

The Theory of Determinants, Matrices, and Invariants (Third Edition). By H. W. Turnbull. Dover, New York, 1960. xviii+374 pp. $2.00 (paper).

To the second edition (1945) an appendix has been added to fill out the sketch of a proof, to incorporate recent work on symmetric determinants, and to give the resolution of a square matrix into triangular factors.

Principia Mathematica (to *56). By A. N. Whitehead and B. Russell. Cambridge University Press, New York, 1962. x+vi+410 pp. $1.95 (paper).

An abridged text of Volume 1. Where else can one obtain these days the writing of *two* such eminent scholars for less than one-half cent per page?

Elements of the Topology of Plane Sets of Points. By M. H. A. Newman. Cambridge University Press, New York, 1962. vii+214 pp. $2.95 (paper).

A reprint of the second edition (1951).

Differential Equations. By Kaj L. Nielsen. Barnes and Noble, New York, 1962. xxi+262 pp. $1.75 (paper).

College Outline Series.

Summation of Series (Second Revised Edition). By L. B. W. Jolley. Dover, New York, 1961. xii+251 pp. $2.00 (paper).

A collection of 1146 formulas, including bibliographical references in each case.

Theory and Applications of Finite Groups. By G. A. Miller, H. F. Blichfeldt, L. E. Dickson. Dover, New York, 1961. xvii+390 pp. $2.00 (paper).

Republication of the last (1938) edition.

Application of Probability Theory and Mathematical Statistics in the Design and Manufacture of Radio-Electronic Equipment. By P. P. Mesyatsev. Office of Technical Services, U. S. Department of Commerce, Washington, D. C., 1961. 250 pp. $2.50 (paper).

Translated from the Russian, by the Israel Program for Scientific Translations, for the National Science Foundation and the Department of Commerce.

Science and the Nation, Policy and Politics. By J. Stefan Dupré and Sanford A. Lakoff. Prentice-Hall, Englewood Cliffs, N. J., 1962. x+181 pp. $1.95 (paper).

An excellent summary of fact and opinion.

Principles of Physics. By Earnest S. Greene. Prentice-Hall, Englewood Cliffs, N. J., 1962. 832 pp. $9.75.

An elementary college text with a minimum of mathematics.

Principles of College Physics. By George Shortley and Dudley Williams. Prentice-Hall, Englewood Cliffs, N. J., 1959. Vol. 1, 474 pp. $6.00; Vol. 2, 400 pp. $6.00.

An extensively revised noncalculus version of a standard freshman text.

Introduction to Engineering Mechanics. By John V. Huddleston. Addison-Wesley, Reading, Mass., 1961. 493 pp. $9.75.

Covers elementary statics, dynamics, and strength of materials at about the sophomore level.

Introduction to Electrical Engineering Science. By Harold A. Foecke. Prentice-Hall, Englewood Cliffs, N. J., 1961. 778 pp. $16.00.

A text "integrating" analytical treatments of electricity and magnetism and circuit theory.

Physics in the Soviet Union. By A. S. Kompanayets. Philosophical Library, Inc., New York, N. Y., 1962. 592 pp. $7.50.

A relatively brief theoretical exposition of the elements of mechanics, electrodynamics, quantum mechanics, and statistical physics.

Tables of Indefinite Integrals. By G. Petit Bois. Dover, New York, 1961. 151 pp. $1.65.

A photographic reprint of "Tafeln der unbestimmten Integrale" (B. G. Teubner, 1906).

Introduction to Elliptic Functions. By F. Bowman. Dover, New York, 1961. 115 pp. $1.25.

Except for the addition of three references and a few minor corrections this is a reprint or the first edition (English Universities Press Ltd., 1953). According to the author's preface, this is "a short practical introduction to some applications of the . . . Jacobian [elliptic] functions; the Weierstrassian function is not even mentioned."

Vorlesungen über Differentialgeometrie. By G. Vranceanu. Translated into German by M. Pinl. Akademie-Verlag, Berlin, 1961. viii+371 pp. DM 58.

This book is a revision and a translation (from the Rumanian) by Max Pinl of the author's earlier work *Leçons de Géométrie Différentielle I*, which was first published in 1947 in Bucharest. The first edition received a full review in *Mathematical Reviews*, (1948) 532. Although the chapter headings in the revision are the same as those in the original, a considerable amount of new material has been added. Further supplementary theorems are contained in three appendices. According to the publisher there is a Second Part (viii+405 pp. DM78).

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor H. L. Alder, University of California, Davis, represented the Association at the inauguration of Brother Timothy Michael, F.S.C., as President of St. Mary's College on October 4.

Professor W. T. Guy, Jr., University of Texas, represented the Association at the inauguration of Dr. K. S. Pitzer as President of Rice University on October 10.

University of California, Berkeley: Dr. Marvin Shinbrot, University of Chicago, has been appointed Assistant Professor; Dr. R. M. Fesq, Jr., University of Oregon, has been appointed Instructor; Professor Bjarni Jonsson, University of Minnesota, has been appointed Visiting Professor and Research Mathematician; Assistant Professor F. A. Raymond, University of Wisconsin, has been appointed Visiting Associate Professor; Dr. S. A. Robertson, University of Liverpool, England, has been appointed Visiting As-

sistant Professor and Assistant Research Mathematician; Associate Professors E. A. Bishop and Istvan Fáry have been promoted to Professors; Professors Bernard Friedman and D. H. Lehmer have been appointed Miller Research Professors for the academic year 1962–63.

Mr. R. D. Anderson, Jr., University of Idaho, has accepted a position as Engineer with Atomics International, Canoga Park, California.

Dr. A. O. Allen, University of California, Los Angeles, has accepted a position as Advisory Systems Engineer with International Business Machines, Inglewood, California.

Dr. C. E. Aull, University of Colorado, has been appointed Assistant Professor at Kent State University.

Assistant Professor Miriam C. Ayer, University of California, Berkeley, has been appointed Associate Professor at the University of Oklahoma.

Mr. R. F. Baker, University of Oklahoma, has accepted a position as Geophysicist with the Chevron Oil Company, Jackson, Mississippi.

Mr. E. F. Baumgartner, Syracuse University, has been appointed Assistant Professor at Bates College.

Mr. R. E. Beals, Massachusetts Institute of Technology, has been appointed Assistant Professor of Economics at Amherst College.

Dr. J. W. Bebernes, University of Nebraska, has been appointed Assistant Professor at the University of Colorado.

Dr. G. R. Blakley, Harvard University, has been appointed Assistant Professor at the University of Illinois.

Dr. H. G. Bray, Iowa State University, has been appointed Assistant Professor at San Diego State College.

Mr. J. R. Brown, University of Massachusetts, has been appointed Assistant Professor at Oregon State University.

Dr. R. J. Bumcrot, University of Missouri, has been appointed Assistant Professor at Ohio State University.

Professor C. E. Burgess, University of Utah, is on leave as Visiting Member of the Institute for Advanced Study, Princeton, New Jersey.

Mr. J. A. Carpenter, Melpar, Cambridge, Massachusetts, has accepted a position as Associate Mathematician at the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Associate Professor F. L. Celauro, Central Michigan University, has been appointed Professor at Peabody College.

Mr. R. K. Clark, Matson Navigation Company, San Francisco, California, has accepted a position in the Applied Mathematics Division of the Argonne National Laboratory, Argonne, Illinois.

Dr. B. R. Davis, University of Pittsburgh, has been appointed Assistant Professor at the University of Toledo.

Dr. J. D. DePree, University of Colorado, has been appointed Assistant Professor at Oregon State University.

Mr. H. B. Foisy, St. Lawrence Central High School, Brasher Falls, New York, has been appointed Assistant Professor at the State University of New York, Potsdam.

Assistant Professor E. E. Grace, University of Wisconsin, has been appointed Visiting Associate Professor at the University of Georgia.

Dr. H. C. Griffin, Massachusetts Institute of Technology, has accepted a position as Research Associate with the Argonne National Laboratory, Lemont, Illinois.

Mr. H. B. Hanes, Jr., University of Kansas, has been appointed Assistant Professor at Earlham College.

Mr. W. J. Hastings, Electronic Technology Corporation, Ormond Beach, Florida, has accepted a position as Mathematician in the Engineering Department of Hayes International Corporation, Birmingham, Alabama.

Mr. S. K. Hildebrand, Iowa State University, has been appointed Assistant Professor at Midwestern University.

Assistant Professor B. F. Hobbs, Olivet Nazarene College, has been awarded a NSF Science Faculty Fellowship and will be on leave for the academic year 1962–63 at Purdue University.

Mr. F. X. Holzhauer, Bendix Systems, Ann Arbor, Michigan, has accepted a position as Computer Analysis Supervisor in the Computation Laboratory of Chrysler Corporation, Detroit, Michigan.

Associate Professor Aughtum S. Howard, Eastern Kentucky State College, has been promoted to Professor.

Mr. Wei Hsuan, Wisconsin State College, has been appointed Assistant Professor at Ithaca College.

Assistant Professor H. L. Hunzeker, University of Nebraska, has been appointed Associate Professor at the University of Omaha.

Dr. C. C. Hurd, International Business Machines, San Jose, California, has accepted a position as Chairman of the Board of Computer Usage Company.

Dr. Ronald Jacobowitz, Massachusetts Institute of Technology, has been appointed Assistant Professor at the University of Arizona.

Mr. Sidney Kaplan, Radio Corporation of America, Moorestown, New Jersey, has accepted a position as Manager of the Advanced Information Storage and Retrieval Systems, Radio Corporation of America, Bethesda, Maryland.

Professor E. S. Keeping, University of Alberta, has been appointed Visiting Professor at the University of Oklahoma.

Assistant Professor E. P. Kelly, Jr., S. F. Austin State College, has been appointed Professor and Assistant Dean of Arts and Sciences at the University of Southern Mississippi.

Professor J. R. Kinzer, Ohio State University, has been appointed Professor at the University of Arizona.

Professor M. S. Knebelman, Allegheny College, has been appointed Visiting Professor at Bucknell University.

Dr. P. J. Knopp, University of Texas, has been appointed Assistant Professor at the University of Missouri.

Dr. Leo Lapidus, Convair, San Diego, California, has been appointed Associate Professor at Lewis and Clark College.

Assistant Professor Shen Lin, Ohio University, has been appointed Research Associate in the Numerical Computation Laboratory and Lecturer at Ohio State University.

Mr. F. H. Lloyd, Convair, Fort Worth, Texas, has accepted a position as Development Engineer with Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota.

Dr. M. W. Lodato, Rutgers, The State University, has accepted a position as Scientist with Laboratory for Electronics, Monterey, California.

Assistant Professor Kenneth Loewen, Tabor College, has been appointed Assistant Professor at Westmont College.

Assistant Professor E. H. McKinney, University of Pittsburgh, has been appointed Head of the Department of Mathematics at Ball State Teachers College.

Assistant Professor D. G. Malm, State University of New York, Oyster Bay, has been appointed Assistant Professor at Michigan State University Oakland.

Dr. C. J. Maloney, Biological Warfare Laboratories, Fort Detrick, Maryland, has accepted a position in the Division of Biologics Standards of the National Institutes of Health, Bethesda, Maryland.

Mr. R. J. Marshall, University of Buffalo, has accepted a position as Senior Mathematician with the Department of Public Works of New York State.

Associate Professor Mabel D. Montgomery, State University College at Buffalo, has been promoted to Professor.

Mr. A. M. Nagy, Pomfret School, Pomfret, Connecticut, has been appointed Assistant Professor at the University of Hawaii.

Mr. V. D. Norum, Space General Corporation, Glendale, California, has accepted a position as Staff Engineer with Research Systems Technology, Los Angeles, California.

Mr. J. M. Ott, Ball State Teachers College, has been appointed Assistant Professor at Montclair State College.

Associate Professor R. N. Pendergrass, Radford College, has been appointed Professor at Southern Illinois University.

Mr. N. W. Rhodus, Ohio State University, has accepted a position as Research Engineer with General Dynamics-Astronautics, San Diego, California.

Mr. W. H. Richardson, Iowa State University, has been appointed Assistant Professor at the University of Wichita.

Mr. James Riddell, University of Alberta, has been appointed Assistant Professor at Eastern Montana College.

Mr. J. F. Rieger, Oklahoma City University, has accepted a position as Mathematician with the Computer Services Branch of the Federal Aviation Agency, Oklahoma City, Oklahoma.

Dr. D. W. Robinson, on sabbatical leave from Brigham Young University, has received a NSF Faculty Fellowship and has been appointed Senior Research Fellow at the California Institute of Technology.

Mr. R. P. Savage, Union Carbide Nuclear Company, Oak Ridge, Tennessee, has been appointed Assistant Professor at the Tennessee Polytechnic Institute.

Miss Donna J. Seaman, University of Oregon, has accepted a position as Senior Engineer with Sylvania Electronic Defense Laboratories, Mountain View, California.

Mr. Arnold Singer, Massachusetts Institute of Technology, has accepted a position as a Member of the Technical Staff of the Institute of Naval Studies, Cambridge, Massachusetts.

Mr. C. J. Steib, Southeastern Louisiana College, has been appointed Instructor at Louisiana State University in New Orleans.

Dr. Leon Steinberg, University of Pennsylvania, has accepted a position in the Applied Mathematics Department of Space Technology Laboratories, Los Angeles, California.

Professor C. F. Stephens, Morgan State College, has been appointed Professor at New York State University College, Geneseo.

Mr. J. F. Stokes, Auburn University, has been appointed Assistant Professor at Western Kentucky State College.

Professor T. Y. Thomas, Indiana University, has been appointed Visiting Professor at the University of California, San Diego.

Dr. M. D. Thompson, University of Wisconsin, has been appointed Lecturer at Indiana University.

Dr. Shih-Hsiung Tung, Pennsylvania State University, has been appointed Assistant Professor at Miami University.

Dr. Fred Van Vleck, Massachusetts Institute of Technology, has been appointed Assistant Professor at the University of Kansas.

Mr. S. H. Walker, Operations Research Office, Bethesda, Maryland, has been promoted to Chief of the Logistics Simulation Division, Research Analysis Corporation.

Mr. U. V. Ward, Howard University, has accepted a position as a Member of the Technical Staff of Hughes Aircraft Company, Culver City, California.

Mr. W. J. Wayne, Purdue University, has been appointed Instructor at Rose Polytechnic Institute.

Associate Professor J. G. Wendel, University of Michigan, has been appointed Visiting Professor at Aarhus Universitet, Aarhus, Denmark.

Dr. J. T. White, University of Texas, has been appointed Assistant Professor at the University of Kansas.

Assistant Professor J. C. Wilson, Florida Presbyterian College, has been appointed Associate Professor at Central College.

Mr. P. K. Wong, Carnegie Institute of Technology, has been appointed Assistant Professor at Lehigh University.

Professor B. C. Keeler, Webb Institute of Naval Architecture, died May 10, 1962. He was a member of the Association for 23 years.

Professor Emeritus R. B. McClenon, Grinnell College, died July 3, 1962. He was a charter member of the Association.

Professor H. W. Reddick, Syracuse University, died August 9, 1962. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW ASSOCIATE DIRECTOR OF CUPM

CUPM is most fortunate to have obtained the services of Professor Bernard Jacobson, who joined the CUPM central office staff as Associate Director of the Committee. In order to accept the full-time appointment beginning September 1, 1962, Dr. Jacobson has obtained a leave of absence from his regular duties as Associate Professor of Mathematics at Franklin and Marshall College, Lancaster, Pennsylvania. It is a happy circumstance that the college officials thought enough of the work of CUPM to endure the hardships which will obviously be imposed by his absence.

Dr. Jacobson, 34, is a native of Cleveland, Ohio. He received his bachelor's degree from Western Reserve University in 1951 and his Ph.D. in mathematics from Michigan State University in 1956. His mathematical interests lie mainly in number theory. He has been active the past several years in NSF summer institute programs at Franklin and Marshall. He attended the meetings at Vancouver in August.

THE PROGRAM OF VISITING LECTURERS OF THE M.A.A.

ROBERT A. ROSENBAUM, Wesleyan University

For nine years the Association, with the support of the National Science Foundation, has sponsored a successful program of visiting lecturers to colleges. Because the demand for lecturers always outstrips the supply, and because some distinguished mathematicians have too many commitments on their own campuses to permit them to take leave for a lecture tour, a supplementary program was initiated a few years ago: a group of mathematicians agreed to give lectures at nearby institutions, one day at a time. At first, this new program was *under*-subscribed; but during 1961–62, thirty-seven mathematicians made one-day visits to one hundred campuses, and now both phases of the program of visiting lecturers are well established.

The Association is fortunate in having the following men as touring lecturers during the academic year 1962–63:

M. K. Fort, Jr., University of Georgia
Edwin Hewitt, University of Washington
Leo Moser, University of Alberta
Jerzy Neyman, University of California

Messrs. Hewitt and Moser are serving second "tours-of-duty" after notable successes several years ago.

Moreover, the Association has happily persuaded 70 other distinguished mathematicians, from 32 states, to lecture occasionally at nearby institutions. Thus it may prove possible to honor most requests for lecturers this year.

Complete information about both phases of the program is contained in a brochure mailed to department chairmen about Sept. 15. Requests for copies of the brochure should be addressed to Professor Harry M. Gehman, Executive Director, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

NEW MAA PUBLICATIONS

Volume 2: *Studies in Modern Algebra* of the series MAA STUDIES IN MATHEMATICS is now in press. Publication is expected in March 1963. The editor of the volume is A. A. Albert. It contains an introduction by Dean Albert and articles by Saunders MacLane, R. H. Buck, C. W. Curtis, Erwin Kleinfeld, and L. J. Paige.

Carus Monograph 14: *Combinatorial Mathematics* by Herbert J. Ryser will be published early in 1963. Although the subject is an old one and many early problems were studied for their recreational appeal, today it is recognized as a lively branch of modern algebra with important applications.

Each member of the Association may purchase one copy of each of these books (as well as of the other STUDIES and Monographs) at $2 per copy. Orders with remittance should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies and copies for nonmembers may be purchased at $4 per copy from Prentice-Hall, Inc., Englewood Cliffs, New Jersey, in the case of the STUDIES; and from Open Court Publishing Company, LaSalle, Illinois, for Monographs 1–8; and from John Wiley and Sons, 440 Park Avenue, South, New York 16, New York, for Monographs 9–14.

ACKNOWLEDGMENT

The Editorial Board acknowledges with thanks the services of the following mathematicians, not members of the Board, who have kindly assisted by evaluating papers submitted for publication in the MONTHLY.

R. P. Agnew, L. V. Ahlfors, George Albert, R. D. Anderson, N. C. Ankeny, T. M. Apostol, R. J. Arms, D. G. Aronson, M. G. Arsove, Florencio Asenjo, Arthur Babakhanian, William Bade, Fred Bagemihl, Joshua Barlaz, John Barrett, Robert Bartle, P. T. Bateman, Edwin Beckenbach, E. M. Beesley, Richard Bellman, L. Bers, Rev. S.J. Bezuszka, R. H. Bing, Garrett Birkhoff, Amos Black, David Blackwell J. H. Blau, R. P. Boas, Truman Botts, J. W. Bradshaw, Glen Bredon, R. W. Brink, Andrew Bruck-

ner, R. C. Buck, K. A. Bush, S. S. Cairns, Alberto Calderon, E. A. Cameron, Leonard Carlitz, Elizabeth Carlson, Robert Carroll, C. C. Chang, Bruce L. Chilton, S. Chowla, R. V. Churchill, G. F. Clanton, A. H. Clifford, E. A. Coddington, Eckford Cohen, W. Wistar Comfort, Harvey Cohn, N. B. Conkwright, N. A. Court, H. S. M. Coxeter, A. E. Currier, M. L. Curtis, D. A. Darling, M. M. Day, W. E. Deskins, Allen Devinatz, R. DeVogelaere, J. B. Diaz, S. P. Diliberto, R. P. Dilworth, Jim Douglas, Patrick Doyle, M. P. Drazin, L. Dubins, Roy Dubisch, James Dugundji, H. G. Eggleston, A. Erdelyi, Leonard Evens, Ky Fan, H. Federer, Walter Feit, Jacob Feldman, N. J. Fine, Daniel Finkbeiner, H. Flanders, B. A. Fleishman, Walter Fleming, Paul Fong, Lester Ford, George E. Forsythe, Tomlinson Fort, J. S. Frame, Delbert Fulkerson, Watson Fulks, I. S. Gal, David Gale, Sam E. Ganis, Martin Gardner, Leslie Gates, B. R. Gelbaum, Edgar Gilbert, W. Givens, A. M. Gleason, Michael Goldberg, Michael Golomb, Solomon Golomb, R. A. Good, E. J. Goodman, W. H. Gottschalk, Ross E. Graves, Donald Greenspan, R. E. Greenwood, T. N. E. Greville, A. Glen Haddock, J. E. Hafstrom, Theodore Hailperin, Dick Wick Hall, Dilla Hall, P. R. Halmos, J. W. Hamblen, R. W. Hamming, Frank Harary, W. L. Hart, Philip Hartman, Charles Hatfield, Jr., Nicholas Heerema, M. H. Heins, M. J. Hellman, Henry Helson, Leon Henkin, Peter Henrici, J. G. Herriot, I. N. Herstein, S. Herstein, Fritz Herzog, M. R. Hestenes, E. Hewitt, E. L. Hill, W. M. Hirsch, I. I. Hirschman, Gerhard Hochschild, F. E. Hohn, W. W. Hooker, Al Horn, A. S. Householder, G. E. Hudson, N. Jacobson, C. M. Jensen, G. P. Johnson, R. E. Johnson, B. W. Jones, Bjarni Jonsson, Mark Kac, Wilfred Kaplan, I. Kaplansky, Samuel Karlin, Herbert Keller, J. L. Kelley, J. B. Kelly, M. L. Kelly, P. J. Kelly, John Kemeny, Morton Kenner, L. S. Kennison, P. W. Ketchum, F. L. Kiokemeister, A. R. Kirby, V. L. Klee, Erwin Kleinfeld, Fulton Koehler, E. R. Kolchin, Antoni Kosinski, Max Kramer, Joseph Landin, Carl Langenhop, Rudolph Langer, Milton Lees, Sherman Lehman, D. H. Lehmer, Emma Lehmer, J. Lehner, Marguerite Lehr, C. W. Leininger, W. H. Leser, M. E. Levenson, W. J. Leveque, Norman Levinson, D. J. Lewis, Hans Lewy, B. W. Lindgren, A. O. Lindstrum, Jr., Michel Loeve, Lynn Loomis, Paul E. Long, E. R. Lorch, George Lorentz, W. S. Loud, Wilhelmus Luxemburg, S. MacLane, H. M. MacNeille, B. M. McCandless, N. H. McCoy, E. J. McShane, Ralph McWilliams, W. Magnus, M. E. Mahowald, H. B. Mann, Margaret O. Marchand, Marvin Marcus, Morris Marden, Y. Matsuoka, K. O. May, E. Mendelson, B. E. Meserve, A. C. Mewborn, Don Miller, K. S. Miller, Louis Milne-Thomson, E. E. Moise, Calvin Moore, Marian Moore, A. P. Morse, Leo Moser, T. S. Motzkin, M. E. Munroe, W. D. Munro, F. J. Murray, Zeev Nehari, Morris Newman, Kaj L. Nielson, Louis Nirenberg, Ivan Niven, Steven Orey, R. C. Osborn, Gordon Pall, F. D. Parker, Emanuel Parzen, L. E. Payne, B. J. Pettis, R. S. Pierce, Edmund Pinney, George Piranian, Richard Pollack, H. Pollard, G. Polya, Walter Prenowitz, W. V. Quine, M. O. Rabin, Hans Rademacher, Earl Rainville, M. O. Reade, I. Reiner, Eric Reissner, B. E. Rhoades, J. R. Rice, D. E. Richmond, R. F. Rinehart, John Riordan, R. M. Robinson, Alex Rosenberg, M. Rosenlicht, A. E. Ross, H. Samelson, Frances M. Sand, R. D. Schafer, M. Schechter, Henry Scheffe, Alfred Schild, Abraham Schwartz, Manuel Schwartz, H. W. E. Schwerdtfeger, Dana Scott, A. Seidenberg, Bernard Shaffer, Merrill Shanks, H. S. Shapiro, V. L. Shapiro, Allen Shields, M. C. Sholander, Edward Silverman, W. H. Simons, Michael Skalsky, Andrew Sobczyk, Robert Sorgenfrey, Murray Spiegel, Abraham Spitzbart, F. L. Spitzer, J. G. Stampfli, T. H. Starks, Rothwell Stephens, B. M. Stewart, E. G. Straus, D. J. Struik, Patrick Suppes, G. Szegö, Olga Taussky-Todd, A. E. Taylor, Henry Thielman, G. P. Thomas, P. D. Thomas, T. Y. Thomas, Dmitri Thoro, R. M. Thrall, W. J. Thron, Marion F. Tinsley, John Todd, L. V. Toralballa, J. F. Traub, Howard Tucker, Atwell Turquette, R. S. Underwood, H. S. Vandiver, H. E. Vaughan, R. L. Vaught, T. L. Wade, R. J. Walker, H. S. Wall, A. D. Wallace, Morgan Ward, Milo W. Weaver, K. W. Wegner, Hans Weinberger, Arnold Wendt, J. T. White, A. L.

Whiteman, J. S. Wholey, D. V. Widder, Albert Wilansky, H. S. Wilf, J. E. Wilson, R. J. Wisner, Jacob Wolfowitz, E. M. Wright, Gail Young, H. Zassenhaus, M. Zedek, David Zeitlin, H. S. Zuckerman, A. Zygmund.

CALENDAR OF FUTURE MEETINGS

Forty-sixth Annual Meeting, University of California, Berkeley, January 26–28, 1963.

Forty-fourth Summer Meeting, University of Colorado, Boulder, August 26–28, 1963.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Executive Director.

ALLEGHENY MOUNTAIN, Pennsylvania State University, University Park, May 4, 1963.

ILLINOIS, Northern Illinois University, De Kalb, May 10–11, 1963.

INDIANA

IOWA, Iowa State University, Ames, April 19–20, 1963.

KANSAS. Sterling College, Sterling, April 27, 1963.

KENTUCKY, University of Louisville, April 27, 1963.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1963.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University, East Lansing, March 23, 1963.

MINNESOTA

MISSOURI, Southwest Missouri State College, Springfield, April 1963.

NEBRASKA, University of Nebraska, Lincoln, May 3–4, 1963.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, January 27, 1963.

OHIO, Ohio State University, Columbus, May 4, 1963.

OKLAHOMA

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA

ROCKY MOUNTAIN, Brigham Young University, Provo, Utah, May 3–4, 1963.

SOUTHEASTERN, University of Chattanooga, Chattanooga, Tennessee, March 29–30, 1963.

SOUTHERN CALIFORNIA, University of California, Riverside, March 9, 1963.

SOUTHWESTERN, Arizona State College, Flagstaff, April 1963.

TEXAS, North Texas State University, Denton, April 19–20, 1963.

UPPER NEW YORK STATE, University of Buffalo, April 27, 1963.

WISCONSIN, Carroll College, Waukesha, May 4, 1963.